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Solvability of Mixed Problems for a Fourth-Order Equation with Involution and Fractional Derivative

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Abstract: In the present work, two-dimensional mixed problems with the Caputo fractional order differential operator are studied using the Fourier method of separation of variables. The equation contains a linear transformation of involution in the second derivative. The considered problem generalizes some previous problems formulated for some fourth-order parabolic-type equations. The basic properties of the eigenfunctions of the corresponding spectral problems, when they are defined as the products of two systems of eigenfunctions, are studied. The existence and uniqueness of the solution to the formulated problem is proved.

Keywords: fractional differential operator; differential equation with involution; eigenvalues; eigenfunctions; biorthonormal system; Riesz basis

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1. Introduction and Problem Statement

The transformation S of the function $f(x)$, $x \in [-1,1]$, which satisfies the condition $S^2 f(x) = f(x)$, is called an involution. We consider equations containing an involution of the form $Sf(x) = f(-x)$.

Differential equations with involution are of theoretical and practical interest, as they occupy a special place among differential equations with deviating arguments and functional differential equations. More information on the theoretical developments and applications of such equations can be found in [1–5].

The solvability of direct, inverse problems for the second-order differential equations with involution is considered in [4–13] (see also references therein). Most of these studies are based on the method of separation of variables, which leads to spectral problems with involution. Regarding spectral problems for differential equations with involution, good references are [14–23].

The boundary value problems for the fourth-order partial differential equations are studied in (see [24], and references therein). However, in the scientific literature, there are no works devoted to the study of the initial-boundary value problems for the fourth-order partial differential equations with involution.

This paper is devoted to the study of the existence and uniqueness of a solution to mixed problems for a two-dimensional fourth-order equation with involution

$$ {}_{C}D_{0t}^{\beta}u + \frac{\partial^4}{\partial x^4}u(x,y,t) + \alpha\frac{\partial^4}{\partial x^4}u(-x,y,t) + \frac{\partial^4}{\partial y^4}u(x,y,t) = f(x,y,t), \tag{1} $$

where $-1 < \alpha < 1$. Here ${}_{C}D_{0t}^{\beta}u = I_{0t}^{1-\beta}(u_t),\ \beta \in (0,1]$ is the Caputo derivative of order β [25] where for any integrable function g the left-hand fractional Riemann–Liouville integral of order $\beta > 0$ is defined as

$$I_0^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(\tau)}{(t-\tau)^{1-\beta}} d\tau,$$

where $\Gamma(\beta)$ is the Euler gamma function.

Let $Q = \{(x, y, t) : -1 < x, y < 1\}$ and $\bar{Q}$ be its closure. The space $C_{x,y}^{k,l}(\bar{Q})$ consists of all functions $u(x, y, t)$ having continuous derivatives of the order k with respect to x and continuous derivatives of the order l with respect to y in the domain $\bar{Q}$, $t > 0$.

The purpose of the work is to find, for given functions $f(x, y, t)$ and $\varphi(x, y)$, a function that satisfies the following conditions:

(1) $u(x, y, t) \in C_{x,y}^{3,3}(\bar{Q}) \cap C_{x,y}^{4,4}(Q)$, $t \geq 0$;

(2) $u(x, y, t)$ satisfies Equation (1) in the domain Q, $t > 0$;

(3) $u(x, y, t)$ satisfies the boundary conditions

$$u(x, y, 0) = \varphi(x, y), \ \ -1 \leq x, y \leq 1, \tag{2}$$

$$u(-1, y, t) = 0, \ \ u(1, y, t) = 0, \ \ \frac{\partial^2}{\partial x^2} u\bigg|_{x=-1} = \frac{\partial^2}{\partial x^2} u\bigg|_{x=1} = 0, \ \ -1 \leq y \leq 1, \ t \geq 0, \tag{3}$$

$$\begin{aligned} l_j(u) &\equiv a_j \frac{\partial}{\partial y} u(x, -1, t) + b_j \frac{\partial}{\partial y} u(x, 1, t) + a_{j1} u(x, -1, t) + b_{j1} u(x, 1, t) = 0, \\ \tilde{l}_j(u) &\equiv \tilde{a}_j \frac{\partial^3}{\partial y^3} u(x, -1, t) + \tilde{b}_j \frac{\partial^3}{\partial y^3} u(x, 1, t) + \tilde{a}_{j1} \frac{\partial^2}{\partial y^2} u(x, -1, t) + \tilde{b}_{j1} \frac{\partial^2}{\partial y^2} u(x, 1, t) = 0, \end{aligned} \tag{4}$$

where for $j = 1, 2$, coefficients a_j, a_{j1}, b_j, b_{j1} are some given numbers, $(x, t) \in [-1, 1] \times [0, \infty)$.

As can be seen from conditions (4), we consider boundary conditions of the most general type.

The paper consists of three sections and a list of references. Section 2 is devoted to the study of basic properties of the fourth-order differential operators. In Section 3, we give a solution to the stated problem.

2. Properties of Eigenfunctions of Spectral Problems

Consider a homogeneous equation

$${}_C D_{0t}^\beta u + \frac{\partial^4}{\partial x^4} u(x, y, t) + \alpha \frac{\partial^4}{\partial x^4} u(-x, y, t) + \frac{\partial^4}{\partial y^4} u(x, y, t) = 0 \tag{5}$$

with boundary conditions (2)–(4). For simplicity and without loss of generality, we seek for a nonzero solution of Equation (5) in the separated form $u(x, y, t) = w(x, y)c(t)$. Substituting this into Equation (5), we obtain the spectral problem

$$\frac{\partial^4}{\partial x^4} w(x, y) + \alpha \frac{\partial^4}{\partial x^4} w(-x, y) + \frac{\partial^4}{\partial y^4} w(x, y) = \sigma w(x, y), \ \ -1 < x, y < 1, \tag{6}$$

$$w(-1, y) = w(1, y) = 0, \ \ \frac{\partial^2}{\partial x^2} w\bigg|_{x=-1} = \frac{\partial^2}{\partial x^2} w\bigg|_{x=1} = 0, \ \ -1 \leq y \leq 1, \tag{7}$$

$$\begin{aligned} l_j(w) &= 0, \ j = 1, 2, \\ \tilde{l}_j(w) &= 0, \ j = 1, 2. \end{aligned} \tag{8}$$

The solution to problem (6)–(8) is sought in the form $w(x, y) = X(x)Y(y)$. Then, the spectral problem (6)–(8) generates the following two one-dimensional spectral problems.

Spectral problem 1.

$$X''''(x) + \alpha X''''(-x) = \lambda X(x), \ \ -1 < x < 1, \tag{9}$$

$$X(-1) = X(1) = X''(-1) = X''(1) = 0. \tag{10}$$

Spectral problem 2.

$$Y''''(y) = \mu Y(y), \;\; -1 < y < 1, \tag{11}$$

$$\begin{aligned} l_j(Y) &\equiv a_j Y'(-1) + b_j Y'(1) + a_{j1} Y(-1) + b_{j1} Y(1) = 0, \; j = 1,2, \\ \tilde{l}_{j\theta}(Y) &\equiv \tilde{a}_j Y'''(-1) + \tilde{b}_j Y'''(1) + \tilde{a}_{j1} Y''(-1) + \tilde{b}_{j1} Y''(1) = 0, \; j = 1,2, \end{aligned} \tag{12}$$

where $\mu = \sigma - \lambda$.

Spectral problem 1. To study problems (9), (10), we use the following notation: $\alpha_0 = \sqrt[4]{\frac{1}{1+\alpha}}$, $\alpha_1 = \sqrt[4]{\frac{1}{1-\alpha}}$, $\rho = \sqrt[4]{\lambda}$, (here, $\left(re^{i\varphi}\right)^{\frac{1}{4}} = r^{\frac{1}{4}} e^{\frac{i\varphi}{4}}$, $-\frac{\pi}{2} < \varphi \leq \frac{3\pi}{2}$).

The solution to Equation (9) can be written as

$$X(x) = c_1\left(e^{\alpha_0 \rho x} + e^{-\alpha_0 \rho x}\right) + c_2\left(e^{i\alpha_0 \rho x} + e^{-i\alpha_0 \rho x}\right) + c_3\left(e^{\alpha_1 \rho x} - e^{-\alpha_1 \rho x}\right) + c_4\left(e^{i\alpha_1 \rho x} - e^{-i\alpha_1 \rho x}\right).$$

Using boundary conditions (10), we find the eigenvalues of the first series

$$\lambda_{k1} = (1+\alpha)\left(k - \frac{1}{2}\right)^4 \pi^4, \;\; k = 1,2,\ldots,$$

whose corresponding eigenfunctions are

$$X_{k1} = \cos\left(k - \frac{1}{2}\right)\pi x.$$

The second series of eigenvalues

$$\lambda_{k2} = (1-\alpha)(k\pi)^4, \;\; k = 1,2,3,\ldots,$$

whose corresponding eigenfunctions are $X_{k2} = \sin k\pi x$.

The system of eigenfunctions of spectral problems (9) and (10) can be written as

$$\{X_k(x)\} = \left\{X_{2k}(x) = \cos\left(k - \frac{1}{2}\right)\pi x, \; X_{2k-1}(x) = \sin k\pi x\right\}. \tag{13}$$

Lemma 1. *System* (13) *forms a complete orthonormal system in* $L_2(-1,1)$.

The lemma is proved by the direct calculation of the Fourier coefficients of an arbitrary function $f(x) \in L_2(-1,1)$.

Spectral problem 2. To study the fourth-order spectral problem (11), (12), we introduce the operator $LY(y) = -Y''(\theta y)$, where $\theta = \pm 1$, with the domain $D(L) = \{Y(y) \in W_2^2[-1,1] : l_j(Y) = 0\}$, where $W_2^2[-1,1]$ is the Sobolev space [26]. Hereafter, the operator L acts on functions depending on the variable y. The operator L commutes with the operator S, i.e., $LS = SL$, and the equality $L^2 Y(y) = Y''''(y)$ is valid if $LY(y) \in D(L)$. The domain of the operator L^2 is

$$D\left(L^2\right) = \left\{Y(y) \in W_4^2[-1,1] : l_j(Y) = 0, \tilde{l}_{j\theta}(Y) = 0\right\}.$$

In this case, the coefficients of the expression $\tilde{l}_j(Y)$ in (12) depend on the number θ. Consider the spectral problem for the second-order equation

$$-Y''(\theta y) = \nu Y(y), \;\; -1 < y < 1, \;\; \theta = \pm 1, \tag{14}$$

$$l_j(Y) = 0, \;\; j = 1,2, \tag{15}$$

where (15) is the first equality in (12). Equation (14) is self-conjugate, whereas boundary conditions (15) may be non-self-conjugate. The conjugate boundary conditions are denoted by $l_j^*(Y)$. For $\theta = 1$, we have a classical spectral problem for an ordinary second-order differential operator. For $\theta = -1$, problem (14), (15) was studied in [27,28]. In the case $\theta = 1$, the result is well known [29].

Theorem 1 ([29])**.** *Let $\theta = 1$. Then, the system of eigenfunctions of the spectral problem (14), (15) forms a Riesz basis in $L_2(-1,1)$ if one of the following three conditions A:*

(1) $a_1b_2 - a_2b_1 \neq 0$;

(2) $a_1 = a_2 = b_1 = b_2 = 0$, $a_{11}b_{21} - a_{21}b_{11} \neq 0$;

(3) $a_1b_2 - a_2b_1 = 0$, $|a_1| + |b_1| > 0$, $(a_1^2 - b_1^2)(a_{21}^2 - b_{21}^2) \neq 0$;

is satisfied.

In the case $\theta = -1$, properties of eigenfunctions in problems (14), (15) were studied in [27,28]. Based on the results of [28], the following theorem is formulated.

Theorem 2 ([28])**.** *Let $\theta = -1$. Then, the system of eigenfunctions of spectral problems (14) and (15) forms a Riesz basis in $L_2(-1,1)$ if one of the three conditions A is satisfied. If $a_1b_2 - a_1b_2 = 0$, $|a_1| + |b_1| > 0$, and one of the following four conditions:*

$$\begin{aligned} &(1)\, a_1 = b_1,\ a_{21} + b_{21} \neq 0;\ (2) a_1 = -b_1,\ a_{21} - b_{21} \neq 0; \\ &(3)\, a_{21} = b_{21},\ a_1 + b_1 \neq 0;\ \ (4)\, a_{21} = -b_{21},\ a_1 - b_1 \neq 0, \end{aligned} \tag{16}$$

is satisfied, then the system of eigenfunctions of the spectral problem (14), (15) also forms a Riesz basis in $L_2(-1,1)$.

It follows from the results of [28] that the system of eigenfunctions of problem (14) does not form a basis in $L_2(-1,1)$ for boundary conditions of the type $Y(-1) = \pm Y(-1)$, $Y'(-1) = \mp Y'(-1)$.

It was shown in [27] that if $LY(y) \in D(L)$, then the system of eigenfunctions of the second-order spectral problems (14) and (15) coincides with the system of eigenfunctions of the fourth-order spectral problems (11) and (12). Therefore, Theorems 1 and 2 are also valid for the fourth-order spectral problems (11) and (12).

Remark 1. *If $\theta = 1$, then the expressions $\tilde{l}_{j\theta}(Y)$ in (12) can be represented in the form*

$$\tilde{l}_{j\theta}(Y) \equiv a_jY'''(-1) + b_jY'''(1) + a_{j1}Y''(-1) + b_{j1}Y''(1) = 0,\ j = 1,2.$$

According to this, the second boundary condition in (4) takes the form

$$\tilde{l}_j(u) \equiv a_j\frac{\partial^3}{\partial y^3}u(x,-1,t) + b_j\frac{\partial^3}{\partial y^3}u(x,1,t) + a_{j1}\frac{\partial^2}{\partial y^2}u(x,-1,t) + b_{j1}\frac{\partial^2}{\partial y^2}u(x,1,t) = 0. \tag{17}$$

Remark 2. *If $\theta = -1$, then the expressions $\tilde{l}_{j\theta}(Y)$ in (12) can be represented in the form*

$$\tilde{l}_{j\theta}(Y) \equiv b_jY'''(-1) + a_jY'''(1) - b_{j1}Y''(-1) - a_{j1}Y''(1) = 0,\ j = 1,2.$$

According to this, the second boundary condition in (4) takes the form

$$\tilde{l}_j(u) \equiv b_j\frac{\partial^3}{\partial y^3}u(x,-1,t) + a_j\frac{\partial^3}{\partial y^3}u(x,1,t) - b_{j1}\frac{\partial^2}{\partial y^2}u(x,-1,t) - a_{j1}\frac{\partial^2}{\partial y^2}u(x,1,t) = 0. \tag{18}$$

Let us formulate theorems on the basis property of the eigenfunctions of the spectral problems (11) and (12) for a fourth-order equation.

Theorem 3. *Let $\theta = 1$ and $LY(y) \in D(L)$. Then, the system of eigenfunctions of the spectral problems (11) and (12) forms the Riesz basis in $L_2(-1,1)$ if the coefficients of the boundary conditions (12) satisfy one of the three conditions A.*

Theorem 4. *Let $\theta = -1$ and $LY(y) \in D(L)$. Then, the system of eigenfunctions of the spectral problems (11) and (12) forms the Riesz basis in the following cases:*

(1) if the coefficients of the boundary conditions (12) satisfy one of the three conditions A; or

(2) if $a_1 b_2 - a_2 b_1 = 0$, $|a_1| + |b_1| > 0$, and for the coefficients of the boundary conditions (12), one of the four conditions in (16) is satisfied.

Note that when one of the three conditions A is satisfied, all eigenvalues of the spectral problem for the second-order Equations (14) and (15) are single. If one of the conditions (16) is satisfied, the boundary conditions (12) for the fourth-order Equation (11) are regular in the sense of Birkhoff [30] (but are not strongly regular), the eigenvalues of the fourth order spectral problem (11), (12) may be multiple. For example [27], the boundary value problem (14) with boundary conditions

$$Y(-1) = Y''(1) = 0, \;\; Y'(-1) = Y'(1), \;\; Y'''(-1) = Y'''(1)$$

($a_1 = -b_1$, $a_{21} = 1$, all other coefficients are equal to zero) is not self-conjugate, has an infinite number of double eigenvalues, has no associated functions, and the system of eigenfunctions forms a Riesz basis in $L_2(-1,1)$. Hence, it appears that the system of eigenfunctions of the conjugate problem

$$Y^{*\prime}(-1) = 0, \; Y^{*\prime\prime\prime}(1) = 0, \; Y^{*}(-1) = Y^{*}(1), \; Y^{*\prime\prime}(-1) = Y^{*\prime\prime}(1),$$

also forms the Riesz basis in $L_2(-1,1)$.

Thus, in the case $\theta = -1$, we cover a larger range of boundary conditions for the boundary value problem (11), (12).

All the systems we consider are Riesz bases, and they remain Riesz bases for any enumeration of system elements (unconditional bases). By virtue of [31], the systems we consider are uniformly bounded. Therefore, the coefficients of bi-orthonormalization, in the case of the non-orthogonality of the studied systems, are also uniformly bounded. This means that the bi-orthnormalization coefficients do not affect either the convergence of expansions in terms of eigenfunctions or the convergence of series formed by the expansion coefficients.

Note that boundary conditions (10) are self-conjugate, whereas boundary conditions (12) may be non-self-conjugate. Therefore, problem (6)–(8) in the general case is a non-self-conjugate problem. For specific boundary conditions of the form (8), the conjugate boundary conditions are written in the standard way.

Let us write a problem conjugate to problem (6)–(8):

$$\frac{\partial^4}{\partial x^4} w^*(x,y) + \alpha \frac{\partial^4}{\partial x^4} w^*(-x,y) + \frac{\partial^4}{\partial y^4} w^*(x,y) = \bar{\sigma} w^*(x,y), \;\; -1 < x,y < 1, \tag{19}$$

$$w^*(-1,y) = w^*(1,y) = 0, \;\; \left.\frac{\partial^2 w^*}{\partial x^2}\right|_{x=-1} = \left.\frac{\partial^2 w^*}{\partial x^2}\right|_{x=1} = 0, \;\; -1 \le y \le 1, \tag{20}$$

$$\begin{gathered} a_j^* \frac{\partial}{\partial y} w^*(x,-1) + b_j^* \frac{\partial}{\partial y} w^*(x,1) + a_{j1}^* w^*(x,-1) + b_{j1}^* w^*(x,1) = 0, \; j = 1,2, \\ \tilde{a}_j^* \frac{\partial^3}{\partial y^3} w^*(x,-1) + \tilde{b}_j^* \frac{\partial^3}{\partial y^3} w^*(x,1) + \tilde{a}_{j1}^* \frac{\partial^2}{\partial y^2} w^*(x,-1) + \tilde{b}_{j1}^* \frac{\partial^2}{\partial y^2} w^*(x,1) = 0, \;\; j = 1,2. \end{gathered} \tag{21}$$

Taking the above into account, we assume that all eigenvalues of problems (9)–(12) are numbered as follows: $\lambda_{2n} = (1+\alpha)\left(n-\frac{1}{2}\right)^4\pi^4$, $\lambda_{2n-1} = (1-\alpha)(n\pi)^4$, $n = 1, 2, \ldots$, $|\mu_k| \leq |\mu_{k+1}|$, for any number k, $\sigma_{nk} = \lambda_n + \mu_k$. If an eigenvalue $\mu_k$is double, then $\mu_{k+1} = \mu_k$. The eigenfunctions wiare numbered respectively.

We denote the system of eigenfunctions of problem (6)–(8) as

$$w_{kn}(x,y) = X_k(x)Y_n(y), \tag{22}$$

where $X_k(x)$ are eigenfunctions of problem (9), (10), and $Y_n(y)$ are eigenfunctions of problem (11), (12). We denote the system of eigenfunctions of the corresponding conjugate problem (19)–(21) as

$$w^*{}_{kn}(x,y) = X_k(x)Y_n^*(y), \tag{23}$$

where $Y_n^*(y)$ are eigenfunctions of the problem conjugate to problems (11) and (12).

Systems (22) and (23) are bi-orthogonal as systems of eigenfunctions of mutually conjugate spectral problems (6)–(8) and (19)–(21). Since the system $\{X_k(x)\}$ is an orthonormal basis and the system $\{Y_n(y)\}$ is a Riesz basis, then each of the systems (22) and (23) is a Riesz basis [32] in $L_2((-1,1)\times(-1,1))$.

3. Existence and Uniqueness of the Problem Solution

In this section, we assume that all the conditions of Theorem 4 are satisfied, and the second boundary condition in (4) is of the form (17) or (18).

Then, each of the systems (22) and (23) forms a Riesz basis in $L_2((-1,1)\times(-1,1))$. To solve the inhomogeneous problem (1)–(4), we first consider the homogeneous Equation (5). The solution to problems (5), (2)–(4) is sought in the form of a Fourier series

$$u(x,y,t) = \sum_{k,n=0}^{\infty} C_{kn}(t)w_{kn}(x,y) \tag{24}$$

using the Riesz basis $\{w_{kn}(x,y)\}$, where the unknown functions $C_{kn}(t)$ are defined by the relations

$$C_{kn}(t) = \iint\limits_{\Omega_{xy}} u(x,y,t)w^*{}_{kn}(x,y)dxdy = \int\limits_{-1}^{1}\int\limits_{-1}^{1} u(x,y,t)X_k(x)Y_n^*(y)dxdy. \tag{25}$$

From Equations (5) and (25) and condition (2), we obtain the problems

$${}_CD_{0t}^{\beta}C_{kn}(t) + \sigma_{nk}C_{kn}(t) = 0, C_{kn}(0) = \varphi_{kn},$$

whose solutions are written as [25]

$$C_{kn}(t) = \varphi_{kn}E_\beta\left(-\sigma_{kn}t^\beta\right), \tag{26}$$

where $\varphi_{kn} = (\varphi(x,y), w^*{}_{kn}(x,y))$ in the sense of the scalar product in $L_2((-1,1)\times(-1,1))$. Here, $E_\beta(z)$ is the Mittag–Leffler function [25] satisfying the estimate

$$\left|E_\beta(z)\right| \leq \frac{M}{1+|z|}.$$

Using relation (26), expression (24) can be written as

$$u(x,y,t) = \sum_{k,n=1}^{\infty} \varphi_{kn}E_\beta\left(-\sigma_{kn}t^\beta\right)X_k(x)Y_n(y). \tag{27}$$

It is necessary to show that the formal solution (27) is a classical solution to problems (5), (2)–(4).

Lemma 2. *Let $\varphi(x,y) \in L_2(\bar{Q})$. Then, series (27) converges absolutely and uniformly in the domain$\bar{Q}$, $t > 0$.*

Proof. For any $t \geq t_0 > 0$, the Mittag–Leffler function satisfies the estimate

$$E_\beta\left(-\sigma_{kn}t^\beta\right) \leq \frac{M}{1+\sigma_{kn}t^\beta}.$$

Since $\sigma_{nk} = \lambda_n + \mu_k$, $\lambda_{k1} = (1+\alpha)\left(k - \frac{1}{2}\right)^4 \pi^4$, $\lambda_{k2} = (1-\alpha)(k\pi)^4$, $\mu_k = O\left(k^4\right)$, then

$$\frac{M}{1+\sigma_{kn}t^\beta} \leq \frac{M}{\sigma_{kn}t^\beta} \leq \frac{M}{\sigma_{kn}t_0^\beta} \leq \frac{Mt_0^{-\beta}}{\lambda_k + \mu_k} \leq \frac{Mt_0^{-\beta}}{2\sqrt{\lambda_k}\sqrt{\mu_k}}.$$

Then

$$\begin{aligned} |u(x,y,t)| &= \left|\sum_{k,n=1}^{\infty} \varphi_{kn} E_\beta\left(-\sigma_{kn}t^\beta\right) X_k(x) Y_n(y)\right| \\ \leq \sum_{k,n=1}^{\infty} &\left|\int_{-1}^{1}\int_{-1}^{1} \varphi(x,y) X_k(x) Y_n^*(y) dxdy\right| \left|E_\beta\left(-\sigma_{kn}t^\beta\right)\right| |X_k(x)| |Y_n(y)| \\ &\leq \sum_{k,n=1}^{\infty} \frac{M_0 M t_0^{-\beta}}{2\sqrt{\lambda_k}\sqrt{\mu_k}} \left|\int_{-1}^{1}\int_{-1}^{1} \varphi(x,y) X_k(x) Y_n^*(y) dxdy\right|. \end{aligned}$$

Applying the Cauchy–Schwarz inequality and the Bessel inequality, we obtain

$$\sum_{k,n=0}^{\infty} |\varphi_{kn}| \leq M_1 \|\varphi\|_{L_2(Q)}.$$

The lemma is proved. □

Note that the method of separation of variables imposes stricter requirements on the initial function. However, in this work, the problem of reducing the smoothness of the initial function was not posed.

The main result of the work is the following theorem.

Theorem 5. *Let the function $\varphi(x,y)$ be such that $\varphi(x,y) \in C_{x,y}^{3,0}(\bar{Q}) \cap C_{x,y}^{0,3}(\bar{Q}) \cap C_{x,y}^{4,0}(Q) \cap C_{x,y}^{0,4}(Q) \cap C_{x,y}^{2,2}(\bar{Q})$, $L\varphi(x,y) \in D(L)$, $L^2\varphi(x,y) \in D(L)$, $t > 0$ and satisfy self-conjugate conditions (3) with respect to the variable x. Then, problem (5), (2)–(4) has a unique solution, which can be represented in the form (27).*

Proof. Let us first prove the uniqueness. If the problem has two solutions u_1, u_2, then the function $u = u_1 - u_2$ satisfies $u(x,y,0) = 0$, Equation (5), and boundary conditions (3), (4). Then, from (26), we obtain $C_{kn}(t) = 0$, and from (25) it follows that $u(x,y,t) = 0$ in $\bar{Q}$ due to the completeness of the system $\{w^*{}_{kn}(x,y)\}$ in $L_2((-1,1)\times(-1,1))$. By virtue of Lemma 2, from the continuity of the solution in the entire closed domain, we obtain $u(x,y,t) = 0$ in $\bar{Q}$. The uniqueness of the solution is proved.

Let us prove the existence of a solution. Formula (27) represents a formal solution to problems (5), (2)–(4). In order to justify that the obtained formal solution is indeed a true solution, we need to show that the series ${}_C D_{0t}^\beta u$, $\frac{\partial^p u}{\partial x^p}$, $\frac{\partial^p u}{\partial y^p}$, $p = 1,2,3$ and $\frac{\partial^4 u}{\partial x^4}$, $\frac{\partial^4 u}{\partial y^4}$ converge uniformly in $\bar{Q}$ and Q, $t > 0$, respectively.

It is obvious that for the elements of system (13), the estimates

$$\left|\frac{d^p X_{2k}}{dx^p}\right| \leq \left(\left(k-\frac{1}{2}\right)\pi\right)^p,$$

$$\left|\frac{d^p X_{2k-1}}{dx^p}\right| \leq (k\pi)^p,\ p = 1,2,3,4$$

are valid. Now, we must prove the uniform convergence of the series $\frac{\partial^4 u(x,y,t)}{\partial x^4}$ in Q, $t > 0$. Formally differentiating series (27) four times with respect to x and integrating by parts four times with respect to the variable x, taking into account (13), we get

$$\frac{\partial^4}{\partial x^4}u(x,y,t) = \sum_{k,n=1}^{\infty} \varphi_{kn} E_\beta\left(-\sigma_{kn}t^\beta\right) X_k^{(4)}(x) Y_n(y)$$

$$= \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi(x,y)X_k(x)Y_n^*(y)dxdy\right] E_\beta\left(-\sigma_{kn}t^\beta\right) X_k^{(4)}(x) Y_n(y)$$

$$= \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi(x,y)X_k(x)Y_n^*(y)dxdy\right] E_\beta\left(-\sigma_{kn}t^\beta\right) \lambda_k X_k(x) Y_n(y)$$

$$\leq \alpha_2 \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi_x{}^{(4)}(x,y)\frac{X_k(x)}{\lambda_k}Y_n^*(y)dxdy\right] E_\beta\left(-\sigma_{kn}t^\beta\right) \lambda_k X_k(x) Y_n(y),$$

where $\alpha_2 = max\{\alpha_0{}^{-1}, \alpha_1{}^{-1}\}$. Further, applying the Lemma 2 we get

$$\left|\frac{\partial^4 u(x,y,t)}{\partial x^4}\right| \leq M_0 \left\|\varphi_x^{(4)}\right\|_{L_2(Q)}.$$

where $\varphi_x^{(4)}$ means the fourth derivative with respect to x. Thus, the uniform convergence of the series $\frac{\partial^4 u(x,y,t)}{\partial x^4}$ in Q is proved.

The convergence of the series $\frac{\partial^4 u(x,y,t)}{\partial y^4}$ is proved similarly by replacing the fourth-order derivative $Y^{(4)}(y)$ with the help of Equation (11). In this case, the function inside the integral $Y_n^*(y)$ is represented as its fourth derivative using Equation (11). Here are the corresponding calculations:

$$\frac{\partial^4}{\partial y^4}u(x,y,t) = \sum_{k,n=1}^{\infty} \varphi_{kn} E_\beta(-\sigma_{kn}t^\beta) X_k(x) Y_n^{(4)}(y)$$

$$= \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi(x,y)X_k(x)Y_n^*(y)dxdy\right] E_\beta(-\sigma_{kn}t^\beta) X_k(x) \mu_n Y_n(y)$$

$$= \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi(x,y)X_k(x)\frac{Y_n^{*(4)}(y)}{\mu_n}dxdy\right] E_\beta(-\sigma_{kn}t^\beta) X_k(x) \mu_n Y_n(y)$$

$$= \sum_{k,n=1}^{\infty}\left[\int_{-1}^{1}\int_{-1}^{1}\varphi_y^{(4)}(x,y)X_k(x)\frac{Y_n^*(y)}{\mu_n}dxdy\right] E_\beta(-\sigma_{kn}t^\beta) X_k(x) \mu_n Y_n(y).$$

Hence, as in the previous case, we obtain the estimate

$$\left|\frac{\partial^4 u(x,y,t)}{\partial y^4}\right| \leq M_0 \left\|\varphi_y^{(4)}\right\|_{L_2(Q)}.$$

Thus, the uniform convergence of the series $\frac{\partial^4 u(x,y,t)}{\partial y^4}$ in Q, $t > 0$ is proved. The uniform convergence of the series $\frac{\partial^2 u(x,y,t)}{\partial y^2}$ in $\bar{Q}$, $t > 0$ is proved in a similar way.

Using estimates [33]

$$\left|Y_n^{(s)}(y)\right| \leq const\left(1+\sqrt[4]{|\mu_n|}\right)^s \max|Y_n(y)|,\ s=1,2,3,$$

for the eigenfunctions of the fourth-order differential operator, as above, we prove the uniform convergence of the series $\frac{\partial^p u}{\partial y^p}$, $p=1,3$, in $\bar{Q}$.

Each term of series (27) is a solution of problems (5), (2)–(4). Therefore, boundary conditions (3) and (4) are satisfied. The function (27) satisfies the relations of the type (2). To do this, it sufficient to show that the series (27) converges uniformly in $\bar{Q}$, $t \geq 0$. For $t \geq 0$, the inequality

$$|u(x,y,t)| \leq M \sum_{k,n=1}^{\infty} \left| \int_{-1}^{1}\int_{-1}^{1} \varphi(x,y)X_k(x)Y_n^*(y)dxdy \right| |X_k(x)||Y_n(y)|$$

is satisfied. The function inside the integral $Y_n^*(y)$ is represented as its second derivative using Equation (14). Integrating by parts two times over the variable x and two times over the variable y, applying the Cauchy–Schwarz inequality and the Bessel inequality, we conclude that the series (27) converges absolutely and uniformly in $\bar{Q}$, $t \geq 0$. The theorem is proved. □

Remark 3. *The solution to problem* (2)–(4) *for the inhomogeneous Equation* (1) *is sought in the form of a series* (24)*. We represent the function $f(x,y,t)$ as an expansion in the basis* (22)

$$f(x,y,t) = \sum_{k,n=0}^{\infty} f_{kn}(t)w_{kn}(x,y),\ f_{kn}(t) = (f(x,y,t), {w^*}_{kn}(x,y)).$$

The functions $C_{kn}(t)$ have the form

$$C_{kn}(t) = \varphi_{kn}E_\beta\left(-\sigma_{kn}t^\beta\right) + \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\left(-\sigma_{kn}(t-\tau)^\beta\right)f_{kn}(\tau)d\tau, \tag{28}$$

where $\varphi_{kn} = (\varphi(x,y), {w^*}_{kn}(x,y))$ in the sense of the scalar product in $L_2((-1,1)\times(-1,1))$, $E_{\beta,\gamma}(z)$ is the Mittag–Leffler function [25] defined by formula $E_{\beta,\gamma}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+\gamma)}$ and satisfying the estimate

$$\left|E_{\beta,\gamma}(z)\right| \leq \frac{M}{1+|z|}, 0<\beta<2, \gamma \in R,\ \delta \leq |\arg z| \leq \pi,\ \delta \in \left(\frac{\beta\pi}{2}, \min\{\pi;\beta\pi\}\right).$$

Substituting (28) *into* (24)*, we obtain the solution of problem* (1)–(4)*.*

Example 1. *Let the boundary conditions* (4) *have the form (case $\theta = -1$)*

$$\begin{aligned} l_1(u) \equiv \frac{\partial}{\partial y}u(x,-1,t) + \frac{\partial}{\partial y}u(x,1,t) = 0,\\ l_2(u) \equiv u(x,1,t) = 0,\\ \tilde{l}_1(u) \equiv \frac{\partial^3}{\partial y^3}u(x,-1,t) + \frac{\partial^3}{\partial y^3}u(x,1,t) = 0,\\ \tilde{l}_2(u) \equiv \frac{\partial^2}{\partial y^2}u(x,-1,t) = 0. \end{aligned} \tag{29}$$

Then, the spectral problem (11), (12) *takes the form*

$$Y''''(y) = \mu Y(y),\ \ -1<y<1,$$

$$Y'(-1) + Y'(1) = 0,\ Y(1) = 0,\ Y'''(-1) + Y'''(1) = 0,\ Y''(-1) = 0.$$

where the eigenvalues $\mu_k = \left(k - \frac{1}{2}\right)^4 \pi^4$ *correspond to two series of eigenfunctions*

$$Y_{k1} = \sin\left(k - \frac{1}{2}\right)\pi y + (-1)^{k+1} \frac{e^{\left(k-\frac{1}{2}\right)\pi y} + e^{-\left(k-\frac{1}{2}\right)\pi y}}{e^{\left(k-\frac{1}{2}\right)\pi} + e^{-\left(k-\frac{1}{2}\right)\pi}}, Y_{k2} = \cos\left(k - \frac{1}{2}\right)\pi y, \tag{30}$$

$k = 1, 2, \ldots$ *The conjugate boundary conditions are*

$$Y^{*\prime}(1) = 0, Y^*(-1) + Y^*(1) = 0,\ Y^{*\prime\prime\prime}(-1) = 0,\ Y^{*\prime\prime}(-1) + Y^{*\prime\prime}(1) = 0.$$

The eigenfunctions of the conjugate problem are

$$Y^*_{k1} = \sin\left(k - \frac{1}{2}\right)\pi y, Y^*_{k2} = \cos\left(k - \frac{1}{2}\right)\pi y + (-1)^{k+1} \frac{e^{\left(k-\frac{1}{2}\right)\pi y} - e^{-\left(k-\frac{1}{2}\right)\pi y}}{e^{\left(k-\frac{1}{2}\right)\pi} + e^{-\left(k-\frac{1}{2}\right)\pi}.} \tag{31}$$

Each of systems (30) and (31) forms a Riesz basis in $L_2(-1, 1)$ *[27].*

Theorem 5 is formulated in the following form.

Theorem 6. *Let the function* $\varphi(x, y)$ *be such that* $\varphi(x, y) \in C^{3,0}_{x,y}(\bar{Q}) \cap C^{0,3}_{x,y}(\bar{Q}) \cap C^{4,0}_{x,y}(Q) \cap C^{0,4}_{x,y}(Q) \cap C^{2,2}_{x,y}(\bar{Q})$, *satisfy self-conjugate conditions (3) with respect to the variable x and conditions (29) with respect to the variable y. Then problem (5), (2), (3), (29) has a unique solution, which can be represented in the form (27).*

Example 2. *Considering the problem (5), (2), (3), (29). Let in (2)* $\varphi(x, y) = \sin \pi x \cos \frac{\pi}{2} y$. *Then in (27) coefficient* $\varphi_{11} = 1$, *all other coefficients* $\varphi_{kn}, k, n \neq 1$, *equal to zero. Since* $\lambda_1 = (1 - \alpha)\pi^4$, $\mu_1 = \frac{1}{16}\pi^4$, $\sigma_{11} = \left(\frac{17}{16} - \alpha\right)\pi^4$, *from Equation (26), we get* $C_{11} = E_\beta \left(-\left(\frac{17}{16} - \alpha\right)\pi^4 t^\beta\right)$. *The solution in the form (27) of problem (5), (2), (3), (29) takes the form*

$$u(x, y, t) = E_\beta\left(-\left(\frac{17}{16} - \alpha\right)\pi^4 t^\beta\right) \sin \pi x \cos \frac{\pi}{2} y.$$

4. Conclusions

The existence and uniqueness of regular solutions of a certain class of problems for two-dimensional parabolic equations of fractional order is established. With respect to one variable, the equation contains an involution transformation. The completeness of the eigenfunctions of the boundary value problem for the fourth-order differential equation with involution is proved. The Riesz basis property of the eigenfunctions of boundary value problems for the fourth-order ordinary differential equation is shown for a special class of boundary conditions. In this case, the properties of eigenfunctions of boundary value problems for the second-order differential equation with involution are used.

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