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Article

Fractional Maclaurin-Type Inequalities for Multiplicatively Convex Functions

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Abstract: This paper's major goal is to prove some symmetrical Maclaurin-type integral inequalities inside the framework of multiplicative calculus. In order to accomplish this and after giving some basic tools, we have established a new integral identity. Based on this identity, some symmetrical Maclaurin-type inequalities have been constructed for functions whose multiplicative derivatives are bounded as well as convex. At the end, some applications to special means are provided.

Keywords: non-newtonian calculus; maclaurin's inequality; multiplicatively convex functions

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1. Introduction

The theory of inequality has seen a rise in research activity over the past 20 years in different fields of sciences, both theoretical and applied, including in the study of the qualitative properties of solutions to ordinary, partial, and integral differential equations as well as in numerical analysis, where this tool is essential for estimating quadrature errors, and in a variety of calculation types, including time scale calculus [1–3], fractional calculus [4–7], quantum calculus [8,9], and classical (Newtonian) calculus [10–12].

The term multiplicative calculus originates from the classical calculation of Newton and Leibniz, which was introduced by Grossman and Katz when they presented and examined the first non-Newtonian systems [13].

The multiplicative derivative and integral were presented by Bashirov et al. [14]. Its relationship to the classical derivative and integral, as well as some of its features, are mentioned below.

The multiplicative derivative of the function $\mathcal{G}$ with the notation $\mathcal{G}^*$ is as follows:

Definition 1 ([14]). *For a positive function $\mathcal{G}:\mathbb{R}\to\mathbb{R}^+$. The multiplicative derivative is*

$$\frac{d^*\mathcal{G}}{dt}=\mathcal{G}^*(t)=\lim_{\Delta\to 0}\left(\frac{\mathcal{G}(t+\Delta)}{\mathcal{G}(t)}\right)^{\frac{1}{\Delta}}.$$

Remark 1. *If $\mathcal{G}$ is positive and differentiable at t, then $\mathcal{G}^*$ exists and is related to the standard derivative $\mathcal{G}'$ as follows:*

$$\mathcal{G}^*(t)=e^{(\ln\mathcal{G}(t))'}=e^{\frac{\mathcal{G}'(t)}{\mathcal{G}(t)}}.$$

The multiplicative integral or * integral of the function $\mathcal{G}$ noted $\int_a^b (\mathcal{G}(t))^{dt}$ is as follows:

Proposition 1 ([14]). *Let $\mathcal{G} \in L^1[a,b]$. Then, the * integral of the function $\mathcal{G}$ is*

$$\int_a^b (\mathcal{G}(t))^{dt} = \exp\left\{ \int_a^b \ln(\mathcal{G}(t))dt \right\}.$$

It is also practical to remember the integration-by-parts formula.

Theorem 1 ([14]). *Let $\mathcal{G}, \chi : [a,b] \to \mathbb{R}$, where $\mathcal{G}$ is a multiplicative differentiable function and χ is a differentiable function. So, the function $\mathcal{G}^{\chi}$ is a multiplicative integrable function that satisfies*

$$\int_a^b \left(\mathcal{G}^{*}(t)^{\chi(t)}\right)^{dt} = \frac{\mathcal{G}(b)^{\chi(b)}}{\mathcal{G}(a)^{\chi(a)}} \times \frac{1}{\int_a^b \left(\mathcal{G}(t)^{\chi'(t)}\right)^{dt}}.$$

Lemma 1 ([15]). *Let $\mathcal{G}, k : [a,b] \to \mathbb{R}$, where $\mathcal{G}$ is a differentiable multiplicative function and k is a differentiable function. Suppose $\chi : J \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function, then*

$$\int_a^b \left(\mathcal{G}^{*}(k(t))^{k'(t)\chi(t)}\right)^{dt} = \frac{\mathcal{G}(k(b))^{\chi(b)}}{\mathcal{G}(k(a))^{\chi(a)}} \times \frac{1}{\int_a^b \left(\mathcal{G}(k(t))^{\chi'(t)}\right)^{dt}}.$$

The analogous multiplicative of the Hermite–Hadamard inequality was provided by Ali et al. in [16], as follows:

Theorem 2. *Let $\mathcal{G}$ be a positive and multiplicatively convex function on the interval $[\alpha_1, \alpha_2]$; then, the following double inequality is true:*

$$\mathcal{G}\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \left(\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x)^{dx}\right)^{\frac{1}{\alpha_2 - \alpha_1}} \leq \sqrt{\mathcal{G}(\alpha_1)\mathcal{G}(\alpha_2)}. \tag{1}$$

Since the publication of the aforementioned paper, several works concerning multiplicative inequalities have been published (see, for instance, [15,17–20]).

In [21], Meftah investigated some Maclaurin-type inequalities for multiplicatively convex functions and established the following results.

Theorem 3. *Assume that $\mathcal{G} : [\alpha_1, \alpha_2] \to \mathbb{R}^{+}$ is a multiplicative differentiable map with multiplicative convex derivative $\mathcal{G}^{*}$ on $[\alpha_1, \alpha_2]$. Then, we have*

$$\begin{aligned} &\left| \left(\left(\mathcal{G}\left(\tfrac{5\alpha_1+\alpha_2}{6}\right)\right)^{3} \left(\mathcal{G}\left(\tfrac{\alpha_1+\alpha_2}{2}\right)\right)^{2} \left(\mathcal{G}\left(\tfrac{\alpha_1+5\alpha_2}{6}\right)\right)^{3} \right)^{\frac{1}{8}} \left(\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x)^{dx} \right)^{\frac{1}{\alpha_1-\alpha_2}} \right| \\ \leq &\left((\mathcal{G}^{*}(\alpha_1))^{64} \left(\mathcal{G}^{*}\left(\tfrac{5\alpha_1+\alpha_2}{6}\right)\right)^{379} \left(\mathcal{G}^{*}\left(\tfrac{\alpha_1+\alpha_2}{2}\right)\right)^{314} \left(\mathcal{G}^{*}\left(\tfrac{\alpha_1+5\alpha_2}{6}\right)\right)^{379} (\mathcal{G}^{*}(\alpha_2))^{64} \right)^{\frac{\alpha_2-\alpha_1}{13{,}824}}. \end{aligned}$$

Theorem 4. *Assume that Theorem 3's whole set of hypotheses is true. Then, we have*

$$\begin{aligned}&\left|\left(\left(\mathcal{G}\left(\frac{5\alpha_1+\alpha_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\alpha_1+5\alpha_2}{6}\right)\right)^3\right)^{\frac{1}{8}}\left(\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x)^{dx}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right|\\ \leq\ &\left((\mathcal{G}^*(\alpha_1))^{221}\left(\mathcal{G}^*\left(\frac{5\alpha_1+\alpha_2}{6}\right)\right)^{379}\left(\mathcal{G}^*\left(\frac{\alpha_1+5\alpha_2}{6}\right)\right)^{379}(\mathcal{G}^*(\alpha_2))^{221}\right)^{\frac{\alpha_2-\alpha_1}{13{,}824}}.\end{aligned}$$

Theorem 5. *Assume that Theorem 3's whole set of hypotheses is true. Then, we have*

$$\begin{aligned}&\left|\left(\left(\mathcal{G}\left(\frac{5\alpha_1+\alpha_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\alpha_1+5\alpha_2}{6}\right)\right)^3\right)^{\frac{1}{8}}\left(\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x)^{dx}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right|\\ \leq\ &\left((\mathcal{G}^*(\alpha_1))^{8}\left(\mathcal{G}^*\left(\frac{5\alpha_1+\alpha_2}{6}\right)\right)^{67}\left(\mathcal{G}^*\left(\frac{\alpha_1+5\alpha_2}{6}\right)\right)^{67}(\mathcal{G}^*(\alpha_2))^{8}\right)^{\frac{b-a}{1728}}.\end{aligned}$$

The multiplicative Riemann–Liouville fractional integrals were first introduced by Abdeljawad and Grossman in [4] and satisfies the following relations:

Definition 2. *The left and right multiplicative Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}$, where $Re(\alpha) > 0$, is given as follows:*

$$({}_aI_*^{\alpha}\varphi)(\varkappa) = e^{(J_{a+}^{\alpha}(\ln\circ\varphi))(\varkappa)} \tag{2}$$

and

$$({}_*I_b^{\alpha}\varphi)(\varkappa) = e^{(J_{b-}^{\alpha}(\ln\circ\varphi))(\varkappa)}, \tag{3}$$

where $J_{\mathfrak{r}_1^+}^{\alpha}$ and $J_{\mathfrak{r}_2^-}^{\alpha}$ are the left and right Riemann–Liouville fractional integrals, respectively, defined as follows:

$$(J_{a+}^{\alpha}\varphi)(\xi) = \frac{1}{\Gamma(\alpha)}\int_a^{\xi}(\xi-\mu)^{\alpha-1}\varphi(\mu)d\mu, a < \xi$$

and

$$(J_{b-}^{\alpha}\varphi)(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{b}(\mu-\xi)^{\alpha-1}\varphi(\mu)d\mu, \xi < b.$$

Budak and Özçelik [22] proved some multiplicative fractional Hermite–Hadamard-type inequalities by combining the operators (2) and (3) with the definition of multiplicative convex functions. One can also consult [22–28] concerning fractional multiplicative inequalities.

Very recently, Peng and Du [29] established some non-symmetrical fractional Maclaurin-type inequalities as follows:

Theorem 6. *$\mathcal{G} : [\alpha_1, \alpha_2] \to \mathbb{R}^+$ is an increasing multiplicative differentiable map. If $\mathcal{G}^*$ is multiplicative convex on $[\alpha_1, \alpha_2]$, then for $\alpha > 0$, the following inequality related to multiplicative RL-fractional integrals holds:*

$$
\begin{aligned}
&\left|\frac{\left(\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3\right)^{\frac{1}{8}}}{\mathcal{L}(\mathcal{G})}\right| \\
\leq\ & \left(\mathcal{G}^*(\varepsilon_1)\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}\Delta_1}\left(\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}\left(\frac{1}{2}\Delta_2+\Delta_3+\Delta_4\right)} \\
&\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}(\Delta_5+\Delta_6+\Delta_7+\Delta_8)} \\
&\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}\left(\Delta_9+\Delta_{10}+\frac{1}{2}\Delta_{11}\right)}\left(\mathcal{G}^*(\varepsilon_2)\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}\Delta_{12}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}(\mathcal{G}) &= \left(\left({}_*I^{\alpha}_{\frac{5\varepsilon_1+\varepsilon_2}{6}}\mathcal{G}\right)(\varepsilon_1)\left({}_*I^{\alpha}_{\varepsilon_2}\mathcal{G}\right)\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^{\alpha}}} \\
&\quad\left(\left({}_*I^{\alpha}_{\frac{\varepsilon_1+\varepsilon_2}{2}}\mathcal{G}\right)\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\left({}_*I^{\alpha}_{\frac{\varepsilon_1+5\varepsilon_2}{6}}\mathcal{G}\right)\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^{\alpha}}}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_1 &= \frac{1}{6(\alpha+1)(\alpha+2)}, \\
\frac{1}{2}\Delta_2+\Delta_3+\Delta_4 &= \frac{10-11\alpha-15\alpha}{48(\alpha+1)(\alpha+2)}+\frac{5\alpha}{12(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}-\frac{5\alpha}{24(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}}, \\
\Delta_5+\Delta_6+\Delta_7+\Delta_8 &= \frac{16-8\alpha-8\alpha^2}{48(\alpha+1)(\alpha+2)}+\frac{5\alpha}{24(\alpha+2)}\left(\frac{5}{8}\right)^{\frac{2}{\alpha}}+\frac{\alpha}{4(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{8(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}, \\
\Delta_9+\Delta_{10}+\frac{1}{2}\Delta_{11} &= \frac{10+19\alpha+\alpha^2}{48(\alpha+1)(\alpha+2)}+\frac{\alpha}{8(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}, \\
\Delta_{12} &= \frac{\alpha}{12(\alpha+2)}.
\end{aligned}
$$

Theorem 7. *Under the assumptions of Theorem 6, if $\mathcal{G}^* \leq M$ with $M > 0$, then we have*

$$
\begin{aligned}
&\left|\frac{\left(\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3\right)^{\frac{1}{8}}}{\mathcal{L}(\mathcal{G})}\right| \\
\leq\ & M^{\frac{\varepsilon_2-\varepsilon_1}{6}\left[\frac{5\alpha}{12(\alpha+1)}\left(\frac{5}{8}\right)^{\frac{1}{\alpha}}+\frac{\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}+\frac{5-3\alpha}{6(\alpha+1)}\right]}.
\end{aligned}
$$

The goal of the current study is to construct some new symmetrical fractional Maclaurin-type inequalities for multiplicatively convex functions, which are motivated by the previously stated papers. To address this, we provide a novel integral identity, from which the fractional Maclaurin inequality for bounded multiplicative derivatives is derived initially. The situation when the multiplicative derivatives are convex is then covered. Some applications to special means are provided at the end. The remainder of the current paper is organized as follows: Some symmetrical fractional Maclaurin inequalities are presented in Section 2. Section 3 provides some applications to special means. Section 4 draws the conclusion.

2. Main Results

We begin with the auxiliary result that follows.

Lemma 2. *Assume that $\mathcal{G} : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}^+$ is a multiplicative differentiable mapping with multiplicative integrable derivative $\mathcal{G}^*$ on $[\varepsilon_1, \varepsilon_2]$. Then, we have*

$$
\begin{aligned}
&\left(\left(\mathcal{G}\left(\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3\left(\mathcal{G}\left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2\left(\mathcal{G}\left(\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3\right)^{\frac{1}{8}}(\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^{\alpha}}} \\
= &\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\varepsilon_1+\hbar\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)^{\frac{1}{6}\hbar^{\alpha}}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}} \\
&\times\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\tfrac{5\varepsilon_1+\varepsilon_2}{6}+\hbar\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)^{(\frac{1}{24}(3-8(1-\hbar)^{\alpha}))}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \\
&\times\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+\varepsilon_2}{2}+\hbar\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)^{(\frac{1}{24}(8\hbar^{\alpha}-3))}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \\
&\times\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+5\varepsilon_2}{6}+\hbar\varepsilon_2\right)^{-\frac{1}{6}(1-\hbar)^{\alpha}}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta(\mathcal{G}) = &\left(\left({}_*I^{\alpha}_{\frac{5\varepsilon_1+\varepsilon_2}{6}}\mathcal{G}\right)(\varepsilon_1)\left({}_{\frac{\varepsilon_1+5\varepsilon_2}{6}}I^{\alpha}_*\mathcal{G}\right)(\varepsilon_2)\right)^{2^{\alpha-1}} \\
&\times\left({}_{\frac{5\varepsilon_1+\varepsilon_2}{6}}I^{\alpha}_*\mathcal{G}\right)\left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\left({}_*I^{\alpha}_{\frac{\varepsilon_1+5\varepsilon_2}{6}}\mathcal{G}\right)\left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right).
\end{aligned} \tag{4}
$$

Proof. Let

$$I_1=\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\varepsilon_1+\hbar\frac{5\varepsilon_1+\varepsilon_2}{6}\right)^{\frac{1}{6}\hbar^{\alpha}}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}},$$

$$I_2=\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\frac{5\varepsilon_1+\varepsilon_2}{6}+\hbar\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{(\frac{1}{24}(3-8(1-\hbar)^{\alpha}))}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}},$$

$$I_3=\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\frac{\varepsilon_1+\varepsilon_2}{2}+\hbar\frac{\varepsilon_1+5\varepsilon_2}{6}\right)^{(\frac{1}{24}(8\hbar^{\alpha}-3))}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}}$$

and

$$I_4=\left(\int_0^1\left(\mathcal{G}^*\left((1-\hbar)\frac{\varepsilon_1+5\varepsilon_2}{6}+\hbar\varepsilon_2\right)^{-\frac{1}{6}(1-\hbar)^{\alpha}}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}}.$$

By using the integration by parts for multiplicative integrals, I_1 yields

$$
\begin{aligned}
I_1 &= \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\varepsilon_1 + \hbar \frac{5\varepsilon_1+\varepsilon_2}{6} \right)^{\frac{1}{6}\hbar^\alpha} \right)^{d\hbar} \right)^{\frac{\varepsilon_2-\varepsilon_1}{6}} \\
&= \int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\varepsilon_1 + \hbar \frac{5\varepsilon_1+\varepsilon_2}{6} \right)^{\frac{\varepsilon_2-\varepsilon_1}{6}\frac{1}{6}\hbar^\alpha} \right)^{d\hbar} \\
&= \frac{\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{1}{6}}}{1} \cdot \frac{1}{\int_0^1 \left(\mathcal{G}\left((1-\hbar)\varepsilon_1 + \hbar \frac{5\varepsilon_1+\varepsilon_2}{6} \right)^{\frac{\alpha}{6}\hbar^{\alpha-1}} \right)^{d\hbar}} \\
&= \left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{1}{6}} \frac{1}{\exp\left\{ \int_0^1 \left(\frac{\alpha}{6}\hbar^{\alpha-1} \ln \mathcal{G}\left((1-\hbar)\varepsilon_1 + \hbar \frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right) d\hbar \right\}} \\
&= \left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{1}{6}} \frac{1}{\exp\left\{ \frac{6^{\alpha-1}\alpha}{(\varepsilon_2-\varepsilon_1)^\alpha} \int_{\varepsilon_1}^{\frac{5\varepsilon_1+\varepsilon_2}{6}} (u-\varepsilon_1)^{\alpha-1} \ln \mathcal{G}(u) du \right\}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{1}{6}} \frac{1}{\left(\exp\left\{ \left(\frac{1}{\Gamma(\alpha)} \int_{\varepsilon_1}^{\frac{5\varepsilon_1+\varepsilon_2}{6}} (u-\varepsilon_1)^{\alpha-1} \ln \mathcal{G}(u) du \right) \right\} \right)^{\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}} \\
&= \left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{1}{6}} \left(\left({}_*I^\alpha_{\frac{5\varepsilon_1+\varepsilon_2}{6}} \mathcal{G} \right)(\varepsilon_1) \right)^{-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}.
\end{aligned} \tag{5}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\frac{5\varepsilon_1+\varepsilon_2}{6} + \hbar \frac{\varepsilon_1+\varepsilon_2}{2} \right)^{\left(\frac{1}{24}(3-8(1-\hbar)^\alpha)\right)} \right)^{d\hbar} \right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \\
&= \int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\frac{5\varepsilon_1+\varepsilon_2}{6} + \hbar \frac{\varepsilon_1+\varepsilon_2}{2} \right)^{\frac{\varepsilon_2-\varepsilon_1}{3}\left(\frac{1}{24}(3-8(1-\hbar)^\alpha)\right)} \right)^{d\hbar} \\
&= \frac{\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right)^{\frac{1}{8}}}{\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right)^{-\frac{5}{24}}} \cdot \frac{1}{\int_0^1 \left(\mathcal{G}\left((1-\hbar)\frac{5\varepsilon_1+\varepsilon_2}{6} + \hbar \frac{\varepsilon_1+\varepsilon_2}{2} \right)^{\frac{\alpha}{3}(1-\hbar)^{\alpha-1}} \right)^{d\hbar}} \\
&= \frac{\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right) \right)^{\frac{1}{8}}}{\exp \int_0^1 \left(\frac{\alpha}{3}(1-\hbar)^{\alpha-1} \ln \mathcal{G}\left((1-\hbar)\frac{5\varepsilon_1+\varepsilon_2}{6} + \hbar \frac{\varepsilon_1+\varepsilon_2}{2} \right) \right) d\hbar} \\
&= \frac{\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right) \right)^{\frac{1}{8}}}{\exp\left\{ \left(\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha} \frac{1}{\Gamma(\alpha)} \int_{\frac{5\varepsilon_1+\varepsilon_2}{6}}^{\frac{\varepsilon_1+\varepsilon_2}{2}} \left(\frac{\varepsilon_1+\varepsilon_2}{2} - u \right)^{\alpha-1} \ln \mathcal{G}(u) du \right) \right\}} \\
&= \frac{\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right) \right)^{\frac{1}{8}}}{\left(\exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\frac{5\varepsilon_1+\varepsilon_2}{6}}^{\frac{\varepsilon_1+\varepsilon_2}{2}} \left(\frac{\varepsilon_1+\varepsilon_2}{2} - u \right)^{\alpha-1} \ln \mathcal{G}(u) du \right\} \right)^{\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}} \\
&= \left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6} \right) \right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right) \right)^{\frac{1}{8}} \left(\left({}_{\frac{5\varepsilon_1+\varepsilon_2}{6}} I^\alpha_* \mathcal{G} \right)\left(\frac{\varepsilon_1+\varepsilon_2}{2} \right) \right)^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
I_3 &= \left(\int_0^1 \left(\mathcal{G}^* \left((1-\hbar)\frac{\varepsilon_1+\varepsilon_2}{2} + \hbar\frac{\varepsilon_1+5\varepsilon_2}{6} \right)^{\left(\frac{1}{24}(8\hbar^\alpha-3)\right)} \right)^{d\hbar} \right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \\
&= \int_0^1 \left(\mathcal{G}^* \left((1-\hbar)\frac{\varepsilon_1+\varepsilon_2}{2} + \hbar\frac{\varepsilon_1+5\varepsilon_2}{6} \right)^{\frac{\varepsilon_2-\varepsilon_1}{3}\left(\frac{1}{24}(8\hbar^\alpha-3)\right)} \right)^{d\hbar} \\
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{5}{24}}}{\left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{-\frac{1}{8}}} \cdot \frac{1}{\int_0^1 \left(\mathcal{G}\left((1-\hbar)\frac{\varepsilon_1+\varepsilon_2}{2} + \hbar\frac{\varepsilon_1+5\varepsilon_2}{6} \right)^{\frac{1}{3}\alpha\hbar^{\alpha-1}} \right)^{d\hbar}} \\
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{8}}}{\exp\left\{ \int_0^1 \frac{1}{3}\alpha\hbar^{\alpha-1} \ln \mathcal{G}\left((1-\hbar)\frac{\varepsilon_1+\varepsilon_2}{2} + \hbar\frac{\varepsilon_1+5\varepsilon_2}{6} \right) d\hbar \right\}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{8}}}{\exp\left\{ \frac{3^{\alpha-1}}{(\varepsilon_2-\varepsilon_1)^\alpha}\alpha \int_{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\frac{\varepsilon_1+5\varepsilon_2}{6}} \left(u-\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{\alpha-1} \ln \mathcal{G}(u)\,du \right\}} \\
&= \frac{\left(f\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{5}{24}} \left(f\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{8}}}{\left(\exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\frac{\varepsilon_1+5\varepsilon_2}{6}} \left(u-\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{\alpha-1} \ln \mathcal{G}(u)\,du \right\}\right)^{\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}} \\
&= \left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{5}{24}} \left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{8}} \cdot \left(\left({}_*I^\alpha_{\frac{\varepsilon_1+5\varepsilon_2}{6}} \mathcal{G} \right)\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) \right)^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
I_4 &= \left(\int_0^1 \left(\mathcal{G}^* \left((1-\hbar)\frac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2 \right)^{-\frac{1}{6}(1-\hbar)^\alpha} \right)^{d\hbar} \right)^{\frac{\varepsilon_2-\varepsilon_1}{6}} \\
&= \int_0^1 \left(\mathcal{G}^* \left((1-\hbar)\frac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2 \right)^{-\frac{1}{6}\frac{\varepsilon_2-\varepsilon_1}{6}(1-\hbar)^\alpha} \right)^{d\hbar} \\
&= \frac{1}{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{-\frac{1}{6}}} \cdot \frac{1}{\int_0^1 \left(\mathcal{G}\left((1-\hbar)\frac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2 \right)^{\frac{1}{6}\alpha(1-\hbar)^{\alpha-1}} \right)^{d\hbar}} \\
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{ \int_0^1 \frac{1}{6}\alpha(1-\hbar)^{\alpha-1} \ln \mathcal{G}\left((1-\hbar)\frac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2 \right) d\hbar \right\}} \\
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{1}{6}}}{\exp\left\{ \frac{6^{\alpha-1}}{(\varepsilon_2-\varepsilon_1)^\alpha}\alpha \int_{\frac{\varepsilon_1+5\varepsilon_2}{6}}^{\varepsilon_2} (\varepsilon_2-u)^{\alpha-1} \ln \mathcal{G}(u)\,du \right\}} \\
&= \frac{\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{1}{6}}}{\left(\exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\frac{\varepsilon_1+5\varepsilon_2}{6}}^{\varepsilon_2} (\varepsilon_2-u)^{\alpha-1} \ln \mathcal{G}(u)\,du \right\}\right)^{\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}} \\
&= \left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{1}{6}} \left(\left({}_{\frac{\varepsilon_1+5\varepsilon_2}{6}} I^\alpha_* \mathcal{G} \right)(\varepsilon_2) \right)^{-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}.
\end{aligned}
\tag{8}
$$

Multiplying (5)–(8) yields the desired outcome. □

Theorem 8. *Assume that Lemma 2's hypotheses are all true. If* $|\ln \mathcal{G}^*| \leq \ln \mathcal{M}$ *on* $[\varepsilon_1, \varepsilon_2]$, *then we have*

$$\begin{aligned}
&\left| \left(\left(\mathcal{G}\left(\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3 \left(\mathcal{G}\left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2 \left(\mathcal{G}\left(\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3 \right)^{\frac{1}{8}} (\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}} \right| \\
\leq\ & \mathcal{M}^{\frac{b-a}{18(\alpha+1)}\left(\frac{7-3\alpha}{2}+3\alpha\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)},
\end{aligned}$$

where $\Theta(.)$ *is defined by (4).*

Proof. According to Lemma 2, multiplicative integration, and the hypothesis that $|\ln \mathcal{G}^*| \leq \ln \mathcal{M}$, we have

$$\begin{aligned}
&\left| \left(\left(\mathcal{G}\left(\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3 \left(\mathcal{G}\left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2 \left(\mathcal{G}\left(\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3 \right)^{\frac{1}{8}} (\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}} \right| \\
=\ & \left| \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\varepsilon_1 + \hbar\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)^{\frac{1}{6}\hbar^\alpha}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}} \right| \\
&\times \left| \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\tfrac{5\varepsilon_1+\varepsilon_2}{6} + \hbar\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)^{\left(\frac{1}{24}(3-8(1-\hbar)^\alpha)\right)}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \right| \\
&\times \left| \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+\varepsilon_2}{2} + \hbar\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)^{\left(\frac{1}{24}(8\hbar^\alpha-3)\right)}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{3}} \right| \\
&\times \left| \left(\int_0^1 \left(\mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2\right)^{-\frac{1}{6}(1-\hbar)^\alpha}\right)^{d\hbar}\right)^{\frac{\varepsilon_2-\varepsilon_1}{6}} \right| \\
\leq\ & \exp\left(\int_0^1 \tfrac{\varepsilon_2-\varepsilon_1}{36}\hbar^\alpha \left(\left| \ln \mathcal{G}^*\left((1-\hbar)\varepsilon_1 + \hbar\tfrac{5\varepsilon_1+\varepsilon_2}{6}\right)\right|\right) d\hbar \right) \\
&\times \exp\left(\int_0^1 \tfrac{\varepsilon_2-\varepsilon_1}{72} |3-8(1-\hbar)^\alpha| \left(\ln\left| \mathcal{G}^*\left((1-\hbar)\tfrac{5\varepsilon_1+\varepsilon_2}{6} + \hbar\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\right|\right) d\hbar \right) \\
&\times \exp\left(\int_0^1 \tfrac{\varepsilon_2-\varepsilon_1}{72} |8\hbar^\alpha-3| \left| \ln \mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+\varepsilon_2}{2} + \hbar\tfrac{\varepsilon_1+5\varepsilon_2}{6}\right)\right| d\hbar \right) \\
&\times \exp\left(\int_0^1 \tfrac{\varepsilon_2-\varepsilon_1}{36} (1-\hbar)^\alpha \left(\left| \ln \mathcal{G}^*\left((1-\hbar)\tfrac{\varepsilon_1+5\varepsilon_2}{6} + \hbar\varepsilon_2\right)\right|\right) d\hbar \right) \\
\leq\ & \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36} \ln \mathcal{M} \int_0^1 \hbar^\alpha d\hbar \right) \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{72} \ln \mathcal{M} \int_0^1 |3-8(1-\hbar)^\alpha| d\hbar \right) \\
&\times \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{72} \ln \mathcal{M} \int_0^1 |8\hbar^\alpha-3| d\hbar \right) \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36} \ln \mathcal{M} \int_0^1 (1-\hbar)^\alpha d\hbar \right) \\
=\ & \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)} \ln \mathcal{M} \right) \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)} \left(\tfrac{5-3\alpha}{2} + 3\alpha\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}} \right) \ln \mathcal{M} \right) \\
&\times \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)} \left(\tfrac{5-3\alpha}{2} + 3\alpha\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}} \right) \ln \mathcal{M} \right) \exp\left(\tfrac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)} \ln \mathcal{M} \right) \\
=\ & \mathcal{M}^{\frac{\varepsilon_2-\varepsilon_1}{18(\alpha+1)}\left(\frac{7-3\alpha}{2}+3\alpha\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)}.
\end{aligned}$$

where we have used

$$\int_0^1 |3-8(1-\hbar)^\alpha| d\hbar = \int_0^1 |8\hbar^\alpha - 3| d\hbar = \frac{5-3\alpha}{\alpha+1} + \frac{6\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}.$$

The proof is finished. □

Corollary 1. *By assuming that $\alpha = 1$ in Theorem 6, we obtain*

$$\left| \left(\left(\mathcal{G}\left(\frac{5\varepsilon_1 + \varepsilon_2}{6} \right) \right)^3 \left(\mathcal{G}\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^2 \left(\mathcal{G}\left(\frac{\varepsilon_1 + 5\varepsilon_2}{6} \right) \right)^3 \right)^{\frac{1}{8}} \left(\int_{\varepsilon_1}^{\varepsilon_2} \mathcal{G}(u)^{du} \right)^{\frac{1}{\varepsilon_1 - \varepsilon_2}} \right| \leq \mathcal{M}^{\frac{25(\varepsilon_2 - \varepsilon_1)}{288}}.$$

Theorem 9. *Assume that Lemma 2's hypotheses are all true. If $\mathcal{G}^*$ is multiplicative convex on $[\varepsilon_1, \varepsilon_2]$, then we have*

$$\begin{aligned} &\left| \left(\left(\mathcal{G}\left(\tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right) \right)^3 \left(\mathcal{G}\left(\tfrac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^2 \left(\mathcal{G}\left(\tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right) \right)^3 \right)^{\frac{1}{8}} (\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2 - \varepsilon_1)^{\alpha}}} \right| \\ \leq &\left((\mathcal{G}^*(\varepsilon_1))^{\frac{1}{(\alpha+1)(\alpha+2)}} \left(\mathcal{G}^*\left(\tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right) \right)^{\frac{14-3\alpha}{4(\alpha+2)} + \frac{3\alpha}{2(\alpha+2)} \left(\frac{3}{8} \right)^{\frac{2}{\alpha}}} \right. \\ &\times \left(\mathcal{G}^*\left(\tfrac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^{\left(\frac{16 - 3(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \frac{6\alpha}{\alpha+1} \left(\frac{3}{8} \right)^{\frac{1}{\alpha}} - \frac{3\alpha}{\alpha+2} \left(\frac{3}{8} \right)^{\frac{2}{\alpha}} \right)} \\ &\left. \times \left(\mathcal{G}^*\left(\tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right) \right)^{\frac{14-3\alpha}{4(\alpha+2)} + \frac{3\alpha}{2(\alpha+2)} \left(\frac{3}{8} \right)^{\frac{2}{\alpha}}} (\mathcal{G}^*(\varepsilon_2))^{\frac{1}{(\alpha+1)(\alpha+2)}} \right)^{\frac{\varepsilon_2 - \varepsilon_1}{36}}, \end{aligned}$$

where $\Theta(.)$ is defined by (4).

Proof. According to Lemma 2, multiplicative integration, and the multiplicative convexity of $\mathcal{G}^*$, we have

$$\begin{aligned} &\left| \left(\left(\mathcal{G}\left(\tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right) \right)^3 \left(\mathcal{G}\left(\tfrac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^2 \left(\mathcal{G}\left(\tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right) \right)^3 \right)^{\frac{1}{8}} (\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2 - \varepsilon_1)^{\alpha}}} \right| \\ \leq &\exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{36} \hbar^{\alpha} \left(\left| \ln \mathcal{G}^*\left((1-\hbar)\varepsilon_1 + \hbar \tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right) \right| \right) d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{72} \left| 3 - 8(1-\hbar)^{\alpha} \right| \left(\ln \left| \mathcal{G}^*\left((1-\hbar) \tfrac{5\varepsilon_1 + \varepsilon_2}{6} + \hbar \tfrac{\varepsilon_1 + \varepsilon_2}{2} \right) \right| \right) d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{72} \left| 8\hbar^{\alpha} - 3 \right| \left| \ln \mathcal{G}^*\left((1-\hbar) \tfrac{\varepsilon_1 + \varepsilon_2}{2} + \hbar \tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right) \right| d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{36} (1-\hbar)^{\alpha} \left(\left| \ln \mathcal{G}^*\left((1-\hbar) \tfrac{\varepsilon_1 + 5\varepsilon_2}{6} + \hbar\varepsilon_2 \right) \right| \right) d\hbar \right) \\ \leq &\exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{36} \hbar^{\alpha} \left(\ln\left(\mathcal{G}^*(\varepsilon_1)^{1-\hbar} \mathcal{G}^*\left(\tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right)^{\hbar} \right) \right) d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{72} \left| 3 - 8(1-\hbar)^{\alpha} \right| \left(\ln\left(\mathcal{G}^*\left(\tfrac{5\varepsilon_1 + \varepsilon_2}{6} \right)^{1-\hbar} \mathcal{G}^*\left(\tfrac{\varepsilon_1 + \varepsilon_2}{2} \right)^{\hbar} \right) \right) d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{72} \left| 8\hbar^{\alpha} - 3 \right| \ln\left(\mathcal{G}^*\left(\tfrac{\varepsilon_1 + \varepsilon_2}{2} \right)^{1-\hbar} \mathcal{G}^*\left(\tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right)^{\hbar} \right) d\hbar \right) \\ &\times \exp\left(\int_0^1 \tfrac{\varepsilon_2 - \varepsilon_1}{36} (1-\hbar)^{\alpha} \ln\left(f^*\left(\tfrac{\varepsilon_1 + 5\varepsilon_2}{6} \right)^{1-\hbar} f^*(\varepsilon_2)^{\hbar} \right) d\hbar \right) \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\frac{\varepsilon_2-\varepsilon_1}{36}\int_0^1 \hbar^\alpha\left((1-\hbar)\ln\mathcal{G}^*(\varepsilon_1)+\hbar\ln\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)d\hbar\right)\\
&\quad\times\exp\left(\frac{\varepsilon_2-\varepsilon_1}{72}\int_0^1 \left|3-8(1-\hbar)^\alpha\right|\left((1-\hbar)\ln\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)+\hbar\ln\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)d\hbar\right)\\
&\quad\times\exp\left(\frac{\varepsilon_2-\varepsilon_1}{72}\int_0^1 |8\hbar^\alpha-3|\left((1-\hbar)\ln\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+\hbar\ln\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)d\hbar\right)\\
&\quad\times\exp\left(\frac{\varepsilon_2-\varepsilon_1}{36}\int_0^1 (1-\hbar)^\alpha\left((1-\hbar)\ln\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)+\hbar\ln\mathcal{G}^*(\varepsilon_2)\right)d\hbar\right)\\
&= (\mathcal{G}^*(\varepsilon_1))^{\frac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)(\alpha+2)}}\left(\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{36(\alpha+2)}+\frac{b-a}{72}\left(\frac{10-3\alpha}{2(\alpha+2)}+\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}\right)}\\
&\quad\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{36}\left(\frac{16-3(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{6\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}\right)}\\
&\quad\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{\varepsilon_2-\varepsilon_1}{36(\alpha+2)}+\frac{b-a}{72}\left(\frac{10-3\alpha}{2(\alpha+2)}+\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}\right)}(\mathcal{G}^*(\varepsilon_2))^{\frac{\varepsilon_2-\varepsilon_1}{36(\alpha+1)(\alpha+2)}}\\
&= \left((\mathcal{G}^*(\varepsilon_1))^{\frac{1}{(\alpha+1)(\alpha+2)}}\left(\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^{\frac{14-3\alpha}{4(\alpha+2)}+\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}\right.\\
&\quad\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\left(\frac{16-3(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{6\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}\right)}\\
&\quad\left.\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{14-3\alpha}{4(\alpha+2)}+\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}(\mathcal{G}^*(\varepsilon_2))^{\frac{1}{(\alpha+1)(\alpha+2)}}\right)^{\frac{\varepsilon_2-\varepsilon_1}{36}}.
\end{aligned}
$$

The result follows from the calculation of the following integrals:

$$\int_0^1 \hbar^\alpha(1-\hbar)d\hbar=\int_0^1(1-\hbar)^\alpha\hbar d\hbar=\frac{1}{(\alpha+1)(\alpha+2)},$$

$$\int_0^1 \hbar^{\alpha+1}d\hbar=\int_0^1(1-\hbar)^{\alpha+1}d\hbar=\frac{1}{\alpha+2},$$

$$\int_0^1\left|3-8(1-\hbar)^\alpha\right|(1-\hbar)d\hbar=\int_0^1|8\hbar^\alpha-3|\hbar d\hbar=\frac{10-3\alpha}{2(\alpha+2)}+\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}$$

and

$$
\begin{aligned}
\int_0^1\left|3-8(1-\hbar)^\alpha\right|\hbar d\hbar &= \int_0^1|8\hbar^\alpha-3|(1-\hbar)d\hbar=\\
&= \frac{16-3(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{6\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{3\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}.
\end{aligned}
$$

The proof is completed. □

Remark 2. *If we put $\alpha=1$, Theorem 7 may be simplified to Theorem 3.2 from [4].*

Corollary 2. *Using the multiplicative convexity of $\mathcal{G}^*$, i.e., $\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) \le \sqrt{\mathcal{G}^*(\varepsilon_1)\mathcal{G}^*(\varepsilon_2)}$, Theorem 7 becomes*

$$\left|\left(\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3\right)^{\frac{1}{8}}(\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}\right|$$
$$\le \left((\mathcal{G}^*(\varepsilon_1))^{\frac{14-3\alpha^2-9\alpha}{4(\alpha+1)(\alpha+2)}+\frac{6\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}\left(\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^{\frac{14-3\alpha}{4(\alpha+2)}+\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}\right.$$
$$\left.\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{14-3\alpha}{4(\alpha+2)}+\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}(\mathcal{G}^*(\varepsilon_2))^{\frac{14-3\alpha^2-9\alpha}{4(\alpha+1)(\alpha+2)}+\frac{6\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}-\frac{3\alpha}{2(\alpha+2)}\left(\frac{3}{8}\right)^{\frac{2}{\alpha}}}\right)^{\frac{\varepsilon_2-\varepsilon_1}{36}}.$$

Remark 3. *Corollary 2 will be reduced to Corollary 3.3 from [4], if we take $\alpha = 1$.*

Corollary 3. *Using the multiplicative convexity of $\mathcal{G}^*$, i.e., $\mathcal{G}^*\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) \le \sqrt{\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)}$, Theorem 7 becomes*

$$\left|\left(\left(\mathcal{G}\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^3\left(\mathcal{G}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^2\left(\mathcal{G}\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^3\right)^{\frac{1}{8}}(\Theta(\mathcal{G}))^{-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^\alpha}}\right|$$
$$\le \left((\mathcal{G}^*(\varepsilon_1))^{\frac{1}{(\alpha+1)(\alpha+2)}}\left(\mathcal{G}^*\left(\frac{5\varepsilon_1+\varepsilon_2}{6}\right)\right)^{\frac{12+\alpha-3\alpha^2}{2(\alpha+1)(\alpha+2)}+\frac{6\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\right.$$
$$\left.\times\left(\mathcal{G}^*\left(\frac{\varepsilon_1+5\varepsilon_2}{6}\right)\right)^{\frac{12+\alpha-3\alpha^2}{2(\alpha+1)(\alpha+2)}+\frac{6\alpha}{2(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(\mathcal{G}^*(\varepsilon_2))^{\frac{1}{(\alpha+1)(\alpha+2)}}\right)^{\frac{\varepsilon_2-\varepsilon_1}{36}}.$$

Remark 4. *Corollary 3 will be reduced to Corollary 3.4 from [4], if we take $\alpha = 1$.*

3. Applications to Special Means

Consider the following means of arbitrary real number $\eta_1, \eta_2, \ldots, \eta_n$:
The arithmetic mean: $A(\eta_1, \eta_2, \ldots, \eta_n) = \frac{\eta_1+\eta_2+\ldots+\eta_n}{n}$.
The harmonic mean: $H(\eta_1, \eta_2, \ldots, \eta_n) = \frac{n}{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\ldots+\frac{1}{\eta_n}}$.
The logarithmic means: $L(\eta_1, \eta_2) = \frac{\eta_2-\eta_1}{\ln\eta_2-\ln\eta_1}$, $\eta_1, \eta_2 > 0$, and $\eta_1 \ne \eta_2$.
The k-logarithmic mean: $L_k(\eta_1, \eta_2) = \left(\frac{\eta_2^{k+1}-\eta_1^{k+1}}{(k+1)(\eta_2-\eta_1)}\right)^{\frac{1}{k}}$, $\eta_1, \eta_2 > 0, \eta_1 \ne \eta_2$, and $k \in \mathbb{R}\backslash\{-1, 0\}$.

Proposition 2. *For two positive real numbers $0 < \eta_1 < \eta_2$, we have*

$$e^{\frac{3}{8}A^p(\eta_1,\eta_1,\eta_1,\eta_1,\eta_1,\eta_2)+\frac{1}{4}A^p(\eta_1,\eta_2)+\frac{3}{8}A^p(\eta_1,\eta_2,\eta_2,\eta_2,\eta_2,\eta_2)-L_p^p(\eta_1,\eta_2)} \le e^{\frac{25(\eta_2-\eta_1)p\eta_2^{p-1}}{288}}.$$

Proof. It suffices to apply Corollary 2, taking $\mathcal{G}(\hbar) = e^{\hbar^p}$ as a function with $p \ge 2$ where $\mathcal{G}^*(\hbar) = e^{p\hbar^{p-1}}$, $\mathcal{M} = e^{p\eta_2^{p-1}}$, and $\left(\int_{\eta_1}^{\eta_2}\mathcal{G}(u)^{du}\right)^{\frac{1}{\eta_1-\eta_2}} = \exp\left\{-L_p^p(\eta_1, \eta_2)\right\}$. □

Proposition 3. *For two positive real numbers $0 < \eta_1 < \eta_2$ and $n > 0$, we have*

$$\begin{aligned} & H^{\frac{3n}{8}}(\eta_2,\eta_2,\eta_2,\eta_2,\eta_2,\eta_1) H^{\frac{n}{4}}(\eta_2,\eta_1) H^{\frac{3n}{8}}(\eta_2,\eta_1,\eta_1,\eta_1,\eta_1,\eta_1) \\ \leq \quad & \eta_2^{\frac{n\eta_1}{\eta_1-\eta_2}} \eta_1^{-\frac{n\eta_2}{\eta_1-\eta_2}} e^{1-n\frac{\eta_2-\eta_1}{864}\left(4(\eta_1+\eta_2)+67\frac{\eta_1-\eta_2}{3\eta_1\eta_2}\right)}. \end{aligned}$$

Proof. It suffices to apply Corollary 3 with $\alpha = 1$ on the interval $\left[\frac{1}{\eta_2}, \frac{1}{\eta_1}\right]$ to the function $\mathcal{G}(t) = \frac{1}{\hbar^n}$, whose $\mathcal{G}^*(\hbar) = e^{-\frac{n}{\hbar}}$ and $\left(\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} f(u)^{du}\right)^{\frac{\eta_1\eta_2}{\eta_1-\eta_2}} = \eta_1^{\frac{n\eta_2}{\eta_1-\eta_2}} \eta_2^{-\frac{n\eta_1}{\eta_1-\eta_2}} e^{-n}$. $\square$

4. Conclusions

The conclusions produced in this work are based on a novel identity. We have constructed certain fractional Maclaurin-type integral inequalities for functions whose multiplicative derivatives are both bounded and multiplicatively convex. We have also discussed some particular cases. A few applications of our findings to special means are given. Our results improve those established in [29], and they also recover those established in [21].

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