fractal and fractional

MDPI

Article

A Novel Implementation of Dhage's Fixed Point Theorem to Nonlinear Sequential Hybrid Fractional Differential Equation

Muath Awadalla [1,*], Mohamed Hannabou [2,*], Kinda Abuasbeh [1] and Khalid Hilal [2]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Hafuf 31982, Al Ahsa, Saudi Arabia
[2] Departement of Mathematics, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, Beni Mellal 23000, Morocco
* Correspondence: mawadalla@kfu.edu.sa (M.A.); hnnabou@gmail.com (M.H.)

Abstract: In this work, the existence and uniqueness of solutions to a sequential fractional (Hybrid) differential equation with hybrid boundary conditions were investigated by the generalization of Dhage's fixed point theorem and Banach contraction mapping, respectively. In addition, the U-H technique is employed to verify the stability of this solution. This study ends with two examples illustrating the theoretical findings.

Keywords: existence; fixed point theorems; stability; fractional differential equations

MSC: 26A33; 34B15; 34B18

check for updates

Citation: Awadalla, M.; Hannabou, M.; Abuasbeh, K.; Hilal, K. A Novel Implementation of Dhage's Fixed Point Theorem to Nonlinear Sequential Hybrid Fractional Differential Equation. *Fractal Fract.* **2023**, *7*, 144. https://doi.org/10.3390/fractalfract7020144

Academic Editors: Eva Kaslik and Oana Brandibur

Received: 31 December 2022
Revised: 27 January 2023
Accepted: 30 January 2023
Published: 2 February 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The theory of fractional calculus is an interesting field to explore in recent years. Additionally, this theory has many applications to describe many events in the real world and deal with a group of phenomena in several fields such as blood flow phenomena, mechanics, biophysics, automatic, aerodynamics, some branches of medicine, and electronics. For instance, the authors of [1] discussed the applicability of fractional differential equations in electric circuits, and in 2019, Saqib, M. et al. applied the fractional differential equation to heat transfer in a hybrid nanofluid, see [2]. For more details, one can refer to [3–16].

In addition to the great importance of studying the existence of solutions to fractional differential equations using the many theories of the fixed point, several studies have been conducted over the years to investigate how stability concepts such as the Mittag–Leffler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified previously unknown types of stability known as Ulam-stability [17]. This example is not exclusive, as many similar works can be found in [18–27].

To name a few studies on hybrid fractional equations, recently, the authors in [28] verified the existence and uniqueness of a solution for the following IVP:

$$\begin{cases} {}_a\mathcal{D}^{\hat{v},\varrho,v}\left(\frac{u(\chi_1)}{\Psi(\chi_1,x(\chi_1))}\right) = \Phi(\chi_1,u(\chi_1)), \chi_1 \in J := [a,b], \\ {}_aI^{1-\hat{v},\varrho,v}\left(\frac{u(\chi_1)}{\Psi(\chi_1,x(\chi_1))}\right)_{\chi_1=a} = \lambda \in \mathbb{R}, \end{cases}$$

where $0 < \hat{v} \leq 1, \varrho \in (0,1]$, ${}_a\mathcal{D}^{\hat{v},\varrho,v}$ is the proportional fractional derivative of order $\hat{v}$, with respect to a certain continuously differentiable and increasing function v with $v'(\theta) > 0$ for all $\chi_1 \in J$, ${}_aI^{1-\hat{v},\varrho,v}$ is the proportional fractional integral of order of $(1-\hat{v})$ with respect

to a continuously differentiable and increasing function $v, \Psi : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $\Phi : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus$ are continuous functions.

In [29], the stability, existence, and uniqueness were found for the following hybrid FDE:

$$\begin{aligned} {}^{CH}\mathcal{D}^{p}\left(\frac{u(\chi_1)}{Z_1(\chi_1,u(\chi_1),v(\chi_1))}\right) &= \hat{\Psi}_1(\chi_1,u(\chi_1),v(\chi_1)), \chi_1 \in J := [1,e], p \in (1,2] \\ {}^{CH}\mathcal{D}^{q}\left(\frac{u(\chi_1)}{Z_2(\chi_1,u(\chi_1),v(\chi_1))}\right) &= \hat{\Psi}_2(\chi_1,u(\chi_1),v(\chi_1)), q \in (1,2] \end{aligned}$$

Subject to the following boundary conditions:

$$\begin{aligned} \left(\frac{u(\chi_1)}{Z_1(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=1} &= 0, \\ {}^{CH}D\left(\frac{u(\chi_1)}{Z_1(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=e} &= \Theta_2{}^{CH}D\left(\frac{v(\chi_1)}{Z_1(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=\vartheta_1}, \\ \left(\frac{v(\chi_1)}{Z_2(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=1} &= 0, \\ {}^{CH}D\left(\frac{v(\chi_1)}{Z_2(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=e} &= \Theta_1{}^{CH}D\left(\frac{v(\chi_1)}{Z_1(\chi_1,u(\chi_1),v(\chi_1))}\right)_{\chi_1=\vartheta_2}, \end{aligned}$$

where ${}^{CH}\mathcal{D}^{\gamma}, \gamma \in \{p,q\}$ is the Caputo–Hadamard fractional derivative of order $1 < \gamma \leq 2$, $\Theta_1, \Theta_2 \in [0,1)$, and $\vartheta_1, \vartheta_2 \in (1,e)$.

Recently [30], existence results were obtained via the well-known fixed point theories for the following system:

$$\begin{cases} {}^{c}\mathcal{D}^{\omega}\left(\frac{\mathcal{D}^{\hat{\Psi}}v(\chi_1)-\sum_{i=1}^{m}I^{\beta_i}h_i(\chi_1,x(\chi_1))}{f_1(\chi_1,x(\chi_1))}\right) = g_1(\chi_1,x(\chi_1),I^{\gamma}x(\chi_1)) \quad a.e. \quad \chi_1 \in J = [0,1], \\ {}^{c}\mathcal{D}^{\omega}\left(\frac{\mathcal{D}^{\hat{\Psi}}x(\chi_1)-\sum_{i=1}^{m}I^{\beta_i}h_i(\chi_1,v(\chi_1))}{f_2(\chi_1,v(\chi_1))}\right) = g_2(\chi_1,x(\chi_1),I^{\gamma}x(\chi_1)) \quad a.e. \quad \chi_1 \in J = [0,1], \\ {}^{c}\mathcal{D}^{\hat{\Psi}}v(0) = 0 = {}^{c}\mathcal{D}^{\hat{\Psi}}x(0), v'(0) = 0 = x'(\chi_1), v(1) = \rho_1(x(\vartheta)), x(1) = \rho_2(v(\vartheta)). \end{cases}$$

and the parameters satisfy the condition $0 < \omega \leq, 1 < \hat{\Psi} \leq 2, 0 < \vartheta < 1$, for $j = 1,2$, $f_j : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}, g_j : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and for $i = 1, ;m$ the function $h_j : J \times \mathbb{R} \longrightarrow \mathbb{R}$ fulfill the Caratheodory property, the special boundary function $\rho_j : \mathbb{R} \longrightarrow \mathbb{R}$ my the nonlinear, and the notation v' is used for the first order ordinary derivative of v and ${}^{c}D$ stands for fractional derivative in the sense of Caputo. In the literature, there are other definitions of fractional derivatives. The present paper is a continuation of the work [9], in order to study the existence of a solution for a nonlinear sequential hybrid fractional differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\xi}({}^{c}\mathcal{D}^{\hat{\iota}}+\gamma)\left(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\right) = \hat{\Psi}(\chi_1,\hat{v}(\chi_1)) \quad a.e. \quad t \in J = [0,1] \\ \\ \frac{\hat{v}(0)}{f(0,\hat{v}(0))} + \vartheta\frac{\hat{v}(1)}{f(1,\hat{v}(1))} = \tau_1\int_0^1 \hat{\Psi}_1(\hat{\varkappa},\hat{v}(\varkappa))d\hat{\varkappa}, \\ {}^{c}\mathcal{D}^{\hat{\iota}}\left(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\right) + \vartheta\,{}^{c}\mathcal{D}^{\hat{\iota}}\left(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\right) = \tau_2\int_0^1 \hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}, \\ {}^{c}\mathcal{D}^{2\hat{\iota}}\left(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\right) + \vartheta\,{}^{c}\mathcal{D}^{2\hat{\iota}}\left(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\right) = \tau_3\int_0^1 \hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}, \end{cases} \tag{1}$$

where $0 < \hat{\imath} < 1$, $1 < \xi \leq 2$, $\gamma, \vartheta, \tau_i, \in \mathbb{R}^*$ for $i = 1, 2, 3$, with $\vartheta \neq -1$. ${}^c\mathcal{D}^{\xi}$, ${}^c\mathcal{D}^{\hat{\imath}}$ are the Caputo's fractional derivatives, and $f : [0; 1] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $\hat{\Psi} \in \mathcal{E}([0, 1] \times \mathbb{R}, \mathbb{R})$ and $\hat{\Psi}_1, \hat{\Psi}_2, \hat{\Psi}_3 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are the given continuous functions.

By a solution of the problem (1), we mean a function $\hat{v} \in C(J, \mathbb{R})$ such that

(i) the function $\chi_1 \mapsto \frac{\hat{v}}{f(\chi_1, \hat{v})}$ is continuous for each $\hat{v} \in \mathbb{R}$, and

(ii) x satisfies the equations in (1).

Despite popular belief, researchers have paid less attention to studies of hybrid fractional differential equations associated with hybrid boundary conditions. According to the authors' observations, there is no analytical literature on the existence of mixed hybrid fractional differential equations of a sequential type involving hybrid BCs. Adding to this, Dhage's fixed point theorem is employed in investigating the existence results for the proposed BVP, which gives this work novelty and originality.

The rest of the article is as follows: Section 2 presents the basic definitions, lemmas, and theorems that underpin our main conclusions. In Section 3, we find solutions to the given fractional differential Equation (1) using Dahg's fixed point theorem. Section 4 looks at the Ulam–Hyers stability of the provided fractional differential Equation (1). In Section 5, an example is provided to further clarify the study's findings. In Section 6, a conclusion and a future work are introduced.

2. Preliminaries

Let us introduce some preliminary results that will be useful for proving our results in the subsequent sections.

Let the Banach space denoted by $\mathcal{X} = \mathcal{E}(J, \mathbb{R})$ with the norm

$$\|\hat{v}\| = \sup\{|\hat{v}(\chi_1)|, t \in J\}.$$

Definition 1 ([31]). *The R-L fractional integral of θ with order $\hat{\imath}$ is given by*

$$I_a^{\hat{\imath}} g(\chi_1) = \int_a^{\chi_1} \frac{(\chi_1 - \hat{\varkappa})^{\hat{\imath}-1}}{\Gamma(\hat{\imath})} g(\hat{\varkappa}) d\hat{\varkappa}.$$

Definition 2 ([31]). *The Caputo fractional integral of order $\hat{\imath}$ for a function θ is given by*

$$({}^cD_{a^+}^{\hat{\imath}} g)(\chi_1) = \frac{1}{\Gamma(n - \hat{\imath})} \int_a^{\chi_1} \frac{(\chi_1 - \hat{\varkappa})^{n-\hat{\imath}-1}}{\Gamma(\hat{\imath})} g^{(n)}(\hat{\varkappa}) d\hat{\varkappa}$$

where $n = [\hat{\imath}] + 1$ and $[\hat{\imath}]$ denotes the integer part of $\hat{\imath}$.

Next, we state Dhage's fixed point theorem followed by our main auxiliary lemma.

Theorem 1 ([32]). *For any $\mathcal{S}$ nonempty, a closed convex and bounded subset of a Banach algebra $\mathcal{X}$ and for any operators $\mathcal{P}, \mathcal{E} : \mathcal{X} \longrightarrow \mathcal{X}$ and $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{X}$ such that:*

(i) $\mathcal{P}$ and $\mathcal{E}$ are Lipschitzian with Lipschitz constants $\hat{v}$ and ρ, respectively,

(ii) $\mathcal{F}$ is compact and continuous,

(iii) $\hat{v} = \mathcal{P}\hat{v}\mathcal{F}\rho + \mathcal{E}\hat{v} \Longrightarrow \hat{v} \in \mathcal{S}$ for all $\rho \in, \mathcal{S}$

(iv) $\hat{v}\mathcal{W} + \rho < 1$, where $\mathcal{W} = \|\mathcal{F}(\mathcal{S})\|$.

Then, the equation $\hat{v} = \mathcal{D}\hat{v}\mathcal{F}\rho + \mathcal{E}\hat{v}$ has a solution.

Lemma 1 ([31]). *Let $n \in \mathbb{N}$ and $n - 1 < \hat{\imath} < n$. If f is a continuous function, then we have*

$$I^{\hat{\imath}}\ {}^c\mathcal{D}^{\hat{\imath}} f(\chi_1) = f(\chi_1) + \omega_0 + \omega_1 t + \omega_2 t^2 + ... + \omega_{n-1} t^{n-1}.$$

For the proof of our auxiliary Lemma, we state the following hypothesis:
(H_0) The function $x \longmapsto \frac{\hat{v}}{f(\chi_1,\hat{v})}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

Lemma 2. *Assume that hypothesis (H_0) holds. Then, for any $\rho \in L^1(J;\mathbb{R})$. The function $\hat{v} \in C(J,\mathbb{R})$ is a solution of the problem*

$$\begin{cases} {}^c\mathcal{D}^{\xi}({}^c\mathcal{D}^{\hat{\imath}}+\gamma)\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big)=\rho(\chi_1) \quad a.e. \quad t\in J=[0,1] \\ \\ \frac{\hat{v}(0)}{f(0,\hat{v}(0))}+\vartheta\frac{\hat{v}(1)}{f(1,\hat{v}(1))}=\tau_1\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v})(\hat{\varkappa})d\hat{\varkappa}, \\ {}^c\mathcal{D}^{\hat{\imath}}\Big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\Big)+\vartheta\,{}^c\mathcal{D}^{\hat{\imath}}\Big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\Big)=\tau_2\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}, \\ {}^c\mathcal{D}^{2\hat{\imath}}\Big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\Big)+\vartheta\,{}^c\mathcal{D}^{2\hat{\imath}}\Big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\Big)=\tau_3\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}, \end{cases} \tag{2}$$

is given by

$$\begin{aligned} \hat{v}(\chi_1) &= f(\chi_1,\hat{v}(\chi_1))\Big[\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa} \\ &+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa} \\ &+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}7 \\ &+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}\rho(\hat{\varkappa})d\hat{\varkappa} \\ &+\frac{\Theta_5(\chi_1)}{\Gamma(\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}+\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa} \\ &-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\Big]-\frac{\gamma}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}, \end{aligned}$$

where

$$\begin{aligned} \Theta_1(\chi_1) &= \frac{\chi_1{}^{\hat{\imath}}\tau_2\big(1-\gamma\Gamma(2-\hat{\imath})\big)}{\Gamma(\hat{\imath}+1)(1+\vartheta)}+\frac{\chi_1{}^{\hat{\imath}+1}\gamma\tau_2\Gamma(2-\hat{\imath})}{\Gamma(2+\hat{\imath})\vartheta} \\ &+\Big(\frac{\vartheta}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}-\frac{1}{(1+\vartheta)\Gamma(\hat{\imath}+2)}\Big)\Gamma(2-\hat{\imath})\gamma\tau_2-\frac{\vartheta\tau_2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}, \\ \Theta_2(\chi_1) &= \frac{\chi_1{}^{\hat{\imath}}\gamma\tau_1}{\Gamma(\hat{\imath}+1)(1+\vartheta)}-\frac{\tau_1}{1+\vartheta}+\frac{\vartheta\gamma\tau_1}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}, \\ \Theta_3(\chi_1) &= -\frac{\chi_1{}^{\hat{\imath}}\Gamma(2-\hat{\imath})\tau_3}{\Gamma(\hat{\imath}+1)(1+\vartheta)}+\frac{\chi_1{}^{\hat{\imath}+1}\Gamma(2-\hat{\imath})\tau_3}{\Gamma(\hat{\imath}+2)\vartheta}+\frac{\vartheta\Gamma(2-\hat{\imath})\tau_3}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}-\frac{\Gamma(2-\hat{\imath})\tau_3}{(1+\vartheta)\Gamma(\hat{\imath}+2)}, \\ \Theta_4(\chi_1) &= \frac{\chi_1{}^{\hat{\imath}}\Gamma(2-\hat{\imath})\vartheta}{\Gamma(\hat{\imath}+1)(1+\vartheta)}-\frac{\chi_1{}^{\hat{\imath}+1}\Gamma(2-\hat{\imath})}{\Gamma(\hat{\imath}+2)}+\Gamma(2-\hat{\imath})\Big(\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+2)}-\frac{\vartheta^2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}\Big), \\ \Theta_5(\chi_1) &= -\frac{\chi_1{}^{\hat{\imath}}\vartheta}{\Gamma(\hat{\imath}+1)(1+\vartheta)}-\frac{\vartheta^2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}, \end{aligned}$$

Proof. By Lemma 1, we obtain
$({}^c\mathcal{D}^{\hat{\imath}}+\gamma)\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big)=I^{\xi}\rho(\chi_1)+\omega_0+\omega_1 t,$
${}^c\mathcal{D}^{\hat{\imath}}\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big)=I^{\xi}\rho(\chi_1)+\omega_0+\omega_1 t-\gamma\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1)}\Big),$
$\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}=I^{\hat{\imath}+\xi}\rho(\chi_1)+I^{\hat{\imath}}\omega_0+I^{\hat{\imath}}\omega_1 t-I^{\hat{\imath}}\lambda\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big)+\omega_2,$
where $\omega_0,\omega_1,\omega_2 \in \mathbb{R}$. By using to the condition ${}^c\mathcal{D}^{2\hat{\imath}}\Big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\Big)+\vartheta\,{}^c\mathcal{D}^{2\hat{\imath}}\Big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\Big)=\tau_3\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}$, we find that:

$$\omega_1=\Gamma(2-\hat{\imath})\Big(\frac{\tau_3}{\vartheta}\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\frac{\gamma\tau_2}{\vartheta}\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}-\frac{1}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}_1}\rho(\hat{\varkappa})d\hat{\varkappa}\Big).$$

Using the ${}^c\mathcal{D}^{\hat{\imath}}\Big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\Big)+\vartheta\;{}^c\mathcal{D}^{\hat{\imath}}\Big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\Big)=\tau_2\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}$ and $\Big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\Big)+\vartheta\Big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\Big)=\tau_1\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}$, we have

$$\begin{aligned}
\omega_0 &= \frac{-\Gamma(2-\hat{\imath})\tau_3}{1+\vartheta}\int_0^1 \hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa} + \frac{(1-\gamma\Gamma(2-\hat{\imath}))\tau_2}{1+\vartheta}\int_0^1 \hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&\quad + \tfrac{\gamma\tau_1}{1+\vartheta}\int_0^1 \hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&\quad + \frac{\Gamma(2-\hat{\imath})\vartheta}{(1+\vartheta)\Gamma(\xi-\hat{\imath})}\int_0^1 (1-\hat{\varkappa})^{\xi-\hat{\imath}-1}\rho(\hat{\varkappa})d\hat{\varkappa} - \frac{\vartheta}{(1+\vartheta)\Gamma(\xi)}\int_0^1 (1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa},\\
\omega_2 &= \frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}\int_0^1 (1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa} + \Big(\frac{\tau_1}{1+\vartheta} - \frac{\vartheta\gamma\tau_1}{\Gamma(\hat{\imath}+1)(1+\vartheta)^2}\Big)\int_0^1 \hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&\quad - \frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1 (1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa} + \frac{\vartheta^2}{(1+\vartheta)^2\Gamma(\xi)\Gamma(\hat{\imath}+1)}\int_0^1 (1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\
&\quad + \frac{\Gamma(2-\hat{\imath})}{\Gamma(\xi-\hat{\imath})}\left(\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+2)} - \frac{\vartheta^2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}\right)\int_0^1 (1-\hat{\varkappa})^{\xi-\hat{\imath}-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\
&\quad + \Big[\Big(\frac{\vartheta^2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)} - \frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+2)}\Big)\frac{\Gamma(2-\hat{\imath})\gamma\tau_2}{\vartheta} - \frac{\vartheta\sigma_2}{(1+\vartheta)^2\Gamma(\hat{\imath}+1)}\Big]\\
&\quad \times \int_0^1 \hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}.
\end{aligned}$$

Substituting the value of ω_0, ω_1, and ω_2, we obtain

$$\begin{aligned}
\hat{v}(\chi_1) &= f(\chi_1,\hat{v}(\chi_1))\Big[\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa} + \Theta_1(\chi_1)\int_0^1 \hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&\quad + \Theta_2(\chi_1)\int_0^1 \hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa} + \Theta_3(\chi_1)\int_0^1 \hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa} + \frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\imath})}\int_0^1 (1-\hat{\varkappa})^{\xi-\hat{\imath}-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\
&\quad + \frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1 (1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa} + \frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}\int_0^1 (1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\
&\quad - \frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1 (1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\Big] - \frac{\gamma}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}.
\end{aligned}$$

□

3. Existence Result

Based on Dhage's fixed point theorem, we will present in this section the verification of the existence of a solution to Equation (1).

To simplify the calculations, we set $\Theta_i = \max_{\chi_1\in[0,1]}|\Theta_i(\chi_1)|$ for $i = 1,2,\ldots,5$.

The following assumptions are the necessary conditions to derive our main results:

(H_1)- $f_:[0,1]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ and $\hat{\Psi}_1,\hat{\Psi}_2,\hat{\Psi}_3,: [0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

(H_2)- There exists a constant q_0, q_1 such that

$|f(\chi_1,\hat{v}_1) - f(\chi_1,\hat{v}_2)| \le q_0|\hat{v}_1-\hat{v}_2|$,

$|w(\chi_1,\hat{v}_1) - w(\chi_1,\hat{v}_2)| \le q_1|\hat{v}_1-\hat{v}_2|$, for all $\chi_1\in J$ and $\hat{v}_1,\hat{v}_2\in\mathbb{R}$.

(H_3)- There exist positive constants q_2, q_3, q_4, such that:

$|\hat{\Psi}_1(\chi_1,\hat{v}_1) - \hat{\Psi}_1(\chi_1,\hat{v}_2)| \le q_2|\hat{v}_1-\hat{v}_2|$,

$|\hat{\Psi}_2(\chi_1,\hat{v}_1) - \hat{\Psi}_2(\chi_1,\hat{v}_2)| \le q_3|\hat{v}_1-\hat{v}_2|$,

$|\hat{\Psi}_3(\chi_1,\hat{v}_1) - \hat{\Psi}_3(\chi_1,\hat{v}_2)| \le q_4|\hat{v}_1-\hat{v}_2|$.

(H_4)- $|f(\chi_1,\hat{v})| \le \varphi(\chi_1)$, $|w(\chi_1,\hat{v})| \le m(\chi_1)$, $|\hat{\Psi}_2(\chi_1,\hat{v})| \le \rho(\chi_1)$, $|\hat{\Psi}_3(\chi_1,\hat{v})| \le \psi(\chi_1)$, $|\hat{\Psi}_1(\chi_1,\hat{v})| \le \phi(\chi_1)$, $\forall(\chi_1,\hat{v})\in[0,1]\times\mathbb{R}$ with $m,\varphi,\rho,\phi,\psi\in\mathcal{E}([0,1];\mathbb{R}^+)$.

Theorem 2. *Assume that hypothesis $(H_0)-(H_4)$ holds. Further, if*

$$\begin{aligned}
&q_0\Big(\Big[q_1\Big(\frac{1}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \frac{\Theta_5}{\Gamma(\xi+1)}\\
&+\Big|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}\Big|\Big) + \Big|\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath}+1)}\Big| + \Theta_1 q_3 + \Theta_2 q_2\\
&+\Theta_3 q_4\Big] + \frac{\Upsilon_0}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \Theta_1\Upsilon_2 + A_2\Upsilon_1\\
&+\Theta_3\Upsilon_3 + \frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)} + \Big|\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}\Big|\Big) + \frac{\gamma}{\Gamma(\hat{\imath}+1)} < 1.
\end{aligned}$$

then the problem (1) has a solution defined on $[0,1]$.

Proof. Let the subset $\mathcal{S}$ of $\mathcal{X}$ be defined as

$$\mathcal{S} = \{\hat{v}\in\mathcal{X}/\|\hat{v}\| \le r\}$$

with

$$r \geq \frac{\|\varphi\|\Big(\frac{\|m\|}{\Gamma(\hat{\iota}+\xi+1)}\Big)+\Theta_1\|\rho\|+\Theta_2\|\phi\|+\Theta_3\|\psi\|+\frac{\Theta_4\|m\|}{\Gamma(\xi-\hat{\iota}+1)}+\frac{\Theta_5\|m\|}{\Gamma(\xi+1)}+\frac{|\vartheta|\|m\|}{(|1+\vartheta|)\Gamma(\hat{\iota}+\xi+1)}\Big)}{1-\Big(\frac{\|\varphi\||\vartheta\lambda|}{(|1+\vartheta|)\Gamma(\hat{\iota}+1)}+\frac{|\gamma|}{\Gamma(\hat{\iota}+1)}\Big)}.$$

Now,

$$\begin{aligned}\hat{v}(\chi_1) &= f(\chi_1,\hat{v}(\chi_1))\Big[\frac{1}{\Gamma(\hat{\iota}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\iota}-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\frac{\vartheta\gamma}{(1+\mu)\Gamma(\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\ &-\frac{\mu}{(1+\vartheta)\Gamma(\hat{\iota}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\Big]-\frac{\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}.\end{aligned}$$

Define three operators $\mathcal{P}:\mathcal{X}\longrightarrow\mathcal{X}$, $\mathcal{F}:\mathcal{S}\longrightarrow\mathcal{X}$ and $\mathcal{E}:\mathcal{S}\longrightarrow\mathcal{X}$ by:

$$\mathcal{P}\hat{v}(\chi_1)=f(\chi_1,\hat{v}(\chi_1)),\quad \chi_1\in J \tag{3}$$

$$\begin{aligned}\mathcal{F}\hat{v}(\chi_1) &= \frac{1}{\Gamma(\hat{\iota}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}-\frac{\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\ &+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\iota}-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\ &-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa},\end{aligned}$$

and

$$\mathcal{E}\hat{v}(\chi_1)=\frac{-\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}.$$

Accordingly, Equation (3) can be transformed into the following operator equation:

$$\hat{v}(\chi_1)=\mathcal{P}\hat{v}(\chi_1)\mathcal{F}\hat{v}(\chi_1)+\mathcal{E}\hat{v}(\chi_1)\quad,\quad \chi_1\in[0,1]. \tag{4}$$

To achieve all the conditions of Theorem 1, we split our proof into four claims as follows.
Claim 1. Let $\hat{v},\rho\in\mathcal{X}$ then by hypothesis (H_1),

$$|\mathcal{P}\hat{v}(\chi_1)-\mathcal{P}\rho(\chi_1)|=|f(\chi_1,\hat{v}(\chi_1))-f(\chi_1,\rho(\chi_1))|\leq q_0|\hat{v}(\chi_1)-\rho(\chi_1)|\leq q_0\|\hat{v}-\rho\|$$

for all $\chi_1\in J$. Then,

$$\|\mathcal{P}\hat{v}-\mathcal{P}\rho\|\leq q_0\|\hat{v}-\rho\|\quad \textit{for all}\quad \hat{v},\rho\in\mathcal{X}.$$

Additionally,

$$|\mathcal{E}\hat{v}(\chi_1)-\mathcal{E}\rho(\chi_1)|=\frac{\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}|\hat{v}(\hat{\varkappa})-\rho(\hat{\varkappa})|ds\leq\frac{|\gamma|}{\Gamma(\hat{\iota}+1)}|\hat{v}-\rho|$$

Claim 2. We show the continuity of the operator $\mathcal{F}$ in $\mathcal{S}$.
Let $(\hat{v}_n)_n$ be a sequence in $\mathcal{S}$ converging to a point $\hat{v}\in\mathcal{S}$. Then, by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}=\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\lim_{n\to\infty}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}$$

Then,

$$\begin{aligned}
\lim_{n\to\infty}\mathcal{F}\hat{v}_n(\chi_1) &= \lim_{n\to\infty}\Big[\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},x_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}_n(\hat{\varkappa})d\hat{\varkappa}\\
&-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\Big]\\
&-\frac{\gamma}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}_n(\hat{\varkappa})d\hat{\varkappa},\\
&=\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\lim_{n\to\infty}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_1(\chi_1)\int_0^1\lim_{n\to\infty}\hat{\Psi}_2(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_2(\chi_1)\int_0^1\lim_{n\to\infty}\hat{\Psi}_1(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_3(\chi_1)\int_0^1\lim_{n\to\infty}\hat{\Psi}_3(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}\lim_{n\to\infty}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}\lim_{n\to\infty}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}\lim_{n\to\infty}\hat{v}_n(\hat{\varkappa})d\hat{\varkappa}\\
&-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\lim_{n\to\infty}w(\hat{\varkappa},\hat{v}_n(\hat{\varkappa}))d\hat{\varkappa}\\
&-\tfrac{\gamma}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}\lim_{n\to\infty}\hat{v}_n(\hat{\varkappa})d\hat{\varkappa},\\
&=\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}+\Theta_2(\chi_1)\int_0^1g(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\omega)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\
&-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&-\tfrac{\gamma}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa},\\
&=\mathcal{F}\hat{v}(\chi_1)
\end{aligned}$$

for all $\chi_1\in[0,1]$. This shows the continuity of $\mathcal{F}$ in $\mathcal{S}$.

Claim 3. We show the compactness of $\mathcal{F}$ on $\mathcal{S}$. Let $\hat{v}\in\mathcal{S}$, for all $\chi_1\in J$ define $\sup_{0\le\chi_1\le1}|w(\chi_1,0)|=\Upsilon_0$, $\sup_{0\le t\le1}|g(\chi_1,0)|=\Upsilon_1$, $\sup_{0\le t\le1}|h(\chi_1,0)|=\Upsilon_2$, $\sup_{0\le t\le1}|k(\chi_1,0)|=\Upsilon_3$.

with

$$
\begin{aligned}
r &= \frac{\Upsilon_0}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \Theta_1\Upsilon_2 + \Theta_2\Upsilon_1 + \Theta_3\Upsilon_3 + \frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)} \\
&\quad + |\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}|.
\end{aligned}
$$

First, we show that $\mathcal{F}(\mathcal{S})$ is a uniformly bounded set in $\mathcal{S}$. For $\hat{v} \in \mathcal{S}$, $\chi_1 \in [0,1]$, we have:

$$
\begin{aligned}
|\mathcal{F}\hat{v}(\chi_1)| &\leq \frac{1}{\Gamma(\hat{\imath}+\xi)} \int_0^{\chi_1} (\chi_1 - \hat{\varkappa})^{\hat{\imath}+\xi-1} \Big[|w(\hat{\varkappa},\hat{v}) - w(\hat{\varkappa},0)| + |w(\hat{\varkappa},0)| \Big] d\hat{\varkappa} \\
&\quad + |\Theta_1(\chi_1)| \int_0^1 |\hat{\Psi}_2(s;\hat{v}(\hat{\varkappa})) - \hat{\Psi}_2(\hat{\varkappa};0)| + |\hat{\Psi}_2(\hat{\varkappa};0)| d\hat{\varkappa} \\
&\quad + |\Theta_2(\chi_1)| \int_0^1 |\hat{\Psi}_1(s;\hat{v}(\hat{\varkappa})) - \hat{\Psi}_1(\hat{\varkappa};0)| + |g(\hat{\varkappa};0)| d\hat{\varkappa} \\
&\quad + |\Theta_3(\chi_1)| \int_0^1 |\hat{\Psi}_3(s;\hat{v}(\hat{\varkappa})) - \hat{\Psi}_3(\hat{\varkappa};0)| + |\hat{\Psi}_3(\hat{\varkappa};0)| ds + \frac{|\Theta_4(\chi_1)|}{\Gamma(\xi-\hat{\imath})} \\
&\quad \times \int_0^1 (1-\hat{\varkappa})^{\xi-\hat{\imath}-1} \Big[|w(\hat{\varkappa},\hat{v}) - w(\hat{\varkappa},0)| + |w(\hat{\varkappa},0)| \Big] ds \\
&\quad + \frac{|\Theta_5(\chi_1)|}{\Gamma(\xi)} \int_0^1 (1-\hat{\varkappa})^{\xi-1} \Big[|f(\hat{\varkappa},\hat{v}(\hat{\varkappa})) - f(\hat{\varkappa},0)| + |f(\hat{\varkappa},0)| \Big] d\hat{\varkappa} \\
&\quad + |\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}| \int_0^1 (1-\hat{\varkappa})^{\hat{\imath}} |\hat{v}(\hat{\varkappa})| ds + |\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi)}| \\
&\quad \times \int_0^1 (1-\hat{\varkappa})^{\hat{\imath}+\xi-1} \Big[|w(\hat{\varkappa},\hat{v}) - w(\hat{\varkappa},0)| + |w(\hat{\varkappa},0)| \Big] ds,
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\mathcal{F}\hat{v}\| &\leq \frac{1}{\Gamma(\hat{\imath}+\xi+1)} \Big(q_1\|\hat{v}\| + \Upsilon_0 \Big) + \Theta_1[q_3\|\hat{v}\| + \Upsilon_2] + \Theta_2[q_2\|\hat{v}\| + \Upsilon_1] + \Theta_3[q_4\|\hat{v}\| + \Upsilon_3] \\
&\quad + \frac{\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} \Big(q_1\|\hat{v}\| + \Upsilon_0 \Big) + \frac{\Theta_5}{\Gamma(\xi+1)} \Big(q_1\|\hat{v}\| + \Upsilon_0 \Big) + |\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath}+1)}| \|\hat{v}\| \\
&\quad + |\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}| \Big(q_1\|\hat{v}\| + \Upsilon_0 \Big) \\
&\quad \leq \Big[q_1 \Big(\frac{1}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \frac{\Theta_5}{\Gamma(\xi+1)} + |\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}| \Big) \\
&\quad + |\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath}+1)}| + \Theta_1 q_3 + \Theta_2 q_2 + \Theta_3 q_4 \Big] \|\hat{v}\| \\
&\quad + \frac{\Upsilon_0}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + A_1\Upsilon_2 + \Theta_2\Upsilon_1 \\
&\quad + \Theta_3\Upsilon_3 + \frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)} + |\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}|,
\end{aligned}
$$

thus,

$$
\begin{aligned}
\|\mathcal{F}\hat{v}\| &\leq \Big[q_1 \Big(\frac{1}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \frac{\Theta_5}{\Gamma(\xi+1)} + |\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}| \Big) \\
&\quad + |\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath}+1)}| + \Theta_1 q_3 + \Theta_2 q_2 + \Theta_3 q_4 \Big] r \\
&\quad + \frac{\Upsilon_0}{\Gamma(\hat{\imath}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\imath}+1)} + \Theta_1\Upsilon_2 + \Theta_2\Upsilon_1 \\
&\quad + \Theta_3\Upsilon_3 + \frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)} + |\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}|
\end{aligned}
$$

for all $\hat{v} \in \mathcal{S}$. This shows that $\mathcal{F}$ is uniformly bounded on $\mathcal{S}$. Next, we show the equicontinuity of $\mathcal{F}(\mathcal{S})$ on $\mathcal{X}$. Let $\chi_{11}, \chi_{12} \in J$, then for any $\hat{v} \in \mathcal{S}$

$$\begin{aligned}
|\mathcal{F}\hat{v}(\chi_{12}) - \mathcal{F}\hat{v}(\chi_{11})| &\leq \Big|\frac{1}{\Gamma(\hat{\imath}+\xi)}\Big(\int_0^{\chi_{12}}(\chi_{12}-s)^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))\\
&-\int_0^{\chi_{11}}(\chi_{11}-s)^{\hat{\imath}+\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))\Big)d\hat{\varkappa}\\
&+(\Theta_1(\chi_{12})-\Theta_1(\chi_{11}))\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+(\Theta_2(\chi_{12})-\Theta_2(\chi_{11}))\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+(\Theta_3(\chi_{12})-\Theta_3(\chi_{11}))\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{(\Theta_4(\chi_{12})-\Theta_4(\chi_{11}))}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{(\Theta_5(\chi_{12})-\Theta_5(\chi_{11}))}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\Big|,\\
&\leq \frac{1}{\Gamma(\hat{\imath}+\xi)}\int_{\chi_{11}}^{\chi_{12}}|w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))|ds\\
&+(\Theta_1(\chi_{12})-\Theta_1(\chi_{11}))\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+(\Theta_2(\chi_{12})-\Theta_2(\chi_{11}))\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+(A_3(\chi_{12})-A_3(\chi_{11}))\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{(\Theta_4(\chi_{12})-\Theta_4(\chi_{11}))}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\
&+\frac{(\Theta_5(\chi_{12})-\Theta_5(\chi_{11}))}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}w(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\Big|.
\end{aligned}$$

Hence,

$$\forall\varepsilon>0,\quad \exists\vartheta>0\quad :\quad |\chi_{11}-\chi_{12}|<\vartheta\Longrightarrow|\mathcal{F}\hat{v}(\chi_{11})-\mathcal{F}\hat{v}(\chi_{12})|<\varepsilon$$

for all $\chi_{11},\chi_{12}\in J$ and for all $\hat{v}\in X$.
Therefore, the operator $\mathcal{F}(\mathcal{S})$ is equicontinuously set in $\mathcal{X}$.
By applying the Arzelá–Ascoli theorem, we deduce that the operator $\mathcal{F}$ is compact and continuous.
Claim 4. We show Hypothesis 3 of Theorem 1 .
Let $\hat{v}\in\mathcal{X}$ and $\rho\in\mathcal{S}$ be arbitrary such that $\hat{v}=\mathcal{P}\hat{v}\mathcal{F}\rho+\mathcal{E}\hat{v}$. Then,

$$\begin{aligned}
|\hat{v}(\chi_1)| &= |\mathcal{P}\hat{v}(\chi_1)||\mathcal{F}\rho(\chi_1)|+|\mathcal{E}\hat{v}(\chi_1)|\\
&\leq |f(\chi_1,\hat{v}(\chi_1))|\Big[\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}|w(\hat{\varkappa},\rho(\hat{\varkappa}))|ds+|\Theta_1(\chi_1)|\\
&\times\int_0^1|\hat{\Psi}_2(s;\rho(\hat{\varkappa}))|ds\\
&+|\Theta_2(\chi_1)|\int_0^1|\hat{\Psi}_1(s;\rho(\hat{\varkappa}))|d\hat{\varkappa}\\
&+|\Theta_3(\chi_1)|\int_0^1|\hat{\Psi}_3(s;\rho(\hat{\varkappa}))|ds+\frac{|\Theta_4(\chi_1)|}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}|w(\hat{\varkappa},\rho(\hat{\varkappa}))|d\hat{\varkappa}\\
&+\frac{|\Theta_5(\chi_1)|}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}|w(\hat{\varkappa},\rho(\hat{\varkappa}))|ds\\
&+\frac{|\vartheta|}{|1+\vartheta|\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\omega+\beta-1}|w(\hat{\varkappa},\rho(\hat{\varkappa}))|d\hat{\varkappa}\\
&+|\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath})}|\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}|\rho(\hat{\varkappa})|ds\Big]+\frac{|\gamma|}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}|\hat{v}(\hat{\varkappa})|d\hat{\varkappa}\\
&\leq |\varphi(\chi_1)|\Big[\frac{1}{\Gamma(\hat{\imath}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}+\xi-1}m(\hat{\varkappa})d\hat{\varkappa}\\
&+\Theta_1\int_0^1\rho(\hat{\varkappa})d\hat{\varkappa}+\Theta_2\int_0^1\phi(\hat{\varkappa})d\hat{\varkappa}+\Theta_3\int_0^1\psi(\hat{\varkappa})d\hat{\varkappa}\\
&+\frac{\Theta_4}{\Gamma(\xi-\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\imath}-1}m(\hat{\varkappa})d\hat{\varkappa}+\frac{\Theta_5}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}m(\hat{\varkappa})d\hat{\varkappa}\\
&+\frac{|\vartheta|}{(|1+\vartheta|)\Gamma(\hat{\imath}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}+\xi-1}\,m(\hat{\varkappa})d\hat{\varkappa}\\
&+\frac{|\vartheta\gamma|}{(|1+\vartheta|)\Gamma(\hat{\imath})}\int_0^1(1-\hat{\varkappa})^{\hat{\imath}-1}|\rho(\hat{\varkappa})|ds\Big]+\frac{|\gamma|}{\Gamma(\hat{\imath})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\imath}-1}|\hat{v}(\hat{\varkappa})|d\hat{\varkappa}\\
&\leq ||\varphi||\Big(\frac{\|m\|}{\Gamma(\hat{\imath}+\xi+1)}+\Theta_1\|\rho\|+\Theta_2\|\phi\|+\Theta_3\|\psi\|+\frac{\Theta_4\|m\|}{\Gamma(\xi-\hat{\imath}+1)}\\
&+\frac{\Theta_5\|m\|}{\Gamma(\xi+1)}+\frac{|\vartheta|\|m\|}{(|1+\vartheta|)\Gamma(\hat{\imath}+\xi+1)}+\frac{|\vartheta\lambda|}{(|1+\vartheta|)\Gamma(\hat{\imath}+1)}\|\rho\|\Big)\\
&+\frac{|\gamma|}{\Gamma|(\hat{\imath}+1)}\|\hat{v}\|.
\end{aligned}$$

Further, we obtain

$$\|x\| \leq r$$

Then, $\hat{v} \in \mathcal{S}$ and, hence, the 3 in Theorem 1 is verified.
Lastly, we obtain

$$\begin{aligned} M = \|\mathcal{F}(\mathcal{S})\| & \\ &= \sup\{\|\mathcal{F}\hat{v}\| : \hat{v} \in \mathcal{S}\} \\ &\leq \Big[q_1\Big(\frac{1}{\Gamma(\hat{\iota}+\xi+1)} + \frac{\Theta_4}{\Gamma(\xi-\hat{\iota}+1)} + \frac{\Theta_5}{\Gamma(\xi+1)} \\ &+|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}|\Big) + |\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\gamma+1)}| + \Theta_1 q_3 + \Theta_2 q_2 \\ &+\Theta_3 q_4\Big]r + \frac{\Upsilon_0}{\Gamma(\hat{\iota}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\iota}+1)} + \Theta_1\Upsilon_2 + \Theta_2\Upsilon_1 \\ &+\Theta_3\Upsilon_3 + \frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)} + |\frac{\mu\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}. \end{aligned}$$

and so,

$$\begin{aligned} q_0 M + \frac{\lambda}{\Gamma(\hat{\iota}+1)} &\leq q_0\Big(\Big[q_1\Big(\frac{1}{\Gamma(\hat{\iota}+\xi+1)} + \frac{\Theta_4}{\Gamma(\xi-\hat{\iota}+1)} + \frac{\Theta_5}{\Gamma(\xi+1)} \\ &+|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}|\Big) + |\frac{\vartheta\lambda}{(1+\vartheta)\Gamma(\hat{\iota}+1)}| + \Theta_1 q_3 + \Theta_2 q_2 \\ &+\Theta_3 q_4\Big]r + \frac{\Upsilon_0}{\Gamma(\hat{\iota}+\xi+1)} + \frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\iota}+1)} + \Theta_1\Upsilon_2 + \Theta_2\Upsilon_1 \\ &+\Theta_3\Upsilon_3 + \frac{\gamma_5\Upsilon_0}{\Gamma(\beta+1)} + |\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}\Big) + \frac{\gamma}{\Gamma(\hat{\iota}+1)} < 1. \end{aligned}$$

Hence, the operator $\mathcal{P}\hat{v}\mathcal{F}\hat{v} + \mathcal{E}\hat{v} = \hat{v}$ has at least one solution in $\mathcal{S}$. So, a solution can be defined for the problem (1) on J, and this finalizes the proof. □

4. Stability Results

Consider the nonlinear operator $\mathcal{Z}_1 \in \mathcal{E}([0,1],\mathbb{R}) \to \mathcal{E}([0,1],\mathbb{R})$, where $\hat{v}$ is define by (4)

$$\Big\{{}^c\mathcal{D}^{\xi}({}^c\mathcal{D}^{\hat{\iota}} + \gamma)\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big) - \hat{\Psi}(\chi_1,\hat{v}(\chi_1)) = \mathcal{Z}_1(\hat{v})(\chi_1), \quad \chi_1 \in (0,1),$$

for $\chi_1 \in [0,1]$. For some $\varsigma_1 > 0$, the following inequality is valid.

$$||\mathcal{Z}_1,(\hat{v})|| \leq \varsigma_1, \tag{5}$$

Definition 3. *The problem (1) is Hyers–Ulam stable if $\mathcal{M}_1 > 0$ exist with a unique solution $(\hat{v}) \in \mathcal{E}([0,1],\mathbb{R})$ of problem (1) with*

$$||\hat{v} - (\hat{\hat{v}})|| \leq \mathcal{M}_1\varsigma_1$$

for each solution $(\hat{\hat{v}})$ of the previous inequality in belongs to $\mathcal{E}([0,1],\mathbb{R})$.

Theorem 3. *Let the assumptions of (H_2) hold. Then, problem (1) is Ulam–Hyers stable.*

Proof. Let $(\hat{v}) \in \mathcal{E}([0,1],\mathbb{R})$ be the solution of the problem that satisfies (1). Let $(\hat{x})$ be any satisfying solution for (1).

$$\Big\{{}^c\mathcal{D}^{\xi}({}^c\mathcal{D}^{\hat{\iota}} + \gamma)\Big(\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}\Big) = \hat{\Psi}(\chi_1,\hat{v}(\chi_1)) + \mathcal{Z}_1(\hat{v})(\chi_1), \quad \chi_1 \in (0,1),$$

for $\chi_1 \in [0,1]$. Therefore,

$$\begin{aligned}\hat{\hat{v}}(\chi_1) =& \hat{v}(\hat{x})(\chi_1) + f(\chi_1, \hat{v}(\chi_1))\Big[\frac{1}{\Gamma(\hat{\iota}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_3(\chi_1)\int_0^1\Theta_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{\Theta_4(\chi_1)}{\Gamma(\xi-\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\iota}-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\frac{\vartheta\gamma}{(1+\mu)\Gamma(\omega)}\int_0^1(1-\hat{\varkappa})^{\omega-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\ &-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\Big]\\ &-\frac{\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa},\end{aligned}$$

which implies

$$\begin{aligned}|\hat{v}(\hat{x})(\chi_1)-\hat{\hat{v}}(\chi_1)| \leq& f(\chi_1,\hat{v}(\chi_1))\Big[\frac{1}{\Gamma(\hat{\iota}+\xi)}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\omega+\beta-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\Theta_1(\chi_1)\int_0^1\hat{\Psi}_2(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_2(\chi_1)\int_0^1\hat{\Psi}_1(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\Theta_3(\chi_1)\int_0^1\hat{\Psi}_3(\hat{\varkappa},\hat{v}(\hat{\varkappa}))d\hat{\varkappa}\\ &+\frac{A_4(\chi_1)}{\Gamma(\xi-\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\xi-\hat{\iota}-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\frac{\Theta_5(\chi_1)}{\Gamma(\xi)}\int_0^1(1-\hat{\varkappa})^{\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\\ &+\frac{\vartheta\gamma}{(1+\mu)\Gamma(\hat{\iota})}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa}\\ &-\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi)}\int_0^1(1-\hat{\varkappa})^{\hat{\iota}+\xi-1}\rho(\hat{\varkappa})d\hat{\varkappa}\Big]\\ &-\frac{\gamma}{\Gamma(\hat{\iota})}\int_0^{\chi_1}(\chi_1-\hat{\varkappa})^{\hat{\iota}-1}\hat{v}(\hat{\varkappa})d\hat{\varkappa},\end{aligned}$$

$$|(\hat{v})-(\hat{\hat{v}})| \leq \mathcal{M}_1\varsigma_1.$$

By this, the previously presented problem (1) is U-H stable. □

5. Examples

Example 1. *The following equation is of a fractional hybrid type:*

$$\begin{cases} {}^{c}\mathcal{D}^{\frac{4}{3}}\Big({}^{c}\mathcal{D}^{\frac{1}{3}}+\frac{1}{300}\Big)\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}=\frac{1}{500+t^2}\Big(sin(\hat{v}(\chi_1))+\frac{|\hat{v}(\chi_1)|}{1+|\hat{v}(\chi_1)|}+\hat{v}(\chi_1)\Big),\ \ \chi_1\in[0,1],\\ \frac{\hat{v}(0)}{f(0,\hat{v}(0))}+\frac{\hat{v}(1)}{f(1,\hat{v}(1))}=\frac{1}{200}\int_0^1\frac{|\hat{v}(\hat{\varkappa})|}{300+|\hat{v}(\hat{\varkappa})|}ds,\\ {}^{c}\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^{c}\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big)=\frac{1}{200}\int_0^1\Big(\frac{1}{s+2}\Big)^3\frac{|\hat{v}(\hat{\varkappa})|}{30+|\hat{v}(\hat{\varkappa})|}ds,\\ {}^{c}\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^{c}\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big)=\frac{1}{200}\int_0^1\Big(\frac{1}{s+4}\Big)^2\frac{|\hat{v}(\hat{\varkappa})|}{30+|\hat{v}(\hat{\varkappa})|}ds, \end{cases} \tag{6}$$

where $\xi=\frac{4}{3},\hat{\imath}=\frac{1}{3},\gamma=\frac{1}{300},\vartheta=1,\tau_1=\tau_2=\tau_3=\frac{1}{200}$, *and*

$$\hat{\Psi}(\chi_1,\hat{v})=\frac{1}{500+t^2}\Big(sin(\hat{v}(\chi_1))+\frac{|\hat{v}(\chi_1)|}{1+|\hat{v}(\chi_1)|}+\hat{v}(\chi_1)\Big),$$

$$f(\chi_1,\hat{v})=\frac{(\chi_1+1)^2}{100}\big(\sin\hat{v}(\chi_1)+\frac{|\hat{v}(\chi_1)|}{1+|\hat{v}(\chi_1)|}+3\big)$$

$$\hat{\Psi}_2(\chi_1,\hat{v})=\frac{|\hat{v}(\chi_1)|}{300+|\hat{v}(\chi_1)|},\ \hat{\Psi}_1(\chi_1,\hat{v})=\Big(\frac{1}{\chi_1+2}\Big)^3\frac{|\hat{v}(\chi_1)|}{30+|\hat{v}(\chi_1)|},$$

$$\hat{\Psi}_3(\chi_1,\hat{v})=\Big(\frac{1}{\chi_1+4}\Big)^2\frac{|\hat{v}(\chi_1)|}{30+|\hat{v}(\chi_1)|}.$$

Clearly, $q_0=q_1=q_2=q_3=\frac{1}{500},q_4=\frac{1}{300}$,
furthermore, we have

$$\begin{aligned} &q_0\Big(\Big[q_1\Big(\frac{1}{\Gamma(\hat{\imath}+\xi+1)}+\frac{A_4}{\Gamma(\xi-\hat{\imath}+1)}+\frac{\Theta_5}{\Gamma(\xi+1)}\\ &+|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}|\Big)+|\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\imath}+1)}|+\Theta_1q_3+A_2q_2\\ &+\Theta_3q_4\Big]+\frac{\Upsilon_0}{\Gamma(\hat{\imath}+\xi+1)}+\frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\imath}+1)}+\Theta_1\Upsilon_2+\Theta_2\Upsilon_1\\ &+\Theta_3\Upsilon_3+\frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)}+|\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\imath}+\xi+1)}\Big)+\frac{\gamma}{\Gamma(\hat{\imath}+1)}\approx 0.026215478961<1. \end{aligned}$$

Hence, the problem 6 has a solution, by Theorem 2.

Example 2. *Consider a problem of the following form*

$$\begin{cases} c\mathcal{D}^{\frac{4}{3}}({}^{c}\mathcal{D}^{\frac{1}{3}}+\frac{1}{700})\frac{x(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))}=\frac{1}{1+t^4}\left(\frac{{\chi_1}^4|\hat{v}(\chi_1)|}{(4|\hat{v}(\chi_1)|+10)}+\frac{|\hat{v}(\chi_1)|}{(4|\hat{v}(\chi_1)|+6)}+\frac{1}{1+\hat{v}^2(\hat{\varkappa})}\right),\chi_1\in[0,1],\\ \frac{\hat{v}(0)}{f(0,\hat{v}(0))}+\frac{\hat{v}(1)}{f(1,\hat{v}(1))}=\frac{1}{300}\int_0^1\frac{1}{s+100}\frac{|x(\hat{\varkappa})|}{300+|x(\hat{\varkappa})|}ds,\\ {}^{c}\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^{c}\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big)=\frac{1}{300}\int_0^1\Big(\frac{1}{s+8000}\Big)\frac{|\hat{v}(\hat{\varkappa})|}{30+|\hat{v}(\hat{\varkappa})|}ds,\\ {}^{c}\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^{c}\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big)=\frac{1}{300}\int_0^1\Big(\frac{1}{s+1600}\Big)\frac{|\hat{v}(\hat{\varkappa})|}{30+|\hat{v}(\hat{\varkappa})|}ds. \end{cases} \tag{7}$$

where $\xi=\frac{3}{2},\hat{\imath}=\frac{1}{2},\gamma=\frac{1}{700},\vartheta=1,\tau_1=\tau_2=\tau_3=\frac{1}{300}$, *and*

$$\hat{\Psi}(\chi_1,\hat{v})=\frac{1}{{\chi_1}^2+4}\left(\frac{{\chi_1}^2|x(\chi_1)|}{(3|\hat{v}(\chi_1)|+1)}+\frac{|\hat{v}(\chi_1)|}{(6|\hat{v}(\chi_1)|+10)}+\frac{1}{1+\hat{v}^2(\chi_1)}\right)$$

$$f(\chi_1,\hat{v})=\frac{1}{2(35+t)}(\frac{\hat{v}^2(\chi_1)+|\hat{v}(\chi_1)|}{1+|\hat{v}(\chi_1)|})+t^{\frac{1}{3}}+1$$

$$\hat{\Psi}_1(\chi_1,\hat{v}) = \frac{1}{\chi_1+100}\frac{|\hat{v}(\chi_1)|}{300+|\hat{v}(\chi_1)|},$$

$$\hat{\Psi}_2(\chi_1,\hat{v}) = \Big(\frac{1}{\chi_1+8000}\Big)\frac{|\hat{v}(\chi_1)|}{30+|\hat{v}(\chi_1)|}, \hat{\Psi}_3(\chi_1,\hat{v}) = \Big(\frac{1}{\chi_1+1600}\Big)\frac{|\hat{v}(\chi_1)|}{30+|\hat{v}(\chi_1)|}.$$

After calculating, we obtain

$$\begin{aligned} &q_0\Big(\Big[q_1\Big(\frac{1}{\Gamma(\hat{\iota}+\xi+1)}+\frac{A_4}{\Gamma(\xi-\hat{\iota}+1)}+\frac{\Theta_5}{\Gamma(\xi+1)}\\ &+|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}|\Big)+|\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\iota}+1)}|+\Theta_1 q_3+A_2 q_2\\ &+\Theta_3 q_4\Big]+\frac{\Upsilon_0}{\Gamma(\hat{\iota}+\xi+1)}+\frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\iota}+1)}+\Theta_1\Upsilon_2+\Theta_2\Upsilon_1\\ &+\Theta_3\Upsilon_3+\frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)}+|\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}\Big)+\frac{\gamma}{\Gamma(\hat{\iota}+1)}\approx 0.145781245789<1. \end{aligned}$$

Hence, the problem 8 has a least one solution, by Theorem 2.

Example 3. *For this example, we consider the following problem of the form:*

$$\begin{cases} {}^c\mathcal{D}^{\frac{3}{2}}({}^c\mathcal{D}^{\frac{1}{2}}+\frac{1}{400})\frac{\hat{v}(\chi_1)}{f(\chi_1,\hat{v}(\chi_1))} = \frac{1}{\sqrt{\chi_1}+e}\Big(\frac{(3|\hat{v}(\chi_1)|+1)}{e^t|\hat{v}(\chi_1)|}+\frac{|\hat{v}(\chi_1)|}{(8|\hat{v}^2(\chi_1)|+10)}+e^{-\hat{v}(\chi_1)}\Big),\ \chi_1\in[0,1],\\ \frac{\hat{v}(0)}{f(0,\hat{v}(0))}+\frac{\hat{v}(1)}{f(1,\hat{v}(1))} = \frac{1}{1000}\int_0^1\frac{1}{s+20}\frac{|\hat{v}(\hat{\varkappa})|}{300+|\hat{v}(\hat{\varkappa})|}ds,\\ {}^c\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^c\mathcal{D}^{\frac{1}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big) = \frac{1}{1000}\int_0^1\Big(\frac{1}{s+8000}\Big)\frac{|\hat{v}(\hat{\varkappa})|}{300+|\hat{v}(\hat{\varkappa})|}ds,\\ {}^c\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(0)}{f(0,\hat{v}(0))}\big)+{}^c\mathcal{D}^{\frac{2}{3}}\big(\frac{\hat{v}(1)}{f(1,\hat{v}(1))}\big) = \frac{1}{1000}\int_0^1\Big(\frac{1}{s+1600}\Big)\frac{|\hat{v}(\hat{\varkappa})|}{300+|\hat{v}(\hat{\varkappa})|}ds. \end{cases} \tag{8}$$

where $\xi=\frac{3}{2}, \hat{\iota}=\frac{1}{2}, \gamma=\frac{1}{400}, \vartheta=1, \tau_1=\tau_2=\tau_3=\frac{1}{1000}$, *and*

$$\hat{\Psi}(\chi_1,\hat{v}) = \frac{1}{\sqrt{\chi_1}+e}\Big(\frac{(3|\hat{v}(\chi_1)|+1)}{e^t|\hat{v}(\chi_1)|}+\frac{|\hat{v}(\chi_1)|}{(8|\hat{v}^2(\chi_1)|+10)}+e^{-\hat{v}(\chi_1)}\Big)$$

$$f(\chi_1,\hat{v}) = \frac{1}{{\chi_1}^2}\big(\cos\hat{v}(\chi_1)+\frac{|\hat{v}(\chi_1)|}{1+|\hat{v}(\chi_1)|}+3\big),$$

$$\hat{\Psi}_1(\chi_1,\hat{v}) = \frac{1}{\chi_1+20}\frac{|\hat{v}(\chi_1)|}{300+|\hat{v}(\chi_1)|},$$

$$\hat{\Psi}_2(\chi_1,\hat{v}) = \Big(\frac{1}{\chi_1+8000}\Big)\frac{|\hat{v}(\chi_1)|}{300+|\hat{v}(\chi_1)|}, \hat{\Psi}_3(\chi_1,\hat{v}) = \Big(\frac{1}{\chi_1+1600}\Big)\frac{|\hat{v}(\chi_1)|}{300+|\hat{v}(\chi_1)|}.$$

After calculating, we obtain

$$\begin{aligned} &q_0\Big(\Big[q_1\Big(\frac{1}{\Gamma(\hat{\iota}+\xi+1)}+\frac{A_4}{\Gamma(\xi-\hat{\iota}+1)}+\frac{\Theta_5}{\Gamma(\xi+1)}\\ &+|\frac{\vartheta}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}|\Big)+|\frac{\vartheta\gamma}{(1+\vartheta)\Gamma(\hat{\iota}+1)}|+\Theta_1 q_3+A_2 q_2\\ &+\Theta_3 q_4\Big]+\frac{\Upsilon_0}{\Gamma(\hat{\iota}+\xi+1)}+\frac{\Upsilon_0\Theta_4}{\Gamma(\xi-\hat{\iota}+1)}+\Theta_1\Upsilon_2+\Theta_2\Upsilon_1\\ &+\Theta_3\Upsilon_3+\frac{\Theta_5\Upsilon_0}{\Gamma(\xi+1)}+|\frac{\vartheta\Upsilon_0}{(1+\vartheta)\Gamma(\hat{\iota}+\xi+1)}\Big)+\frac{\gamma}{\Gamma(\hat{\iota}+1)}\approx 0.2569878945<1. \end{aligned}$$

Hence, the problem 8 has a least one solution, by Theorem 2.

6. Conclusions

Most natural phenomena are treated using different types of fractional differential equations. This diversity in this type of equation helps us to scrutinize the integration of many phenomena in various fields. This helps us in creating programs that enable us to

consume rational materials. In this paper, based on the generalization of Dhage's fixed point theorem, Banach contraction mapping, and U-H stability, we treated the existence, uniqueness, and stability of solutions to a sequential fractional (Hybrid) differential equation with hybrid boundary conditions, respectively. This equation plays an important role in the field of the control system. For future work, we suggest using other types of fractional derivative operators such as the generalized Hilfer fractional derivative, and the one who is interested in the subject can also investigate the existence and uniqueness of the solutions for the tripled systems via several fixed points theorems such as Leray–Schuader's alternative, and Mönch's fixed point theorem.

Author Contributions: Methodology, M.H.; Investigation, M.A. and K.H.; Resources, M.H.; Writing—original draft, K.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. 2249].

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Aydi, H.; Arshad, M.; De la Sen, M. Hybrid Ciric type graphic Y, Λ-contraction mappings with applications to electric circuit and fractional differential equations. *Symmetry* **2020**, *12*, 467.
2. Saqib, M.; Khan, I.; Shafie, S. Application of fractional differential equations to heat transfer in hybrid nanofluid: modeling and solution via integral transforms. *Adv. Differ. Equ.* **2019**, *1*, 52. [CrossRef]
3. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
4. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1993.
5. Zhou, Y. *Basic Theory of Fractional Differential Equations*; Xiangtan University: Xiangtan, China, 2014.
6. Wang, J.; Zhang, Y. Analysis of fractional order differential coupled systems. *Math. Methods Appl. Sci.* **2015**, *38*, 3322–3338. [CrossRef]
7. Shah, K.; Khan, R.A. Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions. *Differ. Equations Appl.* **2015**, *7*, 245–262. [CrossRef]
8. Sitho, S.; Ntouyas, S.K.; Tariboon, J. Existence results for hybrid fractional integro-differential equations. *Bound. Value Probl.* **2015**, *2015*, 113. [CrossRef]
9. Hilal, K.; Kajouni, A. Boundary value problems for hybrid differential equations with fractional order. *Adv. Differ. Equ.* **2015**, *2015*, 183. [CrossRef]
10. Shah, K.; Ali, A.; Khan, R.A. Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. *Bound. Value Probl.* **2016**, *1*, 43. [CrossRef]
11. Houas, M. Existence results for a coupled system of fractional differential equations with multi-point boundary value problems. *Mediterrenean J. Model. Simul.* **2018**, *10*, 45–59.
12. Hannabou, M.; Hilal, K. Existence Results for a System of Coupled Hybrid Differential Equations with Fractional Order. *Int. J. Differ. Equ.* **2020**, *2020*, 3038427. [CrossRef]
13. Kamenskii, M.; Petrosyan, G.; Wen, C.F. An existence result for a periodic boundary value problem of fractional semilinear differential equations in a Banach space. *J. Nonlinear Var. Anal.* **2021**, *5*, 155–177.
14. Nuchpong, C.; Khoployklang, T.; Ntouyas, S.K.; Tariboon, J. Boundary value problems for Hilfer-Hadamard fractional differential inclusions with nonlocal integro-multi-point boundary conditions. *J. Nonlinear Funct. Anal.* **2022**, *2022*, 37, .
15. Awadalla, M.; Mahmudov, N.I. On System of Mixed Fractional Hybrid Differential Equations. *J. Funct. Spaces* **2022**, *2022*, 1258823. [CrossRef]
16. Awadalla, M.; Abuasbeh, K. On System of Nonlinear Sequential Hybrid Fractional Differential Equations. *Math. Probl. Eng.* **2022**, *2022*, 8556578. [CrossRef]
17. Elaiw, A.; Manigandan, A.; Awadalla, M.; Abuasbeh, K. Existence results by Mönch's fixed point theorem for a tripled system of sequential fractional differential equations. *AIMS Math.* **2023**, **8**, 3969–3996. [CrossRef]
18. Dhage, B.C.; Lakshmikantham, V. Basic Results on Hybrid Differential Equations. *Nonlinear Anal. Hybrid* **2010**, *4*, 414–424. [CrossRef]
19. Dhage, B.C. Basic results in the theory of hybrid differential equations with mixed perturbations of second type. *Funct. Differ. Equ.* **2012**, *19*, 87–106.
20. Zhao, Y.; Suna, S.; Han, Z.; Li, Q. Theory of fractional hybrid differential equations. *Comput. Math. Appl.* **2011**, *62*, 1312–1324. [CrossRef]
21. Benson, D.A.; Meerschaert, M.M.; Revielle, J. Fractional calculus in hydrologic modeling, A numerical perspective. *Adv. Water Resour.* **2013**, *51*, 479–497. [CrossRef]

22. Bouaouid, M.; Hannabou, M.; Hilal, K. Nonlocal Conformable-Fractional Differential Equations with a Measure of Noncompactness in Banach Spaces. *J. Math.* **2020**, *2020*, 5615080. [CrossRef]
23. Awadalla, M.; Abuasbeh, K.; Subramanian, M.; Manigandan, M. On a System of ψ-Caputo Hybrid Fractional Differential Equations with Dirichlet Boundary Conditions. *Mathematics* **2022**, *10*, 1681. [CrossRef]
24. Agarwal, R.; Hristova, S.; O'Regan, D. Stability of generalized proportional Caputo fractional differential equations by Lyapunov functions. *Fractal Fract.* **2022**, *6*, 34. [CrossRef]
25. Shahmohammadi, M.A.; Mirfatah, S.M.; Salehipour, H.; Azhari, M.; Civalek, Ö. Free vibration and stability of hybrid nanocomposite-reinforced shallow toroidal shells using an extended closed-form formula based on the Galerkin method. *Mech. Adv. Mater. Struct.* **2022**, *29*, 5284–5300. [CrossRef]
26. Abbas, M.I. Ulam stability and existence results for fractional differential equations with hybrid proportional-Caputo derivatives. *J. Interdiscip. Math.* **2022**, *25*, 213–231. [CrossRef]
27. Ganesh, A.; Deepa, S.; Baleanu, D.; Santra, S.S.; Moaaz, O.; Govindan, V.; Ali, R. Hyers-Ulam-Mittag-Leffler stability of fractional differential equations with two caputo derivative using fractional fourier transform . *AIMS Math.* **2022**, *7*, 1791–1810. [CrossRef]
28. Abbas, M.I.; Ragusa, M.A. On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function. *Symmetry* **2021**, *13*, 264. [CrossRef]
29. Arab, M.; Awadalla, M. A Coupled System of Caputo-Hadamard Fractional Hybrid Differential Equations with Three-Point Boundary Conditions. *Math. Probl. Eng.* **2022**, *2022*, 1500577. [CrossRef]
30. Gul, S.; Khan, R.A.; Khan, H.; George, R.; Etemad, S.; Rezapour, S. Analysis on a coupled system of two sequential hybrid BVPs with numerical simulations to a model of typhoid treatment. *Alex. Eng. J.* **2022**, *61*, 10085–10098. [CrossRef]
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amesterdam, The Netherlands, 2006.
32. Dhage, B.C. A fixed point theorem in Banach algebras with applications to functional integral equations. *Kyungpook Math. J.* **2004**, *44*, 145–155.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.