fractal and fractional

Article

On Variable-Order Fractional Discrete Neural Networks: Existence, Uniqueness and Stability

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Abstract: Given the recent advances regarding the studies of discrete fractional calculus, and the fact that the dynamics of discrete-time neural networks in fractional variable-order cases have not been sufficiently documented, herein, we consider a novel class of discrete-time fractional-order neural networks using discrete nabla operator of variable-order. An adequate criterion for the existence of the solution in addition to its uniqueness for such systems is provided with the use of Banach fixed point technique. Moreover, the uniform stability is investigated. We provide at the end two numerical simulations illustrating the relevance of the aforementioned results.

Keywords: discrete fractional variable-order neural networks; discrete nabla variable-orde fractional operators; Banach fixed point theorem; uniform stability

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1. Introduction

Fractional calculus has drawn the attention of an increasing number of researchers over the years, and it has seen applications in a variety of engineering and applied sciences [1–3]. Many fractional models have recently been presented, demonstrating the importance of fractional calculus. In aiming to provide better-behaved kernels of extra fractional operators, several fractional calculus researchers recently presented and studied novel non-local fractional operators with nonsingular kernels which were proved to be applicable in real world issues [4,5].

Despite that many research works in the literature about continuous fractional calculus provide various solid mathematical theories, there has not been much interest when it comes to discrete fractional calculus development (see [6–10]). However, in the last five to seven years, the focus in improving discrete fractional calculus has been remarkably raising. This progression has shown that discrete fractional calculus is now the most modern and extensively used model of fractional calculus which contains a lot of unforeseen challenges and performance difficulties [11–14]. AB-fractional operators are known to be fundamental operators regarding fractional calculus; for that purpose, it has historically been found that they have been used to build current operators besides their related characterizations. These operators have been conceptually presented further by proposing and studying discrete versions of such fractional operators [15,16]. The reader who is intrigued in the purpose and relevance of considering variable order operators should refer to [17,18].

In the simulation of physics and engineering systems, fractional neural networks have shown to be highly efficient and precise. Several engineering problems, such as digital signal processing or industrial system control and diagnostics, rely on such models, and numerous studies have been conducted to investigate various aspects of these systems. For example, in [19], the stability and bifurcation of fractional-order gene regulatory networks was reported. Furthermore, bifurcation in fractional-order genetic regulatory networks with both types of delays was investigated in [20]. In [21] a summary of various fractional neural network dynamics is presented. On the other hand, discrete time fractional-order neural networks are networks that work with data on isolated temporal scales. They are characterized by more distinct and relatively complicated features comparing to continuous neural networks with fractional order. Discrete time fractional-order neural networks have new capabilities and superior performance, allowing them to handle issues that their continuous counterparts can not. As a result, studying the dynamical features of discrete-time neural networks involving fractional-order is crucial (see [22–26]).

Stability theory is a broad field in engineering and various sciences that focuses on studying the behavior of dynamical patterns of systems that are either linear or non-linear. On the one hand, significant advances have been accomplished in the past few years upon the stability theory of fractional order neural networks [27–29]. Nevertheless, the equivalent theory of discrete-time fractional-order neural-network has been interestingly evolving. Indeed, among the various forms of stability addressed for this type of system one can find the finite-time stability which was studied in [30] for a discrete fractional-order neural networks with complex-valued and time delays. In addition, Mittag–Leffler stability for fractional discrete-time neural networks utilizing fixed point approach was explored in [31]. A class of fractional-order discrete-time complex-valued neural networks with temporal delays was presented and a several stability criteria were discussed in [32]. Furthermore, in [33], the asymptotic stability of discrete fractional order neural networks and presents a unique model of variable order neural networks was discussed.

It is important to note that there has not been enough results related to studies of discrete neural-networks with fractional variable-order. A form of variable order recurrent neural network under the Caputo h-discrete fractional operator applying fixed point methods and Mittag–Leffler stability requirements was explored in [34]. Additionally, in [35], a novel variable order discrete neural network using the nabla variable-order operator, as well as an asymptotic stability study of the proposed model was offered. Moreover, another type of stability known as Ulam Hyers stability for a class of discrete variable order neural networks was addressed in [36]. However, there are not enough studies in literature describing the stability of discrete-time fractional variable-order neural-network with nabla discrete Mittag–Leffler kernels. As a result, establishing suitable criteria for such neural networks is both required and difficult. Motivated by the preceding considerations, and since the dynamics of discrete-time fractional variable-order neural networks have been lucking explanation and investigation, this work is one of the papers to do so, the purpose of this research is to offer adequate criteria for studying the uniform stability related to the considered discrete-time neural-networks.

This work includes the following sections: Section 2 involves useful preliminaries, definitions and related lemmas.In Section 3, a few results on both existence and uniqueness of the solutions of the considered discrete-time neural networks are presented via Banach fixed point technique. Section 4 presents our major conclusions, which is a constraint on the uniform stability of the addressed neural networks. Last but not least, in Section 5, we demonstrate the significance of the key findings using two numerical examples.

2. Preliminaries

In this section, the definitions for discrete fractional calculus are presented, together with the following notations:

$$\mathbb{N}_{s_0} = \{s_0, s_0 + 1, s_0 + 2, ...\}, \quad \mathbb{N}_{s_0}^{T} = \{s_0, s_0 + 1, s_0 + 2, ..., T\}. \tag{1}$$

Definition 1 ([37]). *For $0 < \Lambda(s) \leq 1,\ t \in \mathbb{N}_{s_0}$. Let be a function $\Phi : \mathbb{N}_{s_0} \to \mathbb{R}$, We define the nabla fractional sum with order $\Lambda(s)$*

$$ {}_{s_0}\nabla_s^{-\Lambda(s)}\Phi(s) = \frac{1}{\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s}(s-\rho(k))^{\overline{\Lambda(s)-1}}\Phi(k), \quad \rho(k) = k-1, \quad s \in \mathbb{N}_{s_0+1}, \tag{2} $$

where the rising function is as follows

$$ s^{\overline{r}} = \frac{\Gamma(s+r)}{\Gamma(s)},\ \ r \in \mathbb{R}. \tag{3} $$

Definition 2 ([37]). *For $0 < \Lambda(s) < \frac{1}{2},\ \ t \in \mathbb{N}_{s_0}$ and a function $\Phi : \mathbb{N}_{s_0} \to \mathbb{R}$, the left nabla ABC-fractional diffrence is defined by*

$$ \left({}^{ABC}_{s_0}\nabla_t^{\Lambda(s)}\Phi\right)(s) = \frac{B(\Lambda(s))}{1-\Lambda(s)}\sum_{k=s_0+1}^{s}E_{\overline{\Lambda(s)}}\left(\frac{-\Lambda(s)}{1-\Lambda(s)}, s-\rho(k)\right)\nabla\Phi(s), \quad s \in \mathbb{N}_{s_0+1}, \tag{4} $$

where

$$ B(\Lambda(s)) = 1 - \Lambda(s) + \frac{\Lambda(s)}{\Gamma(\Lambda(s))}, \tag{5} $$

and $E_{\overline{a,b}}(\eta, z)$ represents the nabla discrete Mittag–Leffler function

$$ E_{\overline{a,b}}(\eta, z) = \sum_{k\geq 0}\eta^k\frac{z^{\overline{ka+b-1}}}{\Gamma(ak+b)};\ \ |\eta| < 1;\ \ a,b,z \in \mathbb{C}\ \text{and}\ Re(a) > 0. $$

Definition 3 ([37]). *For $\Lambda(s) \in (0,1)$ the left fractional sum of discrete nabla ABC-fractional variable-order operator is described by*

$$ \left({}^{AB}_{s_0}\nabla_s^{-\Lambda(s)}\Phi\right)(s) = \frac{1-\Lambda(s)}{B(\Lambda(s))}\Phi(s) + \frac{\Lambda(s)}{B(\Lambda(s))}\,{}_{s_0}\nabla_s^{-\Lambda(s)}\Phi(s). \tag{6} $$

Lemma 1. *Let be $s \in \mathbb{N}_{s_0+1}$ the following hold*

$$ \sum_{k=s_0+1}^{s}(s-\rho(k))^{\overline{\Lambda(s)-1}} = \frac{(s-s_0)^{\overline{\Lambda(s)}}}{\Lambda(s)}. \tag{7} $$

Proof. To begin, we have

$$ \begin{aligned} \sum_{k=s_0+1}^{s}(s-\rho(k))^{\overline{\Lambda(s)-1}} &= \sum_{k=s_0+1}^{s-1}(s-\rho(k))^{\overline{\Lambda(s)-1}} + 1^{\overline{\Lambda(s)-1}}, \\ &= \sum_{k=s_0+1}^{s-1}\frac{\Gamma(s-k+\Lambda(s))}{\Gamma(s-k+1)} + \Gamma(\Lambda(s)). \end{aligned} $$

Since

$$ \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\iota+1)} = \frac{1}{\iota+1}\left\{\frac{\kappa+2}{\kappa-\iota+1} - \frac{\kappa+1}{\kappa-\iota}\right\}, $$

we set $\kappa = s-k+\Lambda(s)-1,\ \ \iota = \Lambda(s)-1$, and we obtain

$$ \frac{\Gamma(s-k+\Lambda(s))}{\Gamma(s-k+1)} = \frac{1}{\Lambda(s)}\left\{\frac{\Gamma(s-k+\Lambda(s)+1)}{\Gamma(s-k+1)} - \frac{\Gamma(s-k+\Lambda(s))}{\Gamma(s-k)}\right\}. $$

For all $k \in \{s_0+1, ..., s-1\}$ we obtain

$$\sum_{k=s_0+1}^{s-1} \frac{\Gamma(s-k+\Lambda(s))}{\Gamma(s-k+1)} = \frac{1}{\kappa(s)}\Big\{\left(\frac{\Gamma(s-s_0+\Lambda(s))}{\Gamma(s-s_0)} - \frac{\Gamma(s-s_0+\Lambda(s)-1)}{\Gamma(s-s_0-1)}\right)$$
$$+\left(\frac{\Gamma(s-s_0+\Lambda(s)-1)}{\Gamma(s-s_0-1)} - \frac{\Gamma(s-s_0+\Lambda(s)-2)}{\Gamma(s-s_0-2)}\right)$$
$$+\left(\frac{\Gamma(s-s_0+\Lambda(s)-2)}{\Gamma(s-s_0-2)} - \frac{\Gamma(s-s_0+\Lambda(s)-3)}{\Gamma(s-s_0-3)}\right)$$
$$...$$
$$+\left(\frac{\Gamma(\Lambda(s)+2)}{\Gamma(2)} - \frac{\Gamma(\Lambda(s)+1)}{\Gamma(1)}\right)\Big\},$$

Therefore, we obtain

$$\begin{aligned}\sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}} &= \frac{1}{\Lambda(s)}\left(\frac{\Gamma(s-s_0+\Lambda(s))}{\Gamma(s-s_0)} - \Gamma(\Lambda(s)+1)\right) + \Gamma(\Lambda(s)),\\ &= \frac{1}{\Lambda(s)}\frac{\Gamma(s-s_0+\Lambda(s))}{\Gamma(s-s_0)},\\ &= \frac{(s-s_0)^{\Lambda(s)}}{\Lambda(s)}.\end{aligned}$$

Which concludes the proof. □

3. Existence and Uniqueness

The following variable-order fractional discrete-time neural network is proposed

$${}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}\xi(s) = -D\xi(s) + Bg(s,\xi(s)) + I, \tag{8}$$

were ${}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}$ is the ABC discrete nabla difference operator of order $\Lambda(s)$, $0 < \Lambda(s) < 1$, $\xi(s) = (\xi_1(s), \xi_2(s), ..., \xi_n(s))^T \in \mathbb{R}^n$ is the state vector, $D = diag(d_1, d_2, ..., d_n) \in \mathbb{R}^{n*n}$ is the self-feedback connection weight with $d_i > 0$, $B = (b_{ij})_{n*n} \in \mathbb{R}^{n*n}$ is the connection weight matrix, $g(s,\xi(s)) = (g_1(s,\xi(s)), g_2(s,\xi(s)), ..., g_n(s,\xi(s)))^T \in C(\mathbb{N}_{s_0+1}, \mathbb{R}^n)$ is the acctivation function, $I = (I_1, ..., I_n)^T$ the vector of external inputs.

Assumption 1 (A_1)**.** *For all $s \in \mathbb{N}_{s_0+1}$, $g_i(s,u)$ is a lipschitz continuous function with respect to u, i.e.,*

$$\exists l_i \in \mathbb{R}_+^* : \ |g_i(s,u) - g_i(s,v)| \le l_i|u-v|; \ \forall u, v \in \mathbb{R}, \tag{9}$$

where $l = max_{i=1,...,n}\{l_i\}$.

Assumption 2 (A_2)**.** *For all $t \in \mathbb{N}_{s_0+1}^T = \{s_0+1, s_0+2, ..., T\}$, there is $Q \in \mathbb{R}_+^*$ and $Q < 1$ such that*

$$Q = \frac{\Gamma(\gamma)(\gamma_1 + l\gamma_2)}{\Gamma(\gamma)(1-\beta)+\gamma}\left\{(1-\gamma) + \frac{(T-s_0)^{\overline{\beta}}}{\Gamma(\beta)}\right\}, \tag{10}$$

where

$$\gamma_1 = \|D\|_\infty, \ \ \gamma_2 = \|B\|_\infty, \ \ \gamma \le \Lambda(s) \le \beta.$$

Theorem 1. *If (A_1) and (A_2) hold; then, the uniqueness of the solution of (8) is provided.*

Proof. According to the fractional discrete variable-order calculus properties, a solution of (8) is equal to:

$$\xi(s) = \xi_0 + \frac{1-\Lambda(s)}{B(\Lambda(s))}[-D\xi(s) + Bg(t,\xi(s)) + I] + \frac{\Lambda(s)}{B(\Lambda(s))}\, {}_{s_0}\nabla_s^{-\Lambda(s)}[-D\xi(s) + Bg(s,\xi(s)) + I]. \tag{11}$$

With the use of the following norm

$$\|\xi\| = \sup_{s\in\mathbb{N}_{s_0+1}^T} \|\xi(s)\| \text{ and } \|\xi(s)\| = \max_{\{i=1,\ldots,n\}} |\xi_i(s)|,$$

Problem (11) can be converted into a fixed problem. Consider the following mapping:

$$\Phi\xi(s) = \xi_0 + \frac{1-\Lambda(s)}{B(\Lambda(s))}[-D\xi(s) + Bg(t,\xi(s)) + I] \tag{12}$$

$$+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[-D\xi(k) + Bg(k,\xi(k)) + I], \tag{13}$$

where $\Phi\xi = (\Phi_1\xi_1, \Phi_2\xi_2, \ldots, \Phi_n\xi_n)$ and $\Phi_i\xi_i$ is described by:

$$\Phi_i\xi_i(s) = \xi_{i0} + \frac{1-\Lambda(s)}{B(\Lambda(s))}[-d_i\xi_i(s) + \sum_{j=1}^{n} b_{ij}g_j(s,\xi_j(s)) + I_i]$$
$$+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[-d_i\xi_i(k) + \sum_{j=1}^{n} b_{ij}g_j(k,\xi_j(k)) + I_i].$$

For any two distinct functions $\xi, \mu \in \mathbb{R}^n$ we have

$$\begin{aligned}
|\Phi_i\xi_i(s) - \Phi_i\mu_i(s)| = |&\frac{1-\Lambda(s)}{B(\Lambda(s))}[-d_i(\xi_i(s) - \mu_i(s)) + \sum_{j=1}^{n} b_{ij}(g_j(s,\xi_j(s)) - g_j(s,\mu_j(s)))] \\
&+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[-d_i(\xi_i(k) - \mu_i(k)) \\
&+ \sum_{j=1}^{n} b_{ij}(g_j(k,\xi_j(k)) - g_j(k,\mu_j(k)))]|, \\
\le& \frac{1-\Lambda(s)}{B(\Lambda(s))}[d_i|\xi_i(s) - \mu_i(s)| + \sum_{j=1}^{n} |b_{ij}|l_j|\xi_j(s)) - \mu_j(s)|] \\
&+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[d_i|\xi_i(k) - \mu_i(k))| + \sum_{j=1}^{n} |b_{ij}|l_j|\xi_j(k)) - \mu_j(k)|], \\
\le& \frac{1-\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}[d_i|\xi_i(s) - \mu_i(s)| + \sum_{j=1}^{n} |b_{ij}|l_j|\xi_j(s)) - \mu_j(s)|] \\
&+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[d_i|\xi_i(k) - \mu_i(k))| + \sum_{j=1}^{n} |b_{ij}|l_j v\xi_j(k)) - \mu_j(k)|],
\end{aligned}$$

which leads us to

$$\begin{aligned}
\max_{\{i=1,\ldots,n\}} |\Phi_i\xi_i(s) - \Phi_i\mu_i(s)| \le& \frac{1-\Lambda(s)}{B(\Lambda(s))}[\gamma_1 \max_{\{i=1,\ldots,n\}} |\xi_i(s) - \mu_i(s)| \\
&+ \gamma_2 l \max_{\{i=1,\ldots,n\}} |\xi_i(s)) - \mu_i(s)|] \\
&+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))} \sum_{k=s_0+1}^{s} (s-\rho(k))^{\overline{\Lambda(s)-1}}[\gamma_1 \max_{\{i=1,\ldots,n\}} |\xi_i(k) - \mu_i(k))| \\
&+ \gamma_2 l \max_{\{i=1,\ldots,n\}} |\xi_i(k)) - \mu_i(k)|].
\end{aligned}$$

Using Lemma 1, we conclude that

$$\begin{aligned}
\|\Phi\xi - \Phi\mu\| &= \sup_{s\in\mathbb{N}_{s_0+1}^T}\left\{\max_{\{i=1,\dots,n\}}|\Phi_i\xi_i(s) - \Phi_i\mu_i(s)|\right\},\\
&\leq \left\{\frac{(1-\gamma)\Gamma(\gamma)}{\Gamma(\gamma)(1-\beta)+\gamma} + \frac{\beta\Gamma(\gamma)}{\Gamma(\beta)(\Gamma(\gamma+1)(1-\beta)+\gamma)}\sup_{s\in\mathbb{N}_{s_0+1}^T}\sum_{k=s_0+1}^{s}(s-\rho(k))^{\overline{\Lambda(s)-1}}\right\}\\
&\quad [\gamma_1 + l\gamma_2]\|\xi-\mu\|,\\
&\leq \{(1-\gamma) + \frac{1}{\Gamma(\beta)}\sup_{s\in\mathbb{N}_{s_0+1}^T}(s-s_0)^{\overline{\Lambda(s)}}\frac{\Gamma(\gamma)[\gamma_1+l\gamma_2]}{\Gamma(\gamma)(1-\beta)+\gamma}\|\xi-\mu\|,\\
&\leq \left\{(1-\gamma) + \frac{(T-s_0)^{\overline{\beta}}}{\Gamma(\beta)}\right\}\frac{\Gamma(\gamma)[\gamma_1+l\gamma_2]}{\Gamma(\gamma)(1-\beta)+\gamma}\|\xi-\mu\| = Q\|\xi-\mu\|.
\end{aligned}$$

According to (A_2), we know that $Q < 1$, as a result, the mapping Φ is a contraction on $C(\mathbb{N}_{s_0+1}^T, \mathbb{R}^n)$. This means that problem (12) has a unique fixed point according to the Banach fixed point theorem, implying the uniqueness of the solution of (8), which completes our demonstration. □

4. Stability Analysis

Definition 4 ([38])**.** *The initial time of the discrete variable-order neural networks system (8) with nabla discrete Mittag–Leffler kernals is set to s_0. (8) is called uniformly stable if for any $\epsilon > 0$, there are two constants δ_ϵ and T, $0 < \delta_\epsilon < \epsilon$, $T > 0$, so that for $s \in \mathbb{N}_{s_0+1}^T = \{s_0+1, s_0+2, \dots, T\}$, and for any two solutions $\xi(s, s_0, \phi)$ and $\mu(s, s_0, \psi)$ with the initial conditions $\xi_0 = \phi$ and $\mu_0 = \psi$ such that $\|\phi - \psi\| < \delta_\epsilon$ implies $\|\xi - \mu\| < \epsilon$.*

Theorem 2. *Suppose that (A_1) and (A_2) are valid, if*

$$\frac{1}{1-Q} < \frac{\epsilon}{\delta}, \tag{14}$$

then, (8) is uniformly stable.

Proof. Let there be two different solutions $\xi(s), \mu(s) \in \mathbb{R}^n$ of system (8) with distinct initial conditions as ξ_0 and μ_0

Where $\phi = \xi_0$, $\Phi = \mu_0$ which lead us to

$${}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}(\xi(s) - \mu(s)) = \phi - \psi - D(\xi(s) - \mu(s)) + B(g(s, \xi(s)) - g(s, \mu(s))), \tag{15}$$

which is equivalent to

$$\begin{aligned}
\xi(s) - \mu(s) = \phi - \psi &+ \frac{1-\Lambda(s)}{B(\Lambda(s))}[-D(\xi(s)-\mu(s)) + B(g(s,\xi(s) - g(s,\mu(s)))]\\
&+ \frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s}(-\rho(k)^{\overline{\Lambda(s)-1}})[-D(\xi(k)-\mu(k))\\
&+ B(g(k,\xi(k) - g(k,\mu(k)))],
\end{aligned}$$

Then, using (A_1)m we obtain the following

$$
\begin{aligned}
|\xi_i(s)-\mu_i(s)| = {} & |\phi_i-\psi_i+\frac{1-\Lambda(s)}{B(\Lambda(s))}[-d_i(\xi_i(s)-\mu_i(s))+\sum_{j=1}^{n} b_{ij}(g_j(s,\xi_j(s)-g_j(s,\mu_j(s)))] \\
& +\frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s}(s-\rho(k)^{\overline{\Lambda(s)-1}})[-d_i(\xi_i(k)-\mu_i(k)) \\
& +\sum_{j=1}^{n} b_{ij}(g_j(k,\xi_j(k)-g_j(k,\mu_j(k)))]|, \\
& \leq |\phi_i-\psi_i|+\frac{1-\Lambda(s)}{B(\Lambda(s))}|-d_i(\xi_i(s)-\mu_i(s))+\sum_{j=1}^{n} b_{ij}(g_j(s,\xi_j(s)-g_j(s,\mu_j(s)))| \\
& +\frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s} st-\rho(k)^{\overline{\Lambda(s)-1}})|-d_i(\xi_i(k)-\mu_i(k)) \\
& +\sum_{j=1}^{n} b_{ij}(g_j(k,\xi_j(k)-g_j(k,\mu_j(k)))]|, \\
& \leq |\phi_i-\psi_i|+\frac{1-\Lambda(s)}{B(\Lambda(s))}[d_i|\xi_i(s)-\mu_i(s)|+\sum_{j=1}^{n} l_j|b_{ij}||\xi_i(s)-\mu_i(s)|] \\
& +\frac{\Lambda(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s}(s-\rho(k)^{\overline{\Lambda(s)-1}})[d_i|(\xi_i(k)-\mu_i(k)| \\
& +\sum_{j=1}^{n} l_j|b_{ij}||\xi_i(k)-\mu_i(k)|].
\end{aligned}
$$

Therefore, using (A_2) we have

$$
\begin{aligned}
\|\xi-\mu\| \leq {} & \|\phi-\psi\|+\sup_{s\in\mathbb{N}_{s_0+1}^{T}}\left\{\frac{1-\Lambda(s)}{B(\Lambda(s))}\right\}[\gamma_1+l\gamma_2]\|\xi-\mu\| \\
& +\sup_{s\in\mathbb{N}_{s_0+1}^{T}}\left\{\frac{\kappa(s)}{B(\Lambda(s))\Gamma(\Lambda(s))}\sum_{k=s_0+1}^{s}(s-\rho(k)^{\overline{\Lambda(s)-1}})\right\}[\gamma_1+l\gamma_2]\|\xi-\mu\|, \\
& \leq \|\phi-\psi\|+\{\frac{(1-\gamma)\Gamma(\gamma)}{\Gamma(\gamma)(1-\beta+\gamma}[\gamma_1+l\gamma_2] \\
& +\frac{\Gamma(\gamma)}{\Gamma(\beta)(\Gamma(\gamma+1)(1-\beta)+\gamma)}[\gamma_1+l\gamma_2]\sup_{s\in\mathbb{N}_{s_0+1}^{T}}(s-s_0)^{\Lambda(s)}\|\xi-\mu\|, \\
& = \|\phi-\psi\|+\frac{\Gamma(\gamma)(\gamma_1+l\gamma_2)}{\Gamma(\gamma)(1-\beta)+\gamma}\left\{(1-\gamma)+\frac{(T-s_0)^{\overline{\beta}}}{\Gamma(\beta)}\right\}\|\xi-\mu\| \\
& < \frac{1}{1-\frac{\Gamma(\gamma)(\gamma_1+l\gamma_2)}{\Gamma(\gamma)(1-\beta)+\gamma}\left\{(1-\gamma)+\frac{(T-s_0)^{\overline{\beta}}}{\Gamma(\beta)}\right\}}\|\phi-\psi\|,
\end{aligned}
$$

we conclude that

$$
\|\xi-\mu\| \leq \frac{1}{1-Q}\|\phi-\psi\|.
$$

We reach the conclusion that for any $\epsilon > 0$, there exists $\delta_\epsilon = (1-Q)\epsilon$ where if $\|\phi-\psi\| < \delta$ then $\|\xi-\mu\| < \epsilon$ and according to definition 4, (8) is uniformly stable, the proof is achieved. □

5. Numerical Simulations

Example 1. *Let be the two dimensional neural networks of discrete fractional variable-order*

$$
\begin{cases}
{}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}\xi_1(s) = -d_1\xi_1(s) + b_{11}\sin(\xi_1(s)) + b_{12}\sin(\xi_2(s)) + I_1, \\
{}^{ABC}_{t_0}\nabla_s^{\Lambda(s)}\xi_2(s) = -d_2\xi_2(s) + b_{21}\sin(\xi_1(s)) + b_{22}\sin(\xi_2(s)) + I_2.
\end{cases} \tag{16}
$$

with the following parameters $b_{11} = 0.2$, $b_{12} = -0.3$, $b_{21} = 0.4$, $b_{22} = -0.1$, $d_1 = 0.2$, $d_2 = 0.2$, $I_1 = 0.1, I_2 = 0.1$, $\Lambda(s) = \dfrac{|\ln(\frac{3}{s+5})|}{3(s+1)}$, $s \in [0,50]$ *and the initial condition* $\xi_1(0) = -10, \xi_2(0) = -7$.

The numerical solution of (16) is given by

$$
\begin{cases}
\xi_1(i) = \xi_1(0) + \dfrac{1-\Lambda(i)}{B(\Lambda(i))}[-d_1\xi_1(i) + b_{11}\sin(\xi_1(i)) + b_{12}\sin(\xi_2(i)) + I_1] \\
\qquad + \dfrac{\Lambda(i)}{B(\Lambda(i)))}\sum_{k=1}^{i}\dfrac{\Gamma(i-k+\Lambda(i))}{\Gamma(i-k+1)}(-d_1\xi_1(k) + b_{11}\sin(\xi_1(k)) + b_{12}\sin(\xi_2(k)) + I_1), \\
\xi_2(i) = \xi_2(0) + \dfrac{1-\Lambda(i)}{B(\Lambda(i))}[-d_2\xi_2(i) + b_{21}\sin(\xi_1(i)) + b_{22}\sin(\xi_2(i)) + I_2] \\
\qquad + \dfrac{\Lambda(i)}{B(\Lambda(i)))}\sum_{k=1}^{i}\dfrac{\Gamma(i-k+\Lambda(i))}{\Gamma(i-k+1)}(-d_2\xi_2(k) + b_{21}\sin(\xi_1(k)) + b_{22}\sin(\xi_2(k)) + I_2), \\
B(\Lambda(i)) = 1 - \Lambda(i) + \dfrac{\Lambda(i)}{\Gamma(\Lambda(i))}, \qquad i \geq 1.
\end{cases}
$$

We can check that the parameters satisfy assumptions (A_1)–(A_2) *and the condition in Theorems 1 and 2. The behavior of the solutions* $\xi_1(t)$ *and* $\xi_2(t)$ *is illustrated in Figure 1, where we can notes that each tends to zero for* $s \to +\infty$ *and the solution is uniformly stable.*

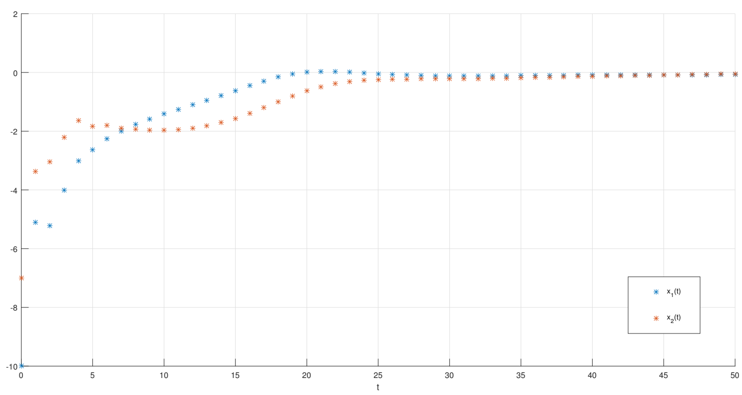

Figure 1. Numerical solution of neural networks (16).

Example 2. *Let us consider the following discrete fractional variable-order neural neutwork:*

$$
\begin{cases}
{}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}\xi_1(s) = -d_1\xi_1(s) + b_{11}\tanh(\xi_1(s)) + b_{12}\tanh(\xi_2(s)) + b_{13}\tanh(\xi_3(s)) + I_1, \\
{}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}\xi_2(s) = -d_2\xi_2(s) + b_{21}\tanh(\xi_1(s)) + b_{22}\tanh(\xi_2(s)) + b_{23}\tanh(\xi_3(s)) + I_2, \\
{}^{ABC}_{s_0}\nabla_s^{\Lambda(s)}\xi_3(s) = -d_3\xi_3(s) + b_{31}\tanh(\xi_1(s)) + b_{32}\tanh(\xi_2(s)) + b_{33}\tanh(\xi_3(s)) + I_3,
\end{cases} \tag{17}
$$

where

$$
D = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.4 & -0.1 & -0.2 \\ 0.1 & -0.4 & 0.1 \\ 0.4 & 0.1 & 0.2 \end{bmatrix}, \quad I = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Since assumptions (A_1) *and* (A_2) *are clearly satisfied, we can use formulas (18) to have the numerical solution shown in Figure 2 along with the initial condition* $\xi(0) = (0.1, 0.1, 0.1)^T$, *which shows the uniform stability of the considered neural network.*

$$\begin{cases} \xi(i) = \xi(0) + \dfrac{1-\Lambda(i)}{B(\Lambda(i))}[-D\xi(i) + B\tanh(\xi(i))] \\ \qquad + \dfrac{\Lambda(i)}{B(\Lambda(i)))}\sum_{k=1}^{i} \dfrac{\Gamma(i-k+\Lambda(i))}{\Gamma(i-k+1)}(-D\xi(k) + B\tanh(\xi(k))), \\ B(\Lambda(i)) = 1 - \Lambda(i) + \dfrac{\Lambda(i)}{\Gamma(\Lambda(i))}, \qquad i \geq 1. \end{cases} \tag{18}$$

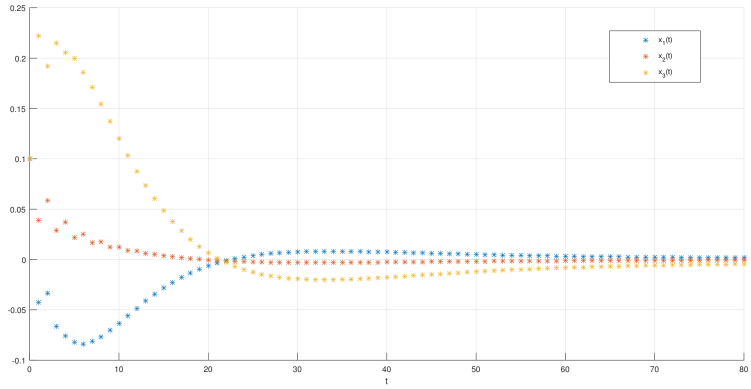

Figure 2. Numerical solution of discrete time variable-order neural networks (17).

6. Conclusions

We address, in this work, the problem of uniform stability for a class of discrete fractional variable-order neural-networks with nabla discrete Mittag–Leffler kernels. In addition, for this type of neural network, we founded essential criterion for the existence and uniqueness of the solution. Moreover, two and three dimensional examples including numerical solutions and simulations are available to highlight the efficacy of our findings. In addition, the system treated in this study is significantly more complex than the ones investigated in previous works, and this form of discrete neural network under the variable-order fractional nabla operator with nonsingular and nonlocal kernel has never been examined before. As a result, when compared to previous results, the discussions in this study are original and improved. Furthermore, this research may result in the development of innovative discrete-time systems with variable order. Furthermore, we will consider additional applications in this research area, such as modeling and analysis in various fields and special features such as systems chaos, stabilization, and systems synchronization in fractional variable-order discrete-time systems, particularly discrete-time neural networks which we will concider in our future works.

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