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Second-Order Dynamic Equations with Noncanonical Operator: Oscillatory Behavior

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Abstract: The present article aims to study the oscillatory properties of a class of second-order dynamic equations on time scales. We consider during this study the noncanonical case, which did not receive much attention compared to the canonical dynamic equations. The approach adopted depends on converting the noncanonical equation to a corresponding canonical equation. By using this transformation and based on several techniques, we create new, more effective, and sharp oscillation criteria. Finally, we explain the effectiveness and importance of the results by applying them to some special cases of the studied equation.

Keywords: second order; nonlinear dynamic equations; oscillation; noncanonical case



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1. Introduction

The study of dynamic equations on time scales dates back to its founder Hilger [1] and has become a prominent area of mathematics. It was created to unify the study of differential and difference equations. Meanwhile, various theoretical aspects of this theory have recently been debated. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . To be thorough, we recall some time scale notions. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\sigma(l) = \inf\{s \in \mathbb{T} \mid s > l\} \quad \text{and} \quad \rho(l) = \sup\{s \in \mathbb{T} \mid s < l\},$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $l \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, left-dense, if $\sigma(l) > l, \sigma(l) = l, \rho(l) < l, \rho(l) = l$ holds, respectively. The set \mathbb{T}^κ is defined to be \mathbb{T} if \mathbb{T} does not have a left-scattered maximum; otherwise it is \mathbb{T} without this left-scattered maximum. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. Hence the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant 1 for $\mathbb{T} = \mathbb{Z}$. However, a time scale \mathbb{T} could have nonconstant graininess. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous and is written $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, provided that f is continuous at right dense points and at left dense points in \mathbb{T} , left hand limits exist and are finite. We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at $l \in \mathbb{T}$ whenever

$$f^\Delta := \lim_{s \rightarrow l} \frac{f(l) - f(s)}{l - s}$$

exists when $\sigma(l) = l$ (here, by $s \rightarrow l$, it is understood that s approaches l in the time scale) and when f is continuous at l and $\sigma(l) > l$ it is

$$f^\Delta := \lim_{s \rightarrow l} \frac{f(\sigma(l)) - f(l)}{\mu(l)}.$$

The product and quotient rules ([2], Theorem 1.20) for the derivative of the product fg and the quotient f/g of two differentiable functions f and g are as follows:

$$(fg)^\Delta(l) = f^\Delta(l)g(l) + f(\sigma(l))g^\Delta = f(l)g^\Delta(l) + f^\Delta(l)g(\sigma(l)), \quad (1)$$

$$\left(\frac{f}{g}\right)^\Delta(l) = \frac{f^\Delta(l)g(l) - f(l)g^\Delta(l)}{g(l)g(\sigma(l))}. \quad (2)$$

The chain rule ([2], Theorem 1.90) for the derivative of the composite function $f \circ g$ of a continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a (delta) differentiable function $g : \mathbb{T} \rightarrow \mathbb{R}$ results in

$$(f \circ g)^\Delta = \left\{ \int_0^1 f'(g + h\mu g^\Delta) dh \right\} g^\Delta. \quad (3)$$

For a great introduction to the fundamentals of time scales, see [2,3].

In this work, we investigate the oscillatory properties of the solutions of noncanonical second-order dynamic equations of the form

$$[a_1(l)(v(l))^\Delta]^\Delta + g_1(l)v^\alpha(q(l)) = 0, \quad l \geq l_0 > 0. \quad (4)$$

The following assumptions will be needed throughout the paper:

- (H1) α is a ratio of two positive integers;
- (H2) $a_1, q \in C_{rd}^1([l_0, \infty)_{\mathbb{T}}, (0, \infty)_{\mathbb{T}})$, $a_1(l) > 0$, $q(l) \leq l$, $q^\Delta(l) \geq 0$, and $\lim_{l \rightarrow \infty} q(l) = \infty$;
- (H3) $g_1(l) \in C_{rd}([l_0, \infty)_{\mathbb{T}})$ is positive.

Following Trench [4], we shall say that Equation (4) is in canonical form if

$$\int_{l_0}^{\infty} \frac{\Delta s}{a_1(s)} = \infty.$$

Conversely, we say that (4) is in noncanonical form if

$$\xi(l_0) := \int_{l_0}^{\infty} \frac{\Delta s}{a_1(s)} < \infty.$$

A solution $v(l)$ of (4) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Equation (4) is said to be oscillatory if all its solutions oscillate. Oscillation phenomena arise in a variety of models based on real-world applications. As a result, much research has been conducted on the oscillatory behaviour of various classes of dynamic equations and their special cases.

For instance, when $\mathbb{T} = \mathbb{Z}$, Chatzarakis et al. [5] obtained new oscillation criteria for the second-order advanced difference equation

$$\Delta(a_1(l)\Delta(v(l))) + g_1(l)v(q(l)) = 0, \quad (5)$$

where $q(l) \geq l + 1$ in the noncanonical form, via a canonical transformation. Saker [6] investigated the delay equation

$$\Delta(a_1(l)(\Delta v(l))^\gamma) + g_1(l)f(v(l - \sigma)) = 0, \quad (6)$$

and established some sufficient conditions for every solution of (6) to be oscillatory in the canonical and noncanonical cases.

On the other hand, for $\mathbb{T} = \mathbb{R}$, Saranya et al. [7] obtained sharp conditions for the oscillation of the delay differential equation

$$(a_1(l)v'(l))' + g_1(l)v^\alpha(\varrho(l)) = 0, \quad (7)$$

where $\alpha > 0$. In [8], Wu et al., examined the oscillatory behaviour of the solutions of the delay dynamic equation

$$(a_1(l)(v^\Delta(l))^\gamma)^\Delta + g_1(l)f(x(\varrho(l))) = 0, \quad (8)$$

in the canonical and noncanonical cases by using integral averaging techniques and generalized Riccati transformations. Recently, in [9] Grace et al. obtained some new oscillation criteria for the oscillation of all solutions of the second order nonlinear dynamic equation with deviating arguments of the form

$$(a_1(l)(v^\Delta(l)))^\Delta + g_1(l)v^\alpha(\varrho(l)) = 0, \quad (9)$$

where $\alpha \in (0, 1]$ is a ratio of odd positive integers.

Many authors have studied the oscillatory behavior of the solutions to Equation (4) (see for example [10,11]). As particular cases, when $\mathbb{T} = \mathbb{R}$ see [12–17], and for $\mathbb{T} = \mathbb{Z}$ see [18–20]. Those authors obtained oscillation criteria in both cases (canonical and noncanonical) by using integral averaging techniques, generalized Riccati transformations and Kneser-type results. It should be noted that the study of the equation in the advanced case drew the most attention, with few results in the case of delay. For more interesting results and improved techniques, one can trace the development of studying the oscillation of solutions of differential equations with the canonical operator through the studies [21–23] and with the noncanonical operator through the studies [24–26].

In canonical form, nonoscillatory (eventually positive) solutions $v(l)$ of (4) have one sign and $v^\Delta(l) > 0$, but in noncanonical equations, both signs of the first derivative $v^\Delta(l)$ of any positive solution $v(l)$ are possible and must be dealt with. An strategy commonly employed in the literature for examining such equations is to extend previous conclusions for canonical equations. Among the drawbacks of this technique are the inclusion of extra conditions or the failure to guarantee the oscillatory of all solutions (for details see [27]).

The goal of this paper depends on finding an appropriate transformation of the delay dynamic Equation (4) from the noncanonical case to the canonical case (11). By using this approach, we present some new sufficient conditions that ensure that all solutions of (4) are oscillatory. The results presented in this paper improve and complement the existing results in the literature even for the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

2. Preliminary Results

In what follows, we need only to consider the eventually positive solutions of Equation (4), because if v satisfies Equation (4), then $-v$ is also a solution. Without loss of generality, we only give proofs for the positive solutions. We begin by presenting some lemmas and related results.

Lemma 1. *Given the dynamic Equation (4), let us assume that (H1)–(H3) hold. Then, it is*

$$(a_1(l)v^\Delta(l))^\Delta = \frac{1}{\xi(\sigma(t))} \left(a_1(l)\xi(\sigma(l))\xi(l) \left(\frac{v(l)}{\xi(l)} \right)^\Delta \right)^\Delta. \quad (10)$$

Proof. We can demonstrate this for any $v(l)$ by using a straightforward calculation :

$$\begin{aligned}
 \frac{\left(a_1(l)\xi(\sigma(l))\xi(l)\left(\frac{v(l)}{\xi(l)}\right)^\Delta\right)^\Delta}{\xi(\sigma(l))} &= \frac{1}{\xi(\sigma(l))} \left(a_1(l)\xi(\sigma(l))\xi(l)\left(\frac{v(l)}{\xi(l)}\right)^\Delta\right)^\Delta \\
 &= \frac{1}{\xi(\sigma(l))} \left(a_1(l)\xi(\sigma(l))\xi(l)\left(\frac{\xi(l)v^\Delta(l) - v(l)\xi^\Delta(l)}{\xi(l)\xi(\sigma(l))}\right)\right)^\Delta \\
 &= \frac{1}{\xi(\sigma(l))} \left(a_1(l)\left(\xi(l)v^\Delta(l) + v(l)\frac{1}{a_1(l)}\right)\right)^\Delta \\
 &= \frac{1}{\xi(\sigma(l))} \left(\xi(l)a_1(l)v^\Delta(l) + v(l)\right)^\Delta \\
 &= \frac{1}{\xi(\sigma(l))} \left(\xi(\sigma(l))(a_1(l)v^\Delta(l))^\Delta + a_1(l)v^\Delta(l)\xi^\Delta(l) + v^\Delta(l)\right) \\
 &= \left(a_1(l)v^\Delta(l)\right)^\Delta.
 \end{aligned}$$

Moreover,

$$\int_{l_0}^{\infty} \frac{\Delta s}{a_1(s)\xi(s)\xi(\sigma(s))} = \lim_{l \rightarrow \infty} \left(\frac{1}{\xi(l)} - \frac{1}{\xi(l_0)}\right) = \infty.$$

□

As a result of Lemma 1, we can see that (4) has the following equivalent form:

$$\left(a(l)y^\Delta(l)\right)^\Delta + g(l)y^\alpha(q(l)) = 0, \quad (11)$$

where

$$a(l) = a_1(l)\xi(\sigma(l))\xi(l), \quad y(l) = \frac{v(l)}{\xi(l)}, \quad \text{and} \quad g(l) = \xi(\sigma(l))\xi^\alpha(q(l))g_1(l).$$

Furthermore, Equation (11) is in canonical form.

The following outcome is directly related to the previous discussion.

Theorem 1. The noncanonical dynamic Equation (4) possesses a solution $v(l)$ if and only if the canonical Equation (11) has the solution $\frac{v(l)}{\xi(l)}$.

Corollary 1. The noncanonical dynamic Equation (4) has an eventually positive solution if and only if the canonical Equation (11) has an eventually positive solution.

It is evident from Corollary 1 that the study of the oscillation in (4) is reduced to that of (11), so we are only concerned with one type of an eventually positive solution, namely

$$y(l) > 0, \quad a(l)y^\Delta(l) > 0 \quad \text{and} \quad \left(a(l)y^\Delta(l)\right)^\Delta < 0, \quad l \geq l_1 \geq l_0. \quad (12)$$

Furthermore, for simplicity, we define

$$\Omega(l, u) = \left(\int_u^l \frac{1}{a(s)} \Delta s\right)^{-1} \left(\int_u^{q(l)} \frac{1}{a(s)} \Delta s\right). \quad (13)$$

The following lemma will play an important role in what follows.

Lemma 2. Let $v(l)$ be a positive solution of (4) on $[l_0, \infty)_{\mathbb{T}}$. Then

$$\frac{y(\varrho(l))}{y(l)} \geq \Omega(l, l_1), \quad l \geq l_1 \quad (14)$$

for sufficiently large l .

Proof. Assume that $v(l) > 0$ and $v(\varrho(l)) > 0$, for all $l \geq l_1$, for some $l_1 \geq l_0$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). According to (12), we conclude that $a(l)(y^\Delta(l))$ is decreasing on $[l_1, \infty)_{\mathbb{T}}$. From (H2) we have $\lim_{l \rightarrow \infty} \varrho(l) = \infty$. Thus, we can choose $l_2 > l_1$ so that $\varrho(l) \geq l_1$ for $l \geq l_2$. Hence,

$$\begin{aligned} y(l) - y(\varrho(l)) &= \int_{\varrho(l)}^l \frac{1}{a(s)} [a(s)y^\Delta(s)] \Delta s \\ &\leq [a(\varrho(l))y^\Delta(\varrho(l))] \int_{\varrho(l)}^l \frac{1}{a(s)} \Delta s, \end{aligned}$$

and thus,

$$\frac{y(l)}{y(\varrho(l))} \leq 1 + \frac{a(\varrho(l))y^\Delta(\varrho(l))}{y(\varrho(l))} \int_{\varrho(l)}^l \frac{1}{a(s)} \Delta s. \quad (15)$$

Additionally, we have, for $l \geq l_2$,

$$\begin{aligned} y(\varrho(l)) &> y(\varrho(l)) - y(l_1) \\ &= \int_{l_1}^{\varrho(l)} \frac{1}{a(s)} [a(s)y^\Delta(s)] \Delta s \\ &\geq [a(\varrho(l))y^\Delta(\varrho(l))] \int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s, \end{aligned}$$

from which it follows that

$$\frac{a(\varrho(l))y^\Delta(\varrho(l))}{y(\varrho(l))} \leq \left(\int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s \right)^{-1}. \quad (16)$$

Combining (16) and (15), we have

$$\begin{aligned} \frac{y(l)}{y(\varrho(l))} &\leq 1 + \left(\int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s \right)^{-1} \int_{\varrho(l)}^l \frac{1}{a(s)} \Delta s \\ &\leq 1 + \left(\int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s \right)^{-1} \left(\int_{l_1}^l \frac{1}{a(s)} \Delta s - \int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s \right) \\ &= \left(\int_{l_1}^{\varrho(l)} \frac{1}{a(s)} \Delta s \right)^{-1} \left(\int_{l_1}^l \frac{1}{a(s)} \Delta s \right), \end{aligned}$$

which leads to

$$\frac{y(\varrho(l))}{y(l)} \geq \Omega(l, l_1).$$

This proves the lemma. \square

Lemma 3 ([8]). Let $G(U) = AU - B(U - R)^{(\gamma+1)/\gamma}$, where $B > 0$, A and R are constants, γ is a ratio of odd positive integers. Then G attains its maximum value on \mathbb{R} at $U^* = R + (\gamma A / ((\gamma + 1)B))^\gamma$ and

$$\max_{U \in \mathbb{R}} G(U) = G(U^*) = AR + \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}} \frac{A^{\gamma+1}}{B^\gamma}. \quad (17)$$

3. Main Results

In this section, we construct some new oscillation criteria for (4). We begin with the oscillation result shown below. For simplicity, in the sequel we will denote

$$0 < G(l, l_1) := \int_l^\infty g(s) \Omega^\alpha(s, l_1) \Delta s < \infty, \quad l_1 \in [l_0, \infty)_{\mathbb{T}}. \quad (18)$$

Theorem 2. *If*

$$\int_{l_0}^\infty g(s) \Omega^\alpha(s, l_1) \Delta s = \infty \quad (19)$$

for some $l_1 \geq l_0$, then (4) is oscillatory on $[l_0, \infty)_{\mathbb{T}}$.

Proof. Assume, for the sake of contradiction, that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). Define the following Riccati substitution

$$w(l) = \frac{a(l)y^\Delta(l)}{y^\alpha(l)}, \quad l \geq l_1. \quad (20)$$

It is clear that $w(l) > 0$ and

$$\begin{aligned} w^\Delta(l) &= (a(l)y^\Delta(l))^\Delta \frac{1}{y^\alpha(l)} + (a(\sigma(l))y^\Delta(\sigma(l))) \left(\frac{1}{y^\alpha(l)} \right)^\Delta \\ &= \frac{(a(l)y^\Delta(l))^\Delta}{y^\alpha(l)} - (a(\sigma(l))y^\Delta(\sigma(l))) \frac{(y^\alpha(l))^\Delta}{y^\alpha(l)y^\alpha(\sigma(l))}. \end{aligned}$$

By virtue of (11) and (14), we get

$$w^\Delta(l) \leq -g(l)\Omega^\alpha(l, l_1) - w(\sigma(l)) \frac{(y^\alpha(l))^\Delta}{y^\alpha(l)}. \quad (21)$$

Applying Pötzsche's chain rule ([2], Theorem 1.87), we have that

$$\begin{aligned} (y^\alpha(l))^\Delta &= \alpha \left[\int_0^1 [y(l) + h\mu(l)y^\Delta(l)]^{\alpha-1} dh \right] y^\Delta(l) \\ &= \alpha \left[\int_0^1 [(1-h)y(l) + hy(\sigma(l))]^{\alpha-1} dh \right] y^\Delta(l) \\ &\geq \begin{cases} \alpha(y(l))^{\alpha-1}y^\Delta(l), & \alpha > 1, \\ \alpha(y(\sigma(l)))^{\alpha-1}y^\Delta(l), & 0 < \alpha \leq 1. \end{cases} \end{aligned}$$

Consequently,

$$\frac{(y^\alpha(l))^\Delta}{y^\alpha(l)} \geq \begin{cases} \alpha \frac{y^\Delta(l)}{y(l)}, & \alpha > 1, \\ \alpha \frac{(y(\sigma(l)))^{\alpha-1}}{y^\alpha(l)} y^\Delta(l), & 0 < \alpha \leq 1. \end{cases}$$

From (12), we know that $y(t)$ is increasing on $[l_0, \infty)_{\mathbb{T}}$, then $y(l) \leq y(\sigma(l))$. Hence,

$$\frac{(y^\alpha(l))^\Delta}{y^\alpha(l)} \geq \frac{y^\Delta(l)}{y(\sigma(l))}. \quad (22)$$

Putting (22) into (21), and taking into account the fact that $y^\Delta(l) > 0$, we obtain

$$w^\Delta(l) \leq -g(l)\Omega^\alpha(l, l_1) - \alpha w(\sigma(l)) \frac{y^\Delta(l)}{y(\sigma(l))} \quad (23)$$

$$< -g(l)\Omega^\alpha(l, l_1). \quad (24)$$

Integrating the last inequality from l_1 to l , we obtain

$$\int_{l_1}^l g(s)\Omega^\alpha(s, l_1)\Delta s < w(l_1) - w(l) < w(l_1) < \infty,$$

which contradicts (19) as $l \rightarrow \infty$. This completes the proof. \square

As a result of Theorem 2, we can now restrict to the case:

$$\int_{l_0}^{\infty} g(s)\Omega^\alpha(s, l_0)\Delta s < \infty, \quad l_1 \geq l_0.$$

Theorem 3. Let assume that $\alpha > 1$ and (18) hold. If

$$\int_{l_0}^{\infty} \frac{1}{a(s)} \left(G(\sigma(l), l_1) + c \int_l^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1)\Delta s \right) \Delta s = \infty, \quad (25)$$

then (4) is oscillatory.

Proof. Assume that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). Following the same steps used to prove Theorem 2, we get

$$w^\Delta(l) \leq -g(l)\Omega^\alpha(l, l_1) - \alpha w(\sigma(l)) \frac{y^\Delta(l)}{y(\sigma(l))}. \quad (26)$$

By using the fact that $a(l)y^\Delta(l)$ is decreasing, we get $a(\sigma(l))y^\Delta(\sigma(l)) \leq a(l)y^\Delta(l)$. Considering the definition of $w(t)$, we obtain

$$\begin{aligned} y^\Delta(l) &\geq \frac{a(\sigma(l))y^\Delta(\sigma(l))}{a(l)} \\ &\geq \frac{1}{a(l)} w(\sigma(l))y^\alpha(\sigma(l)). \end{aligned} \quad (27)$$

Using this inequality, it follows from (26) that

$$w^\Delta(l) \leq -g(l)\Omega^\alpha(l, l_1) - \alpha \frac{1}{a(l)} w^2(\sigma(l))y^{\alpha-1}(\sigma(l)). \quad (28)$$

Integrating (28) from l to ∞ , we obtain

$$\begin{aligned} w(l) &\geq \int_l^{\infty} g(s)\Omega^\alpha(s, l_1)\Delta s + \alpha \int_l^{\infty} \frac{1}{a(s)} w^2(\sigma(s))y^{\alpha-1}(\sigma(s))\Delta s \\ &\geq G(l, l_1) + \alpha \int_{l_1}^{\infty} \frac{1}{a(s)} w^2(\sigma(s))y^{\alpha-1}(\sigma(s))\Delta s. \end{aligned} \quad (29)$$

It's obvious that $w(l) \geq G(l, l_1)$ for $l \geq l_1$, and therefore

$$w(l) \geq G(l, l_1) + \alpha \int_l^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1)y^{\alpha-1}(\sigma(s))\Delta s. \quad (30)$$

Because $y^\Delta(l) > 0$, then there exists $l_1 \geq l_0$ and a positive constant c_1 such that $y^{\alpha-1}(\sigma(s)) \geq c_1$ for $s \geq l_1$. Hence,

$$w(l) \geq G(l, l_1) + c \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s, \quad l \geq l_1, \quad (31)$$

where $c = \alpha c_1$. By virtue of the fact that $a(l)y^\Delta(l)$ is decreasing, and (31), we have

$$\frac{a(l)y^\Delta(l)}{y^\alpha(\sigma(l))} \geq \frac{a(\sigma(l))y^\Delta(\sigma(l))}{y^\alpha(\sigma(l))} = w(\sigma(l)) \geq G(\sigma(l), l_1) + c \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s. \quad (32)$$

This leads to

$$y^{-\alpha}(\sigma(l))y^\Delta(l) \geq \frac{1}{a(l)} \left(G(\sigma(l), l_1) + c \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \right). \quad (33)$$

On the other hand, by using Pötzsche's chain rule, we have for $\alpha > 1$,

$$(y^{1-\alpha}(l))^\Delta \leq (1-\alpha)y^{-\alpha}(\sigma(l))y^\Delta(l). \quad (34)$$

Consequently, it is

$$\frac{(y^{1-\alpha}(l))^\Delta}{1-\alpha} \geq y^{-\alpha}(\sigma(l))y^\Delta(l), \quad l \geq l_1. \quad (35)$$

It follows from (33) and (35) that

$$\frac{(y^{1-\alpha}(l))^\Delta}{1-\alpha} \geq \frac{1}{a(l)} \left(G(\sigma(l), l_1) + c \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \right), \quad l \geq l_1. \quad (36)$$

By integrating (36) from l_1 to l , we get

$$\int_{l_1}^l \frac{1}{a(s)} \left(G(\sigma(s), l_1) + c \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \right) \Delta s \leq \frac{y^{1-\alpha}(l_1)}{\alpha-1}. \quad (37)$$

Letting $l \rightarrow \infty$, we arrive at the intended contradiction. \square

Theorem 4. Let us assume that $\alpha = 1$ and (18) hold. If

$$\limsup_{l \rightarrow \infty} \left[G(l, l_1) + \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \right] \left(\int_{l_1}^l \frac{1}{a(s)} \Delta s \right) > 1, \quad (38)$$

then (4) is oscillatory.

Proof. We proceed by contradiction that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). Proceeding as in the proof of Theorem 3, we arrive at

$$w(l) \geq G(l, l_1) + \int_{l_1}^{\infty} \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s. \quad (39)$$

In view that $a(l)y^\Delta(l)$ is a decreasing function, we deduce for $t \in [t_1, \infty)_{\mathbb{T}}$ that

$$\begin{aligned} y(l) &= y(l_1) + \int_{l_1}^l y^\Delta(s) \Delta s \\ &= y(l_1) + \int_{l_1}^l \frac{1}{a(s)} a(s) y^\Delta(s) \Delta s \\ &\geq a(l) y^\Delta(l) \int_{l_1}^l \frac{1}{a(s)} \Delta s. \end{aligned}$$

Thus, we have

$$\frac{a(l)y^\Delta(l)}{y(l)} \leq \left(\int_{l_1}^l \frac{1}{a(s)} \Delta s \right)^{-1}. \quad (40)$$

In view of the definition of $w(l)$ and (39), from the inequality in (39) we get

$$G(l, l_1) + \int_l^\infty \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \leq \left(\int_l^l \frac{1}{a(s)} \Delta s \right)^{-1}, \quad l \geq l_1. \quad (41)$$

Consequently, it is

$$\left[G(l, l_1) + \int_l^\infty \frac{1}{a(s)} G^2(\sigma(s), l_1) \Delta s \right] \left(\int_l^l \frac{1}{a(s)} \Delta s \right) \leq 1. \quad (42)$$

Taking \limsup as $l \rightarrow \infty$, we obtain a contradiction with (38). This completes the proof. \square

Theorem 5. Let assume that $0 < \alpha < 1$ and (18) hold. If

$$\limsup_{l \rightarrow \infty} G^{\frac{1-\alpha}{\alpha}}(l, l_1) \left(\int_{l_1}^l \frac{\Delta s}{a(s)} \right) \left(G(l, l_1) + \beta \int_l^\infty \frac{1}{a(s)} G^{\frac{\alpha+1}{\alpha}}(\sigma(s), l_1) \Delta s \right) = \infty, \quad (43)$$

where $\beta = \alpha k^{(\alpha-1)/\alpha}$, then (4) is oscillatory.

Proof. Assume that $v(l) > 0$ and $v(\sigma(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). Following the same steps used to prove Theorem 2, we get (29). Therefore, we have $w(l) \geq G(l, l_1)$ for $l \in [l_1, \infty)_{\mathbb{T}}$. In view of definition of $w(l)$, we have

$$a(l)y^\Delta(l) \geq y^\alpha(l)G(l, l_1), \quad l \geq l_1.$$

Since $a(l)y^\Delta(l)$ is a decreasing function, there exist a constant $k > 0$ and $l_2 \geq l_1$ such that

$$k \geq a(l)y^\Delta(l) \geq y^\alpha(l)G(l, l_1), \quad l \geq l_1.$$

Hence, we have

$$y(l) \leq k^{1/\alpha} G^{-1/\alpha}(l, l_1), \quad l \geq l_1.$$

It follows that

$$y^{\alpha-1}(\sigma(l)) \geq k^{(\alpha-1)/\alpha} G^{(1-\alpha)/\alpha}(\sigma(l), l_1), \quad l \geq l_1. \quad (44)$$

Combining (44) with (30), we obtain

$$w(l) \geq G(l, l_1) + \alpha k^{(\alpha-1)/\alpha} \int_l^\infty \frac{1}{a(s)} G^{\frac{\alpha+1}{\alpha}}(\sigma(s), l_1) \Delta s,$$

which can be written as

$$y^{1-\alpha}(l) \frac{a(l)y^\Delta(l)}{y(l)} \geq G(l, l_1) + \beta \int_l^\infty \frac{1}{a(s)} G^{\frac{\alpha+1}{\alpha}}(\sigma(s), l_1) \Delta s, \quad (45)$$

where $\beta = \alpha k^{(\alpha-1)/\alpha}$. From (40), (44) and (45), we conclude that

$$k^{\frac{1-\alpha}{\alpha}} G^{\frac{\alpha-1}{\alpha}}(l, l_1) \left(\int_{l_1}^l \frac{\Delta s}{a(s)} \right)^{-1} \geq G(l, l_1) + \beta \int_l^\infty \frac{1}{a(s)} G^{\frac{\alpha+1}{\alpha}}(\sigma(s), l_1) \Delta s.$$

Thus, we have

$$G^{\frac{1-\alpha}{\alpha}}(l, l_1) \left(\int_{l_1}^l \frac{\Delta s}{a(s)} \right) \left(G(l, l_1) + \beta \int_l^\infty \frac{1}{a(s)} G^{\frac{\alpha+1}{\alpha}}(\sigma(s), l_1) \Delta s \right) \leq k^{\frac{1-\alpha}{\alpha}}, \quad l \geq l_1.$$

Taking \limsup as $l \rightarrow \infty$, we obtain a contradiction with (43). This completes the proof. \square

Theorem 6. Let assume that $0 < \alpha < 1$. If

$$\limsup_{l \rightarrow \infty} \int_{\varrho(l)}^l Q(s, l_1) \Delta s = \infty, \quad (46)$$

then every solution of (4) is oscillatory, where $Q(s, l_1) = g(l) \left(\int_{l_1}^{\varrho(l)} \frac{\Delta s}{a(s)} \right)^\alpha$.

Proof. Assume that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). From (11) and (40), we have

$$(a(l)y^\Delta(l))^\Delta + g(l) \left(a(\varrho(l))y^\Delta(\varrho(l)) \right)^\alpha \left(\int_{l_1}^{\varrho(l)} \frac{\Delta s}{a(s)} \right)^\alpha \leq 0. \quad (47)$$

Set $Y(l) := a(l)y^\Delta(l)$ and $Q(l, l_1) = g(l) \left(\int_{l_1}^{\varrho(l)} \frac{\Delta s}{a(s)} \right)^\alpha$, then (47) takes the form

$$Y^\Delta(l) + Q(l, l_1)Y^\alpha(\varrho(l)) \leq 0. \quad (48)$$

Integrating (48) from $\varrho(l)$ to l , we get

$$\begin{aligned} Y(\varrho(l)) &\geq -Y(l) + Y(\varrho(l)) \\ &\geq \left(\int_{\varrho(l)}^l Q(s, l_1) \Delta s \right) Y^\alpha(\varrho(l)), \end{aligned}$$

which may be written as

$$Y^{1-\alpha}(\varrho(l)) \geq \int_{\varrho(l)}^l Q(s, l_1) \Delta s, \quad l \geq l_1. \quad (49)$$

Taking \limsup as $l \rightarrow \infty$, we get a contradiction and the prove is completed. \square

Theorem 7. If $\alpha = 1$, and there exists a positive function $\varphi(l) \in C_{rd}[l_0, \infty)_{\mathbb{T}}$ such that

$$\limsup_{l \rightarrow \infty} \int_{l_1}^l \left(g(s)\varphi(s)\Omega(s, l_1) - \frac{(\varphi^\Delta(s))^2 a(s)}{4\varphi(s)} \right) \Delta s = \infty, \quad (50)$$

then (4) is oscillatory.

Proof. Assume that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). Define

$$v(l) = \varphi(l) \frac{a(l)y^{\Delta}(l)}{y(l)}, \quad l \geq l_1. \quad (51)$$

It is clear that $v(l) > 0$ and

$$\begin{aligned} v^{\Delta}(l) &= \frac{\varphi(l)}{y(l)} (a(l)y^{\Delta}(l))^{\Delta} + (a(\sigma(l))y^{\Delta}(\sigma(l))) \left(\frac{\varphi(l)}{y(l)} \right)^{\Delta} \\ &\leq -g(l)\varphi(l)\Omega(l, l_1) + (a(\sigma(l))y^{\Delta}(\sigma(l))) \frac{y(l)\varphi^{\Delta}(l) - \varphi(l)y^{\Delta}(l)}{y(l)y(\sigma(l))} \\ &\leq -g(l)\varphi(l)\Omega(l, l_1) + \frac{\varphi^{\Delta}(l)}{\varphi(\sigma(l))} v(\sigma(l)) - \frac{\varphi(l)a(l)y^{\Delta}(l)}{a(l)\varphi(\sigma(l))y(l)} v(\sigma(l)), \end{aligned}$$

and hence

$$\begin{aligned} v^{\Delta}(l) &\leq -g(l)\varphi(l)\Omega(l, l_1) + \frac{\varphi^{\Delta}(l)}{\varphi(\sigma(l))} v(\sigma(l)) - \frac{\varphi(l)}{a(l)\varphi^2(\sigma(l))} v^2(\sigma(l)) \\ &\leq -g(l)\varphi(l)\Omega(l, l_1) + \frac{(\varphi^{\Delta}(l))^2 a(l)}{4\varphi(l)}. \end{aligned} \quad (52)$$

Integrating (52) from l_1 to l , we get

$$\limsup_{l \rightarrow \infty} \int_{l_1}^l \left(g(s)\varphi(s)\Omega(s, l_1) - \frac{(\varphi^{\Delta}(s))^2 a(s)}{4\varphi(s)} \right) \Delta s \leq v(t_1) < \infty, \quad (53)$$

which contradicts the hypothesis, and the proof is complete. \square

The following theorem proposes a new oscillation criterion with less demanding requirements than other results previously appeared in literature.

Theorem 8. Assume that there exists a function $\psi \in C_{rd}^1(\mathbb{T}, \mathbb{R}^+)$, such that

$$\limsup_{l \rightarrow \infty} \left(\psi(l) \int_l^{\infty} g(s) \Delta s + \int_{l_0}^l \left(g(s)\psi(s) - \frac{(\psi_+^{\Delta}(s))^2 \bar{a}(s)}{4\alpha\psi(s)\varrho^{\Delta}(s)} \right) \Delta s \right) = \infty, \quad (54)$$

where $\psi_+^{\Delta}(l) = \max\{\psi^{\Delta}(l), 0\}$ and $\bar{a}(l) = \max\{a(\zeta) | \varrho(l) \leq \zeta \leq \varrho(\sigma(l))\}$. Then (4) is oscillatory.

Proof. Assume that $v(l) > 0$ and $v(\varrho(l)) > 0$ for all $l \geq l_1$ for some $l_1 \in [l_0, \infty)_{\mathbb{T}}$. It follows from Theorem 1 that $\frac{v(l)}{\xi(l)}$ is also a positive solution of (11). From (11) and the fact that $a(l)y^{\Delta}(l)$ is decreasing, we have

$$\begin{aligned} a(l)y^{\Delta}(l) &\geq \int_l^{\infty} g(s)y^{\alpha}(\varrho(s))\Delta s \\ &\geq y^{\alpha}(\varrho(l)) \int_l^{\infty} g(s)\Delta s. \end{aligned} \quad (55)$$

Let define

$$\omega(l) = \psi(l) \frac{a(l)y^{\Delta}(l)}{y^{\alpha}(\varrho(l))}, \quad l \geq l_1. \quad (56)$$

It follows from (55) and (56) that

$$\omega(l) = \psi(l) \frac{a(l)y^\Delta(l)}{y^\alpha(\varrho(l))} \geq \psi(l) \int_l^\infty g(s)\Delta s > 0, \quad l \geq l_1. \quad (57)$$

By using the product rule and the quotient rule, we get

$$\begin{aligned} \omega^\Delta(l) &= (a(l)y^\Delta(l))^\Delta \left(\frac{\psi(l)}{y^\alpha(\varrho(l))} \right) + (a(\sigma(l))y^\Delta(\sigma(l))) \left(\frac{\psi(l)}{y^\alpha(\varrho(l))} \right)^\Delta \\ &= (a(l)y^\Delta(l))^\Delta \left(\frac{\psi(l)}{y^\alpha(\varrho(l))} \right) + (a(\sigma(l))y^\Delta(\sigma(l))) \left(\frac{y^\alpha(\varrho(l))\psi^\Delta(l) - \psi(l)(y^\alpha(\varrho(l)))^\Delta}{y^\alpha(\varrho(l))y^\alpha(\varrho(\sigma(l)))} \right) \\ &\leq -g(l)\psi(l) + \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}\omega(\sigma(l)) - \psi(l) \frac{a(\sigma(l))y^\Delta(\sigma(l))}{y^\alpha(\varrho(l))y^\alpha(\varrho(\sigma(l)))} (y^\alpha(\varrho(l)))^\Delta. \end{aligned} \quad (58)$$

By using Pötzsche's chain rule, we have

$$(y^\alpha(\varrho(l)))^\Delta \geq \begin{cases} \alpha(y(\varrho(l)))^{\alpha-1}y^\Delta(\varrho(l)), & \alpha > 1, \\ \alpha(y(\varrho(\sigma(l))))^{\alpha-1}y^\Delta(\varrho(l)), & 0 < \alpha \leq 1. \end{cases} \quad (59)$$

If $\sigma(l) > l$, it follows from the mean value theorem ([3], Theorem 1.14) that

$$\begin{aligned} y^\Delta(\varrho(l)) &= \frac{y(\sigma(\varrho(l))) - y(\varrho(l))}{\sigma(l) - l} \\ &= \frac{y(\sigma(\varrho(l))) - y(\varrho(l))}{\varrho(\sigma(l)) - \varrho(l)} \varrho^\Delta(l) \\ &\geq y^\Delta(\zeta) \varrho^\Delta(l), \end{aligned} \quad (60)$$

where $\zeta \in [\varrho(l), \varrho(\sigma(l))]$. From (60); (59) takes the form

$$(y^\alpha(\varrho(l)))^\Delta \geq \begin{cases} \alpha(y(\varrho(l)))^{\alpha-1}y^\Delta(\zeta)\varrho^\Delta(l), & \alpha > 1, \\ \alpha(y(\varrho(\sigma(l))))^{\alpha-1}y^\Delta(\zeta)\varrho^\Delta(l), & 0 < \alpha \leq 1. \end{cases} \quad (61)$$

Using this in (58) leads to

$$\begin{aligned} \omega^\Delta(l) &\leq -g(l)\psi(l) + \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}\omega(\sigma(l)) \\ &\quad - \alpha\psi(l)\varrho^\Delta(l)(a(\sigma(l))y^\Delta(\sigma(l))) \left(\begin{cases} \frac{(y(\varrho(l)))^{\alpha-1}y^\Delta(\zeta)}{y^\alpha(\varrho(l))y^\alpha(\varrho(\sigma(l)))}, & \alpha > 1, \\ \frac{(y(\varrho(\sigma(l))))^{\alpha-1}y^\Delta(\zeta)}{y^\alpha(\varrho(l))y^\alpha(\varrho(\sigma(l)))}, & 0 < \alpha \leq 1 \end{cases} \right). \end{aligned} \quad (62)$$

By using the fact that $\varrho(l)$ and $y(l)$ are increasing functions, we conclude that $y(\varrho(\sigma(l))) \geq y(\varrho(l))$. Hence, from (62), we have for $\alpha > 0$

$$\omega^\Delta(l) \leq -g(l)\psi(l) + \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}\omega(\sigma(l)) - \alpha\psi(l)\varrho^\Delta(l) \frac{[a(\sigma(l))y^\Delta(\sigma(l))]y^\Delta(\zeta)}{y^2(\varrho(\sigma(l)))}. \quad (63)$$

Because $a(l)y^\Delta(l)$ is decreasing and $\varrho(l)$ is increasing, we have for $\zeta \in [\varrho(l), \varrho(\sigma(l))]$

$$a(\zeta)(y^\Delta(\zeta)) \geq a(\sigma)(y^\Delta(\varrho(\sigma(l)))) \geq a(\sigma)(y^\Delta(\sigma(l))),$$

and thus

$$y^\Delta(\zeta) \geq \frac{a(\sigma)(y^\Delta(\sigma(l)))}{\bar{a}(l)}. \quad (64)$$

Substituting (64) into (63) results in

$$\begin{aligned}\omega^\Delta(l) &\leq -g(l)\psi(l) + \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}\omega(\sigma(l)) - \frac{\alpha\psi(l)\varrho^\Delta(l)}{\bar{a}(l)} \frac{[a(\sigma(l))y^\Delta(\sigma(l))]^2}{y^2(\varrho(\sigma(l)))} \\ &\leq -g(l)\psi(l) + \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}\omega(\sigma(l)) - \frac{\alpha\psi(l)\varrho^\Delta(l)}{\bar{a}(l)\psi^2(\sigma(l))}\omega^2(\sigma(l)).\end{aligned}\quad (65)$$

Applying Lemma 3 with $A = \frac{\psi_+^\Delta(l)}{\psi(\sigma(l))}$, $B = \frac{\alpha\psi(l)\varrho^\Delta(l)}{\bar{a}(l)\psi^2(\sigma(l))}$ and $C = 0$, we obtain

$$\omega^\Delta(l) \leq -g(l)\psi(l) + \frac{(\psi_+^\Delta(l))^2\bar{a}(l)}{4\alpha\psi(l)\varrho^\Delta(l)}.\quad (66)$$

By integrating (66) from l_1 to l , we get

$$\omega(l) \leq \omega(l_1) + \int_{l_1}^l \left(g(s)\psi(s) - \frac{(\psi_+^\Delta(s))^2\bar{a}(s)}{4\alpha\psi(s)\varrho^\Delta(s)} \right) \Delta s.\quad (67)$$

Combining (67) with (57), we have

$$\psi(l) \int_l^\infty g(s)\Delta s + \int_{l_1}^l \left(g(s)\psi(s) - \frac{(\psi_+^\Delta(s))^2\bar{a}(s)}{4\alpha\psi(s)\varrho^\Delta(s)} \right) \Delta s \leq \omega(l_1).\quad (68)$$

Taking the \limsup on both sides of the above inequality as $l \rightarrow \infty$, we obtain a contradiction to the hypothesis, and the proof is complete. \square

4. Some Illustrative Examples

Example 1. Consider the second order differential equation

$$\left(l^{3/2}y'(l) \right)' + ky(l) = 0, \quad l \geq 1,\quad (69)$$

where $k > 0$. Here $a_1(l) = l^{3/2}$, $\alpha = 1$, $g_1(l) = k$ and $\varrho(l) = l$. It is clear that

$$\zeta(l) = \int_l^\infty s^{-3/2}ds = \frac{2}{\sqrt{l}} < \infty.$$

It follows that $a(l) = 4\sqrt{l}$, $g(l) = \frac{4k}{l}$, and $\Omega(l, l_1) = 1$. The transformed equation into the canonical form is

$$\left(4\sqrt{l}y'(l) \right)' + \frac{4k}{l}y(l) = 0.\quad (70)$$

Choosing $\varphi(l) = l$, condition (50) takes the form

$$\begin{aligned}\limsup_{l \rightarrow \infty} \int_{l_1}^l \left(g(s)\varphi(s)\Omega(s, l_1) - \frac{(\varphi^\Delta(s))^2 a(s)}{4\varphi(s)} \right) \Delta s &= \limsup_{l \rightarrow \infty} \int_{l_1}^l \left(\frac{4k}{s}(s) - \frac{4\sqrt{s}}{4s} \right) ds \\ &= \limsup_{l \rightarrow \infty} \left(4kl - 2\sqrt{l} - 4kl_1 + 2\sqrt{l_1} \right) \\ &= \infty.\end{aligned}$$

Hence, (69) is oscillatory. However, Theorem 3.1 in [27] states that Equation (69) is oscillatory or $\lim_{l \rightarrow \infty} y(l) = 0$.

Example 2. Consider the second order differential equation

$$\left(l^2y'(l) \right)' + l^2y^3(l/2) = 0, \quad l \geq 1.\quad (71)$$

Here $a_1(l) = l^2$, $\alpha = 3$, $g_1(l) = l^2$ and $q(l) = l/2$. It is clear that

$$\zeta(l) = \int_l^\infty s^{-2} ds = \frac{1}{l} < \infty.$$

Because $a(l) = 1$ and $g(l) = \frac{8}{l^2}$, the transformed equation in the canonical form is

$$y''(l) + \frac{8}{l^2} y^3(l/2) = 0. \quad (72)$$

By using the condition (54), and choosing $\psi(l) = l$, we obtain

$$\begin{aligned} \limsup_{l \rightarrow \infty} \psi(l) \int_l^\infty g(s) \Delta s + \int_{l_0}^l \left(g(s) \psi(s) - \frac{(\psi_+^\Delta(s))^2 \bar{a}(s)}{4\alpha \psi(s) q^\Delta(s)} \right) \Delta s \\ = \limsup_{l \rightarrow \infty} \left[l \int_l^\infty \frac{8 ds}{s^2} + \int_{l_0}^l \left(\frac{8s}{s^2} - \frac{1}{6s} \right) ds \right] \\ = \infty. \end{aligned}$$

Hence, (71) is oscillatory.

Example 3. Consider the second order differential equation

$$\left(l^2 y'(l) \right)' + q_0 y(\lambda l) = 0, \quad 0 < \lambda \leq 1, \quad q_0 > 0, \quad l \geq 1. \quad (73)$$

Here $a_1(l) = l^2$, $\alpha = 1$, $g_1(l) = q_0$ and $q(l) = \lambda l$. It is clear that

$$\zeta(l) = \int_l^\infty s^{-2} ds = \frac{1}{l} < \infty.$$

Because $a(l) = 1$ and $g(l) = \frac{q_0}{l^2}$, the transformed equation into the canonical form is

$$y''(l) + \frac{q_0}{l^2} y(\lambda l) = 0. \quad (74)$$

Here $\Omega(l, l_1) = \frac{\lambda l - 1}{l - 1}$. By choosing $\varphi(l) = l$, condition (50) takes the form

$$\begin{aligned} \limsup_{l \rightarrow \infty} \int_{l_1}^l \left(\frac{\lambda s - 1}{s - 1} \frac{q_0}{s^2} s - \frac{1}{4s} \right) ds = \limsup_{l \rightarrow \infty} \left(q_0 \lambda \log(l - 1) - q_0 \log(l - 1) + \left(q_0 - \frac{1}{4} \right) \log(l) \right) \\ = \infty \text{ for } \left(\lambda q_0 - \frac{1}{4} \right) > 0. \end{aligned}$$

Hence, for $\lambda = 0.5$, (73) is oscillatory for $q_0 > 0.5$. The best result in [28] establishes that (73) is oscillatory for $q_0 > 0.70633$, so our results improve Theorem 2.6 of [28]. For $\lambda = 1$, condition $q_0 > \frac{1}{4}$ is a sharp condition of the Euler-type differential equation in (74).

Example 4. Consider the second order differential equation

$$\left(l^2 y'(l) \right)' + l^{1/3} y^{1/3}(l/2) = 0, \quad l \geq 1. \quad (75)$$

Here $a_1(l) = l^2$, $\alpha = 1/3$, $g_1(l) = l^{1/3}$ and $q(l) = l/2$. It is clear that

$$\zeta(l) = \int_l^\infty s^{-2} ds = \frac{1}{l} < \infty.$$

Since $a(l) = 1$ and $g(l) = \frac{2^{1/3}}{l}$, the transformed equation into the canonical form is

$$y''(l) + \frac{2^{1/3}}{l} y(l/2) = 0. \quad (76)$$

Here,

$$Q(l, l_1) = g(l) \left(\int_{l_1}^{q(l)} \frac{ds}{a(s)} \right)^{1/3} = \frac{2^{1/3}}{l} \left(\frac{l}{2} - 1 \right)^{1/3}.$$

Then condition (46) takes the form

$$\limsup_{l \rightarrow \infty} \int_{q(l)}^l Q(s, l_1) ds = \limsup_{l \rightarrow \infty} \int_{l/2}^l \frac{2^{1/3}}{s} \left(\frac{s}{2} - 1 \right)^{1/3} ds = \infty.$$

Hence, (75) is oscillatory.

Example 5. Consider the second-order difference equation

$$\Delta(l(l+1)\Delta y(l)) + q_0 y(l-m) = 0, \quad l \geq l_0 > 1, \quad (77)$$

where q_0 is a positive real number and $m > 1$ is a positive integer. Here $a_1(l) = l(l+1)$, $\alpha = 1$, $g_1(l) = q_0$ and $q(l) = l-m$. It is clear that

$$\xi(l) = \sum_l^{\infty} \frac{1}{s(s+1)} = \frac{1}{l} < \infty.$$

Since $a(l) = 1$, $g(l) = \frac{q_0}{(l+1)(l-m)}$ and $\Omega(s, l_1) = \frac{s-m-l_1}{s-l_1}$, the transformed equation into the canonical form is

$$\Delta^2(y(l)) + \frac{q_0}{(l+1)(l-m)} y(l-m) = 0. \quad (78)$$

Choosing $\varphi(l) = l$, condition (50) takes the form

$$\begin{aligned} \limsup_{l \rightarrow \infty} \sum_{l_1}^l \left(g(s)\varphi(s)\Omega(s, l_1) - \frac{(\Delta\varphi(s))^2 a(s)}{4\varphi(s)} \right) \\ = \limsup_{l \rightarrow \infty} \sum_{l_1}^l \left(\frac{q_0}{(s+1)(s-m)} \left(\frac{s-m-l_1}{s-l_1} \right) - \frac{1}{4s} \right) \\ \geq \limsup_{l \rightarrow \infty} \sum_{l_1}^l \left(\frac{q_0}{(s+1)(s-m)} - \frac{1}{4s} \right) \\ = \infty \quad \text{for } \left(q_0 > \frac{1}{4} \right). \end{aligned}$$

It follows that every solution of (77) is oscillatory for $q_0 > \frac{1}{4}$. By ([29], Theorem 2.1), we see that (77) oscillates if $q_0 > 1$, where they also imposed more restrictions than the ones we used, so our results improve those provided by Theorem 2.1 of [29].

It is worth noting that Equation (77) has also been discussed in [6,30], where it is demonstrated that every solution of (77) oscillates or approaches zero as $q_0 > 1/4$. However, we have demonstrated that under the same conditions, every solution of (77) is oscillatory. As a result, the findings achieved here outperform those found in [6,30].

5. Conclusions

Investigating the oscillatory behavior of solutions of dynamic equations is one of the most important aspects of qualitative theory. Such studies point not only to its importance in many different applied fields, but also to interesting theoretical and analytical questions. In this study, on time scales, we formulate new oscillation conditions for a class of non-canonical dynamic equations with delay. We adopt an approach that links the noncanonical equations with equations in the canonical case. The new oscillation criteria complement and improve some of the previous results in the literature. Extending the results of this paper to neutral equations will be an interesting topic for future work.

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