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On Constructing a Family of Sixth-Order Methods for Multiple Roots

Young Hee Geum

Department of Applied Mathematics, Dankook University, Cheonan 31116, Republic of Korea; conpana@dankook.ac.kr or conpana@empal.com

Abstract: A family of three-point, sixth-order, multiple-zero solvers is developed, and special cases of weight functions are investigated based on polynomials and low-order rational functions. The chosen cases of the proposed iterative method are compared with existing methods. The experiments show the superiority of the proposed schemes in terms of the number of divergent points and the average number of function evaluations per point. The dynamical characteristics of the developed methods, along with their illustrations, are represented with detailed analyses, comparisons, and comments.

Keywords: basins of attraction; blood rheology model; iteration method; multiple root; nonlinear equation

1. Introduction

Nonlinear equations [1–4] occur frequently in various fields of science, artificial intelligence, and engineering. Finding the roots of a nonlinear equation [5–8] involves determining the value of a variable that satisfies a given equation. A zero α of $h(x) = 0$ is called a multiple root with multiplicity m if $h^{(i)}(\alpha) = 0,\ i = 0, 1, 2, \cdots, m-1$ and $h^{(m)}(\alpha) \neq 0$.

It is known that the Newton method is the most used method for solving the equations, given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, \cdots . \tag{1}$$

Researchers [9–15] are interested in finding multiple roots of nonlinear equations [16–20] and investigating the dynamics by exploring the relevant basins of attraction [21–26].

Geum-Kim-Neta [27,28] constructed a class of two-point, sixth-order, multiple-root solvers. Since the following two methods are faster and require fewer iterations per point on average, we have chosen two methods as follows. These two schemes, called here (X1) and (X2), are given by

$$\begin{cases} y_n = x_n - m\frac{f(x)}{f'(x)}, \\ x_{n+1} = y_n - m[2(m-1)(u-s) - u^2 - 2us + 1]h, \end{cases} \tag{2}$$

and

$$\begin{cases} y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{au+m}{+cu^2+1+b*u} \cdot \frac{1}{1+ds}, \end{cases} \tag{3}$$

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where

$$a = \frac{13 + 2m(4m^4 - 16m^3 + 31m^2 - 30m + 13)}{(m-1)(7 + 4m^2 - 8m)}, \quad b = \frac{4(2m^2 - 4m + 3)}{(m-1)(4m^2 - 8m + 7)},$$
$$c = -\frac{3 + 4m^2 - 8m}{7 + 4m^2 - 8m}, \quad d = 2(m-1), \quad u = (\frac{f(y_n)}{f(x_n)})^{\frac{1}{m}},$$
$$s = (\frac{f'(y_n)}{f'(x_n)})^{\frac{1}{m-1}}.$$

In this paper, we propose a three-point, sixth-order method by adding a third step, as follows:

$$\begin{cases} y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - mA_f(k)\frac{f(x_n)}{f'(x_n)}, \; k = (\frac{f'(y_n)}{f'(x_n)})^{\frac{1}{m-1}}, \\ x_n = x_n - mB_f(k,v)\frac{f(x_n)}{f'(x_n)}, \; v = (\frac{f(w_n)}{f(x_n)})^{\frac{1}{m}}, \end{cases} \tag{4}$$

where $A_f : C \to C$ is analytic in a small neighborhood of 0, and $B_f : C^2 \to C$ is holomorphic in a neighborhood of $(0,0)$.

Definition 1. *Let $a_0, a_1, \cdots a_n \cdots$ be a sequence converging to a zero α and $e_n = a_n - \alpha$ be the n-th iterative error. If there exist real numbers $\beta \in R$ and $\delta \in R - \{0\}$, such that the following error equations satisfy*

$$e_{n+1} = \delta {e_n}^{\beta} + O({e_n}^{\beta+1}),$$

then $|\delta|$ or δ is called the asymptotic error constant, and β is defined as the order of convergence [1].

Definition 2. *Suppose that the theoretical asymptotic error constant $\omega = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^{\beta}}$ and the convergence order $\beta \geq 1$ are known [2,3]. The computational convergence order $\beta_n = \frac{\log|e_n/\omega|}{\log|e_{n-1}|}$ is defined. Then $\lim_{n\to\infty} \beta_n = \beta$.*

Definition 3. *If a fixed point ρ of a rational map J is attracting, then all nearby points of ρ are attracted to ρ under a rational map J. The collection of all points whose iterates under J converge to ρ is called the basin of attraction of ρ.*

We investigate the convergence behavior of the proposed scheme in Section 2. The numerical results and the basins of attraction for the typical examples are shown in Section 3. In addition, the basins of attraction for the equation in the blood rheology model are shown, and the convergent regions for selected values are plotted in Section 3. Finally, Section 4 is discussed in the last section.

2. Convergence Analysis

In this section, we determine the maximal convergence order of the developed schemes. We investigate the main theorem describing the convergent behavior of the proposed schemes and address how to construct weight functions A_f and B_f for sixth-order convergence. It is sufficient to consider weight functions A_f and B_f up to the fifth-order terms in e_n in terms of $O(\frac{f(x_n)}{f'(x_n)}) = O(e_n)$.

Let a function $f : C \to C$ have a root α of multiplicity $m > 1$ and be analytic in a small neighborhood of α.

$$
\begin{aligned}
y_n =& x_n - m\frac{f(x_n)}{f'(x_n)},\\
w_n =& x_n - m\cdot A_f(k)\cdot\frac{f(x_n)}{f'(x_n)},\ k=(\frac{f'(y_n)}{f'(x_n)})^{\frac{1}{m-1}},\\
x_n =& x_n - m\cdot B_f(k,v)\cdot\frac{f(x_n)}{f'(x_n)},\ v=(\frac{f(w_n)}{f(x_n)})^{\frac{1}{m}},
\end{aligned}
$$

where $A_f: C \to C$ is analytic in a small neighborhood of the origin 0 and $B_f: C^2 \to C$ is holomorphic in a small neighborhood of $(0,0)$.

Using the Taylor expansion about α, we have

$$f(x_n) = e^m\Gamma(1+\theta_2 e_n + \theta_3 {e_n}^2 + \theta_4 {e_n}^3 + \theta_5 {e_n}^4 + \theta_6 {e_n}^5 + \theta_7 {e_n}^6 + O({e_n}^7)), \tag{5}$$

$$
\begin{aligned}
f'(x_n) =& e^{m-1}\Gamma(m + (1+m)\theta_2 e_n + (2+m)\theta_3 {e_n}^2 + (3+m)\theta_4 {e_n}^3 + (4+m)\theta_5 {e_n}^4\\
&+ (5+m)\theta_6 {e_n}^5 + (6+m)\theta_7 {e_n}^6 + O({e_n}^7)),
\end{aligned} \tag{6}
$$

where $\Gamma = \frac{f^{(m)}(\alpha)}{m!}$, $\theta_j = \frac{m!}{(m+j-1)!}\frac{f^{(m+j-1)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in N - \{1\}$.

For convenience, we denote e_n by e without subscript n. Dividing (5) by (6), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{e}{m} - \frac{\theta_2 e^2}{m^2} + \frac{t_3 e^3}{m^3} + \frac{t_4 e^4}{m^4} + \frac{t_5 e^5}{m^5} + \frac{t_6 e^6}{m^6} + O(e^7), \tag{7}$$

where

$$
\begin{aligned}
t_3 =& ((1+m){\theta_2}^2 - 2m\theta_3),\\
t_4 =& (-(1+m)^2{\theta_2}^3 + m(4+3m)\theta_2\theta_3 - 3m^2\theta_4),\\
t_5 =& (1+m)^3{\theta_2}^4 - 2m(3+5m+2m^2){\theta_2}^2\theta_3 + 2m^2(3+2m)\theta_2\theta_4\\
&+ 2m^2((2+m){\theta_3}^2 - 2m\theta_5),
\end{aligned}
$$

and

$$
\begin{aligned}
t_6 =& -(1+m)^4{\theta_2}^5 + m(1+m)^2(8+5m){\theta_2}^3\theta_3 - m^2(9+14m+5m^2){\theta_2}^2\theta_4\\
&+ m^2\theta_2(-((12+16m+5m^2){\theta_3}^2) + m(8+5m)\theta_5) + m^3((12+5m)\theta_3\theta_4 - 5m\theta_6).
\end{aligned}
$$

By Taylor's expansion, we have expression k, as follows:

$$
\begin{aligned}
k =& (\frac{f'(y_n)}{f'(x_n)})^{\frac{1}{m-1}}, = \frac{\theta_2}{m}e + \frac{-m(1+m){\theta_2}^2 - 2m(1-m)\theta_3}{m(-1+m)}e^2\\
&+ \frac{w_3}{2(m-1)^2}e^3 + \frac{w_4}{6(-1+m)^3}e^4 + \frac{w_5}{24(-1+m)^4}e^5 + \frac{w_6}{120(-1+m)^5}e^6 + O(e^7)),
\end{aligned} \tag{8}
$$

where $w_i = w_i(\theta_2, \theta_3, \cdots \theta_6)$ for $3 \le i \le 6$.

Using k in (8), and expanding the Taylor series of $A_f(k)$ about 0 up to the fifth-order term, we find

$$A_f(k) = A_0 + A_1 k + A_2 k^2 + A_3 k^3 + A_4 k^4 + A_5 k^5 + O(e^6), \tag{9}$$

where $A_j = \frac{{A_f}^{(j)}(0)}{j!}$ for $0 \le j \le 5$.

Substituting (5)–(9) into w_n, we have

$$w_n = \alpha + (1-A_0)e + \frac{(A_0 - A_1)}{m}\theta_2 e^2 + z_3 e^3 + z_4 e^4 + z_5 e^5 + z_6 e^6 + O(e^7), \tag{10}$$

where
$z_3 = \frac{(A_0 - A_1 + A_2 + 2A_1 m - A_2 m - A_0 m^2 + A_1 m^2)\theta_2{}^2 + (-2A_0 m + 2A_1 m + 2A_0 m^2 - 2A_2 m^2)\theta_3}{m^2(-1+m)}$,
$z_i = z_i(\theta_2, \cdots \theta_n, A_0, A_1, \cdots A_4)$ for $4 \leq i \leq 6$.

Choosing $A_0 = A_1 = 1$, $A_2 = \frac{2m}{-1+m}$, we have

$$\begin{aligned} w_n =& \alpha + \frac{((2+m+m^3-2m^2(-4+A_3)-2A_3+4mA_3)\theta_2{}^3 - 2(-1+m)m^2\theta_2\theta_3)}{2m^3(-1+m)^2} e^4 \\ &+ z_5 e^5 + z_6 e^6 + O(e^7). \end{aligned} \tag{11}$$

Then we obtain

$$\begin{aligned} f(w_n) =& \frac{e^{4m}\Gamma}{(-1+m)3^m} (\frac{1}{2}(2+m+m^3-2m^2(-4+A_3)-2A_3+4mA_3)\theta_2{}^3 - (-1+m)m^2\theta_2\theta_3{}^m \\ &+ \frac{1}{3}((2+m+m^3-2m^2(-4+A_3)-2A_3+4mA_3)\theta_2{}^3 - 2(-1+m)m^2\theta_2\theta_3)^{-1+m} \\ &((-7m^5+2m^4(-35+9A_3)+m(17+36A_3-18A_4) \\ &+6(2-A_3+A_4)-6m^3(9+2A_3+A_4)+m^2(14-36A_3+18A_4))\theta_2{}^4 \\ &+12(-1+m)m(2+m+2m^3-3m^2(-4+A_3)-3A_3+6mA_3)\theta_2{}^2\theta_3 \\ &-12(-1+m)^2m^3\theta_3{}^2 - 12(-1+m)^2m^3\theta_2\theta_4)e + w_2e^2 + w_3e^3 + O(e^4)), \end{aligned} \tag{12}$$

where $w_i = w_i(\theta_2, \cdots \theta_n, A_3, A_4), i = 2, 3$.

With the use of (5) and (12), we have v as follows:

$$\begin{aligned} v =& \frac{((2+m+m^3-2m^2(-4+A_3)-2A_3+4mA_3)\theta_2{}^3 - 2(-1+m)m^2\theta_2\theta_3}{2(-1+m)^2m^3} e^3 \\ &+ \frac{1}{6(-1+m)^3m^4}((-7m^5+m^4(-73+18A_3)+2m(7+27A_3-9A_4) \\ &+6(3-2A_3+A_4)-3m^3(25+2A_3+2A_4)+m^2(35-54A_3+18A_4))\theta_2{}^4 \\ &+6(-1+m)m(4+m+4m^3+m^2(25-6A_3)-6A_3+12mA_3)\theta_2{}^2\theta_3 \\ &-12(-1+m)^2m^3\theta_3{}^2 - 12(-1+m)^2m^3\theta_2\theta_4)e^4 + v_5e^5 + O(e^6), \end{aligned} \tag{13}$$

where $v_5 = v_5(\theta_2, \cdots \theta_n, A_1, \cdots A_4)$.

Expanding the Taylor series of $B_f(k, v)$ about (0,0) up to the fifth-order term, we find

$$B_f(k, v) = B_{00} + B_{30}k^3 + B_{40}k^4 + B_{50}k^5 + B_{01}v + k(B_{10} + B_{11}v) + k^2(B_{20} + B_{21}v) + O(e^6), \tag{14}$$

where $B_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial k^i \partial v^j} B_f(k, v)|_{(k=0, v=0)}$ for $0 \leq i \leq 5, 0 \leq j \leq 1$.

By Substituting (5)–(14) into the proposed scheme (4), we have

$$x_{n+1} - \alpha = x_n - \alpha - B_f(k, v)\frac{f(x_n)}{f'(x_n)} \tag{15}$$

$$= C_1 e + C_2 e^2 + C_3 e^3 + C_4 e^4 + C_5 e^5 + C_6 e^6 + O(e^7), \tag{16}$$

where $C_1 = 1 - B_{00}$ and the coefficients $C_i (2 \leq i \leq 6)$ depend of m, $A_j (j = 0, 1, \cdots 5)$, and $\theta_i (i = 1, 2, \cdots)$.

Solving C_1 for B_{00}, we have

$$B_{00} = 1. \tag{17}$$

Using $B_{00} = 1$ into C_2, we have

$$B_{10} = 1. \tag{18}$$

Substituting $B_{00} = B_{10} = 1$ into C_3, we obtain $\frac{(B_{20} + 2m - B_{20}m)\theta_2{}^2}{m(m-1)} = 0$.

From which we find

$$B_{20} = \frac{2m}{-1+m}. \tag{19}$$

Substituting $B_{00} = B_{10} = 1$ and $B_{20} = \frac{2m}{-1+m}$ into $C_4 = 0$, we have

$$\begin{aligned} C_4 =& \frac{1}{2(-1+m)^2} m^{-\frac{3m}{-1+m}} (2(4+(-4+A_3)B_{01} - B_{30})m^{2+\frac{3}{-1+m}} \\ & - 2(-1+B_{01} - A_3 B_{01} + B_{30})m^{\frac{3}{-1+m}} + (1-B_{01})m^{\frac{3m}{-1+m}} \\ & + (-1 - B_{01} - 4A_4 B_{01} + 4B_{30})m^{\frac{2+m}{-1+m}})\theta_2{}^3 \\ & + \frac{m^{-\frac{3m}{-1+m}}(2m^{2+\frac{3}{-1+m}} - 2B_{01}m^{2+\frac{3}{-1+m}} - 2(-1-B_{01})m^{\frac{3m}{-1+m}})\theta_2\theta_3}{2(-1+m)^2}. \end{aligned}$$

and

$$B_{01} = 1,\ B_{30} = A_3. \tag{20}$$

Substituting $B_{00} = B_{10} = 1$, $B_{20} = \frac{2m}{-1+m}$ $B_{01} = 1, B_{30} = A_3$ into $C_5 = 0$, we have

$$\begin{aligned} C_5 =& \frac{1}{2(-1+m)^3} m^{-\frac{4m}{-1+m}} ((-14 - 6A_4 - 6A_3(-2+B_{11}) + 7B_{11} + 6B_{40}))m^{2+\frac{4}{-1+m}} + \\ & (14 + 2A_4 + 2A_3(-2+B_{11}) - 7B_{11} - 2B_{40})m^{3+\frac{4}{-1+m}} \\ & + 2(-2 - A_4 - A_3(-2+B_{11}) + B_{11} + B_{40})m^{\frac{4}{-1+m}}) \\ & + (2 - B_{11})m^{\frac{4m}{-1+m}} + (2 + 6A_4 + 6A_3(-2+B_{11}) - B_{11} - 6B_{40})m^{\frac{3+m}{-1+m}}\theta_2{}^4 \\ & + \frac{m^{-\frac{4m}{-1+m}}(2(-2+B_{11})m^{2+\frac{4}{-1+m}} - 4(-2+B_{11})m^{3+\frac{4}{-1+m}} - 2(2-B_{11})m^{\frac{4m}{-1+m}})\theta_2{}^2\theta_3}{2(-1+m)^3}, \end{aligned}$$

and

$$B_{40} = A_4,\ B_{11} = 2. \tag{21}$$

Putting (17)–(21) into C_6, we obtain

$$C_6 = \frac{p_1\theta_2{}^5 + p_2\theta_2{}^3\theta_3 + p_3\theta_2\theta_3{}^2}{4(-1+m)^3 m^5}, \tag{22}$$

where

$$\begin{aligned} p_1 =& 4A_5(-1+m)^3 + 4A_3B_{21}(-1+m)^3 - 4B_{50}(-1+m)^3 - 2B_{21}(-1+m) \\ & (2 + m + 8m^2 + m^3) + (2 - 2A_3(-1+m)^2 + m + m^2(8+m))(-3 + m(10+m)), \\ p_2 =& -4(-1+m)m(1 + m^3 - A_3 - m^2(-9 + B_{21} + A_3) + m(-1 + B_{21} + 2A_3)), \\ p_3 =& 4(-1+m)^2 m^3. \end{aligned}$$

Then the proposed method, (4), is of the sixth order and possesses the following error equation:

$$e_{n+1} = \frac{p_1\theta_2{}^5 + p_2\theta_2{}^3\theta_3 + p_3\theta_2\theta_3{}^2}{4(-1+m)^3 m^5} e_n{}^6 + O(e_n{}^7), \tag{23}$$

where

$$\begin{aligned}p_1 =&4A_5(-1+m)^3+4A_3B_{21}(-1+m)^3-4B_{50}(-1+m)^3-2B_{21}(-1+m)\\&(2+m+8m^2+m^3)+(2-2A_3(-1+m)^2+m+m^2(8+m))(-3+m(10+m)),\\p_2 =&-4(-1+m)m(1+m^3-A_3-m^2(-9+B_{21}+A_3)+m(-1+B_{21}+2A_3)),\\p_3 =&4(-1+m)^2m^3.\end{aligned}$$

Theorem 1. *Let $m \in N - \{1\}$. Let a function $f : C \to C$ have a multiple root α with multiplicity m. Assume that f is analytic in a neighborhood of α. Let $\theta_j = \frac{m!}{(m+j-1)!}\frac{f^{(n+j-1)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in N - \{1\}$. Let x_0 be an initial value in a neighborhood of zero α. Let A_j $(0 \le j \le 5)$ be defined in (9) and let B_{ij} $(0 \le j \le 5, 0 \le j \le 1)$ be defined in (14). Suppose $A_0 = A_1 = 1$, $A_2 = \frac{2m}{-1+m}$, $|A_3| < \infty, |A_4| < \infty, |A_5| < \infty$ and $B_{00} = B_{10} = B_{01} = 1, B_{20} = \frac{2m}{-1+m}, B_{30} = A_3, B_{40} = A_4, B_{11} = 2, |B_{50}| < \infty, |B_{21}| < \infty$ hold. Then the iterative scheme, (4), is of the sixth-order and has the following error equation:*

$$e_{n+1} = \frac{p_1{\theta_2}^5 + p_2{\theta_2}^3\theta_3 + p_3\theta_2{\theta_3}^2}{4(-1+m)^3m^5}{e_n}^6 + O({e_n}^7),$$

where p_i $(i = 1, 2, 3)$ is defined in (22).

3. Numerical Experiments

Based on the convergence analysis, Taylor-polynomial forms of $A_f(k)$ and $B_f(k, v)$ are given by

$$\begin{cases} A_f(k) = A_0 + A_1k + A_2k^2 + A_3k^3 + A_4k^4 + A_5k^5, \\ B_f(k, v) = B_{00} + B_{10}k + B_{20}k^2 + B_{30}k^3 + B_{40}k^4 + B_{50}k^5 + (B_{01} + B_{11}k + B_{21}k^2)v, \end{cases} \tag{24}$$

where $A_0 = A_1 = 1, A_2 = 2m/(-1+m), A_3, A_4, A_5(free)$ and $B_{00} = B_{01} = B_{10} = 1$, $B_{20} = 2m/(-1+m), B_{11} = 2, B_{30} = A_3, B_{40} = A_4$.

Even if various forms of $A_f(k)$ and $B_f(k, v)$ are possible, we limit ourselves to consider four forms, as follows:

$(M1)$ $B_{50} = B_{21} = 0$

$$\begin{aligned}A_f(k) =&1 + k + \frac{2m}{-1+m}k^2,\\B_k(k, v) =&1 + k + \frac{2m}{-1+m}k^2 + (1 + 2k)v.\end{aligned}$$

$(M2)$ $B_{50} = 0, B_{21} = 1$

$$\begin{aligned}A_f(k) =&1 + k + \frac{2m}{-1+m}k^2,\\B_k(k, v) =&1 + k + \frac{2m}{-1+m}k^2 + (1 + 2k + k^2)v.\end{aligned}$$

$(M3)$ $B_{50} = 1, B_{21} = 0$

$$\begin{aligned}A_f(k) =&1 + k + \frac{2m}{-1+m}k^2,\\B_k(k, v) =&1 + k + \frac{2m}{-1+m} + k^5 + (1 + 2k)v.\end{aligned}$$

$(M4)$ The product of two univariate functions

$$A_f(k) = \frac{1 + \frac{2m}{-1+m}k^2}{1-k},$$
$$B_k(k,v) = \frac{1 + \frac{2m}{-1+m}k^2}{1-k-v}.$$

The numerical experiments are carried out using Mathematica programming to confirm the developed theory. For the experiments, we maintain a minimum of 100 digits of precision to achieve the specified accuracy. If the root is not exact, it is approximated by a value with greater precision, possessing more significant digits than the specified number of precision.

Various numerical experiments have been conducted with Mathematica software 13.0 [29] to confirm the developed theory. In Table 1, the computational convergence order approaches 6. They confirm the sixth-order convergence with test functions $h_1(x), h_2(x), h_3(x)$, and $h_4(x)$, as follows:

$$h_1(x) = (\cos[\frac{\pi}{2}x] + x^2 - \pi)^5,\ m = 5,\ \alpha \approx -2.0347.$$
$$h_2(x) = (\cos[x^2-1] + 1 - x\log[x^2-\pi]))^2(x^2-1-\pi),\ m = 3,\ \alpha = \sqrt{1+\pi}.$$
$$h_3(x) = (\sin^{-1}[x-1)] + x^{x^2} - 3)^3,\ m = 3,\ \alpha \approx 1.0414818.$$
$$h_4(x) = (\cos 2x + 9 - 2x - 2x^4)(-\sin^2 x + 5 - x - x^4)\ m = 2,\ \alpha \approx 1.29173329.$$

Table 1. Convergent process for $g_i(x), (i = 1, 2, 3, 4)$.

Method	Function	n	x_n	$\lvert x_n - \alpha\rvert$	$\lvert e_n/e_{n-1}{}^6\rvert$	p_n
$M1$	h_1	0	−2.1	0.0652751		
		1	−2.0372492017726	4.913×10^{-8}	0.3089431095	6.07933
		2	−2.0372476627913	5.378×10^{-45}	0.428220700	6.00000
		3	−2.0372476627913	0.0×10^{-98}		
$M2$	h_2	0	2.0	0.0350903		
		1	2.0350902813353	2.389×10^{-7}	26.31721953	6.10370
		2	2.0350903306632	7.978×10^{-45}	38.017167	6.00000
		3	2.0350903306632	0.0×10^{-99}		
$M3$	h_3	0	1.084	0.0425181		
		1	1.04148199694193	1.263×10^{-7}	21.38733354	6.06612
		2	1.041481808433	1.076×10^{-40}	26.44205449	6.00000
		3	1.041481808433	0.0×10^{-99}		
$M4$	h_4	0	1.35	0.0582667		
		1	1.29173359504767	3.07×10^{-7}	7.7330687	6.23175
		2	1.29173329265770	9.550×10^{-40}	12.73465793	6.00000
		3	1.29173329265770	0.0×10^{-99}		

Methods $(M1), (M2), (M3), (M4), (X1)$, and $(X2)$ have been applied to the test functions $f_1(x), f_2(x), f_3(x)$ below:

$$f_1(x) = \exp[\frac{(1+x^3)^2}{7\cos(x^3+1)+x^5}] - 1,\ \alpha = \frac{1+i\sqrt{3}}{2}, m = 2, x_0 = 0.51 + 0.83i, i = \sqrt{-1}$$
$$f_2(x) = (-\pi + 2x + \cos[x]\log(x^2+1))^3,\ \alpha = \pi/2, m = 3, x_0 = 1.5,$$
$$f_3(x) = (x - \sqrt{3}x^3\cos[\frac{\pi x}{6}] + \frac{1}{x^2+1} - \frac{11}{5} + 4\sqrt{3})(x-2)^3,\ \alpha = 2, m = 4, x_0 = 1.93,$$

In Table 2, we compare errors $|x_n - \alpha|$ of methods $(M1), (M2), (M3), (M4), (X1)$, and $(X2)$. The least errors within the prescribed error bound are marked in boldface. Method $(M1)$ shows slightly better convergence in this experiment. The local convergence depends on the function, an initial guess, the root, the multiplicity, and the weight functions. So, for a selected set of test functions, one scheme is hardly expected to show better achievement than the others.

Table 2. Comparison of $|x_n - \alpha|$ for $f_1(x)$, $f_2(x)$ and $f_3(x)$.

Function	$\|x_n - \alpha\|$	*M1*	*M2*	*M3*	*M4*	*X1*	*X2*
f_1	$\|x_1 - \alpha\|$	2.31×10^{-11}	2.61×10^{-7}	7.76×10^{-10}	1.21×10^{-8}	3.11×10^{-7}	2.63×10^{-6}
	$\|x_2 - \alpha\|$	$\mathbf{3.71 \times 10^{-62}}$	2.35×10^{-43}	2.42×10^{-55}	3.51×10^{-47}	8.52×10^{-40}	7.16×10^{-41}
f_2	$\|x_1 - \alpha\|$	5.16×10^{-7}	5.57×10^{-8}	3.12×10^{-7}	2.12×10^{-7}	5.32×10^{-6}	2.22×10^{-5}
	$\|x_2 - \alpha\|$	$\mathbf{2.31 \times 10^{-41}}$	1.66×10^{-40}	4.16×10^{-39}	3.51×10^{-38}	2.12×10^{-37}	5.31×10^{-39}
f_3	$\|x_1 - \alpha\|$	2.71×10^{-6}	3.46×10^{-6}	1.67×10^{-5}	1.88×10^{-6}	7.62×10^{-5}	4.42×10^{-8}
	$\|x_2 - \alpha\|$	$\mathbf{3.71 \times 10^{-42}}$	5.23×10^{-42}	5.11×10^{-41}	6.75×10^{-40}	6.51×10^{-35}	4.12×10^{-41}

We study the dynamics of numerical methods $(M1), (M2), (M3), (M4), (X1)$, and $(X2)$. In Table 3, abbreviations **CPU**, **tcon**, **avg**, and **tdiv** denote the value of the CPU time for sixth-order convergence, the total number of convergent points, the value of the average iterative number for sixth-order convergence, and the number of divergent points. Table 3 shows the statistical data for the basins of attraction.

Table 3. Comparison of CPU time, tcon, avg, and tdiv.

p_m	Method	CPU	tcon	avg	tdiv
$p_1(z)$	M1	120.922	360,000	8.73961	0
	M2	165.969	360,000	8.96783	0
	M3	177.515	360,000	8.96783	0
	M4	183.984	354,440	8.15107	5560
	X1	1234.78	0	-	360,000
	X2	333.578	0	-	360,000
$p_2(z)$	M1	274.985	360,000	9.09136	0
	M2	340.906	360,000	9.19766	0
	M3	266.187	360,000	9.07998	0
	M4	306.485	360,000	7.92103	0
	X1	10.822	20,880	142.582	339,120
	X2	1575.34	6118	37.9915	353,882
$p_3(z)$	M1	3104.25	360,000	9.30217	107,540
	M2	3584.16	249,022	8.94829	110,978
	M3	3660.17	249,022	8.94829	110,978
	M4	497.906	287,654	5.48361	72,346
	X1	519.563	0	-	360,000
	X2	3443.61	1054	26.759	358,946

In Figures 1–4, a six-by-six square region centered at the origin includes all the roots of the test functions. A 600×600 uniform grid in this region is used to mark initial values for the iterative schemes to draw the basins of attraction. The grid point of a square is colored differently according to the number of iterations required for convergence. The point is colored in black if the methods do not converge.

Example 1. *We choose a quadratic polynomial raised to the power of two:*

$$p_1(z) = (2z^2 + 1)^2$$

with roots $\alpha = \pm 0.707107i$ with a multiplicity of 2. Basins of attraction for (M1), (M2), (M3), (M4), (X1), and (X2) are illustrated in Figure 1. Each basin is painted in various colors. Basins for $\alpha = +0.707107i$ are painted purple and basins for $\alpha = -0.707107i$ are painted green in Figure 1 (a)–(c). At zero, its color is light (white), while becoming darker for more iterative numbers required for convergence within the set iteration. Method (M1) performs better in terms of ***CPU****, and M4 is* *better in terms of **avg** for $p_1(z)$. There are similar lobes along the boundaries of the basins in (a), (b), and (c) . Picture (d) has colored (yellow, magenta, green, blue, and so on) carrot shapes around the boundary. Methods (X1) and (X2) do not have the value of the total convergent point, and pictures (e) and (f) show the black region.*

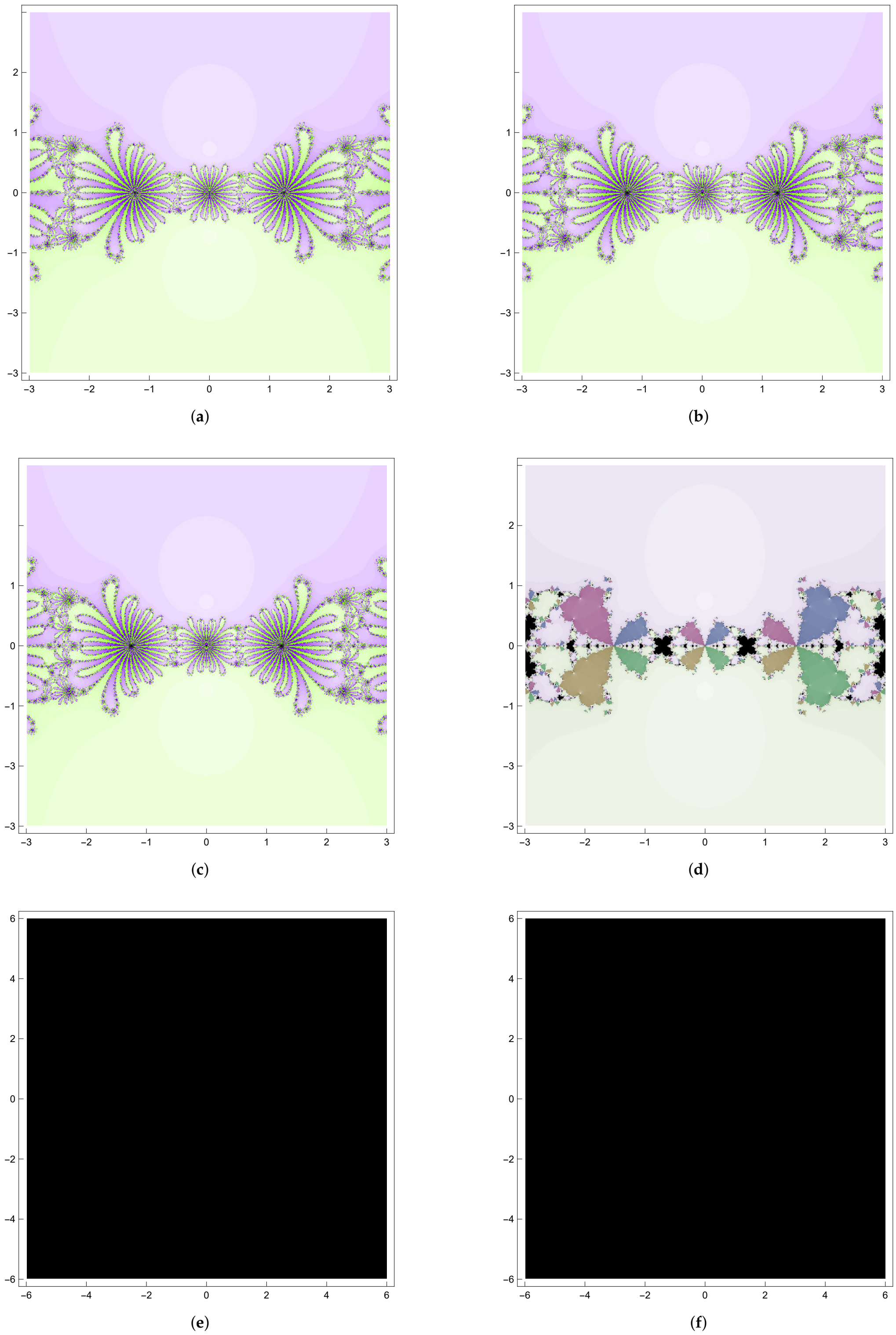


Figure 1. Basins of attraction for $P_1(z) = (2z^2 + 1)^2$. **(a)** $M1$; **(b)** $M2$; **(c)** $M3$; **(d)** $M4$; **(e)** $X1$; **(f)** $X2$.

Example 2. *As a second test example, we selected a quadratic polynomial raised to the power of three with all real roots:*

$$p_2(z) = (z^2 - 1)^3$$

*The roots are $\alpha = \pm 1$ with a multiplicity of 3. Basins of attraction for (M1)-(X2) are illustrated in Figure 2. Method (M4) performs better in terms of **avg**. Basins for $\alpha = -1$ are painted red and basins for $\alpha = 1$ are blue. There are similar lobes along the boundaries of the basins in Figure 2a–d has colored (blue, green, magenta) carrot shapes around the boundary. Methods (X1) and (X2) have considerable black points and circular patterns made up of blue diamond shapes in Figure 2e,f.*

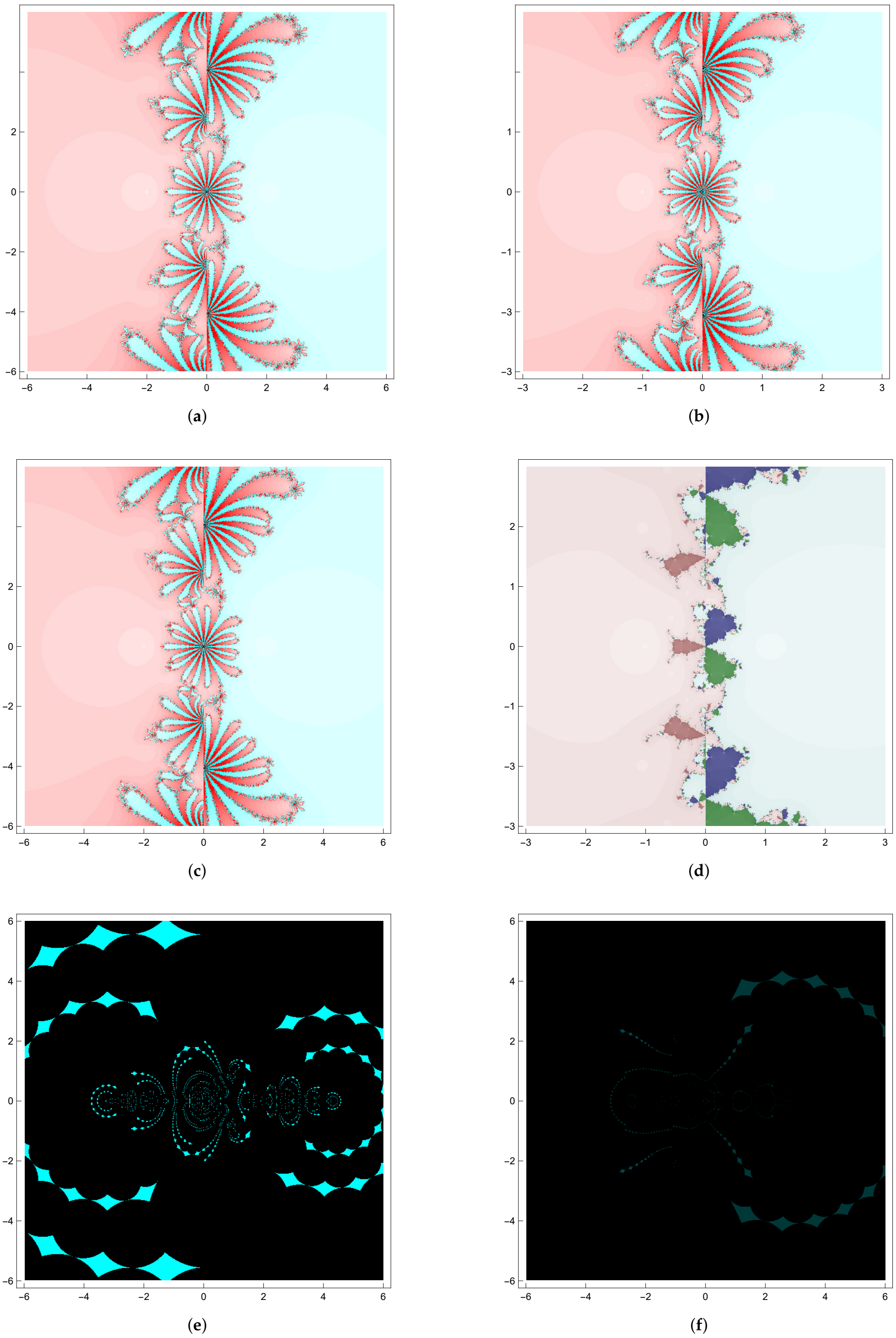


Figure 2. Basins of attraction for $P_2(z) = (z^2 - 1)^3$. (**a**) $M1$; (**b**) $M2$; (**c**) $M3$; (**d**) $M4$; (**e**) $X1$; (**f**) $X2$.

Example 3. *We choose a cubic polynomial raised to the power of two:*

$$p_3(z) = (z^3 - 1)^2,$$

with roots $\alpha = -0.5 \pm 0.866025i, \; 1$. *Method (M4) is better regarding* ***CPU*** *and* ***avg****. The basins of method (X1) are represented by black regions and the basins of method (X2) are represented by circular light blue patterns in Figure 3.*

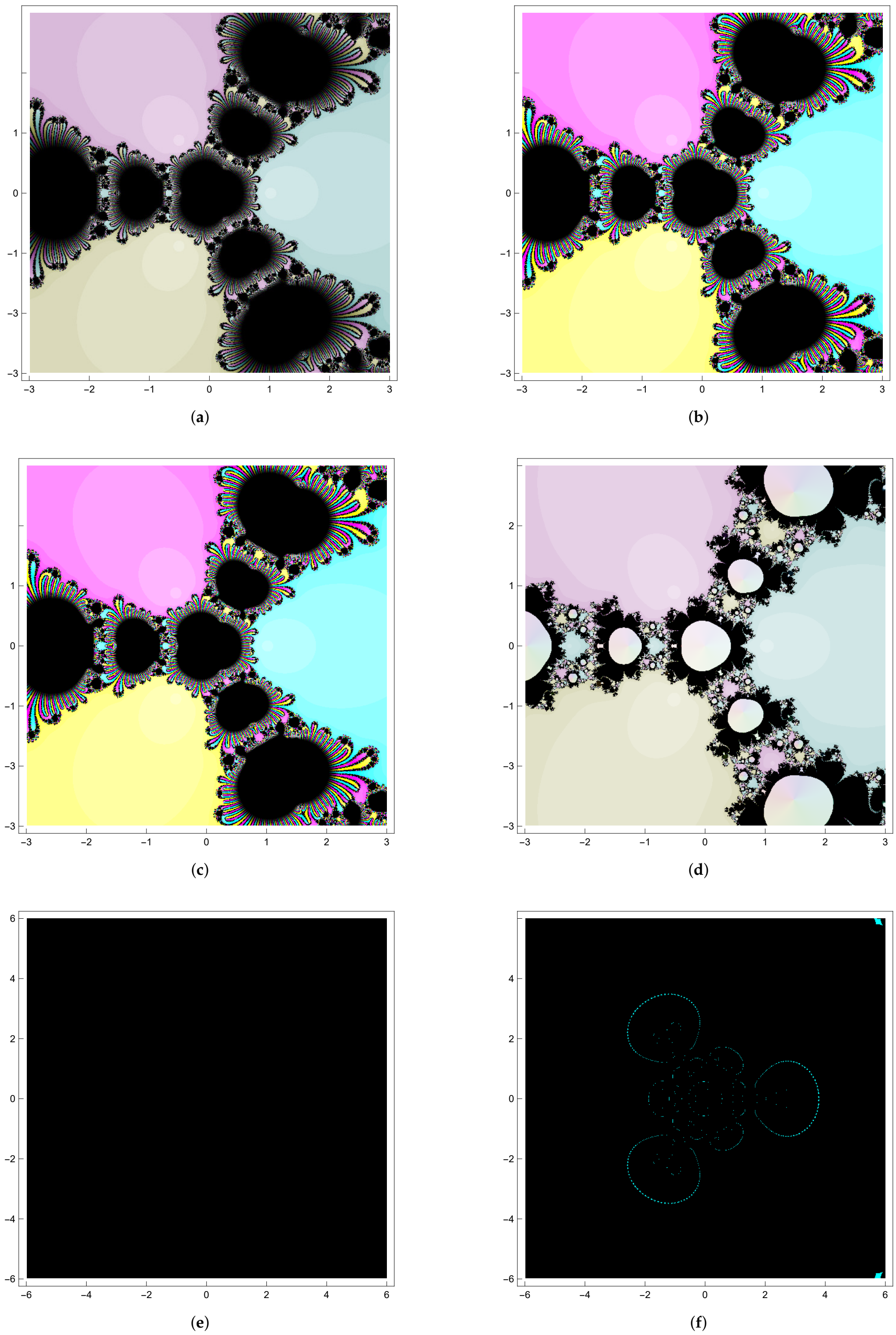


Figure 3. Basins of attraction for $P_3(z) = (z^3 - 1)^2$. (**a**) $M1$; (**b**) $M2$; (**c**) $M3$; (**d**) $M4$; (**e**) $X1$; (**f**) $X2$.

The local convergence depends on the function, an initial value, the multiplicity, and a root. It cannot be said that one iterative method is always better than other methods. It is essential to choose an initial guess that guarantees the convergence of the numerical method. Deciding how close the initial guess should be to the root is not easy, as it depends on computational precision, the selected function, and the error bound. To choose a stable initial value, we can use basins of attraction in Figures 1–4.

We use the equation in the blood rheology model [30]

$$f(x) = (\frac{x^8}{441} + \frac{8x^5}{63} - \frac{2,857,144,357x^4}{50,000,000,000} + \frac{16x^2}{9} - \frac{906,122,449x}{250,000,000} + \frac{3}{10})^4$$

to carry out the experiment (M1) for the basins in Figure 4.

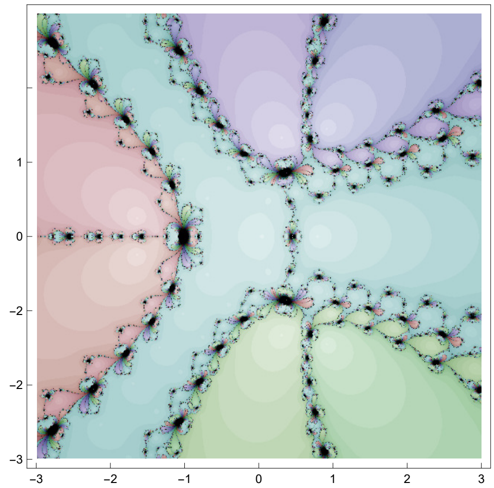
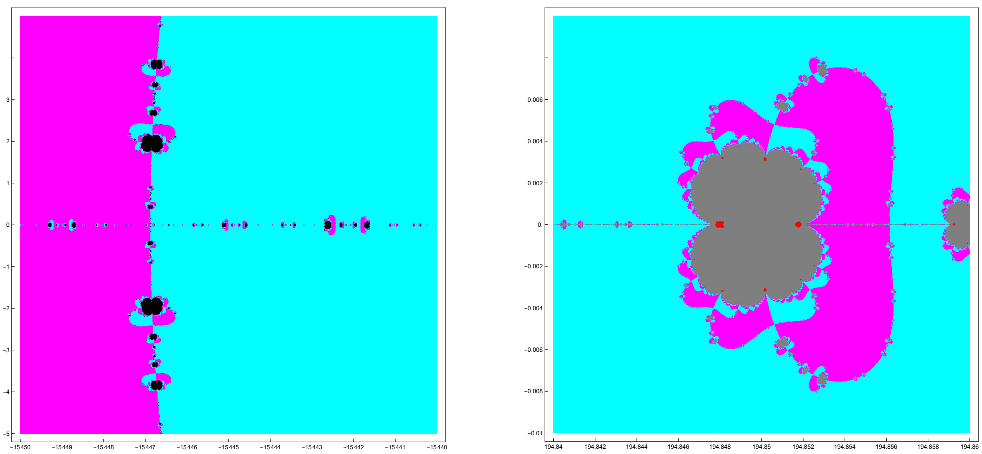


Figure 4. Basins of attraction for the blood rheology model

Substituting $A_0 = A_1 = 1, A_2 = 2m/(-1+m), A_3 = \lambda, A_4 = 0, A_5 = 0$ and $B_{00} = B_{01} = B_{10} = 1, B_{20} = 2m/(-1+m), B_{11} = 2, B_{21} = 0, B_{30} = A_3 = \lambda, B_{40} = A_4 = 0$ into (24) to plot the convergent region, we select the following method, called (M5):

$$\begin{cases} A_f(k) = 1 + k + (2m/(-1+m))k^2 + \lambda k^3, \\ B_f(k,v) = 1 + k + (2m/(-1+m))k^2 + \lambda k^3 + (1+2k)v, \end{cases} \tag{25}$$

Figure 5 shows the convergence region [31] of (M5) according to $\lambda = -2.375$ and $\lambda = -3.3305 + 0.0712i$.



(**a**) (**b**)

Figure 5. Convergent region for (M5). (**a**) $\lambda = -2.375$; (**b**) $\lambda = -3.3305 + 0.0712i$.

4. Conclusions

Even though the proposed methods (M1)–(M4) are of the sixth order, like the existing methods (X1) and (X2), the number of divergent points for (M1)–(M4) is much smaller than for (X1) and (X2). The basins for (X1) and (X2) are black for $p_1(z)$ and the basins for (X1) and (X2) for $p_2(z)$ have some blue patterns but also a lot of black points. In summary, we find that the proposed methods are faster and require fewer iterations per point on average. It is important to choose the type of initial guess for the iterative schemes, ensuring it is chosen near the root for convergence. In future work, the fractal behavior behind the improved methods will be studied in more detail.

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