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Some Results on Fractional Boundary Value Problem for Caputo-Hadamard Fractional Impulsive Integro Differential Equations

Ymnah Alruwaily [1,*], Kuppusamy Venkatachalam [2] and El-sayed El-hady [1,*]

[1] Mathematics Department, College of Science, Jouf University, Sakaka, P.O. Box 2014, Saudi Arabia
[2] Department of Mathematics, Nandha Engineering College (Autonomous), Erode 638052, India; venkatachalam.k@nandhaengg.org
* Correspondence: ymnah@ju.edu.sa (Y.A.); eaelhady@ju.edu.sa (E.-s.E.-h.)

Abstract: The results for a new modeling integral boundary value problem (IBVP) using Caputo-Hadamard impulsive fractional integro-differential equations (C-HIFI-DE) with Banach space are investigated, along with the existence and uniqueness of solutions. The Krasnoselskii fixed-point theorem (KFPT) and the Banach contraction principle (BCP) serve as the basis of this unique strategy, and are used to achieve the desired results. We develop the illustrated examples at the end of the paper to support the validity of the theoretical statements.

Keywords: impulsive; integro differential equations; Caputo-Hadamard fractional derivative; boundary value problems

MSC: 26A33; 34A09; 34A12; 47H10

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1. Introduction

In engineering, physics, chemistry, control theory, signal, image processing, and biology, the study of fractional differential equations (FDE) (see e.g., [1–4]), which is connected to fractional calculus (FC), is significant. The integer-order derivatives are less helpful and useful for characterizing the memory and heredity characteristics of various materials and processes than the fractional derivatives and integrals of arbitrary order; see [5–12].

The investigation of IBVP has advanced in the past few decades. It has also been extremely useful to develop a variety of applied mathematical models of actual processes in applied sciences and engineering. Tian and Bai in [12] stated a few existing findings from IBVP involving fractional derivatives of the Caputo type. Using the fixed-point theorem (FPT), existence and uniqueness results (E-UR) have been developed. Recently, it has been noted that many of the materials on the subject focus on FDEs of the Caputo and Riemann-Liouville types with various situations, including time delays, impulses, and boundary value conditions (BVC) [5,10,13–25].

Along with the Riemann-Liouville and Caputo derivatives, another kind of FD that is mentioned in the literature is the Hadamard FD, which first appeared in 1892; see e.g., [26]. It differs from the previous ones in that it includes an arbitrary logarithm function; see [13–15] for additional details.

The fundamental fractional calculus theorem was subsequently included in the C-H in [16], wher they also suggested a Caputo-type version of the Hadamard FD. Impulsive differential equations with Hadamard and C-H derivatives have been the focus of recent studies (see [11,17–20] and the references therein).

The authors of [20] discussed the following form of the C-H FDE with the impulsive boundary condition:

$$\begin{aligned}
&{}^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\alpha}_{\tau_{\mathscr{M}}}\mathscr{X}(\tau) = \mathscr{F}(\tau, \mathscr{X}(\tau)),\ \tau, \tau_{\mathscr{M}} \in [1, \mathscr{E}],\ \tau \neq \tau_{\mathscr{M}},\\
&\Delta\mathscr{X}(\tau_{\mathscr{M}}) = \mu_{\mathscr{M}}(\mathscr{X}(\tau_{\mathscr{M}})), \mathscr{K} = 1, 2, ..., \mathscr{M},\\
&\Delta\delta\mathscr{X}(\tau_{\mathscr{M}}) = \mu^{-}_{\mathscr{M}}(\mathscr{X}(\tau_{\mathscr{M}})), \mathscr{K} = 1, 2, ..., \mathscr{M},\\
&\mathscr{X}(1) = \mathscr{H}(\mathscr{X}),\ \mathscr{X}(\mathscr{E}) = \mathscr{G}(\mathscr{X}),
\end{aligned}$$

In [24], the authors investigated the following FDEs

$$\begin{aligned}
&{}^{\mathscr{C}}\mathscr{D}^{\alpha}_{\mathtt{a}^+}\varpi(\tau) = \mathscr{F}(\tau, \varpi(\tau), \mathscr{I}^{\alpha}_{\mathtt{a}^+}\varpi(\tau)),\ \ \tau \in [\mathtt{a}, \mathtt{b}]\\
&\varpi(\mathtt{b}) = \varpi_{\mathtt{b}}.
\end{aligned}$$

The E-UR of the solution of some fractional integro differential equations involving non-instantaneous impulsive boundary conditions have been studied in [27] by some FPTs. See also [28], where the sequential Caputo-Hadamard FDE with fractional boundary conditions have been examined using FPTs.

In [29], W. Yukunthorn et.al. studied the H-FDEs for impulsive multi-order form,

$$\begin{aligned}
&{}^{\mathscr{C}}\mathscr{D}^{\mathscr{Q}_{\mathscr{K}}}_{\tau_{\mathscr{K}}}\varpi(\tau) = \mathscr{F}(\tau, \varpi(\tau)),\ \ \tau \in \mathscr{J}_{\mathscr{K}} \subset [\tau_0, \mathscr{T}],\ \tau \neq \tau_{\mathscr{K}},\\
&\Delta_1\varpi(\tau_{\mathscr{K}}) = \phi 1_{\mathscr{K}}(\varpi(\tau_{\mathscr{K}})), \mathscr{K} = 1, 2, ..., M,\\
&\alpha_1\varpi(\tau_0) + \beta_1\varpi(\mathscr{T}) = \sum_{\mathtt{i}=0}^{\mathscr{M}}\gamma 1_{\mathtt{i}}\,\mathscr{I}^{\mathscr{R}_{\mathtt{i}}}_{\tau_{\mathtt{i}}}\varpi(\tau_{\mathtt{i+1}}),
\end{aligned}$$

In [30], W. Benhamida et.al. discussed the BVP,

$$\begin{aligned}
&{}^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\alpha_1}\varpi(\tau) = \mathscr{F}(\tau, \varpi(\tau)),\ \tau \in [1, \mathscr{T}],\ \ 0 < \alpha_1 \leq 1,\\
&\mathscr{A}\mathscr{Y}(1) + \mathscr{B}\mathscr{Y}(\mathscr{T}) = \mathscr{C},
\end{aligned}$$

The literature described above served as our inspiration as we considered a C-HIF I-DE which involves fractional BCs:

$$
{}^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\mathscr{P}}\varpi(\tau) = \mathscr{F}(\tau, \varpi(\tau), \mathscr{B}\varpi(\tau)),\ \tau \in \mathscr{J} : [1, \mathscr{T}], 1 < \mathscr{P} \leq 2 \tag{1}
$$

$$
\varpi(\tau^{+}_{\mathscr{K}}) = \varpi(\tau^{-}_{\mathscr{K}}) + \mathscr{Y}_{\mathscr{K}},\ \mathscr{Y}_{\mathscr{K}} \in \mathbb{R},\ \mathscr{K} = 1, 2, ..., \mathscr{M}, \tag{2}
$$

$$
\varpi(1) = 0, \alpha\,{}_{\mathscr{H}}\mathscr{I}^{\mathscr{Q}}\varpi(\eta) + \beta^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\mathscr{R}}\varpi(\mathscr{T}) = \lambda,\ \ \mathscr{Q}, \gamma \in (0, 1], \tag{3}
$$

where ${}^{\mathscr{C}\mathscr{H}}\mathscr{D}(.)$ is the C-H FD. ${}_{\mathscr{H}}\mathscr{I}^{\mathscr{Q}}$ is the standard Hadamard fractional integral. $\mathscr{F} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. α, β, λ are real constants and $\eta \in (1, \mathscr{T})$, where $\mathscr{B}\varpi(\tau) = \int_0^{\tau}\mathscr{K}(\tau, \mathtt{s}, \varpi(\mathtt{s}))\mathtt{ds}$, $\mathscr{K} : \Delta \times [1, \mathscr{T}] \to \mathbb{R}, \Delta = \{(\tau, \mathtt{s}) : 1 \leq \mathtt{s} \leq \tau \leq \mathscr{T}\}, 1 = \tau_0 < \tau_1 < \tau_2 ... < \tau_{\mathtt{m}} = \mathscr{T}$, $\Delta\varpi = \varpi(\tau^{+}_{\mathscr{K}}) - \varpi(\tau^{-}_{\mathscr{K}})$, $\varpi(\tau^{+}_{\mathscr{K}}) = \lim_{\mathscr{H}\to 0^+}\varpi(\tau_{\mathscr{K}} + \mathscr{H})$ and $\varpi(\tau^{-}_{\mathscr{K}}) = \lim_{\mathscr{H}\to 0^-}\varpi(\tau_{\mathscr{K}} + \mathscr{H})$ represent the right and left limits of $\varpi(\tau)$ at $\tau = \tau_{\mathscr{K}}$.

Motivations:

1. This study uses the C-HFD to develop a new class of impulsive C-HIFI-DE with BCs.
2. We additionally verify the E-UR of the solutions to Equations (1)–(3) using BCP and KFPT, respectively.
3. We extend the C-HFD, nonlinear integral terms, and impulsive conditions to the results discussed in [25].

The rest of the paper is organized as follows. Section 2 discusses the basic concepts and lemmas that will be used to support findings. In Section 3, we prove the uniqueness of solutions (1)–(3) and the existence of the system under suitable assumptions. Applications are also presented in Section 4.

2. Supporting Notes

Let the space $\mathscr{PC}(\mathscr{J},\mathbb{R})=\{\varpi:\mathscr{J}\to\mathbb{R}:\varpi\in\mathscr{PC}(\tau_{\mathscr{K}},\tau_{\mathscr{K}+1}],\mathbb{R}\},\mathscr{K}=1,2,...,\mathscr{M}$ and there $\varpi(\tau^-_{\mathscr{K}})$ and $\varpi(\tau^+_{\mathscr{K}})$ exist with $\varpi(\tau^-_{\mathscr{K}})=\varpi(\tau^+_{\mathscr{K}})$ and endowed with the norm

$$\|\varpi\|_{\mathscr{PC}}=\sup\{|\varpi(\tau)|:0\le\tau\le1\}.$$

Definition 1. *Given a continuous function $\mathscr{G}:[1,+\infty)\to\mathbb{R}$, its Hadamard fractional integral of order $\alpha_1>0$, is as follows:*

$$_{\mathscr{H}}\mathscr{I}^{\alpha_1}\mathscr{G}(\tau)=(\Gamma(\alpha_1))^{-1}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{\alpha_1-1}\mathscr{G}(s)\frac{ds}{s}$$

where Γ is the Euler gamma function and $\log(.)=\log_e(.)$

Definition 2. *Given a function $\mathscr{G}\in\mathscr{PC}([a,b],\mathbb{R})$the C-HFD of order α_1 is follows:*

$$^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\alpha_1}_1\mathscr{G}(\tau)=\frac{1}{\Gamma(\pi-\alpha_1)}\left(\tau\frac{d}{d\tau}\right)^\pi\int_a^\tau\left(\log\frac{\tau}{s}\right)^{\pi-\alpha_1-1}\mathscr{G}(s)\frac{ds}{s},\ \pi-1<\alpha<\pi,$$

where $\delta^\pi=\left(\tau\frac{d}{d\tau}\right)^\pi$, $\pi=[\alpha_1]+1$ and $[\alpha_1]$ is the integer part of α_1.

Lemma 1. *Let $\mathscr{G}\in\mathscr{PC}^\pi_\delta[a,b]$ (or)$\mathscr{G}\in\mathscr{PC}^\pi_\delta[a,b]$ and $\alpha_1\in\mathbb{R}$. Then*

$$_{\mathscr{H}}\mathscr{I}^{\alpha_1}_a(^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\alpha_1}_a\mathscr{G})(\tau)=\mathscr{G}(\tau)-\sum_{\mathscr{K}=0}^{\pi-1}\frac{\delta(\mathscr{K})\mathscr{G}(a)}{\mathscr{K}!}(\log\frac{\tau}{a})^{\mathscr{K}}.$$

Proof. Let $\alpha_1>0,\beta_1>0,\pi=[\alpha_1]+1$ and $a>0$, then

$$_{\mathscr{H}}\mathscr{I}^{\alpha_1}_{a^+}(\log\frac{\varpi}{a})^{\beta_1-1}(\varpi)=\frac{\Gamma(\beta_1)}{\Gamma(\beta_1+\alpha_1)}(\log\frac{\varpi}{a})^{\beta_1+\alpha_1-1},$$
$$^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\alpha_1}_{a^+}(\log\frac{\varpi}{a})^{\beta_1-1}(\varpi)=\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)}(\log\frac{\varpi}{a})^{\beta_1-\alpha_1-1},\ \beta_1>\alpha_1.$$

□

Lemma 2. *The function ϖ is a solution of the BVP*

$$^{\mathscr{CH}}\mathscr{D}^{\mathscr{P}}\varpi(\tau)=\mathscr{H}(\tau),\quad \mathscr{J}:[1,\mathscr{T}],1<\mathscr{P}\le2 \tag{4}$$
$$\varpi(\tau^+_{\mathscr{K}})=\varpi(\tau^-_{\mathscr{K}})+\mathscr{Y}_{\mathscr{K}},\ \mathscr{Y}_{\mathscr{K}}\in\mathbb{R},\ \mathscr{K}=1,2,...,M,, \tag{5}$$
$$\varpi(1)=0,\alpha\,_{\mathscr{H}}\mathscr{I}^{\mathscr{Q}}\varpi(\eta)+\beta^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\mathscr{R}}\varpi(\mathscr{T})=\lambda,\ \ \mathscr{Q},\gamma\in(0,1], \tag{6}$$

if and only if

$$\varpi(\tau)=\begin{cases}\frac{1}{\Gamma\mathscr{P}}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\mathscr{P}-1}\mathscr{H}(s)\frac{ds}{s}+\frac{\log\tau}{\Lambda}[\lambda-\alpha(\Gamma(\mathscr{P}+\mathscr{Q}))^{-1}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(s)\frac{ds}{s}]\\ -\frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{s}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(s)\frac{ds}{s},\ \ for\ \tau\in(1,\tau_1]\\ \mathscr{Y}_1+\frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\mathscr{P}-1}\mathscr{H}(s)\frac{ds}{s}+\frac{\log\tau}{\Lambda}[\lambda-\frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(s)\frac{ds}{s}]\\ -\frac{\beta}{\Gamma(\mathscr{R}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{s}\right)^{\mathscr{R}-\gamma-1}\mathscr{H}(s)\frac{ds}{s},\ \ for\ \tau\in(\tau_1,\tau_2)\\ .\\ .\\ .\\ \sum_{\mathscr{K}=1}^{\mathscr{M}}\mathscr{Y}_i+(\Gamma(\mathscr{P}))^{-1}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\mathscr{P}-1}\mathscr{H}(s)\frac{ds}{s}\\ +\log\tau(\Lambda)^{-1}[\lambda-\frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(s)\frac{ds}{s}]\\ -\frac{\beta}{\Gamma(\mathscr{R}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{s}\right)^{\mathscr{R}-\gamma-1}\mathscr{H}(s)\frac{ds}{s},\ \ for\ \tau\in(\tau_{\mathscr{M}},\mathscr{T}]\end{cases}$$

where

$$\Lambda=\frac{\alpha(\log\eta)^{\mathscr{Q}+1}}{\Gamma(\mathscr{Q}+2)}+\frac{\beta(\log\mathscr{T})^{1-\gamma}}{\Gamma(2-\gamma)}$$

Proof. Assume that ϖ satisfies (4)–(6). If $\tau\in[0,\tau_1)$,

$$^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\mathscr{P}}\varpi(\tau)=\mathscr{H}(\tau),\ \ \mathscr{J}:[1,\mathscr{T}],$$
$$\varpi(1)=0,\alpha_{\mathscr{H}}\mathscr{I}^{\mathscr{Q}}\varpi(\eta)+\beta^{\mathscr{C}}_{\mathscr{H}}\mathscr{D}^{\mathscr{R}}\varpi(\mathscr{T})=\lambda,\ \ \mathscr{Q},\gamma\in(0,1].$$

We can obtain

$$\begin{aligned}\varpi(\tau)=&\Gamma(\mathscr{P})^{-1}\int_1^{\tau}\left(\log\frac{\tau}{s}\right)^{\mathscr{P}-1}\mathscr{H}(s)\frac{ds}{s}\\ &+\frac{\log\tau}{\Lambda}[\lambda-\frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(s)\frac{ds}{s}]\\ &-\frac{\beta}{\Gamma(\mathscr{R}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{s}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(s)\frac{ds}{s}.\end{aligned}$$

If $\tau\in(\tau_1,\tau_2)$, then

$$^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\mathscr{P}}\varpi(\tau)=\mathscr{H}(\tau),\ \ \varpi(\tau^+_{\mathscr{K}})=\varpi(\tau^-_{\mathscr{K}})+\mathscr{Y}_{\mathscr{K}},$$

we have

$$
\begin{aligned}
\varpi(\tau) =&\mathscr{Y}(\tau_1^+) - \Gamma(\mathscr{R})^{-1}\int_1^{\tau_1}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}, \\
&= \mathscr{Y}(\tau_1^+) + \mathscr{Y}_1 - \frac{1}{\Gamma(\mathscr{R})}\int_1^{\tau_1}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}, \\
&= \mathscr{Y}_1 - \frac{1}{\Gamma(\mathscr{R})}\int_1^{\tau_1}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}.
\end{aligned}
$$

If $\tau \in (\tau_2, \tau_3)$, then

$$
\begin{aligned}
\varpi(\tau) =& \mathscr{Y}(\tau_2^+) - (\Gamma(\mathscr{R}))^{-1}\int_1^{\tau_2}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}},
\end{aligned}
$$

$$
\begin{aligned}
&= \mathscr{Y}(\tau_2^+) + \mathscr{Y}_2 - \frac{1}{\Gamma(\mathscr{R})}\int_1^{\tau_2}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}, \\
&= \mathscr{Y}_1 + \mathscr{Y}_2 + \frac{1}{\Gamma(\mathscr{R})}\int_1^{\tau_2}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{R}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} + \int_1^{\tau}\left(\log\frac{\tau}{\mathtt{s}}\right)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\left(\log\frac{\eta}{\mathtt{s}}\right)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{P}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}.
\end{aligned}
$$

If $\tau \in [\tau_{\mathscr{M}}, \mathscr{T}]$,

$$\begin{aligned}\varpi(\tau) = \sum_{\mathscr{K}=1}^{\mathscr{M}} \mathscr{Y}_{\mathtt{i}} + \frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau}\Big(\log\frac{\tau}{\mathtt{s}}\Big)^{\mathscr{P}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}} \\ + \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}\Big(\log\frac{\eta}{\mathtt{s}}\Big)^{\mathscr{P}+\mathscr{Q}-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}] \\ - \frac{\beta}{\Gamma(\mathscr{R}-\gamma)}\int_1^{\mathscr{T}}\left(\log\frac{\mathscr{T}}{\mathtt{s}}\right)^{\mathscr{R}-\gamma-1}\mathscr{H}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}.\end{aligned} \tag{7}$$

Suppose that ϖ fulfills the impulsive FBC of (7). □

3. Main Results

The following hypotheses are needed for the main results.

Hypothesis 1. *$\exists$ a constant $\mathscr{L}_1 > 0$, $0 < \mathscr{L}_2 < 1$:*

$$|\mathscr{F}(\tau, \iota, v) - \mathscr{F}(\tau, \iota_1, v_1)| \leq \mathscr{L}_1|\iota - \iota_1| + \mathscr{L}_2|v - v_1|$$

for $\iota, v, \iota_1, v_1 \in \mathbb{R}$ and $\tau \in \mathscr{J}$.

Hypothesis 2. *A constant $\mathscr{G}_1 > 0$ exists:*

$$|\mathscr{K}(\tau, \nu, \mathscr{U}) - \mathscr{K}(\tau, \nu, \mathscr{V})| \leq \mathscr{G}_1|\mathscr{U} - \mathscr{V}|.$$

For $\mathscr{U} - \mathscr{V} \in \mathbb{R}$ and $\tau, \nu \in \mathscr{J}$.

Hypothesis 3. *Let $\mathscr{F} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function and $\exists$ a function $\mu \in \mathscr{PC}([1, \mathscr{T}], \mathbb{R})$:*

$$|\mathscr{F}(\tau, \varpi, \mathscr{Y})| \leq \mu(\tau), \quad \textit{for any } (\tau, \varpi, \mathscr{Y}) \in [1, \mathscr{T}] \times \mathbb{R}.$$

Hypothesis 4. *$\exists$ a constant $\mathscr{M}^* > 0$: $\sum_{i=1}^{m} |\mathscr{Y}_i| \leq \mathscr{M}^*$.*

Theorem 3. *If Hypothesis 1 and 2 are satisfied and if*

$$(\mathscr{L}_1 \models +\mathscr{L}_2\mathscr{G}_1)\left[\frac{(\log\mathscr{T})^{\mathscr{P}}}{\Gamma(\mathscr{P}+1)} + \frac{|\alpha|(\log\mathscr{T})(\log\eta)^{\mathscr{P}+\mathscr{Q}}}{|\Lambda|\Gamma(\mathscr{P}+\mathscr{Q}+1)} + \frac{|\beta|(\log\mathscr{T})^{\mathscr{P}-\gamma+1}}{|\Lambda|(\mathscr{P}-\gamma+1)}\right] < 1 \tag{8}$$

then the problems (1)–(3) have a unique solution on $[1, \mathscr{T}]$.

Proof. Take a look at the following operator $\mathscr{W} : \mathscr{PC}(\mathscr{J}, \mathbb{R}) \to \mathscr{PC}(\mathscr{J}, \mathbb{R})$ defined by

$$\begin{aligned}\mathscr{W}\varpi(\tau) = \sum_{\mathscr{K}=1}^{\mathscr{M}} \mathscr{Y}_{\mathtt{i}} + \frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau}(\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{F}(\mathtt{s}, \varpi(\mathtt{s}), \mathscr{B}(\varpi(\mathtt{s}))))\frac{\mathtt{ds}}{\mathtt{s}} \\ + \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^{\eta}(\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}\mathscr{F}(\mathtt{s}, \varpi(\mathtt{s}), \mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}] \\ - \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}(\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}\mathscr{F}(\mathtt{s}, \varpi(\mathtt{s}), \mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}.\end{aligned}$$

Use the BCP to demonstrate that $\mathscr{W}$ is contraction.
Let $(\varpi, \mathscr{Y}) \in \mathscr{PC}_1^{\vartheta}([1, \mathscr{T}], \mathbb{R})$, we have

$$|(\mathscr{W}\varpi)\tau-(\mathscr{W}\mathscr{Y})(\tau)| \leq \frac{1}{\Gamma(\mathscr{P})}\int_1^\tau (\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\mathscr{F}(\mathtt{s},\mathscr{Y}(\mathtt{s}),\mathscr{B}(\mathscr{Y}(\mathtt{s})))|\,\frac{\mathtt{ds}}{\mathtt{s}}$$
$$+\frac{\log\tau}{\Lambda}[\lambda-\frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^\eta(\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))-\mathscr{F}(\mathtt{s},(\mathtt{s}),\mathscr{B}\mathscr{Y}((\mathtt{s})))|\,\frac{\mathtt{ds}}{\mathtt{s}}]$$
$$-\frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}}(\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}\varpi((\mathtt{s})))-\mathscr{F}(\mathtt{s},\mathscr{Y}(\mathtt{s}),\mathscr{B}\mathscr{Y}((\mathtt{s})))|\,\frac{\mathtt{ds}}{\mathtt{s}},$$
$$\leq\frac{(\log\mathscr{T})^{\mathscr{P}}}{\Gamma(\mathscr{P})+1}(\mathscr{L}_1+\mathscr{L}_2\mathscr{G}_1)|\varpi(\mathtt{s})-\mathscr{Y}(\mathtt{s})|+\frac{|\alpha|(\log\mathscr{T})(\log\eta)^{\mathscr{P}+\mathscr{Q}}}{|\Lambda|\Gamma(\mathscr{P}+\mathscr{Q}+1)}(\mathscr{L}_1+\mathscr{L}_2\mathscr{G}_1)|\varpi(\mathtt{s})-\mathscr{Y}(\mathtt{s})|$$
$$+\frac{|\beta|(\log\mathscr{T})^{\mathscr{P}-\gamma+1}}{|\Lambda|\Gamma(\mathscr{P}-\gamma+1)}|\varpi(\mathtt{s})-\mathscr{Y}(\mathtt{s})\leq(\mathscr{L}_1+\mathscr{L}_2\mathscr{G}_1)|\left[\frac{(\log\mathscr{T})^{\mathscr{P}}}{\Gamma(\mathscr{P}+1)}+\frac{|\alpha|(\log\mathscr{T})(\log\eta)^{\mathscr{P}+\mathscr{Q}}}{|\Lambda|\Gamma(\mathscr{P}+\mathscr{Q}+1)}\right.$$
$$\left.+\frac{|\beta|(\log\mathscr{T})^{\mathscr{P}-\gamma+1}}{|\Lambda|(\mathscr{P}-\gamma+1)}\right]||\varpi(\mathtt{s})-\mathscr{Y}(\mathtt{s})|$$
$$\leq(\mathscr{L}_1+\mathscr{L}_2\mathscr{G}_1)\Theta|\varpi(\mathtt{s})-\mathscr{Y}(\mathtt{s})|\,.$$

where

$$\Theta=\left[\frac{(\log\mathscr{T})^{\mathscr{P}}}{\Gamma(\mathscr{P}+1)}+\frac{|\alpha|(\log\mathscr{T})(\log\eta)^{\mathscr{P}+\mathscr{Q}}}{|\Lambda|\Gamma(\mathscr{P}+\mathscr{Q}+1)}+\frac{|\beta|(\log\mathscr{T})^{\mathscr{P}-\gamma+1}}{|\Lambda|(\mathscr{P}-\gamma+1)}\right].$$

By (3), consequences are expressed as $\mathscr{W}$, a contraction. As a result of the Banach FPT, we obtain the result that $\mathscr{W}$ has a FP that is a solution to the problem (1)–(3). □

Theorem 4 ((Krasnoselkii's FPT) [31,32] **).** *Let a bounded, closed, and convex set $\emptyset\neq\mathbb{M}_1\subset\mathbb{M}$ with Banach space $\mathbb{M}$. Take operators Γ and Δ: (a) $\Gamma x_1+\Delta x_2\in\mathbb{M}_1$, $x_1,x_2\in\mathbb{M}_1$; (b) Γ is compact and continuous; (c) Δ is a contraction mapping. Therefore, $\exists\, z\in\mathbb{M}_1$: $z=\Gamma z+\Delta z$.*

The following Theorem is based on existence results.

Theorem 5. *If Hypothesis 3 and 4 hold, then the problem (1)–(3) has at least one solution for on $[1,\mathscr{T}]$.*

Proof. Introduce the new operator $\mathscr{E}_1$ and $\mathscr{E}_2$ are

$$(\mathscr{E}_1\varpi)(\tau)=\frac{1}{\Gamma(\mathscr{P})}\int_1^\tau(\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s}))))\frac{\mathtt{ds}}{\mathtt{s}} \tag{9}$$

and

$$(\mathscr{E}_2\varpi)(\tau)=\sum_{\mathscr{K}=1}^{\mathscr{M}}\mathscr{Y}_{\mathtt{i}}+\frac{\log\tau}{\Lambda}[\lambda-\frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^\eta(\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}]$$
$$-\beta(\Gamma(\mathscr{P}-\gamma))^{-1}\int_1^{\mathscr{T}}(\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}. \tag{10}$$

Consider

$$\mathscr{B}_{\mathtt{d}}=\{\varpi\in\mathscr{PC}:||\varpi||<\mathtt{d}\}.$$

For any $\varpi,\mathscr{Y}\in\mathscr{B}_{\mathtt{d}}$ the $\mathscr{E}_1\varpi+\mathscr{E}_2\mathscr{Y}\in\mathscr{B}_{\mathtt{d}}$ where $\mathscr{E}_1$and $\mathscr{E}_2$ is denoted by (3.2) and (3.3).

$$
\begin{aligned}
\mathscr{E}_1\varpi + \mathscr{E}_2\mathscr{Y}| = &|\frac{1}{\Gamma(\mathscr{P})}\int_1^\tau (\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s}))))\frac{\mathtt{ds}}{\mathtt{s}} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^\eta (\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}} (\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}| + \sum_{\mathscr{K}=1}^{\mathscr{M}}\mathscr{Y}_{\mathtt{i}} \\
&\leq |\frac{1}{\Gamma(\mathscr{P})}\int_1^\tau (\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s}))))\frac{\mathtt{ds}}{\mathtt{s}}| \\
&+ |\frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^\eta (\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}} (\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}(\varpi(\mathtt{s})))\frac{\mathtt{ds}}{\mathtt{s}}| + \sum_{\mathscr{K}=1}^{\mathscr{M}}\mathscr{Y}_{\mathtt{i}}, \\
&\leq \frac{1}{\Gamma(\mathscr{P})}\int_1^\tau (\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}\varpi((\mathtt{s})))|\frac{ds}{s} \\
&+ \frac{\log\tau}{\Lambda}[\lambda - \frac{\alpha}{\Gamma(\mathscr{P}+\mathscr{Q})}\int_1^\eta (\log\frac{\eta}{\mathtt{s}})^{\mathscr{P}+\mathscr{Q}-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}\varpi((\mathtt{s})))|\frac{\mathtt{ds}}{\mathtt{s}}] \\
&- \frac{\beta}{\Gamma(\mathscr{P}-\gamma)}\int_1^{\mathscr{T}} (\log\frac{\mathscr{T}}{\mathtt{s}})^{\mathscr{P}-\gamma-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}\varpi((\mathtt{s})))|\frac{\mathtt{ds}}{\mathtt{s}} + \sum_{\mathscr{K}=1}^{\mathscr{M}}\mathscr{Y}_{\mathtt{i}}, \\
&\leq \frac{1}{\Gamma(\mathscr{P}+1)}(\log\mathscr{T})^{\mathscr{P}}\mu(\tau)\frac{\mathtt{ds}}{\mathtt{s}} + \frac{\log\mathscr{T}}{|\Lambda|} \\
&\left[\lambda - \frac{|\alpha|}{\Gamma(\mathscr{P}+\mathscr{Q}+1)}(\log\eta)^{\mathscr{P}+\mathscr{Q}}\mu(\tau)\frac{\mathtt{ds}}{\mathtt{s}} + \frac{\beta}{\Gamma(\mathscr{P}-\gamma+1)}(\log\mathscr{T})^{\mathscr{P}-\gamma}\mu(\tau)\frac{\mathtt{ds}}{\mathtt{s}}\right] + \mathscr{M}^*, \\
&\leq \mu(\tau)\frac{1}{\Gamma(\mathscr{P}+1)}(\log\mathscr{T})^{\mathscr{P}} + \frac{\log\mathscr{T}}{|\Lambda|}[\lambda - \frac{|\alpha|}{\Gamma(\mathscr{P}+\mathscr{Q}+1)} + \frac{\beta}{\Gamma(\mathscr{P}-\gamma+1)}] + \mathscr{M}^*, \\
&\leq \mathtt{d}.
\end{aligned}
$$

Thus

$$\mathscr{E}_1\varpi + \mathscr{E}_2\mathscr{Y} \in \mathscr{B}_{\mathtt{d}},$$

using $\mathscr{H}_4$, $\mathscr{E}_2$ is a contraction, and when using $\mathscr{E}_1$ the operator $(\mathscr{E}_1\varpi)(\tau)$ is continuous. Additionally, we notice

$$
\begin{aligned}
(\mathscr{E}_1\varpi)(\tau) &= \frac{1}{\Gamma(\mathscr{P})}\int_1^\tau (\log\frac{\tau}{\mathtt{s}})^{\mathscr{P}-1}|\mathscr{F}(\mathtt{s},\varpi(\mathtt{s}),\mathscr{B}\varpi((\mathtt{s})))\mathtt{ds}(\mathtt{s})^{-1} \\
&\leq (\Gamma(\mathscr{P}+1))^{-1}(\log\mathscr{T})^{\mathscr{P}}\mu(\tau).
\end{aligned}
$$

$\mathscr{E}_1$ is uniformly bounded on $\mathscr{B}_{\mathtt{d}}$. Let us now demonstrate that the function $(\mathscr{A}_1\varpi)(\tau)$ is equicontinuous.

$$\sup_{(\tau,\varpi,\mathscr{Y})} \in [1,\mathscr{T}] \times \mathscr{B}_{\mathtt{d}}|\mathscr{F}(\tau,\varpi(\tau),\mathscr{B}(\varpi(\tau)))| < \mathscr{C}_0 < \infty,$$

We will obtain

$$
\begin{aligned}
|(\mathscr{E}_1\varpi)(\tau_2)+(\mathscr{E}_2\mathscr{Y})(\tau_1)| &= |\frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau_1}(\log\frac{\tau_1}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{G}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}-\frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau_2}(\log\frac{\tau_2}{\mathtt{s}})^{\mathscr{P}-1}\mathscr{G}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}| \\
&\leq \frac{1}{\Gamma(\mathscr{P})}\int_1^{\tau_1}[(\log\frac{\tau_2}{\mathtt{s}})^{\mathscr{P}-1}-(\log\frac{\tau_1}{\mathtt{s}})^{\mathscr{P}-1}]|\mathscr{G}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}| \\
&+\frac{1}{\Gamma(\mathscr{P})}\int_{\tau_2}^{\tau_1}(\log\frac{\tau_2}{\mathtt{s}})^{\mathscr{P}-1}|\mathscr{G}(\mathtt{s})\frac{\mathtt{ds}}{\mathtt{s}}| \\
&\leq \frac{\mathscr{C}_0}{\Gamma(\mathscr{P}+1)}[|(\log\tau_1)^{\mathscr{P}}+\log(\frac{\tau_2}{\tau_1})^{\mathscr{P}}-(\log\tau_2)^{\mathscr{P}}|+|\log(\frac{\tau_2}{\tau_1})^{\mathscr{P}}|] \\
&\leq \frac{\mathscr{C}_0}{\Gamma(\mathscr{P}+1)}[|(\log\tau_1)^{\mathscr{P}}-(\log\tau_2)^{\mathscr{P}}|].
\end{aligned}
$$

Consequently, $\mathscr{E}_1(\mathscr{B}_{\mathtt{d}})$ is relatively compact. Therefore, according to the Ascoli-Arzela theorem, $\mathscr{E}_1$ is compact. Therefore, the problems (1)–(3) under consideration have at least one FP on $\mathscr{J}$. □

4. Example

Consider the following BVP:

$$
{}^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\frac{3}{2}}\varpi(\tau)=\frac{\cos^2\tau}{(\mathtt{e}^{-\tau+2})^2|\varpi(\tau)|}+\int_0^{\tau}\frac{(\mathtt{s}+|\varpi(\mathtt{s})|)}{(2+\tau)^2(1+|\varpi(\mathtt{s}))|}\mathtt{ds}, \tag{11}
$$

$$
\varpi(\tau_{\mathscr{K}}^{+})=\varpi(\tau_{\mathscr{K}}^{-})+\frac{1}{6}, \tag{12}
$$

$$
\varpi(1)=0,\ \frac{1}{2}{}_{\mathscr{H}}\mathscr{I}^{\frac{1}{2}}\varpi(2)+2^{\mathscr{C}\mathscr{H}}\mathscr{D}^{\frac{1}{3}}\varpi(\mathtt{e})=\frac{3}{4}, \tag{13}
$$

$$
\begin{aligned}
\mathscr{F}(\tau,\iota,v) &= \frac{\cos^2\tau}{(\mathtt{e}^{-\tau+2})^2|\varpi(\tau)|} \\
\mathscr{K}(\tau,\mathtt{s},\varpi) &= \int_0^{\tau}\frac{(\mathtt{s}+|\varpi(\mathtt{s})|)}{(2+\tau)^2(1+|\varpi(\mathtt{s}))|}\mathtt{ds},
\end{aligned}
$$

where $\mathscr{P}=\frac{3}{2},\ \mathscr{Q}=\frac{1}{2},\gamma=\frac{1}{3},\eta=2,\ \alpha=\frac{1}{2},\ \beta=2,\ \lambda=\frac{3}{4},\ \mathscr{T}=\mathtt{e},\ \mathscr{L}_1+\mathscr{L}_2=\frac{1}{9},\ \mathscr{G}_1=\frac{1}{9}$. Hence Hypothesis 1 and 2 hold. We check the condition

$$
(\mathscr{L}_1+\mathscr{L}_2\mathscr{G}_1)\left[\frac{(\log\mathscr{T})^{\mathscr{P}}}{\Gamma(\mathscr{P}+1)}+\frac{|\alpha|(\log\mathscr{T})(\log\eta)^{\mathscr{P}+\mathscr{Q}}}{|\Lambda|\Gamma(\mathscr{P}+\mathscr{Q}+1)}+\frac{|\beta|(\log\mathscr{T})^{\mathscr{P}-\gamma+1}}{|\Lambda|(\mathscr{P}-\gamma+1)}\right]\approx 0.047509<1
$$

Hence, problems (11)–(13) have a unique solution $[1,\mathscr{T}]$.

Proof. Using Theorem 3 to derive a unique solution, since Hypothesis 1 and 2 are satisfied. Then Theorem 3 implies the uniqueness solution. □

5. Conclusions

In this work, results for a new modeling of IBVP using C-HIFI-DE with Banach space are investigated, along with the E-UR of solutions. The KFPT and the BCP serve as the basis of this unique strategy, and are used to achieve the desired results. We develop the illustrated examples at the end of the paper to support the validity of the theoretical statements. Potential future works could be to examine much more complicated fractional systems and employ some other tools.

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