fractal and fractional

Article

Solitary Waves Propagation Analysis in Nonlinear Dynamical System of Fractional Coupled Boussinesq-Whitham-Broer-Kaup Equation

M. Mossa Al-Sawalha [1], Safyan Mukhtar [2,*], Rasool Shah [3,*], Abdul Hamid Ganie [4] and Khaled Moaddy [5]

1 Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; m.alswalha@uoh.edu.sa
2 Department of Basic Sciences, General Administration of the Preparatory Year Deanship, King Faisal University, Al Ahsa 31982, Saudi Arabia
3 Department of Computer Science and Mathematics, Lebanese American University, Beirut P.O. Box 13-5053, Lebanon
4 Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; a.ganie@seu.edu.sa
5 Department of Mathematics, College of Science and Humanities, Shaqra University, Shaqra 11691, Saudi Arabia; moaddy@yahoo.com
* Correspondence: smahmad@kfu.edu.sa (S.M.); rasool.shah@lau.edu.lb (R.S.)

Abstract: The primary goal of this study is to create and characterise solitary wave solutions for the conformable Fractional Coupled Boussinesq-Whitham-Broer-Kaup Equations (FCBWBKEs), a model that governs shallow water waves. Through wave transformations and the chain rule, the authors used the modified Extended Direct Algebraic Method (mEDAM) for transforming FCBWBKEs into a more manageable Nonlinear Ordinary Differential Equation (NODE). This accomplishment is particularly noteworthy because it surpasses the drawbacks linked to both the Caputo and Riemann–Liouville definitions in complying to the chain rule. The study uses visual representations such as 3D, 2D, and contour graphs to demonstrate the dynamic nature of solitary wave solutions. Furthermore, the investigation of diverse wave phenomena such as kinks, shock waves, periodic waves, and bell-shaped kink waves highlights the range of knowledge obtained in the study of shallow water wave behavior. Overall, this study introduces novel methodologies that produce valuable and consistent results for the problem under consideration.

Keywords: nonlinear fractional partial differential equations; fractional Boussinesq-Whitham-Broer-Kaup equation; mEDAM; wave transformation; conformable fractional derivatives; solitary waves; shallow water wave

Citation: Al-Sawalha, M.M.; Mukhtar, S.; Shah, R.; Ganie, A.H.; Moaddy, K. Solitary Waves Propagation Analysis in Nonlinear Dynamical System of Fractional Coupled Boussinesq-Whitham-Broer-Kaup Equation. *Fractal Fract.* **2023**, *7*, 889. https://doi.org/10.3390/fractalfract7120889

Academic Editors: John R. Graef and Bipan Hazarika

Received: 28 September 2023
Revised: 3 December 2023
Accepted: 8 December 2023
Published: 18 December 2023

1. Introduction

Nonlinear Fractional Partial Differential Equations (NFPDEs) are an assortment of mathematical models that are significant in many disciplines of science and engineering because of their capacity to represent complicated events using fractional derivatives and nonlinear components [1–3]. These equations are significant because they represent complicated behaviors that are not captured by standard integer-order Nonlinear Partial Differential Equation (NPDEs). For example, in fluid dynamics, the Navier–Stokes equations expanded to include fractional derivatives that can more properly represent non-Newtonian fluid flow [4]. In biology, fractional FPDEs have been used to simulate the spread of illnesses with abnormal diffusion patterns [5]. Furthermore, NFPDEs have applications in finance for modeling price variations as well as image processing for edge identification and picture denoising [6,7]. The importance of nonlinear FPDEs stems from their ability to provide a more complete framework for modeling real-world processes, with more accuracy and predictive power than integer-order equivalents. These equations require researchers to devise novel numerical methods and analytical approaches in order to solve them, making them an important topic of study in modern mathematics and science.

Due to the fact that numerical methods such as the Finite Element Method (FEM) [8], Numerical Laplace inversion technique [9], Finite Difference Method (FDM) [10], and many more [11–13] have inherent errors, complicated computational prerequisites, and a heavy reliance on computer resources, researchers frequently opt for analytical solutions when handling NFPDEs. Analytical solutions give closed-form formulations that provide a thorough insight into the system's underlying dynamics without the need for considerable computing labour. Analytical approaches such as the fractional transform method [14], homotopy perturbation method [15], and natural transform method [16] have emerged as useful tools in this setting. These approaches allow researchers to obtain insights into the complicated behavior of NFPDEs, allowing for a more detailed investigation and understanding of the mathematical models they represent. Researchers may investigate NFPDEs with accuracy by using analytical tools, giving insight into the basic principles driving complicated processes while avoiding the restrictions and constraints associated with numerical approaches.

The investigation of solitary wave solutions for NFPDEs has long piqued the curiosity of physicists and applied mathematicians. To construct solitary wave solutions for NFPDEs, a variety of analytical methodologies, including the tan-function method [17], the Sardar sub-equation method [18], the sub-equation method [19], the Kudryashov method [20], the Khater method [21], the exp-function method [22], and the mEDAM method [23], among others, have been proposed.

Among these analytical strategies for obtaining solitary wave solutions, mEDAM [24,25] is a cutting-edge method that may be applied to both NPDEs and NFPDEs. This technique employs a transformational procedure to turn NFPDEs or NPDEs into NODEs, which are then addressed using series-based solutions. The resultant NODE is then utilized to create a set of algebraic equations that, when solved, yield solitary wave solutions for the FPDE. mEDAM stands out for its amazing effectiveness in creating a larger range of solitary wave solutions' families. For example, Sayed et al. successfully used mEDAM to build multiple solitary wave solutions' families for three different types of Tzitzeica-type PDEs found in nonlinear optics [26]. Similarly, Yasmin et al. investigated 32 distinct families of symmetric solitary wave solutions for the fractional coupled Konno–Onno system [25], as well as 33 distinct families containing 131 optical solitary wave solutions for the fractionally perturbed Radhakrishnan-Kundu-Lakshmanan model [24], all accomplished through the expert use of mEDAM.

We employ mEDAM in conjunction with conformable fractional derivatives to discover solitary wave solutions for FCBWBKEs in this study. In 1872, Boussinesq laid out a set of equations for evaluating the propagation of small and large waves in water [27]. In 1965, Whitham used a Lagrangian approach for detecting both linear and nonlinear dispersive waves, and with Lighthill in 1967, he developed a theory applicable to slowly evolving wave trains [28,29]. CBWBKEs are generalized into NFPDEs using conformable fractional derivatives as follows [30]:

$$\begin{aligned} &D_t^{\alpha} w + D_x^{\beta} z + w D_x^{\beta} w = 0, \\ &D_t^{\alpha} z + D_x^{\beta}(wz) + D_x^{\beta}(D_x^{\beta}(D_x^{\beta} w)) = 0, \end{aligned} \tag{1}$$

where $w \equiv w(x,t)$ is the velocity, $z \equiv z(x,t)$ is the free wave surface height for fluid in the trough, and the operators $D_t^{\alpha}(\cdot)$ and $D_x^{\beta}(\cdot)$ denote the conformable fractional partial derivatives. Prior to this research work, many researchers have addressed this model with different strategies. For instance, the auxiliary equation method was employed by Atilgan et al. to construct travelling wave solutions for FCBWBKEs with conformable derivatives [30]. Similarly, Jin and Kim have addressed FCBWBKEs with variable coefficients for constructing travelling wave solutions [31]. Finally, Yu's research employs the Lie symmetry analysis technique for transforming the FCBWBKEs into a more manageable system of fractional ODEs [32]. The study includes novel contributions such as a conservation theorem and the generalization of Noether operators, both of which are required for the construction of conservation laws. These analytical tools provide a comprehensive approach to solving equations, significantly contributing to physics' broader knowledge. Influenced by the existing literature, this study aims to establish the efficacy of mEDAM for producing solitary wave solutions for FCBWBKEs with conformable derivatives. The method involves transforming FCBWBKEs into a NODE, which afterwards transforms into a set of nonlinear algebraic equations. Solving these equations yields various families of solitary wave solutions, including kinks, shock waves, periodic waves, and bell-shaped kink solitary wave solutions.

The remainder of the paper is organized as follows: Section 2 describes the mEDAM materials and approach. Section 3 gives FCBWBKEs's solitary wave solutions. Section 4 includes a discussion and several graphics, and Section 5 ends our research.

2. Methodology and Materials

2.1. Conformable Fractional Derivative

We may derive solitary wave solutions for NFPDEs using the superiority of conformable fractional derivative over alternative fractional derivative operators. The solitary wave solutions of FCBWBKEs presented in (1), for example, cannot be achieved using alternative fractional derivative formulations due to the fact that they do not obey the chain rule [33,34]. As a result, we have defined the fractional derivatives used in (1) in the sense of conformable fractional derivatives. This derivative operator of order α is described as stated in [35]:

$$D_\chi^\alpha w(\chi) = \lim_{\gamma \to 0} \frac{w(\gamma \chi^{1-\alpha} + \chi) - w(\chi)}{\gamma}, \quad \alpha \in (0,1]. \tag{2}$$

In this investigation, the following properties of this derivative are utilized:

$$D_\chi^\alpha \chi^r = r\chi^{r-\alpha}, \tag{3}$$

$$D_\chi^\alpha (r_1 \rho(\chi) \pm r_2 \eta(\chi)) = r_1 D_\chi^\alpha (\rho(\chi)) \pm r_2 D_\chi^\alpha (\eta(\chi)), \tag{4}$$

$$D_\chi^\alpha \varphi[\zeta(\chi)] = \varphi'_\zeta(\zeta(\chi)) D_\chi^\alpha \zeta(\chi), \tag{5}$$

where $\rho(\chi)$, $\eta(\chi)$, $\varphi(\chi)$, and $\zeta(\chi)$ are arbitrary differentiable functions, whereas r, r_1 and r_2 signify constants.

2.2. The Working Procedure of mEDAM

This section provides an overview of the mEDAM method. Consider the FPDE in the following form:

$$E(v, \partial_t^\alpha v, \partial_{\sigma_1}^\beta v, \partial_{\sigma_2}^\gamma v, v\partial_{\sigma_1}^\beta v, \dots) = 0, \ \ 0 < \alpha, \beta, \gamma \le 1, \tag{6}$$

where $v = v(t, \sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$.

The procedures below are employed to address Equation (6):

1. Equation (6) is initially subjected to a variable transformation of the type $v(t, \sigma_1, \sigma_2, \sigma_3, ; \sigma_n) = V(\chi)$, where χ signifies a function of $t, \sigma_1, \sigma_2, \sigma_3, ; \sigma_n$ and can be written in several ways. As a result of this transformation, Equation (6) is transformed into a NODE having the following structure:

$$F(V, V', VV', \dots) = 0, \tag{7}$$

In Equation (7), the variable V has derivatives with respect to χ. Equation (7) may be integrated one or more times to determine the constant(s) of integration.

2. Following that, we assume the following closed form series solution to Equation (7):

$$V(\chi) = \sum_{s=-j}^{j} F_s(U(\chi))^s, \tag{8}$$

In this case, F_s (with $s = -j, \ldots, 0, 1, 2, \ldots, j$) represents the parameters that must be driven. In addition, $U(\chi)$ fulfils another NODE of the form:

$$U'(\chi) = \ln(\mu)(d + eU(\chi) + f(U(\chi))^2), \tag{9}$$

where $\mu \neq 0, 1$ and d, e, f are constants.

3. We acquire a positive integer j (shown in Equation (8)) by looking for the homogeneous balance between the predominant nonlinear component and the largest order derivative in Equation (7).

4. We next substitute (8) into (7) or the equation obtained by integrating (7), and lastly we arrange all the terms of $U(\chi)$ in the same order, yielding a polynomial in $U(\chi)$. When the coefficients of the resulting polynomial are all set to zero, a system of algebraic equations is created for $F_s(s = -j, \ldots, 0, 1, 2, \ldots, j)$ and additional parameters.

5. To solve this set of nonlinear algebraic equations, we use the Maple-13 tool.

6. The solitary wave solutions to Equation (6) are determined by determining the unknown parameters and plugging them into Equation (8) together with the corresponding solution $U(\chi)$ from Equation (9). Using the generic solution of Equation (9), we may obtain the families of solitary wave solutions given below.

Family. 1: For $\psi < 0$ and $f \neq 0$,

$$U_1(\chi) = -\frac{e}{2f} + \frac{\sqrt{-\psi}\tan_\mu(1/2\sqrt{-\psi}\chi)}{2f}$$

$$U_2(\chi) = -\frac{e}{2f} - \frac{\sqrt{-\psi}\cot_\mu(1/2\sqrt{-\psi}\chi)}{2f}$$

$$U_3(\chi) = -\frac{e}{2f} + \frac{\sqrt{-\psi}\big(\tan_\mu(\sqrt{-\psi}\chi) \pm (\sec_\mu(\sqrt{-\psi}\chi))\big)}{2f}$$

$$U_4(\chi) = -\frac{e}{2f} - \frac{\sqrt{-\psi}\big(\cot_\mu(\sqrt{-\psi}\chi) \pm (\csc_\mu(\sqrt{-\psi}\chi))\big)}{2f}$$

and

$$U_5(\chi) = -\frac{e}{2f} + \frac{\sqrt{-\psi}\big(\tan_\mu(1/4\sqrt{-\psi}\chi) - \cot_\mu(1/4\sqrt{-\psi}\chi)\big)}{4f}$$

Family. 2: For $\psi > 0$ and $f \neq 0$,

$$U_6(\chi) = -\frac{e}{2f} - \frac{\sqrt{\psi}\tanh_\mu(1/2\sqrt{\psi}\chi)}{2f}$$

$$U_7(\chi) = -\frac{e}{2f} - \frac{\sqrt{\psi}\coth_\mu(1/2\sqrt{\psi}\chi)}{2f}$$

$$U_8(\chi) = -\frac{e}{2f} - \frac{\sqrt{\psi}\big(\tanh_\mu(\sqrt{\psi}\chi) \pm i(sech_\mu(\sqrt{\psi}\chi))\big)}{2f}$$

$$U_9(\chi) = -\frac{e}{2f} - \frac{\sqrt{\psi}\big(\coth_\mu(\sqrt{\psi}\chi) \pm (csch_\mu(\sqrt{\psi}\chi))\big)}{2f}$$

and

$$U_{10}(\chi) = -\frac{e}{2f} - \frac{\sqrt{\psi}\big(\tanh_\mu(1/4\sqrt{\psi}\chi) - \coth_\mu(1/4\sqrt{\psi}\chi)\big)}{4f}$$

Family. 3: For $df > 0$ and $e = 0$,

$$U_{11}(\chi) = \sqrt{\frac{d}{f}} \tan_{\mu}\left(\sqrt{df}\chi\right)$$

$$U_{12}(\chi) = -\sqrt{\frac{d}{f}} \cot_{\mu}\left(\sqrt{df}\chi\right)$$

$$U_{13}(\chi) = \sqrt{\frac{d}{f}}\left(\tan_{\mu}\left(2\sqrt{df}\chi\right) \pm \left(\sec_{\mu}\left(2\sqrt{df}\chi\right)\right)\right)$$

$$U_{14}(\chi) = -\sqrt{\frac{d}{f}}\left(\cot_{\mu}\left(2\sqrt{df}\chi\right) \pm \left(\csc_{\mu}\left(2\sqrt{df}\chi\right)\right)\right)$$

and

$$U_{15}(\chi) = \frac{1}{2}\sqrt{\frac{d}{f}}\left(\tan_{\mu}\left(1/2\sqrt{df}\chi\right) - \cot_{\mu}\left(1/2\sqrt{df}\chi\right)\right)$$

Family. 4: For $df < 0$ and $e = 0$,

$$U_{16}(\chi) = -\sqrt{-\frac{d}{f}} \tanh_{\mu}\left(\sqrt{-df}\chi\right)$$

$$U_{17}(\chi) = -\sqrt{-\frac{d}{f}} \coth_{\mu}\left(\sqrt{-df}\chi\right)$$

$$U_{18}(\chi) = -\sqrt{-\frac{d}{f}}\left(\tanh_{\mu}\left(2\sqrt{-df}\chi\right) \pm \left(isech_{\mu}\left(2\sqrt{-df}\chi\right)\right)\right)$$

$$U_{19}(\chi) = -\sqrt{-\frac{a}{c}}\left(\coth_{\mu}\left(2\sqrt{-df}\chi\right) \pm \left(csch_{\mu}\left(2\sqrt{-df}\chi\right)\right)\right)$$

and

$$U_{20}(\chi) = -\frac{1}{2}\sqrt{-\frac{d}{f}}\left(\tanh_{\mu}\left(1/2\sqrt{-df}\chi\right) + \coth_{\mu}\left(1/2\sqrt{-df}\chi\right)\right)$$

Family. 5: For $f = d$ and $e = 0$,

$$U_{21}(\chi) = \tan_{\mu}(d\chi)$$

$$U_{22}(\chi) = -\cot_{\mu}(d\chi)$$

$$U_{23}(\chi) = \tan_{\mu}(2\,d\chi) \pm \left(\sec_{\mu}(2\,d\chi)\right)$$

$$U_{24}(\chi) = -\cot_{\mu}(2\,d\chi) \pm \left(\csc_{\mu}(2\,d\chi)\right)$$

and

$$U_{25}(\chi) = \frac{1}{2}\tan_{\mu}(1/2\,d\chi) - 1/2\,\cot_{\mu}(1/2\,a\chi)$$

Family. 6: For $f = -d$ and $e = 0$,

$$U_{26}(\chi) = -\tanh_{\mu}(d\chi)$$

$$U_{27}(\chi) = -\coth_{\mu}(d\chi)$$

$$U_{28}(\chi) = -\tanh_{\mu}(2\,d\chi) \pm \left(isech_{\mu}(2\,d\chi)\right)$$

$$U_{29}(\chi) = -\coth_{\mu}(2\,d\chi) \pm \left(csch_{\mu}(2\,d\chi)\right)$$

and

$$U_{30}(\chi) = -\frac{1}{2}\tanh_{\mu}(1/2\,d\chi) - 1/2\,\coth_{\mu}(1/2\,d\chi)$$

Family. 7: For $\psi = 0$,

$$U_{31}(\chi) = -2\,\frac{d(e\chi\ln r + 2)}{e^2\chi\ln r}$$

Family. 8: For $e = \tau$, $d = p\tau$ (with $n \neq 0$), and $f = 0$,

$$U_{32}(\chi) = r^{\tau\chi} - p$$

Family. 9: For $e = f = 0$,

$$U_{33}(\chi) = d\chi\ln r$$

Family. 10: For $e = d = 0$,

$$U_{34}(\chi) = -\frac{1}{f\chi\ln r}$$

Family. 11: For $d = 0$, $e \neq 0$, and $f \neq 0$,

$$U_{35}(\chi) = -\frac{e}{f\big(\cosh_{\mu}(e\chi) - \sinh_{\mu}(e\chi) + 1\big)}$$

and

$$U_{36}(\chi) = -\frac{e\big(\cosh_{\mu}(e\chi) + \sinh_{\mu}(e\chi)\big)}{f\big(\cosh_{\mu}(e\chi) + \sinh_{\mu}(e\chi) + 1\big)}$$

Family. 12: For $e = \tau$, $f = p\tau$ (with $p \neq 0$), and $d = 0$,

$$U_{37}(\chi) = \frac{\mu^{\tau\chi}}{1 - pr^{\tau\chi}}$$

In above solutions, $\psi = e^2 - 4df$ and the generalized trigonometric and hyperbolic functions are defined as follows:

$$\begin{aligned}
\sin_{\mu}(\chi) &= \frac{\mu^{i\chi} - \mu^{-i\chi}}{2i}, \quad \cos_{\mu}(\chi) = \frac{\mu^{i\chi} + \mu^{-i\chi}}{2},\\
\tan_{\mu}(\chi) &= \frac{\sin_{\mu}(\chi)}{\cos_{\mu}(\chi)}, \quad \cot_{\mu}(\chi) = \frac{\cos_{\mu}(\chi)}{\sin_{\mu}(\chi)},\\
\sec_{\mu}(\chi) &= \frac{1}{\cos_{\mu}(\chi)}, \quad \csc_{\mu}(\chi) = \frac{1}{\sin_{\mu}(\chi)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sinh_{\mu}(\chi) &= \frac{\mu^{\chi} - \mu^{-\chi}}{2}, \quad \cosh_{\mu}(\chi) = \frac{\mu^{\chi} + \mu^{-\chi}}{2},\\
\tanh_{\mu}(\chi) &= \frac{\sinh_{\mu}(\chi)}{\cosh_{\mu}(\chi)}, \quad \coth_{\mu}(\chi) = \frac{\cosh_{\mu}(\chi)}{\sinh_{\mu}(\chi)},\\
sech_{\mu}(\chi) &= \frac{1}{\cosh_{\mu}(\chi)}, \quad csch_{\mu}(\chi) = \frac{1}{\sinh_{\mu}(\chi)}.
\end{aligned}$$

3. Execution of mEDAM

Our goal in this section is to create solitary wave solutions for FCBWBKEs. To do this, we begin the procedure with the wave transformation:

$$w(t,x) = W(\chi), \quad z(t,x) = Z(\chi), \quad \chi = \frac{\omega t^{\alpha}}{\alpha} + \frac{kx^{\beta}}{\beta}, \tag{10}$$

which transforms (1) into the subsequent system of NODEs:

$$\begin{aligned} &\omega W' + kZ' + kWW' = 0, \\ &\omega Z' + k(WZ)' + k^3 W''' = 0, \end{aligned} \tag{11}$$

where k symbolizes the wave's number, ω shows the velocity of the wave, and primes denotes the derivatives of W and Z with respect to χ. Integrating both the equations in (11) with respect to χ and setting the constant of integration equal to zero yields:

$$\begin{aligned} &\omega W + kZ + \frac{kW^2}{2} = 0, \\ &\omega Z + kWZ + k^3 W'' = 0. \end{aligned} \tag{12}$$

The first equation in (12) implies:

$$Z = -\frac{\omega}{k} W - \frac{W^2}{2}. \tag{13}$$

Thus with (13), the system in (12) is reduced to the following single NODE:

$$2\omega^2 W + 3\omega k W^2 + k^2 W^3 - 2k^4 W'' = 0. \tag{14}$$

Establishing the homogenous balance between W^3 and W'' yields $j = 1$. Substituting $j = 1$ in (8) suggests the following closed form solution for (14):

$$V(\chi) = \sum_{s=-1}^{1} F_s (U(\chi))^s = F_{-1}(U(\chi))^{-1} + F_0 + F_1 (U(\chi))^1. \tag{15}$$

We develop an expression in $U(\chi)$ by inserting (15) into (14) and collecting all terms with the same powers of $U(\chi)$. Equating all coefficients to zero results in a system of nonlinear algebraic equations. Using Maple to solve the problem provides with the subsequent two distinct sets of solutions:

Case 1

$$F_{-1} = 0, F_0 = \left(e + \sqrt{\psi}\right) \ln(\mu) k, F_1 = 2\, k \ln(\mu) f, \omega = \frac{\ln(\mu) k^2 \left(4\, d e f + 2\, f d \sqrt{\psi} - e^3 - e^2 \sqrt{\psi}\right)}{e^2 + e\sqrt{\psi} - 2\, f d}, k = k. \tag{16}$$

Case 2

$$F_{-1} = \frac{(e - \sqrt{\psi}) F_0}{2f}, F_0 = F_0, F_1 = 0, \omega = \frac{-F_0{}^2 \left(-4\, d e f + 2\, f d \sqrt{\psi} + e^3 - e^2 \sqrt{\psi}\right)}{8 d^2 f^2 \ln(\mu)}, k = \frac{(-e + \sqrt{\psi}) F_0}{4 f d \ln(\mu)}. \tag{17}$$

Taking into account case 1, and utilizing (10), (13), and (15) together with the corresponding general solutions of (9), we obtain the following class of solitary wave solutions for FCBWBK given in (1):

Family. 1.1: When $\psi < 0 \quad f \neq 0$,

$$\begin{aligned} &w_{1,1}(x,t) = \ln(\mu) k \left(\sqrt{\psi} + \sqrt{-\psi} \tan_\mu \left(\frac{1}{2} \sqrt{-\psi} \chi \right) \right), \\ &z_{1,1}(x,t) = -\frac{\omega}{k} \left(\ln(\mu) k \left(\sqrt{\psi} + \sqrt{-\psi} \tan_\mu \left(\frac{1}{2} \sqrt{-\psi} \chi \right) \right) \right) \\ &- \frac{1}{2} \left(\ln(\mu) k \left(\sqrt{\psi} + \sqrt{-\psi} \tan_\mu \left(\frac{1}{2} \sqrt{-\psi} \chi \right) \right) \right)^2, \end{aligned} \tag{18}$$

$$
\begin{aligned}
w_{1,2}(x,t) &= -\ln(\mu)k\left(-\sqrt{\psi}+\sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right),\\
z_{1,2}(x,t) &= -\frac{\omega}{k}\left(-\ln(\mu)k\left(-\sqrt{\psi}+\sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)\right)\\
&\quad-\frac{1}{2}\left(-\ln(\mu)k\left(-\sqrt{\psi}+\sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)\right)^{2},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
w_{1,3}(x,t) &= \frac{\ln(\mu)k(\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\cos_{\mu}(\sqrt{-\psi}\chi)},\\
z_{1,3}(x,t) &= -\frac{\omega}{k}\left(\frac{\ln(\mu)k(\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\cos_{\mu}(\sqrt{-\psi}\chi)}\right)\\
&\quad-\frac{1}{2}\left(\frac{\ln(\mu)k(\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\cos_{\mu}(\sqrt{-\psi}\chi)}\right)^{2},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
w_{1,4}(x,t) &= -\frac{\ln(\mu)k(-\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\sin_{\mu}(\sqrt{-\psi}\chi)},\\
z_{1,4}(x,t) &= -\frac{\omega}{k}\left(-\frac{\ln(\mu)k(-\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\sin_{\mu}(\sqrt{-\psi}\chi)}\right)\\
&\quad-\frac{1}{2}\left(-\frac{\ln(\mu)k(-\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi)+\sqrt{-\psi})}{\sin_{\mu}(\sqrt{-\psi}\chi)}\right)^{2},
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
w_{1,5}(x,t) &= -\frac{1}{2}\frac{\ln(\mu)k\left(-2\sqrt{\psi}\Psi-\sqrt{-\psi}+2\sqrt{-\psi}\left(\cos_{\mu}\left(\frac{1}{4}\sqrt{-\psi}\chi\right)\right)^{2}\right)}{\Psi},\\
z_{1,5}(x,t) &= -\frac{\omega}{k}\left(-\frac{1}{2}\frac{\ln(\mu)k\left(-2\sqrt{\psi}\Psi-\sqrt{-\psi}+2\sqrt{-\psi}\left(\cos_{\mu}\left(\frac{1}{4}\sqrt{-\psi}\chi\right)\right)^{2}\right)}{\Psi}\right)\\
&\quad-\frac{1}{2}\left(-\frac{1}{2}\frac{\ln(\mu)k\left(-2\sqrt{\psi}\Psi-\sqrt{-\psi}+2\sqrt{-\psi}\left(\cos_{\mu}\left(\frac{1}{4}\sqrt{-\psi}\chi\right)\right)^{2}\right)}{\Psi}\right)^{2},
\end{aligned}
\tag{22}
$$

where $\Psi=\cos_{\mu}\left(\frac{1}{4}\sqrt{-\psi}\chi\right)\sin_{\mu}\left(\frac{1}{4}\sqrt{-\psi}\chi\right)$.

Family. 1.2: When $\psi>0\quad f\neq 0$,

$$
\begin{aligned}
w_{1,6}(x,t) &= -\ln(\mu)k\sqrt{\psi}\left(-1+\tanh_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right),\\
z_{1,6}(x,t) &= -\frac{\omega}{k}\left(-\ln(\mu)k\sqrt{\psi}\left(-1+\tanh_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right)\right)\\
&\quad-\frac{1}{2}\left(-\ln(\mu)k\sqrt{\psi}\left(-1+\tanh_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right)\right)^{2},
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
w_{1,7}(x,t) &= -\ln(\mu)k\sqrt{\psi}\left(-1+\coth_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right),\\
z_{1,7}(x,t) &= -\frac{\omega}{k}\left(-\ln(\mu)k\sqrt{\psi}\left(-1+\coth_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right)\right)\\
&\quad-\frac{1}{2}\left(-\ln(\mu)k\sqrt{\psi}\left(-1+\coth_{\mu}\left(\frac{1}{2}\sqrt{\psi}\chi\right)\right)\right)^{2},
\end{aligned}
\tag{24}
$$

$$w_{1,8}(x,t) = -\frac{\ln(\mu)k\sqrt{\psi}\big(-\cosh_\mu(\sqrt{\psi}\chi) + \sinh_\mu(\sqrt{\psi}\chi) + i\big)}{\cosh_\mu(\sqrt{\psi}\chi)},$$

$$z_{1,8}(x,t) = -\frac{\omega}{k}\left(-\frac{\ln(\mu)k\sqrt{\psi}\big(-\cosh_\mu(\sqrt{\psi}\chi) + \sinh_\mu(\sqrt{\psi}\chi) + i\big)}{\cosh_\mu(\sqrt{\psi}\chi)}\right) \tag{25}$$

$$-\frac{1}{2}\left(-\frac{\ln(\mu)k\sqrt{\psi}\big(-\cosh_\mu(\sqrt{\psi}\chi) + \sinh_\mu(\sqrt{\psi}\chi) + i\big)}{\cosh_\mu(\sqrt{\psi}\chi)}\right)^2,$$

$$w_{1,9}(x,t) = \frac{\ln(\mu)k\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi) - \cosh_\mu(\sqrt{\psi}\chi)\big)}{\sinh_\mu(\sqrt{\psi}\chi)},$$

$$z_{1,9}(x,t) = -\frac{\omega}{k}\left(\frac{\ln(\mu)k\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi) - \cosh_\mu(\sqrt{\psi}\chi)\big)}{\sinh_\mu(\sqrt{\psi}\chi)}\right) \tag{26}$$

$$-\frac{1}{2}\left(\frac{\ln(\mu)k\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi) - \cosh_\mu(\sqrt{\psi}\chi)\big)}{\sinh_\mu(\sqrt{\psi}\chi)}\right)^2,$$

and

$$w_{1,10}(x,t) = \frac{1}{2}\frac{\ln(\mu)k\sqrt{\psi}\left(2\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right) + 1\right)}{\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)},$$

$$z_{1,10}(x,t) = -\frac{\omega}{k}\left(\frac{1}{2}\frac{\ln(\mu)k\sqrt{\psi}\left(2\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right) + 1\right)}{\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)}\right) \tag{27}$$

$$-\frac{1}{2}\left(\frac{1}{2}\frac{\ln(\mu)k\sqrt{\psi}\left(2\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right) + 1\right)}{\cosh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)\sinh_\mu\left(\frac{1}{4}\sqrt{\psi}\chi\right)}\right)^2.$$

Family. 1.3: When $df > 0$ and $e = 0$,

$$w_{1,11}(x,t) = 2\ln(\mu)k\left(\sqrt{-fd} + \sqrt{df}\tan_\mu\left(\sqrt{fd}\chi\right)\right),$$

$$z_{1,11}(x,t) = -\frac{\omega}{k}\left(2\ln(\mu)k\left(\sqrt{-fd} + \sqrt{df}\tan_\mu\left(\sqrt{fd}\chi\right)\right)\right) \tag{28}$$

$$-\frac{1}{2}\left(2\ln(\mu)k\left(\sqrt{-fd} + \sqrt{df}\tan_\mu\left(\sqrt{fd}\chi\right)\right)\right)^2,$$

$$w_{1,12}(x,t) = 2\ln(\mu)k\left(\sqrt{-fd} - \sqrt{df}\cot_\mu\left(\sqrt{fd}\chi\right)\right),$$

$$z_{1,12}(x,t) = -\frac{\omega}{k}\left(2\ln(\mu)k\left(\sqrt{-fd} - \sqrt{df}\cot_\mu\left(\sqrt{fd}\chi\right)\right)\right) \tag{29}$$

$$-\frac{1}{2}\left(2\ln(\mu)k\left(\sqrt{-fd} - \sqrt{df}\cot_\mu\left(\sqrt{fd}\chi\right)\right)\right)^2,$$

$$w_{1,13}(x,t) = 2\,\frac{k\ln(\mu)\big(\sqrt{-fd}\cos_\mu(2\sqrt{fd}\chi) + \sqrt{df}\sin_\mu(2\sqrt{fd}\chi) + \sqrt{df}\big)}{\cos_\mu(2\sqrt{fd}\chi)},$$

$$z_{1,13}(x,t) = -\frac{\omega}{k}\left(2\,\frac{k\ln(\mu)\big(\sqrt{-fd}\cos_\mu(2\sqrt{fd}\chi) + \sqrt{df}\sin_\mu(2\sqrt{fd}\chi) + \sqrt{df}\big)}{\cos_\mu(2\sqrt{fd}\chi)}\right) \tag{30}$$

$$-\frac{1}{2}\left(2\,\frac{k\ln(\mu)\big(\sqrt{-fd}\cos_\mu(2\sqrt{fd}\chi) + \sqrt{df}\sin_\mu(2\sqrt{fd}\chi) + \sqrt{df}\big)}{\cos_\mu(2\sqrt{fd}\chi)}\right)^2,$$

$$w_{1,14}(x,t) = 2\,\frac{k\ln(\mu)\left(\sqrt{-fd}\sin_\mu(2\sqrt{fd}\chi) - \sqrt{df}\cos_\mu(2\sqrt{fd}\chi) - \sqrt{df}\right)}{\sin_\mu(2\sqrt{fd}\chi)},$$
$$z_{1,14}(x,t) = -\frac{\omega}{k}\left(2\,\frac{k\ln(\mu)\left(\sqrt{-fd}\sin_\mu(2\sqrt{fd}\chi) - \sqrt{df}\cos_\mu(2\sqrt{fd}\chi) - \sqrt{df}\right)}{\sin_\mu(2\sqrt{fd}\chi)}\right)$$
$$-\frac{1}{2}\left(2\,\frac{k\ln(\mu)\left(\sqrt{-fd}\sin_\mu(2\sqrt{fd}\chi) - \sqrt{df}\cos_\mu(2\sqrt{fd}\chi) - \sqrt{df}\right)}{\sin_\mu(2\sqrt{fd}\chi)}\right)^2, \tag{31}$$

and

$$w_{1,15}(x,t) = \frac{\ln(\mu)k\left(2\sqrt{-fd}\cos_\mu(1/2\sqrt{fd}\chi)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) + \sqrt{df} - 2\sqrt{df}\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)},$$
$$z_{1,15}(x,t) = -\frac{\omega}{k}\left(\frac{\ln(\mu)k\left(2\sqrt{-fd}\cos_\mu(1/2\sqrt{fd}\chi)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) + \sqrt{df} - 2\sqrt{df}\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)}\right)$$
$$-\frac{1}{2}\left(\frac{\ln(\mu)k\left(2\sqrt{-fd}\cos_\mu(1/2\sqrt{fd}\chi)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) + \sqrt{df} - 2\sqrt{df}\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)}\right)^2. \tag{32}$$

Family. 1.4: When $df > 0$ and $e = 0$,

$$w_{1,16}(x,t) = -2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-df}\tanh_\mu\left(\sqrt{-fd}\chi\right)\right),$$
$$z_{1,16}(x,t) = -\frac{\omega}{k}\left(-2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-df}\tanh_\mu\left(\sqrt{-fd}\chi\right)\right)\right)$$
$$-\frac{1}{2}\left(-2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-df}\tanh_\mu\left(\sqrt{-fd}\chi\right)\right)\right)^2, \tag{33}$$

$$w_{1,17}(x,t) = -2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-fd}\coth_\mu\left(\sqrt{-fd}\chi\right)\right),$$
$$z_{1,17}(x,t) = -\frac{\omega}{k}\left(-2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-fd}\coth_\mu\left(\sqrt{-fd}\chi\right)\right)\right)$$
$$-\frac{1}{2}\left(-2\ln(\mu)k\left(-\sqrt{-fd} + \sqrt{-fd}\coth_\mu\left(\sqrt{-fd}\chi\right)\right)\right)^2, \tag{34}$$

$$w_{1,18}(x,t) = -2\,\frac{\ln(\mu)k\left(-\sqrt{-fd}\cosh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}\sinh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}i\right)}{\cosh_\mu(2\sqrt{-fd}\chi)},$$
$$z_{1,18}(x,t) = -\frac{\omega}{k}\left(-2\,\frac{\ln(\mu)k\left(-\sqrt{-fd}\cosh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}\sinh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}i\right)}{\cosh_\mu(2\sqrt{-fd}\chi)}\right)$$
$$-\frac{1}{2}\left(-2\,\frac{\ln(\mu)k\left(-\sqrt{-fd}\cosh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}\sinh_\mu(2\sqrt{-fd}\chi) + \sqrt{-df}i\right)}{\cosh_\mu(2\sqrt{-fd}\chi)}\right)^2, \tag{35}$$

$$w_{1,19}(x,t) = 2\,\frac{\ln(\mu)k\left(\sqrt{-fd}\sinh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\cosh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\right)}{\sinh_\mu(2\sqrt{-fd}\chi)},$$
$$z_{1,19}(x,t) = -\frac{\omega}{k}\left(2\,\frac{\ln(\mu)k\left(\sqrt{-fd}\sinh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\cosh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\right)}{\sinh_\mu(2\sqrt{-fd}\chi)}\right)$$
$$-\frac{1}{2}\left(2\,\frac{\ln(\mu)k\left(\sqrt{-fd}\sinh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\cosh_\mu(2\sqrt{-fd}\chi) - \sqrt{-df}\right)}{\sinh_\mu(2\sqrt{-fd}\chi)}\right)^2, \tag{36}$$

and

$$\begin{aligned} w_{1,20}(x,t) &= -\frac{\ln(\mu)k\left(-2\sqrt{-fd}\Lambda+2\sqrt{-df}\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-\sqrt{-df}\right)}{\Lambda}, \\ z_{1,20}(x,t) &= -\frac{\omega}{k}\left(-\frac{\ln(\mu)k\left(-2\sqrt{-fd}\Lambda+2\sqrt{-df}\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-\sqrt{-df}\right)}{\Lambda}\right) \\ &-\frac{1}{2}\left(-\frac{\ln(\mu)k\left(-2\sqrt{-fd}\Lambda+2\sqrt{-df}\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-\sqrt{-df}\right)}{\Lambda}\right)^2, \end{aligned} \tag{37}$$

where $\Lambda = \cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\sinh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)$.

Family. 1.5: When $f = d$ and $e = 0$,

$$\begin{aligned} w_{1,21}(x,t) &= 2\ln(\mu)k\left(di+d\tan_\mu(d\chi)\right), \\ z_{1,21}(x,t) &= -\frac{\omega}{k}\left(2\ln(\mu)k\left(di+d\tan_\mu(d\chi)\right)\right)-\frac{1}{2}\left(2\ln(\mu)k\left(di+d\tan_\mu(d\chi)\right)\right)^2, \end{aligned} \tag{38}$$

$$\begin{aligned} w_{1,22}(x,t) &= -2\ln(\mu)k\left(-di+d\cot_\mu(d\chi)\right), \\ z_{1,22}(x,t) &= -\frac{\omega}{k}\left(-2\ln(\mu)k\left(-di+d\cot_\mu(d\chi)\right)\right)-\frac{1}{2}\left(-2\ln(\mu)k\left(-di+d\cot_\mu(d\chi)\right)\right)^2, \end{aligned} \tag{39}$$

$$\begin{aligned} w_{1,23}(x,t) &= 2\,\frac{\ln(\mu)k\left(di\,\cos_\mu(2\,d\chi)+d\sin_\mu(2\,d\chi)+d\right)}{\cos_\mu(2\,d\chi)}, \\ z_{1,23}(x,t) &= -\frac{\omega}{k}\left(2\,\frac{\ln(\mu)k\left(di\,\cos_\mu(2\,d\chi)+d\sin_\mu(2\,d\chi)+d\right)}{\cos_\mu(2\,d\chi)}\right) \\ &-\frac{1}{2}\left(2\,\frac{\ln(\mu)k\left(di\,\cos_\mu(2\,d\chi)+d\sin_\mu(2\,d\chi)+d\right)}{\cos_\mu(2\,d\chi)}\right)^2, \end{aligned} \tag{40}$$

$$\begin{aligned} w_{1,24}(x,t) &= 2\,\frac{\ln(\mu)k\left(di\,\sin_\mu(2\,d\chi)-d\cos_\mu(2\,d\chi)-d\right)}{\sin_\mu(2\,d\chi)}, \\ z_{1,24}(x,t) &= -\frac{\omega}{k}\left(2\,\frac{\ln(\mu)k\left(di\,\sin_\mu(2\,d\chi)-d\cos_\mu(2\,d\chi)-d\right)}{\sin_\mu(2\,d\chi)}\right) \\ &-\frac{1}{2}\left(2\,\frac{\ln(\mu)k\left(di\,\sin_\mu(2\,d\chi)-d\cos_\mu(2\,d\chi)-d\right)}{\sin_\mu(2\,d\chi)}\right)^2, \end{aligned} \tag{41}$$

and

$$\begin{aligned} w_{1,25}(x,t) &= \frac{\ln(\mu)k\left(2\,di\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)+d-2\,d\left(\cos_\mu(1/2\,d\chi)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)}, \\ z_{1,25}(x,t) &= -\frac{\omega}{k}\left(\frac{\ln(\mu)k\left(2\,di\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)+d-2\,d\left(\cos_\mu(1/2\,d\chi)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)}\right) \\ &-\frac{1}{2}\left(\frac{\ln(\mu)k\left(2\,di\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)+d-2\,d\left(\cos_\mu(1/2\,d\chi)\right)^2\right)}{\cos_\mu\left(\frac{1}{2}d\chi\right)\sin_\mu\left(\frac{1}{2}d\chi\right)}\right)^2. \end{aligned} \tag{42}$$

Family. 1.6: When $f = -d$ and $e = 0$,

$$\begin{aligned} w_{1,26}(x,t) &= 2\,d\ln(\mu)k\left(1+\tanh_\mu(d\chi)\right), \\ z_{1,26}(x,t) &= -\frac{\omega}{k}\left(2\,d\ln(\mu)k\left(1+\tanh_\mu(d\chi)\right)\right)-\frac{1}{2}\left(2\,d\ln(\mu)k\left(1+\tanh_\mu(d\chi)\right)\right)^2, \end{aligned} \tag{43}$$

$$
\begin{aligned}
w_{1,27}(x,t) &= 2\,d\ln(\mu)k\big(1+\coth_\mu(d\chi)\big),\\
z_{1,27}(x,t) &= -\frac{\omega}{k}\Big(2\,d\ln(\mu)k\big(1+\coth_\mu(d\chi)\big)\Big)-\frac{1}{2}\Big(2\,d\ln(\mu)k\big(1+\coth_\mu(d\chi)\big)\Big)^2,
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
w_{1,28}(x,t) &= 2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\big)}{\cosh_\mu(2\,d\chi)},\\
z_{1,28}(x,t) &= -\frac{\omega}{k}\left(2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\big)}{\cosh_\mu(2\,d\chi)}\right)\\
&-\frac{1}{2}\left(2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\big)}{\cosh_\mu(2\,d\chi)}\right)^2,
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
w_{1,29}(x,t) &= 2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+1\big)}{\sinh_\mu(2\,d\chi)},\\
z_{1,29}(x,t) &= -\frac{\omega}{k}\left(2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+1\big)}{\sinh_\mu(2\,d\chi)}\right)\\
&-\frac{1}{2}\left(2\,\frac{d\ln(\mu)k\big(\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+1\big)}{\sinh_\mu(2\,d\chi)}\right)^2,
\end{aligned} \tag{46}
$$

and

$$
\begin{aligned}
w_{1,30}(x,t) &= \frac{\ln(\mu)kd\Big(2\,\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)+2\,\big(\cosh_\mu(1/2\,d\chi)\big)^2-1\Big)}{\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)},\\
z_{1,30}(x,t) &= -\frac{\omega}{k}\left(\frac{\ln(\mu)kd\Big(2\,\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)+2\,\big(\cosh_\mu(1/2\,d\chi)\big)^2-1\Big)}{\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)}\right)\\
&-\frac{1}{2}\left(\frac{\ln(\mu)kd\Big(2\,\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)+2\,\big(\cosh_\mu(1/2\,d\chi)\big)^2-1\Big)}{\cosh_\mu\big(\tfrac{1}{2}d\chi\big)\sinh_\mu\big(\tfrac{1}{2}d\chi\big)}\right)^2.
\end{aligned} \tag{47}
$$

Family. 1.7: When $\psi = 0$,

$$
\begin{aligned}
w_{1,31}(x,t) &= \frac{k\big(e^3\ln(\mu)\chi-4\,fd(e\chi\,\ln(\mu)+2)\big)}{e^2\chi},\\
z_{1,31}(x,t) &= -\frac{\omega}{k}\left(\frac{k\big(e^3\ln(\mu)\chi-4\,fd(e\chi\,\ln(\mu)+2)\big)}{e^2\chi}\right)\\
&-\frac{1}{2}\left(\frac{k\big(e^3\ln(\mu)\chi-4\,fd(e\chi\,\ln(\mu)+2)\big)}{e^2\chi}\right)^2.
\end{aligned} \tag{48}
$$

Family. 1.8: When $d = 0$, $f \neq 0$, and $e \neq 0$,

$$
\begin{aligned}
w_{1,32}(x,t) &= 2\,\frac{e\ln(\mu)k\big(\cosh_\mu(e\chi)-\sinh_\mu(e\chi)\big)}{\cosh_\mu(e\chi)-\sinh_\mu(e\chi)+1},\\
z_{1,32}(x,t) &= -\frac{\omega}{k}\left(2\,\frac{e\ln(\mu)k\big(\cosh_\mu(e\chi)-\sinh_\mu(e\chi)\big)}{\cosh_\mu(e\chi)-\sinh_\mu(e\chi)+1}\right)\\
&-\frac{1}{2}\left(2\,\frac{e\ln(\mu)k\big(\cosh_\mu(e\chi)-\sinh_\mu(e\chi)\big)}{\cosh_\mu(e\chi)-\sinh_\mu(e\chi)+1}\right)^2,
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
&w_{1,33}(x,t) = 2\,\frac{e\ln(\mu)k}{\cosh_{\mu}(e\chi) + \sinh_{\mu}(e\chi) + 1},\\
&z_{1,33}(x,t) = -\frac{\omega}{k}\left(2\,\frac{e\ln(\mu)k}{\cosh_{\mu}(e\chi) + \sinh_{\mu}(e\chi) + 1}\right)\\
&-\frac{1}{2}\left(2\,\frac{e\ln(\mu)k}{\cosh_{\mu}(e\chi) + \sinh_{\mu}(e\chi) + 1}\right)^{2}.
\end{aligned}
\tag{50}
$$

Family. 1.9: When $e = \nu$, $f = p\tau (p \neq 0)$, and $d = 0$,

$$
\begin{aligned}
&w_{1,34}(x,t) = -2\,\frac{\tau\,\ln(\mu)k(1 - p\mu^{\tau\chi} + p\mu^{\tau\chi})}{-1 + p\mu^{\tau\chi}},\\
&z_{1,34}(x,t) = -\frac{\omega}{k}\left(-2\,\frac{\tau\,\ln(\mu)k(q_1 - pq_2\mu^{\tau\chi} + p\mu^{\tau\chi})}{-1 + p\mu^{\tau\chi}}\right)\\
&-\frac{1}{2}\left(-2\,\frac{\tau\,\ln(\mu)k(1 - p\mu^{\tau\chi} + p\mu^{\tau\chi})}{-1 + p\mu^{\tau\chi}}\right)^{2}.
\end{aligned}
\tag{51}
$$

where $\chi = \frac{\left(\frac{\ln(\mu)k^2\left(4def + 2fd\sqrt{\psi} - e^3 - e^2\sqrt{\psi}\right)}{e^2 + e\sqrt{\psi} - 2fd}\right)t^{\alpha}}{\alpha} + \frac{kx^{\beta}}{\beta}$.

Taking into account case 2, and utilizing (10), (13), and (15) together with the corresponding general solutions of (9), we obtain the following class of solitary wave solutions for FCBWBK given in (1):

Family. 2.1: When $\psi < 0 \quad f \neq 0$,

$$
\begin{aligned}
&w_{2,1}(x,t) = \frac{F_0\left(-\sqrt{\psi} + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{-e + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)},\\
&z_{2,1}(x,t) = -\frac{\omega}{k}\left(\frac{F_0\left(-\sqrt{\psi} + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{-e + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)}\right) - \frac{1}{2}\left(\frac{F_0\left(-\sqrt{\psi} + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{-e + \sqrt{-\psi}\tan_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)}\right)^{2},
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&w_{2,2}(x,t) = \frac{F_0\left(\sqrt{\psi} + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{e + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)},\\
&z_{2,2}(x,t) = -\frac{\omega}{k}\left(\frac{F_0\left(\sqrt{\psi} + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{e + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)}\right) - \frac{1}{2}\left(\frac{F_0\left(\sqrt{\psi} + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)\right)}{e + \sqrt{-\psi}\cot_{\mu}\left(\frac{1}{2}\sqrt{-\psi}\chi\right)}\right)^{2},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
&w_{2,3}(x,t) = \frac{F_0(-\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{-e\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}},\\
&z_{2,3}(x,t) = -\frac{\omega}{k}\left(\frac{F_0(-\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{-e\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}}\right)\\
&-\frac{1}{2}\left(\frac{F_0(-\sqrt{\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{-e\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}}\right)^{2},
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
&w_{2,4}(x,t) = \frac{F_0(\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{e\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}},\\
&z_{2,4}(x,t) = -\frac{\omega}{k}\left(\frac{F_0(\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{e\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}}\right)\\
&-\frac{1}{2}\left(\frac{F_0(\sqrt{\psi}\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi})}{e\sin_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}\cos_{\mu}(\sqrt{-\psi}\chi) + \sqrt{-\psi}}\right)^{2},
\end{aligned}
\tag{55}
$$

and

$$
\begin{aligned}
w_{2,5}(x,t) &= \frac{F_0\Big(2\sqrt{\psi}\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2\Big)}{2\,e\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2},\\
z_{2,5}(x,t) &= -\frac{\omega}{k}\Bigg(\frac{F_0\Big(2\sqrt{\psi}\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2\Big)}{2\,e\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2}\Bigg)\\
&-\frac{1}{2}\Bigg(\frac{F_0\Big(2\sqrt{\psi}\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2\Big)}{2\,e\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\sin_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big) - \sqrt{-\psi} + 2\sqrt{-\psi}\Big(\cos_\mu\Big(\frac{1}{4}\sqrt{-\psi}\chi\Big)\Big)^2}\Bigg)^2.
\end{aligned}
\tag{56}
$$

Family. 2.2: When $\psi > 0 \quad f \neq 0$,

$$
\begin{aligned}
w_{2,6}(x,t) &= \frac{F_0\sqrt{\psi}\Big(1+\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)},\\
z_{2,6}(x,t) &= -\frac{\omega}{k}\Bigg(\frac{F_0\sqrt{\psi}\Big(1+\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)}\Bigg) - \frac{1}{2}\Bigg(\frac{F_0\sqrt{\psi}\Big(1+\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\tanh_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)}\Bigg)^2,
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
w_{2,7}(x,t) &= \frac{F_0\sqrt{\psi}\Big(1+\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)},\\
z_{2,7}(x,t) &= -\frac{\omega}{k}\Bigg(\frac{F_0\sqrt{\psi}\Big(1+\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)}\Bigg) - \frac{1}{2}\Bigg(\frac{F_0\sqrt{\psi}\Big(1+\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)\Big)}{e+\sqrt{\psi}\coth_\mu\Big(\frac{1}{2}\sqrt{\psi}\chi\Big)}\Bigg)^2,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
w_{2,8}(x,t) &= \frac{F_0\sqrt{\psi}\big(\cosh_\mu(\sqrt{\psi}\chi)+\sinh_\mu(\sqrt{\psi}\chi)+i\big)}{e\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}i},\\
z_{2,8}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0\sqrt{\psi}\big(\cosh_\mu(\sqrt{\psi}\chi)+\sinh_\mu(\sqrt{\psi}\chi)+i\big)}{e\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}i}\right)\\
&-\frac{1}{2}\left(\frac{F_0\sqrt{\psi}\big(\cosh_\mu(\sqrt{\psi}\chi)+\sinh_\mu(\sqrt{\psi}\chi)+i\big)}{e\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}i}\right)^2,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
w_{2,9}(x,t) &= \frac{F_0\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi)+\cosh_\mu(\sqrt{\psi}\chi)+1\big)}{e\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}},\\
z_{2,9}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi)+\cosh_\mu(\sqrt{\psi}\chi)+1\big)}{e\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}}\right)\\
&-\frac{1}{2}\left(\frac{F_0\sqrt{\psi}\big(\sinh_\mu(\sqrt{\psi}\chi)+\cosh_\mu(\sqrt{\psi}\chi)+1\big)}{e\sinh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}\cosh_\mu(\sqrt{\psi}\chi)+\sqrt{\psi}}\right)^2,
\end{aligned}
\tag{60}
$$

and

$$w_{2,10}(x,t) = -\frac{F_0\sqrt{\psi}\Big(2\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) - 1\Big)}{-2\,e\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) + \sqrt{\psi}},$$

$$z_{2,10}(x,t) = -\frac{\omega}{k}\Bigg(- \frac{F_0\sqrt{\psi}\Big(2\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) - 1\Big)}{-2\,e\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) + \sqrt{\psi}}\Bigg) \quad (61)$$

$$-\frac{1}{2}\Bigg(- \frac{F_0\sqrt{\psi}\Big(2\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) - 1\Big)}{-2\,e\cosh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big)\sinh_\mu\Big(\frac{1}{4}\sqrt{\psi}\chi\Big) + \sqrt{\psi}}\Bigg)^2.$$

Family. 2.3: When $df > 0$ and $e = 0$,

$$w_{2,11}(x,t) = -\frac{F_0\big(i - \tan_\mu\big(\sqrt{fd}\chi\big)\big)}{\tan_\mu\big(\sqrt{fd}\chi\big)},$$

$$z_{2,11}(x,t) = -\frac{\omega}{k}\Bigg(- \frac{F_0\big(i - \tan_\mu\big(\sqrt{fd}\chi\big)\big)}{\tan_\mu\big(\sqrt{fd}\chi\big)}\Bigg) - \frac{1}{2}\Bigg(- \frac{F_0\big(i - \tan_\mu\big(\sqrt{fd}\chi\big)\big)}{\tan_\mu\big(\sqrt{fd}\chi\big)}\Bigg)^2, \quad (62)$$

$$w_{2,12}(x,t) = \frac{F_0\big(i + \cot_\mu\big(\sqrt{fd}\chi\big)\big)}{\cot_\mu\big(\sqrt{fd}\chi\big)},$$

$$z_{2,12}(x,t) = -\frac{\omega}{k}\Bigg(\frac{F_0\big(i + \cot_\mu\big(\sqrt{fd}\chi\big)\big)}{\cot_\mu\big(\sqrt{fd}\chi\big)}\Bigg) - \frac{1}{2}\Bigg(\frac{F_0\big(i + \cot_\mu\big(\sqrt{fd}\chi\big)\big)}{\cot_\mu\big(\sqrt{fd}\chi\big)}\Bigg)^2, \quad (63)$$

$$w_{2,13}(x,t) = \frac{F_0\big(-i\cos_\mu\big(2\sqrt{fd}\chi\big) + \sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)},$$

$$z_{2,13}(x,t) = -\frac{\omega}{k}\Bigg(\frac{F_0\big(-i\cos_\mu\big(2\sqrt{fd}\chi\big) + \sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}\Bigg) \quad (64)$$

$$-\frac{1}{2}\Bigg(\frac{F_0\big(-i\cos_\mu\big(2\sqrt{fd}\chi\big) + \sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\sin_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}\Bigg)^2,$$

$$w_{2,14}(x,t) = \frac{F_0\big(i\sin_\mu\big(2\sqrt{fd}\chi\big) + \cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)},$$

$$z_{2,14}(x,t) = -\frac{\omega}{k}\Bigg(\frac{F_0\big(i\sin_\mu\big(2\sqrt{fd}\chi\big) + \cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}\Bigg) \quad (65)$$

$$-\frac{1}{2}\Bigg(\frac{F_0\big(i\sin_\mu\big(2\sqrt{fd}\chi\big) + \cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}{\big(\cos_\mu\big(2\sqrt{fd}\chi\big) + 1\big)}\Bigg)^2,$$

and

$$w_{2,15}(x,t) = \frac{F_0\left(2\,i\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) - +2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\left(-1+2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)},$$
$$z_{2,15}(x,t) = -\frac{\omega}{k}\left(\frac{F_0\left(2\,i\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) - +2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\left(-1+2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}\right) \tag{66}$$
$$-\frac{1}{2}\left(\frac{F_0\left(2\,i\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\sin_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right) - +2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}{\left(-1+2\left(\cos_\mu\left(\frac{1}{2}\sqrt{fd}\chi\right)\right)^2\right)}\right)^2.$$

Family. 2.4: When $df > 0$ and $e = 0$,

$$\begin{aligned} w_{2,16}(x,t) &= \frac{F_0(1+\tanh_\mu(\sqrt{-fd}\chi))}{\tanh_\mu(\sqrt{-fd}\chi)},\\ z_{2,16}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0(1+\tanh_\mu(\sqrt{-fd}\chi))}{\tanh_\mu(\sqrt{-fd}\chi)}\right) - \frac{1}{2}\left(\frac{F_0(1+\tanh_\mu(\sqrt{-fd}\chi))}{\tanh_\mu(\sqrt{-fd}\chi)}\right)^2, \end{aligned} \tag{67}$$

$$\begin{aligned} w_{2,17}(x,t) &= \frac{F_0(1+\coth_\mu(\sqrt{-fd}\chi))}{\coth_\mu(\sqrt{-fd}\chi)},\\ z_{2,17}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0(1+\coth_\mu(\sqrt{-fd}\chi))}{\coth_\mu(\sqrt{-fd}\chi)}\right) - \frac{1}{2}\left(\frac{F_0(1+\coth_\mu(\sqrt{-fd}\chi))}{\coth_\mu(\sqrt{-fd}\chi)}\right)^2, \end{aligned} \tag{68}$$

$$\begin{aligned} w_{2,18}(x,t) &= \frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+i)}{\sinh_\mu(2\sqrt{-fd}\chi)+i},\\ z_{2,18}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+i)}{\sinh_\mu(2\sqrt{-fd}\chi)+i}\right)\\ &-\frac{1}{2}\left(\frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+i)}{\sinh_\mu(2\sqrt{-fd}\chi)+i}\right)^2, \end{aligned} \tag{69}$$

$$\begin{aligned} w_{2,19}(x,t) &= \frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+1)}{\cosh_\mu(2\sqrt{-fd}\chi)+1},\\ z_{2,19}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+1)}{\cosh_\mu(2\sqrt{-fd}\chi)+1}\right)\\ &-\frac{1}{2}\left(\frac{F_0(\cosh_\mu(2\sqrt{-fd}\chi)+\sinh_\mu(2\sqrt{-fd}\chi)+1)}{\cosh_\mu(2\sqrt{-fd}\chi)+1}\right)^2, \end{aligned} \tag{70}$$

and

$$w_{2,20}(x,t) = \frac{F_0\left(2\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\sinh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)+2\left(\cosh_\mu(1/2\sqrt{-fd}\chi)\right)^2-1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-1},$$
$$z_{2,20}(x,t) = -\frac{\omega}{k}\left(\frac{F_0\left(2\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\sinh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)+2\left(\cosh_\mu(1/2\sqrt{-fd}\chi)\right)^2-1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-1}\right) \tag{71}$$
$$-\frac{1}{2}\left(\frac{F_0\left(2\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\sinh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)+2\left(\cosh_\mu(1/2\sqrt{-fd}\chi)\right)^2-1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}\sqrt{-fd}\chi\right)\right)^2-1}\right)^2.$$

Family. 2.5: When $f = d$ and $e = 0$,

$$\begin{aligned} w_{2,21}(x,t) &= -\frac{F_0\big(i - \tan_\mu(d\chi)\big)}{d\tan_\mu(\chi)}, \\ z_{2,21}(x,t) &= -\frac{\omega}{k}\bigg(-\frac{F_0\big(di - d\tan_\mu(d\chi)\big)}{d\tan_\mu(d\chi)}\bigg) - \frac{1}{2}\bigg(-\frac{F_0\big(di - d\tan_\mu(d\chi)\big)}{d\tan_\mu(d\chi)}\bigg)^2, \end{aligned} \tag{72}$$

$$\begin{aligned} w_{2,22}(x,t) &= \frac{F_0\big(i + \cot_\mu(d\chi)\big)}{\cot_\mu(d\chi)}, \\ z_{2,22}(x,t) &= -\frac{\omega}{k}\bigg(\frac{F_0\big(i + \cot_\mu(d\chi)\big)}{\cot_\mu(d\chi)}\bigg) - \frac{1}{2}\bigg(\frac{F_0\big(i + \cot_\mu(d\chi)\big)}{\cot_\mu(d\chi)}\bigg)^2, \end{aligned} \tag{73}$$

$$\begin{aligned} w_{2,23}(x,t) &= \frac{F_0\big(i\cos_\mu(2\,d\chi) + \sin_\mu(2\,d\chi) + 1\big)}{\sin_\mu(2\,d\chi) + 1}, \\ z_{2,23}(x,t) &= -\frac{\omega}{k}\bigg(\frac{F_0\big(i\cos_\mu(2\,d\chi) + \sin_\mu(2\,d\chi) + 1\big)}{\sin_\mu(2\,d\chi) + 1}\bigg) \\ &- \frac{1}{2}\bigg(\frac{F_0\big(i\cos_\mu(2\,d\chi) + \sin_\mu(2\,d\chi) + 1\big)}{\sin_\mu(2\,d\chi) + 1}\bigg)^2, \end{aligned} \tag{74}$$

$$\begin{aligned} w_{2,24}(x,t) &= \frac{F_0\big(i\sin_\mu(2\,d\chi) + \cos_\mu(2\,d\chi) + 1\big)}{\cos_\mu(2\,d\chi) + 1}, \\ z_{2,24}(x,t) &= -\frac{\omega}{k}\bigg(\frac{F_0\big(i\sin_\mu(2\,d\chi) + \cos_\mu(2\,d\chi) + 1\big)}{\cos_\mu(2\,d\chi) + 1}\bigg) \\ &- \frac{1}{2}\bigg(\frac{F_0\big(i\sin_\mu(2\,d\chi) + \cos_\mu(2\,d\chi) + 1\big)}{\cos_\mu(2\,d\chi) + 1}\bigg)^2, \end{aligned} \tag{75}$$

and

$$\begin{aligned} w_{2,25}(x,t) &= \frac{F_0\Big(2\,i\cos_\mu\big(\tfrac{1}{2}d\chi\big)\sin_\mu\big(\tfrac{1}{2}d\chi\big) - 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2\Big)}{-1 + 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2}, \\ z_{2,25}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0\Big(2\,i\cos_\mu\big(\tfrac{1}{2}d\chi\big)\sin_\mu\big(\tfrac{1}{2}d\chi\big) - 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2\Big)}{-1 + 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2}\right) \\ &- \frac{1}{2}\left(\frac{F_0\Big(2\,i\cos_\mu\big(\tfrac{1}{2}d\chi\big)\sin_\mu\big(\tfrac{1}{2}d\chi\big) - 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2\Big)}{-1 + 2\big(\cos_\mu\big(\tfrac{1}{2}d\chi\big)\big)^2}\right)^2. \end{aligned} \tag{76}$$

Family. 2.6: When $f = -d$ and $e = 0$,

$$\begin{aligned} w_{2,26}(x,t) &= -\frac{F_0\big(1 - \tanh_\mu(d\chi)\big)}{\tanh_\mu(d\chi)}, \\ z_{2,26}(x,t) &= -\frac{\omega}{k}\bigg(-\frac{F_0\big(1 - \tanh_\mu(d\chi)\big)}{\tanh_\mu(d\chi)}\bigg) - \frac{1}{2}\bigg(-\frac{F_0\big(1 - \tanh_\mu(d\chi)\big)}{\tanh_\mu(d\chi)}\bigg)^2, \end{aligned} \tag{77}$$

$$\begin{aligned} w_{2,27}(x,t) &= -\frac{F_0\big(1 - \coth_\mu(d\chi)\big)}{\coth_\mu(d\chi)}, \\ z_{2,27}(x,t) &= -\frac{\omega}{k}\bigg(-\frac{F_0\big(1 - \coth_\mu(d\chi)\big)}{\coth_\mu(d\chi)}\bigg) - \frac{1}{2}\bigg(-\frac{F_0\big(1 - \coth_\mu(d\chi)\big)}{\coth_\mu(d\chi)}\bigg)^2, \end{aligned} \tag{78}$$

$$
\begin{aligned}
w_{2,28}(x,t) &= \frac{F_0\left(-\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\right)}{\sinh_\mu(2\,d\chi)+i}, \\
z_{2,28}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0\left(-\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\right)}{\sinh_\mu(2\,d\chi)+i}\right) \\
&\quad -\frac{1}{2}\left(\frac{F_0\left(-\cosh_\mu(2\,d\chi)+\sinh_\mu(2\,d\chi)+i\right)}{\sinh_\mu(2\,d\chi)+i}\right)^2,
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
w_{2,29}(x,t) &= \frac{F_0\left(-\sinh_\mu(2\,d\chi)+\cosh_\mu(2\,d\chi)+1\right)}{\cosh_\mu(2\,d\chi)+1}, \\
z_{2,29}(x,t) &= -\frac{\omega}{k}\left(\frac{F_0\left(-\sinh_\mu(2\,d\chi)+\cosh_\mu(2\,d\chi)+1\right)}{\cosh_\mu(2\,d\chi)+1}\right) \\
&\quad -\frac{1}{2}\left(\frac{F_0\left(-\sinh_\mu(2\,d\chi)+\cosh_\mu(2\,d\chi)+1\right)}{\cosh_\mu(2\,d\chi)+1}\right)^2,
\end{aligned} \tag{80}
$$

and

$$
\begin{aligned}
w_{2,30}(x,t) &= -\frac{F_0\left(2\,\cosh_\mu\left(\frac{1}{2}d\chi\right)\sinh_\mu\left(\frac{1}{2}d\chi\right)-2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2+1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2-1}, \\
z_{2,30}(x,t) &= -\frac{\omega}{k}\left(-\frac{F_0\left(2\,\cosh_\mu\left(\frac{1}{2}d\chi\right)\sinh_\mu\left(\frac{1}{2}d\chi\right)-2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2+1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2-1}\right) \\
&\quad -\frac{1}{2}\left(-\frac{F_0\left(2\,\cosh_\mu\left(\frac{1}{2}d\chi\right)\sinh_\mu\left(\frac{1}{2}d\chi\right)-2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2+1\right)}{2\left(\cosh_\mu\left(\frac{1}{2}d\chi\right)\right)^2-1}\right)^2.
\end{aligned} \tag{81}
$$

Family. 2.7: When $\psi = 0$,

$$
\begin{aligned}
w_{2,31}(x,t) &= -\frac{1}{4}\frac{F_0\left(e^3\chi\,\ln(\mu)-4\,fd(e\chi\,\ln(\mu)+2)\right)}{fd(e\chi\,\ln(\mu)+2)}, \\
z_{2,31}(x,t) &= -\frac{\omega}{k}\left(-\frac{1}{4}\frac{F_0\left(e^3\chi\,\ln(\mu)-4\,fd(e\chi\,\ln(\mu)+2)\right)}{fd(e\chi\,\ln(\mu)+2)}\right) \\
&\quad -\frac{1}{2}\left(-\frac{1}{4}\frac{F_0\left(e^3\chi\,\ln(\mu)-4\,fd(e\chi\,\ln(\mu)+2)\right)}{fd(e\chi\,\ln(\mu)+2)}\right)^2.
\end{aligned} \tag{82}
$$

where $\chi = \dfrac{\left(\frac{-F_0{}^2\left(-4\,def+2\,fd\sqrt{\psi}+e^3-e^2\sqrt{\psi}\right)}{8d^2f^2\ln(\mu)}\right)t^\alpha}{\alpha} + \dfrac{\left(\frac{(-e+\sqrt{\psi})F_0}{4fd\ln(\mu)}\right)x^\beta}{\beta}$.

4. Discussion and Graphs

This section examines the ensemble of solitary wave solutions derived from our research of the FCBWBKEs. Using the unique mEDAM technique, we deduce these solitary wave solutions, which allows us to obtain a complete grasp of the intricate dynamics of the FCBWBKEs. Visual representations clearly demonstrate the spectrum of solitary wave behavior, including kinks, shock waves, periodic waves, and bell-shaped kink waves.

Figure 1, the (a) three dimensional, (b) contour (c) two dimensional (when t = 0) and representations of the kink solitary wave solution $w_{1,16}$ stated in Equation (33) are shown with the parameters' values $d = -5, e = 0, f = 15, k = 10, \alpha = 1, \beta = 1, \mu =$ e. Figure 2, the (a) 3D representation of kink wave solution $z_{1,17}$ stated in Equation (34) is shown with the parameters' values $d = -1, e = 0, f = 9, k = 5, \alpha = 1, \beta = 0.9, \mu =$ e. Simultaneously, the (b) two-dimensional graph is constructed for the scenario where $t = 100$ and the same values of parameters. Figure 3, the (a) 3D representation of $w_{1,27}$ stated in Equation (44) is shown with the parameters' values $d = 5, e = 0, f = -5, k = 11, \alpha = 0.8, \beta = 1, \mu = \mathrm{e}$. Simultaneously, the (b) two-dimensional graph is constructed for the scenario where $t = 0$ and the same values of parameters. Overall, the graph shows shock wave profile. Figure 4, the (a) 2D representations of $w_{1,31}$ and (b) $z_{1,31}$ stated in Equation (48) are shown with the parameters' values $d = 4, e = 4, f = 1, k = 2, \alpha = 0.6, \mu = 2$. Overall, the figure represents shock wave profiles. Figure 5, the (a) 3D representation of shock solitary wave solution $w_{2,9}$ stated in Equation (60) is shown with the parameters' values $d = 1, e = 4, f = 2, F_0 = 2$, $\alpha = 1, \beta = 1, \mu = \mathrm{e}$. Simultaneously, (b) a two-dimensional graph is constructed for the scenario where $t = 1$ and the same values of parameters. Figure 6, the (a) 3D representations of shock wave solution $w_{2,12}$, (b) the two-dimensional graphs are constructed for the scenario where $t = 1$ and the same values of parameters and (c) periodic bell-shaped solitary wave solution $z_{2,12}$ stated in Equation (63) are shown with the parameters' values $d = -25, e = 0$, $f = -1, F_0 = 5, \alpha = 0.9, \beta = 0.9, \mu = \mathrm{e}$. Simultaneously, (d) two-dimensional graphs are constructed for the scenario where $t = 1$ and the same values of parameters. Figure 7, the (a) 3D representation of kink wave solution $w_{2,20}$ stated in Equation (71) is shown with the parameters' values $d = -95, e = 0, f = 40, F_0 = 200, \alpha = 1, \beta = 1, \mu = \mathrm{e}$. Simultaneously, (b) two-dimensional graph is constructed for the scenario where $t = 3$ and the same values of parameters. Figure 8, the (a) 2D representations of $w_{2,31}$ and (b) $z_{2,31}$ stated in Equation (82) are shown with the parameters' values $d = 1, e = 1, f = 1/4, F_0 = 10, \mu = 5$ and different values of α.

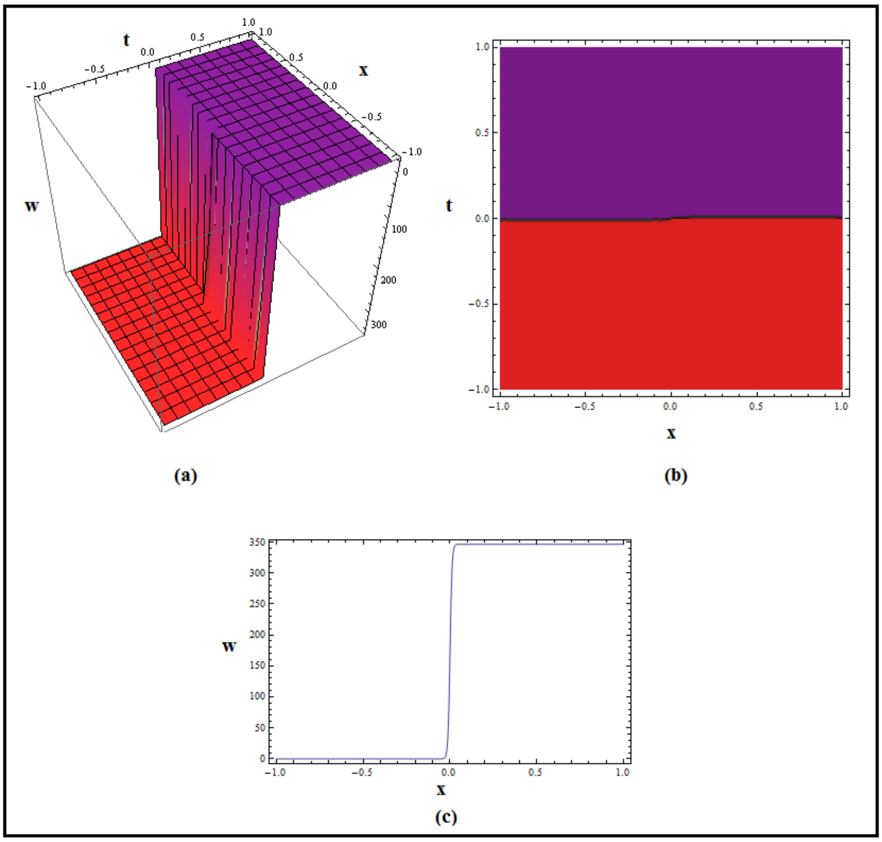

Figure 1. The (**a**) three dimensional, (**b**) contour (**c**) two dimensional (when $t = 0$) and representations of the kink solitary wave solution $w_{1,16}$ stated in Equation (33) are shown with the parameters' values $d = -5, e = 0, f = 15, k = 10, \alpha = 1, \beta = 1, \mu = \mathrm{e}$.

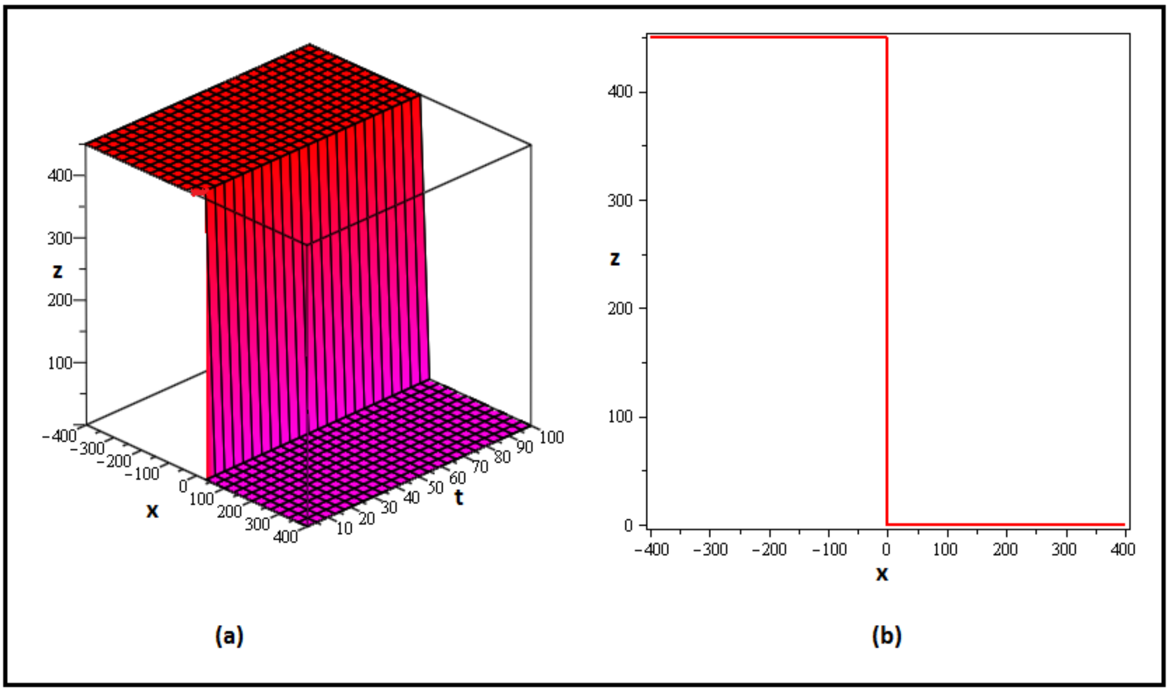

Figure 2. The (**a**) 3D representation of kink wave solution $z_{1,17}$ stated in Equation (34) is shown with the parameters' values $d = -1, e = 0, f = 9, k = 5, \alpha = 1, \beta = 0.9, \mu =$ e. Simultaneously, the (**b**) two-dimensional graph is constructed for the scenario where $t = 100$ and the same values of parameters.

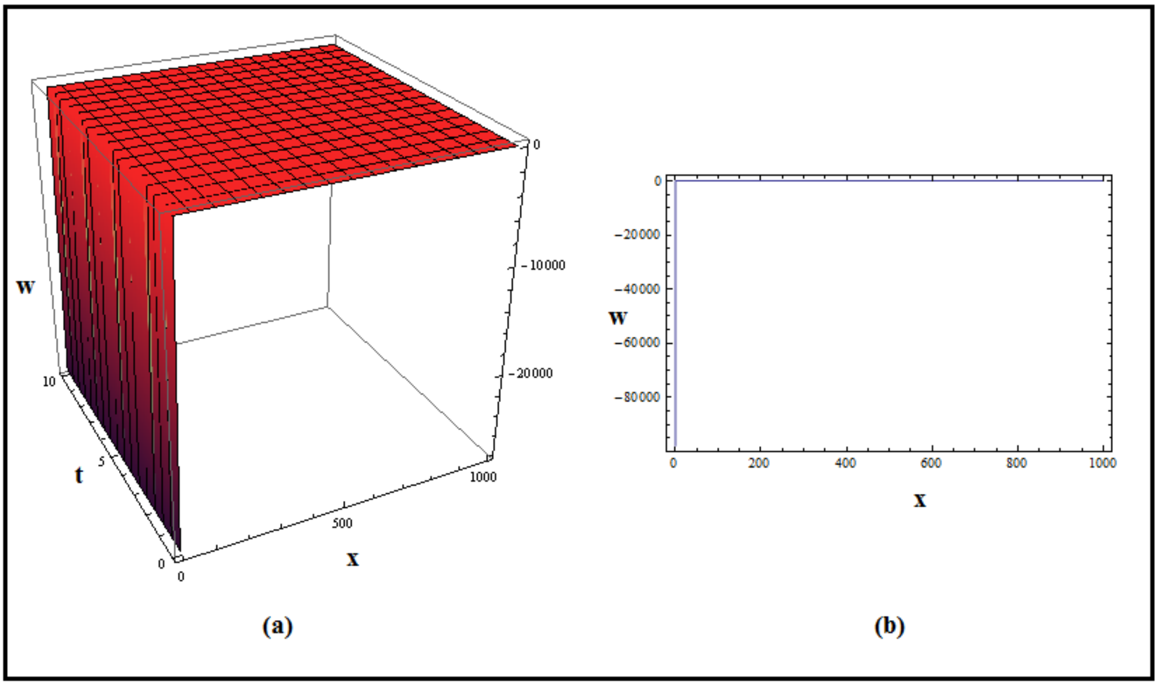

Figure 3. The (**a**) 3D representation of $w_{1,27}$ stated in Equation (44) is shown with the parameters' values $d = 5, e = 0, f = -5, k = 11, \alpha = 0.8, \beta = 1, \mu =$ e. Simultaneously, the (**b**) two-dimensional graph is constructed for the scenario where $t = 0$ and the same values of parameters. Overall, the graph shows shock wave profile.

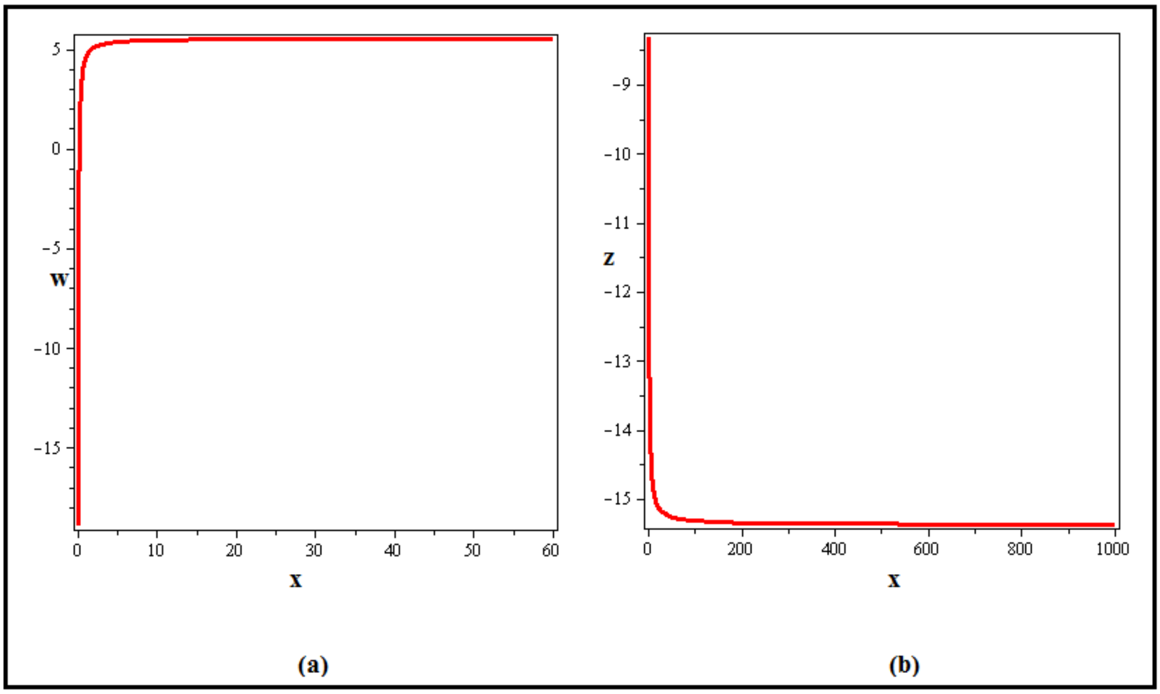

Figure 4. The (**a**) 2D representations of $w_{1,31}$ and (**b**) $z_{1,31}$ stated in Equation (48) are shown with the parameters' values $d = 4, e = 4, f = 1, k = 2, \alpha = 0.6, \mu = 2$. Overall, the figure represents shock wave profiles.

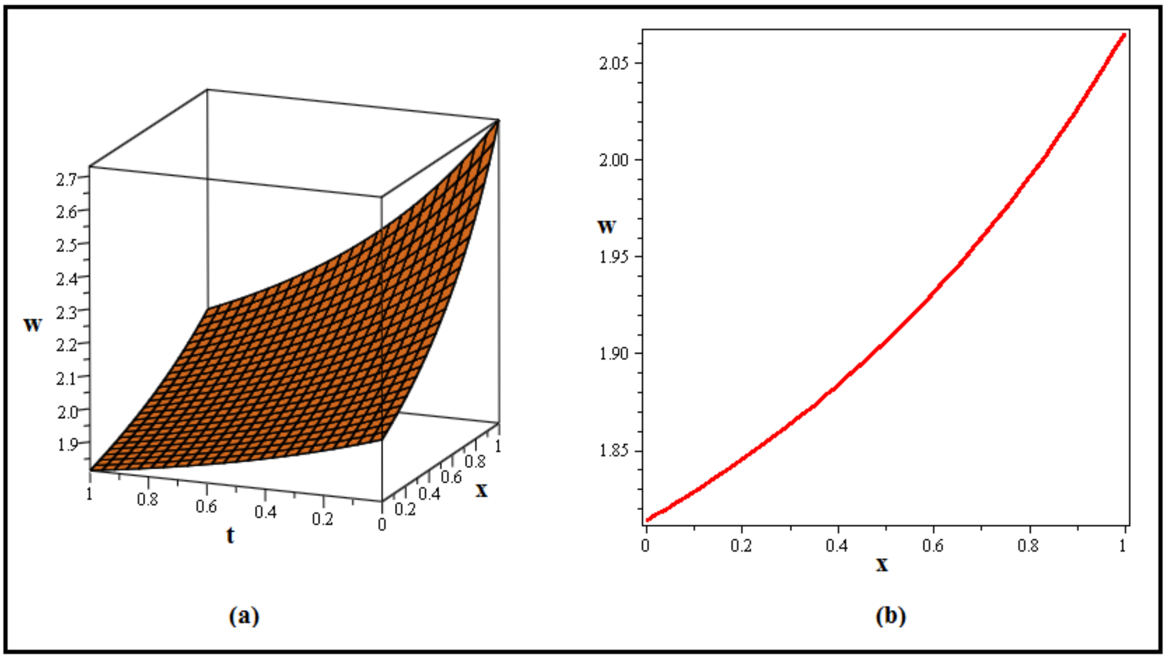

Figure 5. The (**a**) 3D representation of shock solitary wave solution $w_{2,9}$ stated in Equation (60) is shown with the parameters' values $d = 1, e = 4, f = 2, F_0 = 2, \alpha = 1, \beta = 1, \mu = \mathrm{e}$. Simultaneously, (**b**) a two-dimensional graph is constructed for the scenario where $t = 1$ and the same values of parameters.

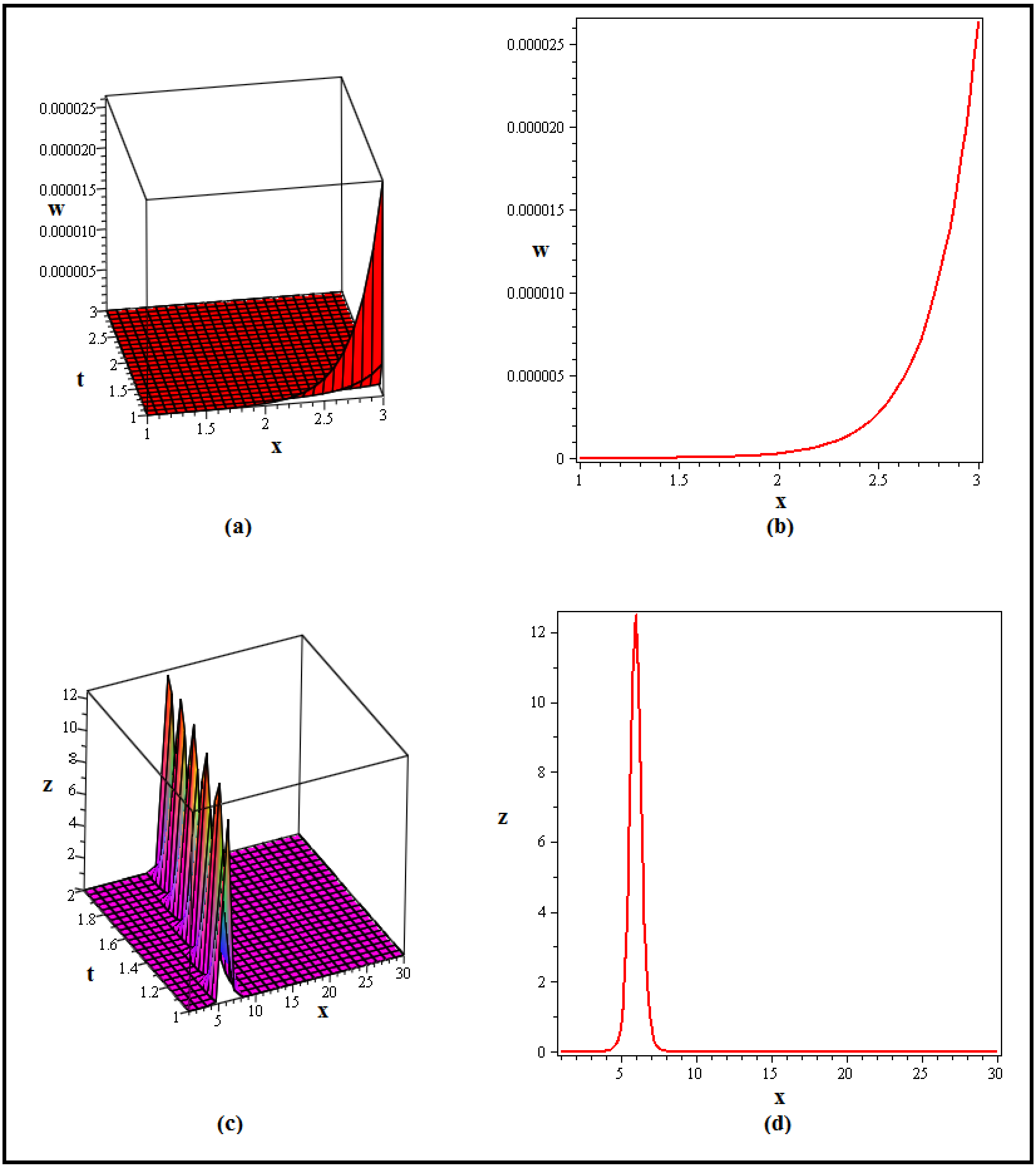

Figure 6. The (**a**) 3D representations of shock wave solution $w_{2,12}$, (**b**) the two-dimensional graphs are constructed for the scenario where $t = 1$ and the same values of parameters and (**c**) periodic bell-shaped solitary wave solution $z_{2,12}$ stated in Equation (63) are shown with the parameters' values $d = -25, e = 0,$ $f = -1, F_0 = 5, \alpha = 0.9, \beta = 0.9, \mu = \mathrm{e}$. Simultaneously, (**d**) two-dimensional graphs are constructed for the scenario where $t = 1$ and the same values of parameters.

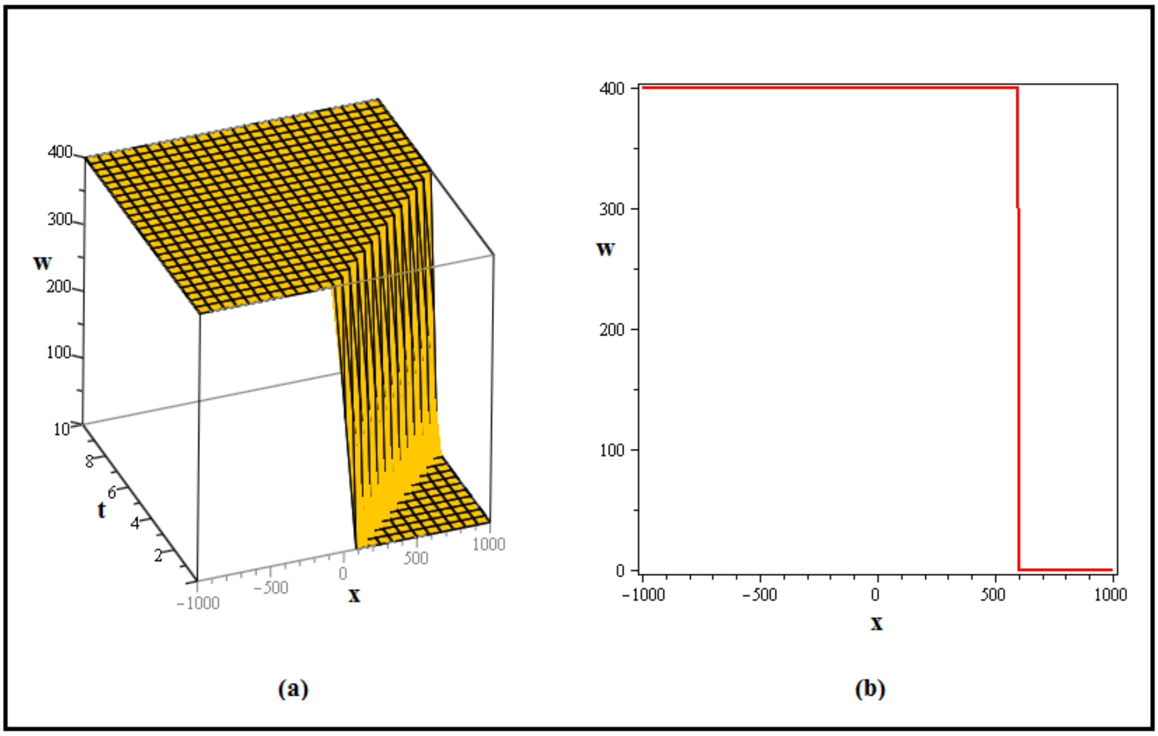

Figure 7. The (**a**) 3D representation of kink wave solution $w_{2,20}$ stated in Equation (71) is shown with the parameters' values $d = -95, e = 0, f = 40, F_0 = 200, \alpha = 1, \beta = 1, \mu =$ e. Simultaneously, (**b**) two-dimensional graph is constructed for the scenario where $t = 3$ and the same values of parameters.

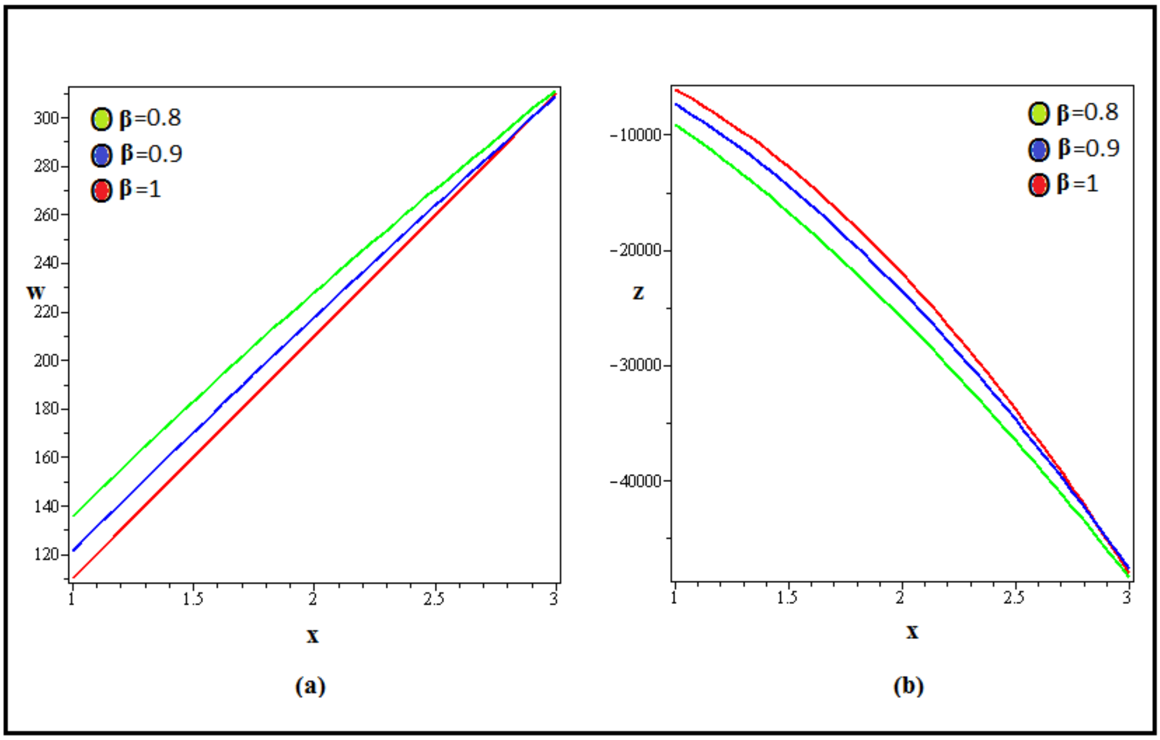

Figure 8. The (**a**) 2D representations of $w_{2,31}$ and (**b**) $z_{2,31}$ stated in Equation (82) are shown with the parameters' values $d = 1, e = 1, f = 1/4, F_0 = 10, \mu = 5$ and different values of α.

A solitary wave is a single, sustainable wave disturbance that keeps its shape as it moves through a medium, resulting from a delicate balance of nonlinearity and dispersion. It can take many forms, including solitons, kink, shock, and periodic. Different types of solitary waves have distinct properties. Kinks, which are characterized by localized amplitude or phase deviations, represent a change in the wave's behavior. Shock solitary waves, also known as shock waves, are abrupt changes in wave amplitude caused by nonlinear interactions. Rogue solitary waves are uncommon, extreme events with unusually large amplitudes that frequently appear unexpectedly in nonlinear systems. Periodic solitary waves are stable, repetitive waves with regularly spaced crests and troughs that exhibit periodicity in their oscillatory behavior. Each type of solitary wave reflects a distinct set of nonlinear dynamics in a given medium.

In the context of shallow water waves, multiple fascinating phenomena appear. Singular kinks are sudden variations in wave amplitude caused by rapid changes in water depth or other contributing variables. Kinks, on the other hand, appear as smooth, localized aberrations within wave patterns, and their presence is frequently ascribed to the complex interplay of bathymetry and wave interactions. Periodic waves, caused by prolonged winds or other steady driving factors, beautify shallow seas with their regular and repeating patterns. Shock waves, on the other hand, appear as abrupt and steep wave fronts in shallow water habitats, caused by occurrences such as undersea landslides, violent currents, or quick changes in water depth. Bell-shaped waves arise when waves converge or interference patterns emerge in shallow water and have centre peaks with slow amplitude reductions.

Some 2D, 3D, and contour plots are used to graphically depict the interdependence of distinct solitary wave kinds, propagation patterns, and interactions. This graphical assessment highlights the significance of our findings and justifies the mEDAM technique for untangling complicated nonlinear systems. Finally, this graphical depiction demonstrates the mEDAM approach's groundbreaking contributions to resolving complex nonlinear events while also expanding our understanding of solitonic behavior in the domain of FCBWBKEs. The suggested mEDAM's effectiveness stems from its ability to generate a wide range of solitary wave solutions, including hyperbolic, periodic, exponential, and rational functional solutions. These solutions provide a more in-depth understanding of the specified model's inherent behavior in shallow water. Furthermore, by balancing linear and nonlinear terms, the proposed method converts the NFPDE into a solvable system of nonlinear algebraic equations without the use of linearization or other processes. Furthermore, imposing certain constraint conditions on the mEDAM results for NFPDEs allows for the derivation of solutions obtained via alternative methods such as the (G′/G)-expansion method and the tan-function method. This observation suggests that the proposed approach functions as a generalization of the aforementioned methods, with the latter regarded as specific instances of the former.

5. Conclusions

In conclusion, this study successfully developed and discovered solitary wave solutions within the context of the FCBWBKE, a shallow water wave model. The mEDAM was used to convert a complex NFPDE into a more manageable NODE, overcoming the difficulties associated with fractional calculus definitions. In addition, the use of visual representations such as 2D, 3D, and contour graphs has improved our understanding of solitary wave dynamics in shallow water wave situations and provided a mechanism for effectively expressing difficult mathematical ideas. Furthermore, the study of various wave phenomena such as kinks, shock waves, periodic waves, and bell-shaped kink waves demonstrates the breadth of knowledge gained in the study of shallow water wave behavior. This research contributes a novel methodology and significant and consistent discoveries, and it ultimately advances our comprehension of wave physics and mathematical procedures, making a major contribution to the field. Moreover, since the suggested approach demonstrated its effectiveness in producing multiple families of solitary wave solutions for FCBWBKEs, we intend to extend its application to other complex NFPDEs in the future. We seek to investigate its effectiveness in dealing with NFPDEs with variable coefficients in particular, marking a new frontier in mEDAM research. This extension has the potential to usher in a new era of research in the field of mEDAM methodologies.

Author Contributions: Conceptualization, M.M.A.-S.; Methodology, M.M.A.-S.; Software, S.M.; Validation, S.M.; Formal analysis, R.S.; Investigation, R.S. and A.H.G.; Resources, A.H.G. and K.M.; Writing—review & editing, K.M; Funding, S.M. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. 5258]

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Acknowledgments: The authors would to thank the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia, Grant No. 5258 for supporting.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Singh, J.; Kumar, D.; Kılıçman, A. Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations. *Abstr. Appl. Anal.* **2014**, *2014*, 535793. [CrossRef]
2. Kumar, S. A new fractional modeling arising in engineering sciences and its analytical approximate solution. *Alex. Eng. J.* **2013**, *52*, 813–819. [CrossRef]
3. Dubey, V.P.; Kumar, R.; Kumar, D.; Khan, I.; Singh, J. An efficient computational scheme for nonlinear time fractional systems of partial differential equations arising in physical sciences. *Adv. Differ. Equ.* **2020**, *2020*, 46. [CrossRef]
4. Fareed, A.F.; Elsisy, M.A.; Semary, M.S.; Elbarawy, M.T. Controlled Picard's Transform Technique for Solving a Type of Time Fractional Navier–Stokes Equation Resulting from Incompressible Fluid Flow. *Int. J. Appl. Comput. Math.* **2022**, *8*, 184. [CrossRef]
5. Murphy, M.L.; Frodl, T. Meta-analysis of diffusion tensor imaging studies shows altered fractional anisotropy occurring in distinct brain areas in association with depression. *Biol. Mood Anxiety Disord.* **2011**, *1*, 3. [CrossRef]
6. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Phys. A Stat. Mech. Its Appl.* **2000**, *284*, 376–384. [CrossRef]
7. Wang, Q.; Ma, J.; Yu, S.; Tan, L. Noise detection and image denoising based on fractional calculus. *Chaos Solitons Fractals* **2020**, *131*, 109463. [CrossRef]
8. Burrage, K.; Hale, N.; Kay, D. An efficient implicit FEM scheme for fractional-in-space reaction-diffusion equations. *SIAM J. Sci. Comput.* **2012**, *34*, A2145–A2172. [CrossRef]
9. Javidi, M.; Ahmad, B. Numerical solution of fractional partial differential equations by numerical Laplace inversion technique. *Adv. Differ. Equ.* **2013**, *2013*, 375. [CrossRef]
10. Fu, H.; Ng, M.K.; Wang, H. A divide-and-conquer fast finite difference method for space–time fractional partial differential equation. *Comput. Math. Appl.* **2017**, *73*, 1233–1242. [CrossRef]
11. Baeumer, B.; Kovács, M.; Meerschaert, M.M. Numerical solutions for fractional reaction–diffusion equations. *Comput. Math. Appl.* **2008**, *55*, 2212–2226. [CrossRef]
12. Chen, Y.; Sun, Y.; Liu, L. Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions. *Appl. Math. Comput.* **2014**, *244*, 847–858. [CrossRef]
13. Marom, O.; Momoniat, E. A comparison of numerical solutions of fractional diffusion models in finance. *Nonlinear Anal. Real World Appl.* **2009**, *10*, 3435–3442. [CrossRef]
14. Mahor, T.C.; Mishra, R.; Jain, R. Analytical solutions of linear fractional partial differential equations using fractional Fourier transform. *J. Comput. Appl. Math.* **2021**, *385*, 113202. [CrossRef]
15. Ganji, Z.Z.; Ganji, D.D.; Ganji, A.D.; Rostamian, M. Analytical solution of time-fractional Navier–Stokes equation in polar coordinate by homotopy perturbation method. *Numer. Methods Partial. Differ. Equ. Int. J.* **2010**, *26*, 117–124. [CrossRef]
16. Shah, K.; Khalil, H.; Khan, R.A. Analytical solutions of fractional order diffusion equations by natural transform method. *Iran. J. Sci. Technol. Trans. A Sci.* **2018**, *42*, 1479–1490. [CrossRef]
17. Manafian, J.; Foroutan, M. Application of $\tan(\phi(\xi)/2)tan(\phi(\xi)/2)$-expansion method for the time-fractional Kuramoto–Sivashinsky equation. *Opt. Quantum Electron.* **2017**, *49*, 272. [CrossRef]
18. Alsharidi, A.K.; Bekir, A. Discovery of New Exact Wave Solutions to the M-Fractional Complex Three Coupled Maccari's System by Sardar Sub-Equation Scheme. *Symmetry* **2023**, *15*, 1567. [CrossRef]
19. Zheng, B.; Wen, C. Exact solutions for fractional partial differential equations by a new fractional sub-equation method. *Adv. Differ. Equ.* **2013**, *2013*, 199. [CrossRef]
20. Gaber, A.; Ahmad, H. Solitary wave solutions for space-time fractional coupled integrable dispersionless system via generalized kudryashov method. *Facta Univ. Ser. Math. Inform.* **2021**, *35*, 1439–1449. [CrossRef]
21. Bibi, S.; Mohyud-Din, S.T.; Khan, U.; Ahmed, N. Khater method for nonlinear Sharma Tasso-Olever (STO) equation of fractional order. *Results Phys.* **2017**, *7*, 4440–4450. [CrossRef]

22. Zheng, B. Exp-function method for solving fractional partial differential equations. *Sci. World J.* **2013**, *2013*, 465723. [CrossRef] [PubMed]
23. Yasmin, H.; Aljahdaly, N.H.; Saeed, A.M.; Shah, R. Investigating Families of Soliton Solutions for the Complex Structured Coupled Fractional Biswas–Arshed Model in Birefringent Fibers Using a Novel Analytical Technique. *Fractal Fract.* **2023**, *7*, 491. [CrossRef]
24. Yasmin, H.; Aljahdaly, N.H.; Saeed, A.M.; Shah, R. Probing Families of Optical Soliton Solutions in Fractional Perturbed Radhakrishnan–Kundu–Lakshmanan Model with Improved Versions of Extended Direct Algebraic Method. *Fractal Fract.* **2023**, *7*, 512. [CrossRef]
25. Yasmin, H.; Aljahdaly, N.H.; Saeed, A.M.; Shah, R. Investigating Symmetric Soliton Solutions for the Fractional Coupled Konno–Onno System Using Improved Versions of a Novel Analytical Technique. *Mathematics* **2023**, *11*, 2686. [CrossRef]
26. Mirhosseini-Alizamini, S.M.; Rezazadeh, H.; Eslami, M.; Mirzazadeh, M.; Korkmaz, A. New extended direct algebraic method for the Tzitzica type evolution equations arising in nonlinear optics. *Comput. Methods Differ. Equ.* **2020**, *8*, 28–53.
27. Boussinesq, J. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.* **1872**, *17*, 55–108.
28. Whitham, G.Á. A general approach to linear and non-linear dispersive waves using a Lagrangian. *J. Fluid Mech.* **1965**, *22*, 273–283. [CrossRef]
29. Whitham, G.B. Variational methods and applications to water waves. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1967**, *299*, 6–25.
30. Atilgan, E.; Senol, M.; Kurt, A.; Tasbozan, O. New wave solutions of time-fractional coupled Boussinesq–Whitham–Broer–Kaup equation as a model of water waves. *China Ocean. Eng.* **2019**, *33*, 477-483. [CrossRef]
31. Choi, J.H.; Kim, H. Coupled Fractional Traveling Wave Solutions of the Extended Boussinesq–Whitham–Broer–Kaup-Type Equations with Variable Coefficients and Fractional Order. *Symmetry* **2021**, *13*, 1396. [CrossRef]
32. Yu, J. Lie symmetry analysis, power series solutions and conservation laws of time fractional coupled Boussinesq-Whitham-Broer-Kaup equations. *Authorea Prepr.* **2022**. [CrossRef]
33. Tarasov, V.E. On chain rule for fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *30*, 1–4. [CrossRef]
34. He, J.H.; Elagan, S.K.; Li, Z.B. Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. *Phys. Lett. A* **2012**, *376*, 257–259. [CrossRef]
35. Sarikaya, M.Z.; Budak, H.; Usta, H. On generalized the conformable fractional calculus. *TWMS J. Appl. Eng. Math.* **2019**, *9*, 792–799.