fractal and fractional

Article

Approximate Analytical Methods for a Fractional-Order Nonlinear System of Jaulent–Miodek Equation with Energy-Dependent Schrödinger Potential

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Abstract: In this paper, we study the numerical solution of fractional Jaulent–Miodek equations with the help of two modified methods: coupled fractional variational iteration transformation technique and the Adomian decomposition transformation technique. The Jaulent–Miodek equation has applications in several related fields of physics, including control theory of dynamical systems, anomalous transport, image and signal processing, financial modelings, nanotechnology, viscoelasticity, nanoprecipitate growth in solid solutions, random walk, modeling for shape memory polymers, condensed matter physics, fluid mechanics, optics and plasma physics. The results are presented as a series of quickly converging solutions. Analytical solutions have been performed in absolute error to confirm the proposed methodologies are trustworthy and accurate. The generated solutions are visually illustrated to guarantee the validity and applicability of the taken into consideration algorithm. The study's findings show that, compared to alternative analytical approaches for analyzing fractional non-linear coupled Jaulent–Miodek equations, the Adomian decomposition transform method and the variational iteration transform method are computationally very efficient and accurate

Keywords: fractional Jaulent–Miodek equations; Caputo operator; variational iteration transform method; Adomian decomposition transform method

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1. Introduction

In recent years, fractional differential equations have risen to prominence due to their proved usefulness in a number of seemingly unrelated scientific and engineering fields. For example, the nonlinear oscillations of an earthquake can be characterized by a fractional derivative, and the fractional derivative of the traffic fluid dynamics model can solve the insufficiency resulting from the assumption of continuous traffic flows [1,2]. Numerous chemical processes, mathematical biology, engineering, and scientific problems [3–6] are also modeled with fractional differential problems. Nonlinear partial differential equations (NPDEs) characterize a variety of physical, biological, and chemical phenomena. Current research focuses on developing precise traveling wave solutions for such equations. Exact solution replies help scientists understand the complicated physical phenomena and dynamical processes portrayed by NPDEs [7–9]. In the past four decades, numerous important methodologies for attaining accurate solutions to NPDEs have been proposed [10,11]. The Jaulent–Miodek equation was developed in 1979 by Jaulent and Miodek as an application to energy-dependent potentials [12,13]. The JM equation first came into existence in several related fields of physics, including classical dynamics [14], condensed material physics [15], optics [16], and fluid mechanics [17]. The goal of this study is to investigate the numerical results of the following difficult and anomalous physical model,

$$\begin{aligned}
&\frac{\partial^{\varpi}\nu}{\partial\tau^{\varpi}}+\frac{\partial^{3}\nu}{\partial\psi^{3}}+\frac{3}{2}\varphi\frac{\partial^{3}\varphi}{\partial\psi^{3}}+\frac{9}{2}\frac{\partial\varphi}{\partial\psi}\frac{\partial^{2}\varphi}{\partial\psi^{2}}-6\nu\frac{\partial\nu}{\partial\psi}-6\nu\varphi\frac{\partial\varphi}{\partial\psi}-\frac{3}{2}\frac{\partial\nu}{\partial\psi}\varphi^{2}=0,\\
&\frac{\partial^{\varpi}\varphi}{\partial\tau^{\varpi}}+\frac{\partial^{3}\varphi}{\partial\psi^{3}}-6\frac{\partial\nu}{\partial\psi}\varphi-6\nu\frac{\partial\varphi}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi}{\partial\psi}\varphi^{2}=0,
\end{aligned} \tag{1}$$

With initial conditions

$$\begin{aligned}
&\nu(\psi,0)=\frac{1}{8}\lambda^{2}\left(1-4sech^{2}\left(\frac{\lambda\psi}{2}\right)\right),\\
&\varphi(\psi,0)=\lambda sech\left(\frac{\lambda\psi}{2}\right).
\end{aligned} \tag{2}$$

where λ is the arbitrary constant and $0<\varpi\leq 1$. Numerous writers have looked at the system of nonlinear JM equation in the past using various methods, including the Sumudu transformation homotopy perturbation technique (STHPM) [18], Hermite Wavelets and OHAM [19], Invariant subspace technique [20], and many others [21–23]. On the other hand, new methods for explaining nonlinear differential systems with fractional orders have been discovered throughout the past three decades, along with novel computer algorithms and symbolic programming. Fundamental significance is attached to the nonlinear fractional differential equations analytical and numerical solutions. Since nonlinear fractional differential equations are used to mathematically model most complex processes to solve both linear and nonlinear differential equations, Keskin and Oturance [24] devised the Reduced Differential Transform Method (RDTM) in 2009 [25].

2. Basic Preliminaries

In this section, we give some definitions of fractional calculus used in our present work.

Definition 1. *The derivative of order fractional in Caputo sense is defined as*

$$D_{\tau}^{\varpi}\varphi(\varpi,\tau)=\frac{1}{\Gamma(k-\varpi)}\int_{0}^{\tau}(\tau-\rho)^{k-\varpi-1}\varphi^{(k)}(\varpi,\rho)d\rho,\quad k-1<\varpi\leq k,\quad k\in N. \tag{3}$$

Definition 2. *For a function $\varphi(\tau)$, the Yang transform is denoted by $Y\{\varphi(\tau)\}$ or $\mathcal{M}(u)$ and is defined as*

$$Y\{\varphi(\tau)\}=\mathcal{M}(u)=\int_{0}^{\infty}e^{\frac{-\tau}{u}}\varphi(\tau)d\tau,\ \ \tau>0,\ \ u\in(-\tau_{1},\tau_{2}). \tag{4}$$

and inverse transform is as

$$Y^{-1}\{\mathcal{M}(u)\}=\varphi(\tau). \tag{5}$$

Laplace–Yang duality Property:

If the Laplace Transform of the function $\varphi(\tau)$ is $F(u)$, then

$$F(u)=\mathcal{L}\{\varphi(\tau)\}=\int_{0}^{\infty}e^{-u(\tau)}\varphi(\tau)d(\tau)$$

Substitute $(\tau)=\frac{1}{u}$ in the integral on right-hand side we get

$$F(u)=\mathcal{L}\{\varphi(\tau)\}=\frac{1}{u}\int_{0}^{\infty}e^{-u}\varphi\left(\frac{1}{u}\right)d(\tau)$$

We get,

$$F(u)=T\left(\frac{1}{u}\right)$$

Additionally, from the above equations, we get

$$T(u) = F\left(\frac{1}{u}\right)$$

Definition 3. *The Yang transformation in terms of fractional-order derivative is as*

$$Y\{\varphi^{\varpi}(\tau)\} = \frac{\mathcal{M}(u)}{u^{\gamma}} - \sum_{k=0}^{n-1} \frac{\varphi^{k}(0)}{u^{\varpi-(k+1)}}, \;\; 0 < \varpi \leq n. \tag{6}$$

Definition 4. *The Yang transform in terms of nth derivative is as*

$$Y\{\varphi^{n}(\tau)\} = \frac{\mathcal{M}(u)}{u^{n}} - \sum_{k=0}^{n-1} \frac{\varphi^{k}(0)}{u^{n-k-1}}, \;\; \forall \;\; n = 1, 2, 3, \cdots \tag{7}$$

3. The Methodology of Adomian Decomposition Transform Method

In this section, the Adomian decomposition transform method solution system for fractional partial differential fractional equations.

$$\begin{aligned} &D_{\tau}^{\varpi}\nu(\psi,\tau) + \bar{\mathcal{M}}_{1}(\nu,\varphi) + \mathcal{N}_{1}(\nu,\varphi) - \mathcal{P}_{1}(\psi,\tau) = 0, \\ &D_{\tau}^{\varpi}\varphi(\psi,\tau) + \bar{\mathcal{G}}_{2}(\nu,\varphi) + \mathcal{N}_{2}(\nu,\varphi) - \mathcal{P}_{2}(\psi,\tau) = 0, \;\; \kappa - 1 < \varpi \leq \kappa, \end{aligned} \tag{8}$$

with initial conditions

$$\nu(\psi,0) = g_{1}(\psi), \quad \varphi(\psi,0) = g_{2}(\psi). \tag{9}$$

where $D_{\tau}^{\varpi} = \frac{\partial^{\varpi}}{\partial\tau^{\varpi}}$ the Caputo fractional derivative of order ϖ, $\bar{\mathcal{M}}_1$, $\bar{\mathcal{G}}_2$ and $\mathcal{N}_1, \mathcal{N}_2$ are linear and non-linear functions, respectively, and $\mathcal{P}_1$,$\mathcal{P}_2$ are source operators.

The Yang transformation is applied to Equation (8), we have

$$\begin{aligned} &Y[D_{\tau}^{\varpi}\nu(\psi,\tau)] + Y[\bar{\mathcal{M}}_{1}(\nu,\varphi) + \mathcal{N}_{1}(\nu,\varphi) - \mathcal{P}_{1}(\psi,\tau)] = 0, \\ &Y[D_{\tau}^{\varpi}\varphi(\psi,\tau)] + Y[\bar{\mathcal{G}}_{2}(\nu,\varphi) + \mathcal{N}_{2}(\nu,\varphi) - \mathcal{P}_{2}(\psi,\tau)] = 0. \end{aligned} \tag{10}$$

Using the Yang transform differentiation property, we get

$$\begin{aligned} Y[\nu(\psi,\tau)] &= s^{\varpi}\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^{J}\nu(\psi,\tau)}{\partial^{k}\tau}|_{\tau=0} + s^{\varpi}Y[\mathcal{P}_{1}(\psi,\tau)] - s^{\varpi}Y\{\bar{\mathcal{M}}_{1}(\nu,\varphi) + \mathcal{N}_{1}(\nu,\varphi)\}], \\ Y[\varphi(\psi,\tau)] &= s^{\varpi}\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^{J}\varphi(\psi,\tau)}{\partial^{k}\tau}|_{\tau=0} + s^{\varpi}Y[\mathcal{P}_{2}(\psi,\tau)] - s^{\varpi}Y\{\bar{\mathcal{G}}_{2}(\nu,\varphi) + \mathcal{N}_{2}(\nu,\varphi)\}], \end{aligned} \tag{11}$$

ADTM defines the result of infinite series $\nu(\psi,\tau)$ and $\varphi(\psi,\tau)$,

$$\nu(\psi,\tau) = \sum_{\kappa=0}^{\infty}\nu_{\kappa}(\psi,\tau), \quad \varphi(\psi,\tau) = \sum_{\kappa=0}^{\infty}\varphi_{\kappa}(\psi,\tau) \tag{12}$$

Adomian polynomials decomposition of nonlinear terms of $\mathcal{N}_1$ and $\mathcal{N}_2$ are described as

$$\mathcal{N}_{1}(\nu,\varphi) = \sum_{\kappa=0}^{\infty}\mathcal{A}_{\kappa}, \quad \mathcal{N}_{2}(\nu,\varphi) = \sum_{\kappa=0}^{\infty}\mathcal{B}_{\kappa}, \tag{13}$$

All forms of nonlinearity the Adomian polynomials can be represented as

$$\begin{aligned} \mathcal{A}_\kappa &= \frac{1}{\kappa!}\left[\frac{\partial^\kappa}{\partial\lambda^\kappa}\left\{\mathcal{N}_1\left(\sum_{J=0}^{\infty}\lambda^J\nu_J,\sum_{J=0}^{\infty}\lambda^J\varphi_J\right)\right\}\right]_{\lambda=0}, \\ \mathcal{B}_\kappa &= \frac{1}{\kappa!}\left[\frac{\partial^\kappa}{\partial\lambda^\kappa}\left\{\mathcal{N}_2\left(\sum_{J=0}^{\infty}\lambda^J\nu_J,\sum_{J=0}^{\infty}\lambda^J\varphi_J\right)\right\}\right]_{\lambda=0}. \end{aligned} \tag{14}$$

Putting Equations (12) and (14) into Equation (11), gives

$$\begin{aligned} Y\left[\sum_{\kappa=0}^{\infty}\nu_\kappa(\psi,\tau)\right] =& s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^J\nu(\psi,\tau)}{\partial^k\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_1(\psi,\tau)\} \\ &-s^\varpi Y\left\{\bar{\mathcal{M}}_1(\sum_{\kappa=0}^{\infty}\nu_\kappa,\sum_{\kappa=0}^{\infty}\varphi_m)+\sum_{m=0}^{\infty}\mathcal{A}_\kappa\right\}, \\ Y\left[\sum_{\kappa=0}^{\infty}\varphi_\kappa(\psi,\tau)\right] =& s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^J\varphi(\psi,\tau)}{\partial^k\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_2(\psi,\tau)\} \\ &-s^\varpi Y\left\{\bar{\mathcal{G}}_2(\sum_{\kappa=0}^{\infty}\nu_\kappa,\sum_{\kappa=0}^{\infty}\varphi_\kappa)+\sum_{\kappa=0}^{\infty}\mathcal{B}_\kappa\right\}. \end{aligned} \tag{15}$$

Using the inverse Yang transform of Equation (15), we get

$$\begin{aligned} \sum_{\kappa=0}^{\infty}\nu_\kappa(\psi,\tau) =& Y^{-1}\left[s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-k-1}}\frac{\partial^J\nu(\psi,\tau)}{\partial^J\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_1(\psi,\tau)\}\right. \\ &\left.-s^\varpi Y\left\{\bar{\mathcal{M}}_1\left(\sum_{\kappa=0}^{\infty}\nu_\kappa,\sum_{\kappa=0}^{\infty}\varphi_\kappa\right)+\sum_{\kappa=0}^{\infty}\mathcal{A}_\kappa\right\}\right], \\ \sum_{\kappa=0}^{\infty}\varphi_\kappa(\psi,\tau) =& Y^{-1}\left[s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^k\varphi(\psi,\tau)}{\partial^J\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_2(\psi,\tau)\}\right. \\ &\left.-s^\varpi Y\left\{\bar{\mathcal{G}}_2\left(\sum_{\kappa=0}^{\infty}\nu_\kappa,\sum_{\kappa=0}^{\infty}\varphi_\kappa\right)+\sum_{\kappa=0}^{\infty}\mathcal{B}_\kappa\right\}\right], \end{aligned} \tag{16}$$

we define the following terms,

$$\begin{aligned} \nu_0(\psi,\tau) &= Y^{-1}\left[s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^J\nu(\psi,\tau)}{\partial^J\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_1(\psi,\tau)\}\right], \\ \varphi_0(\psi,\tau) &= Y^{-1}\left[s^\varpi\sum_{J=0}^{\kappa-1}\frac{1}{s^{\varpi-J-1}}\frac{\partial^J\varphi(\psi,\tau)}{\partial^J\tau}|_{\tau=0}+s^\varpi Y\{\mathcal{P}_2(\psi,\tau)\}\right], \end{aligned} \tag{17}$$

$$\begin{aligned} \nu_1(\psi,\tau) &= -Y^{-1}\left[s^\varpi Y\{\bar{\mathcal{M}}_1(\nu_0,\varphi_0)+\mathcal{A}_0\}\right], \\ \varphi_1(\psi,\tau) &= -Y^{-1}\left[s^\varpi Y\{\bar{\mathcal{G}}_2(\nu_0,\varphi_0)+\mathcal{B}_0\}\right], \end{aligned}$$

the general for $\kappa \geq 1$, is given by

$$\begin{aligned} \nu_{\kappa+1}(\psi,\tau) &= -Y^{-1}\left[s^\varpi Y\{\bar{\mathcal{M}}_1(\nu_\kappa,\varphi_\kappa)+\mathcal{A}_\kappa\}\right], \\ \varphi_{\kappa+1}(\psi,\tau) &= -Y^{-1}\left[s^\varpi Y\{\bar{\mathcal{G}}_2(\nu_\kappa,\varphi_\kappa)+\mathcal{B}_\kappa\}\right], \end{aligned}$$

4. The Producer of Yang Variational Iteration Method

In this section explain the Yang variational iteration method the solution of FPDEs.

$$
\begin{aligned}
&D_\tau^\varpi \nu(\psi,\tau) + \bar{\mathcal{M}}_1(\nu,\varphi) + \mathcal{N}_1(\nu,\varphi) - \mathcal{P}_1(\psi,\tau) = 0,\\
&D_\tau^\varpi \varphi(\psi,\tau) + \bar{\mathcal{G}}_2(\nu,\varphi) + \mathcal{N}_2(\nu,\varphi) - \mathcal{P}_2(\psi,\tau) = 0, \quad m-1 < \varpi \leq m,
\end{aligned} \tag{18}
$$

with initial conditions

$$
\nu(\psi,0) = g_1(\psi), \quad \varphi(\psi,0) = g_2(\psi). \tag{19}
$$

where is $D_\tau^\varpi = \frac{\partial^\varpi}{\partial\tau^\varpi}$ the Caputo fractional derivative of order ϖ, $\bar{\mathcal{M}}_1$, $\bar{\mathcal{G}}_2$ and $\mathcal{N}_1$, $\mathcal{N}_2$ are linear and non-linear functions, respectively, and $\mathcal{P}_1$,$\mathcal{P}_2$ are source operators.

The Yang transformation is applied to Equation (18),

$$
\begin{aligned}
&Y[D_\tau^\varpi \nu(\psi,\tau)] + Y[\bar{\mathcal{M}}_1(\nu,\varphi) + \mathcal{N}_1(\nu,\varphi) - \mathcal{P}_1(\psi,\tau)] = 0,\\
&Y[D_\tau^\varpi \varphi(\psi,\tau)] + Y[\bar{\mathcal{G}}_2(\nu,\varphi) + \mathcal{N}_2(\nu,\varphi) - \mathcal{P}_2(\psi,\tau)] = 0,
\end{aligned} \tag{20}
$$

Using the differentiation property Yang transform, we get

$$
\begin{aligned}
&Y[\nu(\psi,\tau)] - \sum_{\kappa=0}^{J-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \nu(\psi,\tau)}{\partial^J \tau}|_{\tau=0} = -Y\left[\bar{\mathcal{M}}_1(\nu,\varphi) + \mathcal{N}_1(\nu,\varphi) - \mathcal{P}_1(\psi,\tau)\right],\\
&Y[\varphi(\psi,\tau)] - \sum_{\kappa=0}^{J-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \varphi(\psi,\tau)}{\partial^J \tau}|_{\tau=0} = -Y\left[\bar{\mathcal{G}}_2(\nu,\varphi) + \mathcal{N}_2(\nu,\varphi) - \mathcal{P}_2(\psi,\tau)\right],
\end{aligned} \tag{21}
$$

The procedure iteration method is define as

$$
\begin{aligned}
Y[\nu_{\kappa+1}(\psi,\tau)] =& Y[\nu_\kappa(\psi,\tau)] + \lambda(s)\left[s^\varpi \nu_\kappa(\psi,\tau) - \sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \nu(\psi,\tau)}{\partial^J \tau}|_{\tau=0} - Y[\mathcal{P}_1(\psi,\tau)]\right.\\
&\left. - Y\{\bar{\mathcal{M}}_1(\nu,\varphi) + \mathcal{N}_1(\nu,\varphi)\}\right],\\
Y[\varphi_{\kappa+1}(\psi,\tau)] =& Y[\varphi_\kappa(\psi,\tau)] + \lambda(s)\left[s^\varpi \varphi_\kappa(\psi,\tau) - \sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \varphi(\psi,\tau)}{\partial^J \tau}|_{\tau=0} - Y[\mathcal{P}_2(\psi,\tau)]\right.\\
&\left. - Y\{\bar{\mathcal{G}}_2(\nu,\varphi) + \mathcal{N}_2(\nu,\varphi)\}\right],
\end{aligned} \tag{22}
$$

A Lagrange multiplier as

$$
\lambda(s) = -s^\varpi, \tag{23}
$$

the inverse Yang transformation L^{-1}, the iteration method Equation (22) can be given as

$$
\begin{aligned}
\nu_{\kappa+1}(\psi,\tau) &= \nu_\kappa(\psi,\tau) - Y^{-1}\left[s^\varpi\left[\sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \nu(\psi,\tau)}{\partial^J \tau}|_{\tau=0} - Y[\mathcal{P}_1(\psi,\tau)] - Y\{\bar{\mathcal{M}}_1(\nu,\varphi) + \mathcal{N}_1(\nu,\varphi)\}\right]\right],\\
\varphi_{\kappa+1}(\psi,\tau) &= \varphi_\kappa(\psi,\tau) - Y^{-1}\left[s^\varpi\left[\sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \varphi(\psi,\tau)}{\partial^J \tau}|_{\tau=0} - Y[\mathcal{P}_2(\psi,\tau)] - Y\{\bar{\mathcal{G}}_2(\nu,\varphi) + \mathcal{N}_2(\nu,\varphi)\}\right]\right],
\end{aligned} \tag{24}
$$

the initial iteration can be find as

$$
\begin{aligned}
\nu_0(\psi,\tau) &= Y^{-1}\left[s^\varpi\left\{\sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \nu(\psi,\tau)}{\partial^J \tau}|_{\tau=0}\right\}\right],\\
\varphi_0(\psi,\tau) &= Y^{-1}\left[s^\varpi\left\{\sum_{J=0}^{\kappa-1} \frac{1}{s^{\varpi-J-1}} \frac{\partial^J \varphi(\psi,\tau)}{\partial^J \tau}|_{\tau=0}\right\}\right].
\end{aligned} \tag{25}
$$

5. Implementation of Techniques

Problem 1. *Consider the fractional-order non-linear Jaulent–Miodek equation is given as*

$$\begin{aligned}
&\frac{\partial^{\varpi}\nu}{\partial\tau^{\varpi}}+\frac{\partial^3\nu}{\partial\psi^3}+\frac{3}{2}\varphi\frac{\partial^3\varphi}{\partial\psi^3}+\frac{9}{2}\frac{\partial\varphi}{\partial\psi}\frac{\partial^2\varphi}{\partial\psi^2}-6\nu\frac{\partial\nu}{\partial\psi}-6\nu\varphi\frac{\partial\varphi}{\partial\psi}-\frac{3}{2}\frac{\partial\nu}{\partial\psi}\varphi^2=0\\
&\frac{\partial^{\varpi}\nu}{\partial\tau^{\varpi}}+\frac{\partial^3\varphi}{\partial\psi^3}-6\frac{\partial\nu}{\partial\psi}\varphi-6\nu\frac{\partial\varphi}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi}{\partial\psi}\varphi^2=0
\end{aligned}\tag{26}$$

With initial conditions

$$\begin{aligned}
\nu(\psi,0)&=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right),\\
\varphi(\psi,0)&=\lambda sech\left(\frac{\lambda\psi}{2}\right).
\end{aligned}\tag{27}$$

Taking Yang transform of (26),

$$\begin{aligned}
&\frac{1}{s^{\varpi}}Y[\nu(\psi,\tau)]-\frac{1}{s^{\varpi-1}}\nu(\psi,0)=-Y\left[\frac{\partial^3\nu}{\partial\psi^3}+\frac{3}{2}\varphi\frac{\partial^3\varphi}{\partial\psi^3}+\frac{9}{2}\frac{\partial\varphi}{\partial\psi}\frac{\partial^2\varphi}{\partial\psi^2}-6\nu\frac{\partial\nu}{\partial\psi}-6\nu\varphi\frac{\partial\varphi}{\partial\psi}-\frac{3}{2}\frac{\partial\nu}{\partial\psi}\varphi^2\right],\\
&\frac{1}{s^{\varpi}}Y[\varphi(\psi,\tau)]-\frac{1}{s^{\varpi-1}}\varphi(\psi,0)=-Y\left[\frac{\partial^3\varphi}{\partial\psi^3}-6\frac{\partial\nu}{\partial\psi}\varphi-6\nu\frac{\partial\varphi}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi}{\partial\psi}\varphi^2\right].
\end{aligned}$$

Using inverse Yang transformation, we get

$$\begin{aligned}
\nu(\psi,\tau)=&Y^{-1}\left[\frac{\nu(\psi,0)}{s}-\frac{1}{s^{\varpi}}Y\left\{\frac{\partial^3\nu}{\partial\psi^3}+\frac{3}{2}\varphi\frac{\partial^3\varphi}{\partial\psi^3}+\frac{9}{2}\frac{\partial\varphi}{\partial\psi}\frac{\partial^2\varphi}{\partial\psi^2}-6\nu\frac{\partial\nu}{\partial\psi}-6\nu\varphi\frac{\partial\varphi}{\partial\psi}-\frac{3}{2}\frac{\partial\nu}{\partial\psi}\varphi^2\right\}\right],\\
\varphi(\psi,\tau)=&Y^{-1}\left[\frac{\varphi(\psi,0)}{s}-\frac{1}{s^{\varpi}}Y\left\{\frac{\partial^3\varphi}{\partial\psi^3}-6\frac{\partial\nu}{\partial\psi}\varphi-6\nu\frac{\partial\varphi}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi}{\partial\psi}\varphi^2\right\}\right].
\end{aligned}$$

Applying the Adomian procedure, we have

$$\begin{aligned}
\nu_0(\psi,\tau)&=Y^{-1}\left[\frac{\nu(\psi,0)}{s}\right]=Y^{-1}\left[\frac{\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right)}{s}\right],\\
\varphi_0(\psi,\tau)&=Y^{-1}\left[\frac{\varphi(\psi,0)}{s}\right]=Y^{-1}\left[\frac{\lambda sech\left(\frac{\lambda\psi}{2}\right)}{s}\right],
\end{aligned}$$

$$\begin{aligned}
\nu_0(\psi,\tau)&=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right),\\
\varphi_0(\psi,\tau)&=\lambda sech\left(\frac{\lambda\psi}{2}\right).
\end{aligned}\tag{28}$$

$$\begin{aligned}
&\sum_{\kappa=0}^{\infty}\nu_{\kappa+1}(\psi,\tau)=-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left\{\sum_{\kappa=0}^{\infty}\frac{\partial^3\nu_\kappa}{\partial\psi^3}+\frac{3}{2}\sum_{\kappa=0}^{\infty}A_\kappa+\frac{9}{2}\sum_{\kappa=0}^{\infty}B_\kappa-6\sum_{\kappa=0}^{\infty}C_\kappa-6\sum_{\kappa=0}^{\infty}D_\kappa-\frac{3}{2}\sum_{\kappa=0}^{\infty}E_\kappa\right\}\right],\quad\kappa=0,1,2,\cdots\\
&\sum_{\kappa=0}^{\infty}\varphi_{\kappa+1}(\psi,\tau)=-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left\{\sum_{\kappa=0}^{\infty}\frac{\partial^3\varphi_\kappa}{\partial\psi^3}-6\sum_{\kappa=0}^{\infty}F_\kappa-6\sum_{\kappa=0}^{\infty}G_\kappa-\frac{15}{2}\sum_{\kappa=0}^{\infty}H_\kappa\right\}\right],\quad\kappa=0,1,2,\cdots
\end{aligned}$$

The nonlinear terms find with the help of Adomain polynomials

For $\kappa = 1$

$$\begin{aligned}\nu_1(\psi,\tau) &= \frac{-2\lambda^5 \csc h^3(\lambda\psi)\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)}\\ \varphi_1(\psi,\tau) &= \frac{\lambda^4 \csc h^2(\lambda\psi)\sec h^3\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)}\end{aligned} \tag{29}$$

for $\kappa = 2$

$$\begin{aligned}\nu_2(\psi,\tau) &= -\frac{\lambda^8(-2+\cos h(\lambda\psi))\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{2\varpi}}{16\Gamma(2\varpi+1)}\\ \varphi_2(\psi,\tau) &= \frac{\lambda^7(-3+\cos h(\lambda\psi))\sec h^3\left(\frac{\lambda\psi}{2}\right)(\tau)^{2\varpi}}{32\Gamma(2\varpi+1)}\end{aligned} \tag{30}$$

for $\kappa = 2$

$$\begin{aligned}\nu_3(\psi,\tau) &= -\frac{1}{1024(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{11}\Bigg\{2\bigg(-165+28\cosh(\lambda\psi)+\cosh(2\lambda\psi)(\Gamma(\varpi+1)\bigg)^2\\ &+3\bigg(43-20\cosh(\lambda\psi)+\cosh(2\lambda\psi)\Gamma(2\varpi+1)\sec h^6\left(\frac{\lambda\psi}{2}\right)\tanh(\frac{\lambda\psi}{2})(\tau)^{3\varpi}\bigg)\bigg)\Bigg\}\\ \varphi_3(\psi,\tau) &= \frac{1}{512(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{10}\Bigg\{(147-92\cosh(\lambda\psi)+\cosh(2\lambda\psi))(\Gamma(\varpi+1))^2\\ &+12(-7+3\cosh(\lambda\psi))\Gamma(2\varpi+1)\sec h^5\left(\frac{\lambda\psi}{2}\right)\tanh\left(\frac{\lambda\psi}{2}\right)(\tau)^{3\varpi}\Bigg\}\end{aligned} \tag{31}$$

The ADTM solution is

$$\begin{aligned}\nu(\psi,\tau) &= \nu_0(\psi,\tau)+\nu_1(\psi,\tau)+\nu_2(\psi,\tau)+\nu_3(\psi,\tau)+\cdots,\\ \varphi(\psi,\tau) &= \varphi_0(\psi,\tau)+\varphi_1(\psi,\tau)+\varphi_2(\psi,\tau)+\varphi_3(\psi,\tau)+\cdots,\end{aligned}$$

$$\begin{aligned}\nu(\psi,\tau) &= \frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right)-\frac{2\lambda^5\csc h^3(\lambda\psi)\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)}-\frac{\lambda^8(-2+\cos h(\lambda\psi))\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{2\varpi}}{16\Gamma(2\varpi+1)}\\ &-\frac{1}{1024(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{11}\Bigg\{2\bigg(-165+28\cosh(\lambda\psi)+\cosh(2\lambda\psi)(\Gamma(\varpi+1)\bigg)^2\\ &+3\bigg(43-20\cosh(\lambda\psi)+\cosh(2\lambda\psi)\Gamma(2\varpi+1)\sec h^6\left(\frac{\lambda\psi}{2}\right)\tanh\left(\frac{\lambda\psi}{2}\right)(\tau)^{3\varpi}\bigg)\bigg)\Bigg\}-\cdots.\\ \varphi(\psi,\tau) &= \lambda sech\left(\frac{\lambda\psi}{2}\right)+\frac{\lambda^4\csc h^2(\lambda\psi)\sec h^3\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)}+\frac{\lambda^7(-3+\cos h(\lambda\psi))\sec h^3\left(\frac{\lambda\psi}{2}\right)(\tau)^{2\varpi}}{32\Gamma(2\varpi+1)}\\ &+\frac{1}{512(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{10}\Bigg\{(147-92\cosh(\lambda\psi)+\cosh(2\lambda\psi))(\Gamma(\varpi+1))^2\\ &+12(-7+3\cosh(\lambda\psi))\Gamma(2\varpi+1)\sec h^5\left(\frac{\lambda\psi}{2}\right)\tanh\left(\frac{\lambda\psi}{2}\right)(\tau)^{3\varpi}\Bigg\}+\cdots\end{aligned} \tag{32}$$

The approximate solution by VITM
Apply the iteration technique, we have

$$\sum_{\kappa=0}^{\infty}\nu_{\kappa+1}(\psi,\tau)=\nu_{\kappa}(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\sum_{\kappa=0}^{\infty}\frac{\partial^3\nu_\kappa}{\partial\psi^3}+\frac{3}{2}\sum_{\kappa=0}^{\infty}\varphi_\kappa\frac{\partial^3\varphi_\kappa}{\partial\psi^3}+\frac{9}{2}\sum_{\kappa=0}^{\infty}\frac{\partial\varphi_\kappa}{\partial\psi}\frac{\partial^2\varphi_\kappa}{\partial\psi^2}\right.\right.$$
$$\left.\left.-6\sum_{\kappa=0}^{\infty}\nu_\kappa\frac{\partial\nu_\kappa}{\partial\psi}-6\sum_{\kappa=0}^{\infty}\nu_\kappa\varphi_\kappa\frac{\partial\varphi_\kappa}{\partial\psi}-\frac{3}{2}\sum_{\kappa=0}^{\infty}\frac{\partial\nu_\kappa}{\partial\psi}\varphi_\kappa^2\right]\right],\quad\kappa=0,1,2,\cdots \tag{33}$$
$$\sum_{\kappa=0}^{\infty}\varphi_{\kappa+1}(\psi,\tau)=\varphi_{\kappa}(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\sum_{\kappa=0}^{\infty}\frac{\partial^3\varphi_\kappa}{\partial\psi^3}-6\sum_{\kappa=0}^{\infty}\frac{\partial\nu_\kappa}{\partial\psi}\varphi_\kappa-6\sum_{\kappa=0}^{\infty}\nu_\kappa\frac{\partial\varphi_\kappa}{\partial\psi}-\frac{15}{2}\sum_{\kappa=0}^{\infty}\frac{\partial\varphi_\kappa}{\partial\psi}\varphi_\kappa^2\right]\right],$$

where

$$\nu_0(\psi,\tau)=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right),$$
$$\varphi_0(\psi,\tau)=\lambda sech\left(\frac{\lambda\psi}{2}\right). \tag{34}$$

For $\kappa=0,1,2,\cdots$

$$\nu_1(\psi,\tau)=\nu_0(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3\nu_0}{\partial\psi^3}+\frac{3}{2}\varphi_0\frac{\partial^3\varphi_0}{\partial\psi^3}+\frac{9}{2}\frac{\partial\varphi_0}{\partial\psi}\frac{\partial^2\varphi_0}{\partial\psi^2}\right.\right.$$
$$\left.\left.-6\nu_0\frac{\partial\nu_0}{\partial\psi}-6\nu_0\varphi_0\frac{\partial\varphi_0}{\partial\psi}-\frac{3}{2}\frac{\partial\nu_0}{\partial\psi}\varphi_0^2\right]\right], \tag{35}$$
$$\varphi_1(\psi,\tau)=\varphi_0(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3\varphi_0}{\partial\psi^3}-6\frac{\partial\nu_0}{\partial\psi}\varphi_0-6\nu_0\frac{\partial\varphi_0}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi_0}{\partial\psi}\varphi_0^2\right]\right],$$

$$\nu_1(\psi,\tau)=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right)-\frac{2\lambda^5\csc h^3(\lambda\psi)\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)}$$
$$\varphi_1(\psi,\tau)=\lambda sech\left(\frac{\lambda\psi}{2}\right)+\frac{\lambda^4\csc h^2(\lambda\psi)\sec h^3\left(\frac{\lambda\psi}{2}\right)(\tau)^{\varpi}}{\Gamma(\varpi+1)} \tag{36}$$

$$\nu_2(\psi,\tau)=\nu_1(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3\nu_1}{\partial\psi^3}+\frac{3}{2}\varphi_1\frac{\partial^3\varphi_1}{\partial\psi^3}+\frac{9}{2}\frac{\partial\varphi_1}{\partial\psi}\frac{\partial^2\varphi_1}{\partial\psi^2}\right.\right.$$
$$\left.\left.-6\nu_1\frac{\partial\nu_1}{\partial\psi}-6\nu_1\varphi_1\frac{\partial\varphi_1}{\partial\psi}-\frac{3}{2}\frac{\partial\nu_1}{\partial\psi}\varphi_1^2\right]\right], \tag{37}$$
$$\varphi_2(\psi,\tau)=\varphi_1(\psi,\tau)-Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3\varphi_1}{\partial\psi^3}-6\frac{\partial\nu_1}{\partial\psi}\varphi_1-6\nu_1\frac{\partial\varphi_1}{\partial\psi}-\frac{15}{2}\frac{\partial\varphi_1}{\partial\psi}\varphi_1^2\right]\right],$$

$$\nu_2(\psi,\tau)=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda\psi}{2}\right)\right)-\frac{2\lambda^5\csc h^3(\lambda\psi)\sec h^4(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)}-\frac{\lambda^8(-2+\cos h(\lambda\psi))\sec h^4\left(\frac{\lambda\psi}{2}\right)(\tau)^{2\varpi}}{16\Gamma(2\varpi+1)}$$
$$\varphi_2(\psi,\tau)=\lambda sech\left(\frac{\lambda\psi}{2}\right)+\frac{\lambda^4\csc h^2(\lambda\psi)\sec h^3(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)}+\frac{\lambda^7(-3+\cos h(\lambda\psi))\sec h^3(\frac{\lambda\psi}{2})(\tau)^{2\varpi}}{32\Gamma(2\varpi+1)} \tag{38}$$

$$\nu_3(\psi,\tau) = \nu_2(\psi,\tau) - Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3 \nu_2}{\partial\psi^3} + \frac{3}{2}\varphi_2\frac{\partial^3\varphi_2}{\partial\psi^3} + \frac{9}{2}\frac{\partial\varphi_2}{\partial\psi}\frac{\partial^2\varphi_2}{\partial\psi^2} - 6\nu_2\frac{\partial\nu_2}{\partial\psi} - 6\nu_2\varphi_2\frac{\partial\varphi_2}{\partial\psi} - \frac{3}{2}\frac{\partial\nu_2}{\partial\psi}\varphi_2^2\right]\right],$$
$$\varphi_3(\psi,\tau) = \varphi_2(\psi,\tau) - Y^{-1}\left[\frac{1}{s^{\varpi}}Y\left[\frac{\partial^3\varphi_2}{\partial\psi^3} - 6\frac{\partial\nu_2}{\partial\psi}\varphi_2 - 6\nu_2\frac{\partial\varphi_2}{\partial\psi} - \frac{15}{2}\frac{\partial\varphi_2}{\partial\psi}\varphi_2^2\right]\right], \tag{39}$$

$$\nu_3(\psi,\tau) = \frac{1}{8}\lambda^2\left(1 - 4sech^2\left(\frac{\lambda\psi}{2}\right)\right) - \frac{2\lambda^5 \csc h^3(\lambda\psi)\sec h^4(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)} - \frac{\lambda^8(-2+\cos h(\lambda\psi))\sec h^4(\frac{\lambda\psi}{2})(\tau)^{2\varpi}}{16\Gamma(2\varpi+1)}$$
$$- \frac{1}{1024(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{11}\left\{2\left(-165 + 28\cosh(\lambda\psi) + \cosh(2\lambda\psi)(\Gamma(\varpi+1)\right)^2\right.$$
$$\left.+3\left(43 - 20\cosh(\lambda\psi) + \cosh(2\lambda\psi)\Gamma(2\varpi+1)\sec h^6(\frac{\lambda\psi}{2})\tanh(\frac{\lambda\psi}{2})(\tau)^{3\varpi}\right)\right)\right\}.$$
$$\varphi_3(\psi,\tau) = \lambda sech\left(\frac{\lambda\psi}{2}\right) + \frac{\lambda^4\csc h^2(\lambda\psi)\sec h^3(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)} + \frac{\lambda^7(-3+\cos h(\lambda\psi))\sec h^3(\frac{\lambda\psi}{2})(\tau)^{2\varpi}}{32\Gamma(2\varpi+1)}$$
$$+ \frac{1}{512(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{10}\left\{(147 - 92\cosh(\lambda\psi) + \cosh(2\lambda\psi))(\Gamma(\varpi+1))^2\right.$$
$$\left.+12(-7+3\cosh(\lambda\psi))\Gamma(2\varpi+1)\sec h^5(\frac{\lambda\psi}{2})\tanh(\frac{\lambda\psi}{2})(\tau)^{3\varpi}\right\} \tag{40}$$

$$\nu(\psi,\tau) = \frac{1}{8}\lambda^2\left(1 - 4sech^2\left(\frac{\lambda\psi}{2}\right)\right) - \frac{2\lambda^5 \csc h^3(\lambda\psi)\sec h^4(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)} - \frac{\lambda^8(-2+\cos h(\lambda\psi))\sec h^4(\frac{\lambda\psi}{2})(\tau)^{2\varpi}}{16\Gamma(2\varpi+1)}$$
$$- \frac{1}{1024(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{11}\left\{2\left(-165 + 28\cosh(\lambda\psi) + \cosh(2\lambda\psi)(\Gamma(\varpi+1)\right)^2\right.$$
$$\left.+3\left(43 - 20\cosh(\lambda\psi) + \cosh(2\lambda\psi)\Gamma(2\varpi+1)\sec h^6(\frac{\lambda\psi}{2})\tanh(\frac{\lambda\psi}{2})(\tau)^{3\varpi}\right)\right)\right\} - \cdots.$$
$$\varphi(\psi,\tau) = \lambda sech\left(\frac{\lambda\psi}{2}\right) + \frac{\lambda^4\csc h^2(\lambda\psi)\sec h^3(\frac{\lambda\psi}{2})(\tau)^{\varpi}}{\Gamma(\varpi+1)} + \frac{\lambda^7(-3+\cos h(\lambda\psi))\sec h^3(\frac{\lambda\psi}{2})(\tau)^{2\varpi}}{32\Gamma(2\varpi+1)}$$
$$+ \frac{1}{512(\Gamma(\varpi+1))^2\Gamma(3\varpi+1)}\lambda^{10}\left\{(147 - 92\cosh(\lambda\psi) + \cosh(2\lambda\psi))(\Gamma(\varpi+1))^2\right.$$
$$\left.+12(-7+3\cosh(\lambda\psi))\Gamma(2\varpi+1)\sec h^5(\frac{\lambda\psi}{2})\tanh\left(\frac{\lambda\psi}{2}\right)(\tau)^{3\varpi}\right\} + \cdots \tag{41}$$

The exact solution of Equation (8) at $\varpi = 1$,

$$\nu(\psi,\tau) = \frac{1}{8}\lambda^2\left(1 - 4sech^2\left(\frac{\lambda}{2}\left(\psi + \frac{1}{2}\lambda^2\tau\right)\right)\right),$$
$$\varphi(\psi,\tau) = \lambda sech\left(\frac{\lambda}{2}\left(\psi + \frac{1}{2}\lambda^2\tau\right)\right). \tag{42}$$

Figure 1, exact and analytical solution of $\nu(\psi,\tau)$ *at* $\varpi = 1$ *and* 0.8 *and Figure 2, analytical solutions of* $\nu(\psi,\tau)$ *at* $\varpi = 0.6$ *and* 0.4*. Figure 3, the different fractional-order of* ϖ *of* $\nu(\psi,\tau)$*. In Tables 1 and 2 the different fractional-order of* $\wp$ *of* $\nu(\psi,\tau)$ *and* $\varphi(\psi,\tau)$ *of Problem 1.*

Figure 4, exact and analytical solution of $\varphi(\psi,\tau)$ at $\varpi = 1$ and 0.8 and Figure 5, the analytical solutions of $\varphi(\psi,\tau)$ at $\varpi = 0.6$ and 0.4. Figure 6, the different fractional-order of ϖ of $\varphi(\psi,\tau)$.

Table 1. The different fractional-order of $\wp$ at of $\nu(\psi,\tau)$ of Problem 1.

x	(τ)	$\wp = 0.6$	$\wp = 0.75$	$\wp = 0.9$	$\wp = 1$
0.2	0.2	1.14864×10^{-4}	6.02179×10^{-5}	2.03492×10^{-5}	1.24335×10^{-10}
	0.4	1.49003×10^{-4}	8.38441×10^{-5}	3.00661×10^{-5}	1.98709×10^{-9}
	0.6	1.61589×10^{-4}	9.58441×10^{-5}	3.59113×10^{-5}	1.00477×10^{-8}
	0.8	1.58816×10^{-4}	9.90276×10^{-5}	3.86768×10^{-5}	3.17168×10^{-8}
	1	1.43066×10^{-4}	9.43676×10^{-5}	3.85778×10^{-5}	7.73351×10^{-8}
0.4	0.2	2.00217×10^{-4}	1.07699×10^{-4}	3.71407×10^{-5}	1.16347×10^{-10}
	0.4	2.40806×10^{-4}	1.39099×10^{-4}	5.10803×10^{-5}	1.85733×10^{-9}
	0.6	2.43904×10^{-4}	1.48381×10^{-4}	5.69845×10^{-5}	9.38090×10^{-9}
	0.8	2.23097×10^{-4}	1.42794×10^{-4}	5.72312×10^{-5}	2.95781×10^{-8}
	1	1.83773×10^{-4}	1.25233×10^{-4}	5.27649×10^{-5}	7.20380×10^{-8}
0.6	0.2	2.81663×10^{-4}	1.53074×10^{-4}	5.32052×10^{-5}	1.03760×10^{-10}
	0.4	3.27938×10^{-4}	1.91648×10^{-4}	7.10983×10^{-5}	1.65425×10^{-9}
	0.6	3.21523×10^{-4}	1.98048×10^{-4}	7.69518×10^{-5}	8.34447×10^{-9}
	0.8	2.83130×10^{-4}	1.83823×10^{-4}	7.46824×10^{-5}	2.62762×10^{-8}
	1	2.21041×10^{-4}	1.53730×10^{-4}	6.59448×10^{-5}	6.39134×10^{-8}
0.8	0.2	3.57726×10^{-4}	1.95516×10^{-4}	6.82488×10^{-5}	8.74434×10^{-11}
	0.4	4.08852×10^{-4}	2.40549×10^{-4}	8.97594×10^{-5}	1.39192×10^{-9}
	0.6	3.93110×10^{-4}	2.43979×10^{-4}	9.54607×10^{-5}	7.01008×10^{-9}
	0.8	3.37931×10^{-4}	2.21425×10^{-4}	9.07319×10^{-5}	2.20392×10^{-8}
	1	2.54330×10^{-4}	1.79417×10^{-4}	7.79054×10^{-5}	5.35217×10^{-8}

Table 2. The different fractional-order of $\wp$ at $\varphi(\psi,\tau)$ Problem 1.

x	(τ)	$\wp = 0.6$	$\wp = 0.75$	$\wp = 0.9$	$\wp = 1$
0.2	0.2	2.30416×10^{-4}	1.20720×10^{-4}	4.07827×10^{-5}	5.21689×10^{-9}
	0.4	2.99561×10^{-4}	1.68382×10^{-4}	6.03594×10^{-5}	4.29844×10^{-8}
	0.6	3.25817×10^{-4}	1.92979×10^{-4}	7.22983×10^{-5}	1.49281×10^{-7}
	0.8	3.21531×10^{-4}	2.00136×10^{-4}	7.82094×10^{-5}	3.63810×10^{-7}
	1	2.91426×10^{-4}	1.91790×10^{-4}	7.85409×10^{-5}	7.29980×10^{-7}
0.4	0.2	4.03498×10^{-4}	2.16852×10^{-4}	7.47504×10^{-5}	1.01172×10^{-8}
	0.4	4.86882×10^{-4}	2.80810×10^{-4}	1.03056×10^{-4}	8.21274×10^{-8}
	0.6	4.95394×10^{-4}	3.00730×10^{-4}	1.15446×10^{-4}	2.81182×10^{-7}
	0.8	4.56224×10^{-4}	2.91168×10^{-4}	1.16744×10^{-4}	6.75959×10^{-7}
	1	3.80107×10^{-4}	2.57923×10^{-4}	1.08885×10^{-4}	1.33863×10^{-6}
0.6	0.2	5.71624×10^{-4}	3.10320×10^{-4}	1.07799×10^{-4}	1.47159×10^{-8}
	0.4	6.68234×10^{-4}	3.89793×10^{-4}	1.44487×10^{-4}	1.18823×10^{-7}
	0.6	6.58907×10^{-4}	4.04795×10^{-4}	1.57175×10^{-4}	4.04709×10^{-7}
	0.8	5.85337×10^{-4}	3.78632×10^{-4}	1.53839×10^{-4}	9.67991×10^{-7}
	1	4.64139×10^{-4}	3.20891×10^{-4}	1.37879×10^{-4}	1.90747×10^{-6}
0.8	0.2	7.32846×10^{-4}	4.00036×10^{-4}	1.39542×10^{-4}	1.88855×10^{-8}
	0.4	8.41543×10^{-4}	4.94077×10^{-4}	1.84172×10^{-4}	1.52057×10^{-7}
	0.6	8.14513×10^{-4}	5.03993×10^{-4}	1.97007×10^{-4}	5.16457×10^{-7}
	0.8	7.07446×10^{-4}	4.61552×10^{-4}	1.89076×10^{-4}	1.23187×10^{-6}
	1	5.42633×10^{-4}	3.80010×10^{-4}	1.65197×10^{-4}	2.42089×10^{-6}
1	0.2	8.85417×10^{-4}	4.85021×10^{-4}	1.69634×10^{-4}	2.25214×10^{-8}
	0.4	1.00496×10^{-3}	5.92545×10^{-4}	2.21684×10^{-4}	1.80999×10^{-7}
	0.6	9.60595×10^{-4}	5.97282×10^{-4}	2.34519×10^{-4}	6.13643×10^{-7}
	0.8	8.21325×10^{-4}	5.39082×10^{-4}	2.22087×10^{-4}	1.46106×10^{-6}
	1	6.14855×10^{-4}	4.34707×10^{-4}	1.90563×10^{-4}	2.86622×10^{-6}

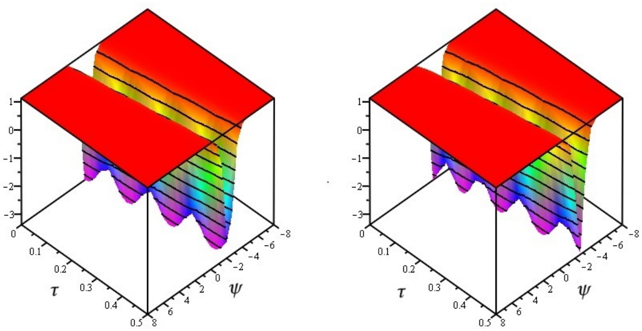

Figure 1. Exact and analytical solution of $\nu(\psi,\tau)$ at $\varpi = 1$ and 0.8.

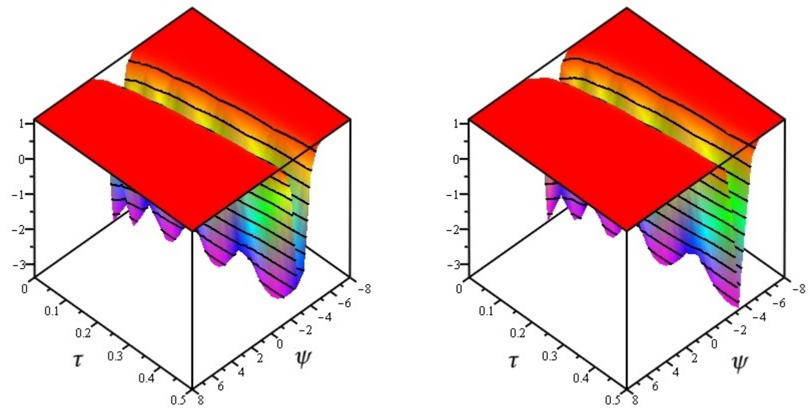

Figure 2. The analytical solutions of $\nu(\psi,\tau)$ at $\varpi = 0.6$ and 0.4.

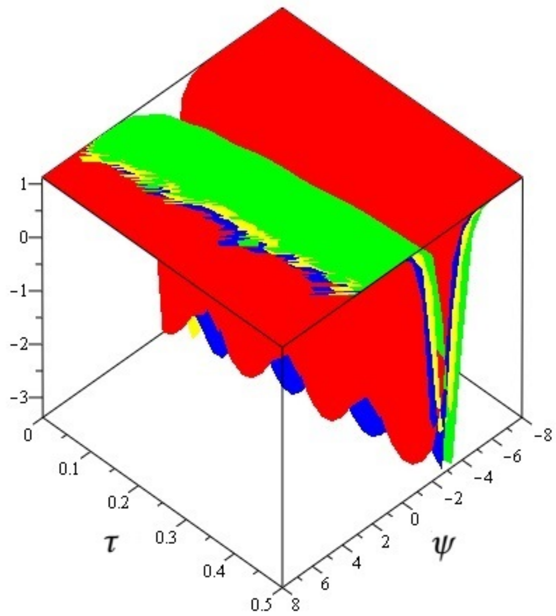

Figure 3. The different fractional-order of ϖ of $\nu(\psi,\tau)$.

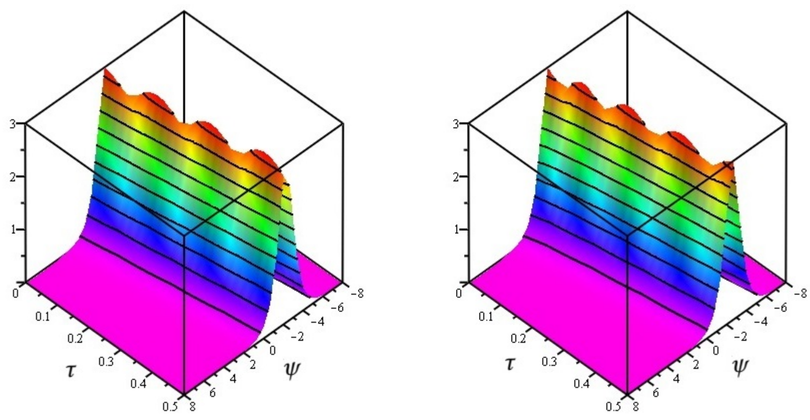

Figure 4. Exact and analytical solution of $\varphi(\psi,\tau)$ at $\varpi = 1$ and 0.8.

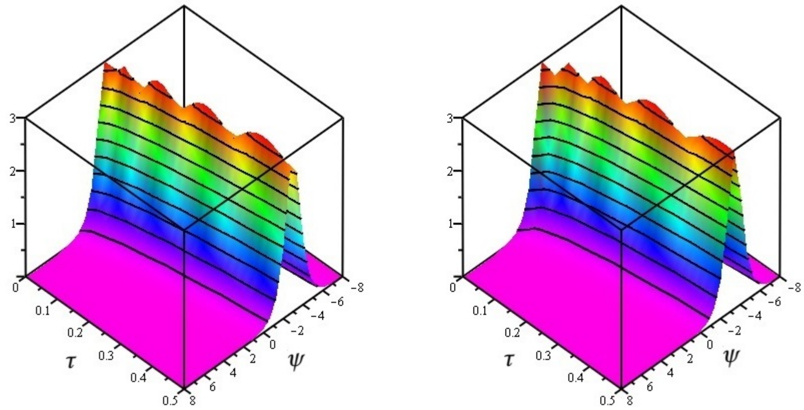

Figure 5. The analytical solutions of $\varphi(\psi, \tau)$ at $\varpi = 0.6$ and 0.4.

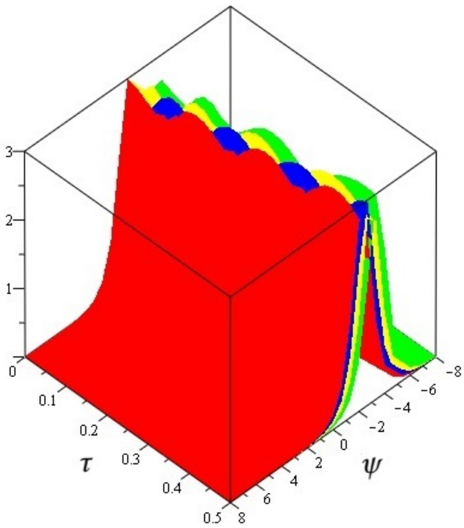

Figure 6. The different fractional-order of ϖ of $\varphi(\psi, \tau)$.

6. Conclusions

In this paper, we have successfully applied two modified methods to investigate the numerical results of fractional system Jaulent–Miodek equations connected with energy dependent Schrodinger potential. Accord between the obtained numerical solution by ADTM and VITM with exact results appears very appreciable by way of illustrative solutions. The suggested algorithms are simple to implement, efficient for achieving the solutions of nonlinear coupled JM equations of fractional order. In addition, both the ADTM and the VITM yield convergent series solutions with easily determinable components without using perturbation, linearization, or limiting assumptions. We can draw a final conclusion that the suggested methods are extremely analytical and more reliable, and that they can be used to analyze nonlinear issues that develop in complicated processes.

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