fractal and fractional

MDPI

Article

Synchronization of Julia Sets in Three-Dimensional Discrete Financial Models

Zhongyuan Zhao [1], Yongping Zhang [1,*] and Dadong Tian [2,*]

[1] School of Mathematics and Statistics, Shandong University, Weihai 264209, China
[2] College of Information Science and Engineering, Shandong Agricultural University, Taian 271018, China
* Correspondence: ypzhang@sdu.edu.cn (Y.Z.); dadong@sdau.edu.cn (D.T.)

Abstract: When aiming to achieve consistency in fractal characteristics between different models, it is crucial to consider the synchronization of Julia sets. This paper studies the synchronization of Julia sets in three-dimensional discrete financial models. First, three-dimensional discrete financial models with different model parameters are proposed and their Julia sets are presented. According to the model forms, two kinds of synchronous couplers that can achieve synchronization of Julia sets between different models are designed by changing the synchronization parameters. The proposed synchronization method is theoretically derived and the efficiency of different synchronous couplers are compared. Finally, the effectiveness is verified by Julia sets graphics. This method has reference value for theoretical research into financial models in the field of fractals.

Keywords: financial model; Julia set; synchronization

Citation: Zhao, Z.; Zhang, Y.; Tian, D. Synchronization of Julia Sets in Three-Dimensional Discrete Financial Models. *Fractal Fract.* **2023**, *7*, 872. https://doi.org/10.3390/fractalfract7120872

Received: 8 November 2023
Revised: 27 November 2023
Accepted: 5 December 2023
Published: 9 December 2023

1. Introduction

The financial system plays an important role in the global economic market, and research on the financial system has developed rapidly in recent decades. In the early years, traditional economic theory usually used the linear analysis method to linearize the system and simplify the analysis. However, with the globalization of the economy, the global economic market presents greater diversity and complexity, and the financial system can no longer be simply analyzed by linear methods. Combined with the progress of computer technology, nonlinear analysis methods have received more attention and gradually become a new discipline called nonlinear dynamics. Scholars have used computer technology tools and methods to conduct research on nonlinear economic phenomena and considered financial systems as complex nonlinear dynamic systems [1–3]. In 1993, Huang [4] proposed a differential equation model for chaotic financial systems in which the long-term behavior is irregular and highly sensitive to initial state values and parameter changes. Based on this nonlinear financial system, Ma and Chen [5,6] proposed a study on the bifurcation chaotic topology and global complexity of the system. Subsequently, many scholars have conducted multifaceted research on this financial system with the aim of analyzing its dynamic behavior. For example, Gao [7] considered a continuous financial system with delayed feedback and studied the stability and Hopf bifurcation of the controlled system by combining bifurcation theory and chaos theory. Kai [8] studied the impact of double-delay feedback on the nonlinear dynamic behavior of financial systems, suggesting that double-delay feedback can effectively control the unstable behavior of financial markets and pointing out that time delay can be an effective method to control the stability of financial markets. The above research results indicate that the study of nonlinear dynamic behavior in the financial system has expanded in many different directions.

Fractal theory is an important branch of nonlinear science. Classic fractal geometry can be divided into three categories: escape-time systems, iterated function systems, and attractors. Escape-time systems typically involve the convergence boundary of complex iterations, such as the Mandelbrot set and Julia set. Well known fractals in iterated function systems include the Cantor set, Koch snowflake, and Peano curve, all of which can be achieved by "substitution" with simple geometries [9]. Attractors are usually structures obtained by iterating points, such as the Lorenz attractor. These unsmoothed sets or curves are generated by simple rules, with infinitely fine and extremely complex structures and self-similarity characteristics [10,11]. Julia sets are important fractal sets in fractal theory, named after the French mathematician Gaston Julia [12]. Their fine and complex structure is widely used in various fields, including biology [13], physics [14], cryptography [15], and others. Julia sets can display the dynamic characteristics of systems, and are often used for theoretical research on nonlinear dynamic systems. Julia sets of several functions have been discussed in recent years [16–18]. In the past two decades, the study of control and synchronization problems of Julia sets for different models has become a new field gradually being explored by many scholars. For example, Sui [19] proposed the control of Julia sets generated by the typical complex iteration function $z_{n+1} = {z_n}^2 + c$, using appropriate mathematical transformations to achieve the amplification and reduction of the Julia set based on complex iterative functions to achieve control of the Julia set. Zhang [20] introduced the visualization of the Julia set of a complex discrete Henon mapping system with two variables, and introduced the optimal control function method into the system to achieve control and synchronization of its Julia set. Sun [21] studied the Brusselator model with forcing terms from the fractal perspective, introduced the Julia set, and used feedback control to control the Julia set of the model. Then, a coupling term was designed to achieve synchronization between two Julia sets with different parameters. These studies on Julia sets are of great significance for the development of nonlinear dynamic systems. In recent years, research related to the synchronization of Julia sets has been extended. In [22], the authors introduced the consensus of Julia sets of multi-agent systems. In [23,24], Wang used adaptive control methods to discuss the synchronization and anti-synchronization problems of Julia sets in different systems.

Based on this research background, it can be seen that the financial system, as a complex nonlinear dynamic system, is in a stage of continuous expansion into various fields. By combining fractal theory, this paper establishes discrete financial models based on a classic three-dimensional financial model, then provides Julia sets of the models with different parameters and discusses the synchronization problem of Julia sets. In Section 2, the necessary definitions of a Julia set are provided. In Section 3, three-dimensional discrete financial models are proposed and Julia sets of the models with different model parameters are presented. In Section 4, two kinds of synchronous couplers are designed to achieve synchronization of the Julia sets and theoretical proofs are provided. Finally the effectiveness of the synchronization couplers is verified through Julia set graphics, then the efficiencies of different synchronous couplers are discussed.

2. Basic Theory

2.1. Julia Set

The essential definitions of the Julia set are provided below.

Assume that $f : D \to D$ is a continuous mapping and denote f^k as the k-th iteration of f, which means

$$f^0(\omega) = \omega, f^1(\omega) = f(\omega), f^2(\omega) = f(f(\omega)), \cdots, f^k(\omega) = f(f(\cdots(f(\omega))\cdots)),$$

where the iterative function graph $\left\{f^k\right\}$ is called a discrete dynamical system.

A point ω is referred to as a fixed point if it satisfies the equation $f(\omega) = \omega$. On the other hand, ω is considered a cyclic point if there exists an integer p such that $f^p(\omega) = \omega$, where p is greater than or equal to 1. Moreover, p is termed the period of ω if there is a minimum p that satisfies the equation $f^p(\omega) = \omega$.

The classical Julia set is defined in the complex plane, as follows.

Definition 1 ([25]). *The filled Julia set of function f is defined as $K(f)$, which satisfies*

$$K(f) = \left\{ z \in \mathbb{C} \middle| f^k(z) \nrightarrow \infty, k \to \infty \right\},$$

where $\mathbb{C}$ is a complex field and $f^k(z)$ is the k-th iteration of the mapping f.

The Julia set of function f is the boundary of $K(f)$, that is

$$J(f) = \partial K(f).$$

2.2. Synchronization of Julia Sets

When different models exhibit different features or performances, it is possible to study the relationship between the Julia sets of different models in order to improve their performance. In this case, it is necessary to consider the problem of synchronizing the Julia sets of the models. The synchronization of Julia sets belongs to fractal synchronization; through appropriate fractal control methods, it is possible to ensure that the geometric structure and fractal features of Julia sets originating from different systems remain consistent [26].

Consider the following two three-dimensional discrete systems H and L, where system H is

$$H = \begin{cases} x_{n+1} = p(x_n, y_n, z_n), \\ y_{n+1} = q(x_n, y_n, z_n), \\ z_{n+1} = s(x_n, y_n, z_n), \end{cases} \tag{1}$$

and system L is

$$L = \begin{cases} X_{n+1} = P(X_n, Y_n, Z_n), \\ Y_{n+1} = Q(X_n, Y_n, Z_n), \\ Z_{n+1} = S(X_n, Y_n, Z_n). \end{cases} \tag{2}$$

When the system parameters are provided, their corresponding Julia set is determined as well. Define the Julia sets of system (1) and system (2) as $J(H)$ and $J(L)$, respectively, and add the coupling term $O(\cdot)$ to system (2) to associate the Julia sets $J(H)$ and $J(L)$ of the two systems. Then, system L^* can be obtained as

$$L^* = L + O(x_n, y_n, z_n, X_n, Y_n, Z_n; k), \tag{3}$$

where k is the uncertain coupling parameter. When the coupling parameter k is determined, the Julia set of the system L^* that added the coupling term $O(\cdot)$ is determined as well, that is, $J(L^*)$.

Definition 2. *For a constant k_0, the synchronization between the Julia sets of system (1) and system (3) occurs if*

$$\lim_{k \to k_0} (J(L^*) \cup J(H) - J(L^*) \cap J(H)) = \varnothing$$

or

$$(J(L^*) \cup J(H) - J(L^*) \cap J(H))|_{k=k_0} = \varnothing.$$

It is evident that the boundedness property of the trajectories of points within the filled Julia set is distinct from those outside of it. Thus, the way to achieve synchronization between $J(H)$ and $J(L^*)$ is to achieve trajectory synchronization. In trajectory synchronization, the variation of the coupling parameter k reflects both the trajectory variation of system (3) and the variation of the Julia set $J(L^*)$ of system (3).

Theorem 1 ([27]). *For a constant k_0, synchronization between the Julia sets of systems (1) and (2) occurs if $\forall v_0 \in R^3$ such that*

$$\lim_{n \to \infty} \lim_{k \to k_0} \left| (L^*)^n(v_0) - H^n(v_0) \right| = 0$$

or

$$\lim_{n\to\infty}\Big|(L^*)^n(v_0)|_{k=k_0} - H^n(v_0)\Big| = 0.$$

3. Julia Sets of Discrete Financial Models

In the literature, it is possible to find established financial models consisting of a production sub-block, monetary sub-block, securities sub-block, and labor sub-block (see [5,6] for more details). These model depict the temporal changes of three state variables: the interest rate x, the investment demand y, and the price exponent z, as follows:

$$\begin{cases} \dot{x} = z + (y-a)x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \end{cases} \tag{4}$$

where a is the amount of savings, b is the cost of investment, and c is the elasticity of commercial demand.

By selecting a suitable coordinate model and assigning appropriate dimensions to each state variable [28], we can derive the simplified finance model:

$$\begin{cases} \dot{x_1} = \alpha(x_1 + x_2), \\ \dot{x_2} = -x_2 - \alpha x_1 x_3, \\ \dot{x_3} = \beta + \alpha x_1 x_2, \end{cases} \tag{5}$$

where α and β are the parameters of the model.

In this paper, our discussions are based on this simplified financial model.

Due to the fact that the Julia set is defined by an iterative algorithm, discrete points are more advantageous for computers to implement the iterative process. At the same time, combined with the good research value of discrete fractional calculus, in this paper we consider the discrete form of the model (5). We discretize the equation using

$$\dot{x_i} \to \frac{x_i(t+\triangle t) - x_i(t)}{\triangle t}, i = 1,2,3.$$

We replace $x_1(t), x_2(t), x_3(t)$ with x_n, y_n, z_n and $x_1(t+\triangle t), x_2(t+\triangle t), x_3(t+\triangle t)$ with $x_{n+1}, y_{n+1}, z_{n+1}$. Then, $\triangle t$ is expressed by γ, obtaining the discrete version of model:

$$\begin{cases} x_{n+1} = (1+\gamma\alpha)x_n + \gamma\alpha y_n, \\ y_{n+1} = (1-\gamma)y_n - \gamma\alpha x_n z_n, \\ z_{n+1} = z_n + \gamma\beta + \gamma\alpha x_n y_n, \end{cases} \tag{6}$$

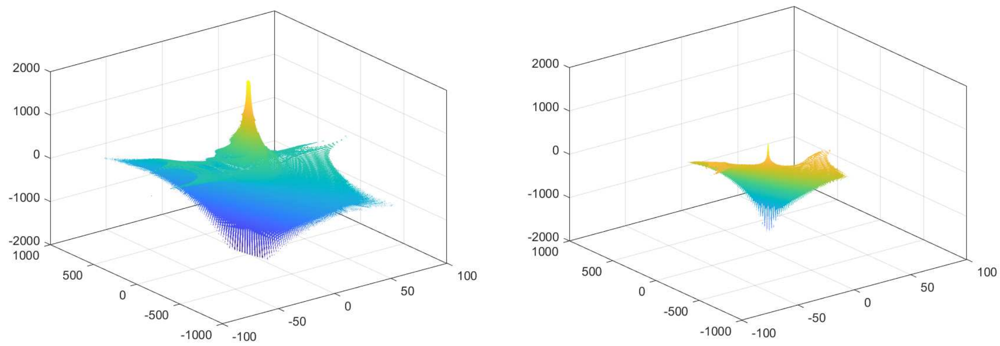
where α, β, and γ are the parameters of the model.

In fractal theory, the definition of the classic Julia set in the complex field is analogized to establish the Julia set of the three-dimensional discrete financial model (6). Therefore, the Julia set of the model (6) can be described as follows.

Definition 3. *The filled Julia set of model (6), represented as K, is defined as the following set:* $K = \{(x,y,z) \in \mathbb{R}^3 \mid$ *the iteration* $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ *remains bounded with the initial points* $(x,y,z)\}$.

The Julia set of model (6), denoted as J, refers to the boundary of the filled Julia set K, i.e., $J = \partial K$.

Figure 1 shows the Julia sets of model (6) when the model parameters are taken to be as follows: (a) $\alpha = 1.1, \beta = 3.5, \gamma = 0.05$*, and (b)* $\alpha = 1.7, \beta = 5000, \gamma = 0.05$*. In this paper, the graphics of the Julia sets uniformly use the spatial coordinate range* $[-100, 100] \times [-1000, 1000] \times [-2000, 2000]$.



(**a**) $\alpha = 1.1, \beta = 3.5, \gamma = 0.05$ (**b**) $\alpha = 1.7, \beta = 5000, \gamma = 0.05$

Figure 1. Julia sets of the financial models (6) with different model parameters.

4. Synchronization of Julia Sets between the Financial Models

For the financial model (6), it is necessary to design a synchronous coupler to achieve synchronization of Julia sets between models with different model parameters in the same structure. For example, Figure 1a,b shows two different Julia sets of financial models with different model parameters. The desired effect is to add a synchronous coupler to one model, causing its Julia set to change to the Julia set of the other model.

The drive system is

$$\begin{cases} x_{n+1} = (1+\gamma\alpha)x_n + \gamma\alpha y_n, \\ y_{n+1} = (1-\gamma)y_n - \gamma\alpha x_n z_n, \\ z_{n+1} = z_n + \gamma\beta + \gamma\alpha x_n y_n, \end{cases} \tag{7}$$

where the model parameters are taken to be $\alpha = 1.1, \beta = 3.5, \gamma = 0.05$.

The response system is

$$\begin{cases} X_{n+1} = (1+\gamma A)X_n + \gamma A Y_n + u_n, \\ Y_{n+1} = (1-\gamma)Y_n - \gamma A X_n Z_n + v_n, \\ Z_{n+1} = Z_n + \gamma B + \gamma A X_n Y_n + w_n, \end{cases} \tag{8}$$

where the model parameters are taken to be $A = 1.7, B = 5000, \gamma = 0.05$ and where u_n, v_n, w_n are the added synchronous coupling terms.

When $u_n = 0, v_n = 0, w_n = 0$, that is, without adding a synchronous coupler, the Julia sets of model (7) and model (8) are shown as (a) and (b) in Figure 1. Next, two kinds of synchronous couplers are designed based on the overall terms and partial nonlinear terms to achieve synchronization of the Julia sets for two financial models.

4.1. Synchronous Coupler (I)

The synchronous coupler (I) is based on the following overall terms:

$$\begin{cases} u_n = k(((1+\gamma\alpha)x_n + \gamma\alpha y_n) - ((1+\gamma A)X_n + \gamma A Y_n)), \\ v_n = k(((1-\gamma)y_n - \gamma\alpha x_n z_n) - ((1-\gamma)Y_n - \gamma A X_n Z_n)), \\ w_n = k((z_n + \gamma\beta + \gamma\alpha x_n y_n) - (Z_n + \gamma B + \gamma A X_n Y_n)), \end{cases} \tag{9}$$

where k is the synchronization parameter.

Theorem 2. *For the synchronous coupler (9), $|X_{n+1} - x_{n+1}| \to 0$, $|Y_{n+1} - y_{n+1}| \to 0$ and $|Z_{n+1} - z_{n+1}| \to 0$ when $n \to \infty$ and $k \to 1$, respectively, that is, synchronization of the trajectories of the drive system (7) and response system (8) is achieved.*

Proof. For model variables x_{n+1} and X_{n+1}, we substitute the coupling terms into

$$\begin{aligned} X_{n+1} &= (1+\gamma A)X_n + \gamma A Y_n + k(((1+\gamma\alpha)x_n + \gamma\alpha y_n) - ((1+\gamma A)X_n + \gamma A Y_n)) \\ &= (1-k)((1+\gamma A)X_n + \gamma A Y_n) + k((1+\gamma\alpha)x_n + \gamma\alpha y_n), \\ x_{n+1} &= (1+\gamma\alpha)x_n + \gamma\alpha y_n. \end{aligned}$$

Subtracting between two equations, we have

$$\begin{aligned} X_{n+1} - x_{n+1} &= (1-k)((1+\gamma A)X_n + \gamma A Y_n) + k((1+\gamma\alpha)x_n + \gamma\alpha y_n) - ((1+\gamma\alpha)x_n + \gamma\alpha y_n) \\ &= (1-k)(((1+\gamma A)X_n + \gamma A Y_n) - ((1+\gamma\alpha)x_n + \gamma\alpha y_n)) \\ &= (1-k)(X_n - x_n + \gamma A X_n - \gamma\alpha x_n + \gamma A Y_n - \gamma\alpha y_n). \end{aligned}$$

Because

$$\begin{aligned} \gamma A X_n - \gamma\alpha x_n &= \gamma(A X_n - A x_n + A x_n - \alpha x_n) \\ &= \gamma A(X_n - x_n) + \gamma(A-\alpha)x_n, \\ \gamma A Y_n - \gamma\alpha y_n &= \gamma(A Y_n - A y_n + A Y_n - \alpha y_n) \\ &= \gamma A(Y_n - y_n) + \gamma(A-\alpha)y_n, \end{aligned}$$

the formula can be simplified as

$$\begin{aligned} X_{n+1} - x_{n+1} &= (1-k)(X_n - x_n + \gamma A X_n - \gamma\alpha x_n + \gamma A Y_n - \gamma\alpha y_n) \\ &= (1-k)((1+\gamma A)(X_n - x_n) + \gamma A(Y_n - y_n) + \gamma(A-\alpha)(x_n + y_n)). \end{aligned}$$

Taking the absolute value, we obtain

$$\begin{aligned} |X_{n+1} - x_{n+1}| &= |(1-k)((1+\gamma A)(X_n - x_n) + \gamma A(Y_n - y_n) + \gamma(A-\alpha)(x_n + y_n))| \\ &\leq |1-k||1+\gamma A||X_n - x_n| + |1-k||\gamma A||Y_n - y_n| + |1-k||\gamma(A-\alpha)||x_n + y_n| \\ &\leq |1-k||1+\gamma A||X_n - x_n| + |1-k||\gamma A|(|Y_n| + |y_n|) \\ &\quad + |1-k||\gamma(A-\alpha)|(|x_n| + |y_n|). \end{aligned}$$

The Julia set is defined based on the iteration of points within a specific bounded region. Let D denote this bounded region and consider the iteration of points within D. It is important to note that, in accordance with the characteristics of the Julia set, only the iteration of points with trajectories that remain within D should be considered. This requirement arises from the fact that if there exists n_0 such that $f^{n_0}(z) \notin D$ for some $z \notin J$, then the Julia set remains unaltered. Additionally, the boundedness property of D guarantees the existence of a constant $W > 0$ such that $|z| < W$ for arbitrary $z \in D$.

Because D is a bounded region and the value of the model parameters γ, A and α are bounded, there exists $W_1 > 0$ and $W_2 > 0$ for all non-negative integers n such that

$$\begin{aligned} |\gamma A|(|Y_n| + |y_n|) &\leq W_1, \\ |\gamma(A-\alpha)|(|x_n| + |y_n|) &\leq W_2. \end{aligned}$$

Then, we have

$$\begin{aligned} |X_{n+1} - x_{n+1}| &\leq |1-k||1+\gamma A||X_n - x_n| + |1-k||\gamma A|(|Y_n| + |y_n|) \\ &\quad + |1-k||\gamma(A-\alpha)|(|x_n| + |y_n|) \\ &\leq |1-k||1+\gamma A||X_n - x_n| + |1-k|W_1 + |1-k|W_2 \\ &= |1-k|(|1+\gamma A||X_n - x_n| + W_1 + W_2) \\ &\leq |1-k|(|1+\gamma A|(|1-k|(|1+\gamma A||\ X_{n-1} - x_{n-1}| + W_1 + W_2)) + W_1 + W_2) \\ &= |1-k|^2|1+\gamma A|^2|X_{n-1} - x_{n-1}| + |1-k|^2|1+\gamma A|(W_1 + W_2) \\ &\quad + |1-k|(W_1 + W_2) \\ &\leq \cdots \\ &\leq |1-k|^n|1+\gamma A|^n|X_1 - x_1| + |1-k|^n|1+\gamma A|^{n-1}(W_1 + W_2) \\ &\quad + \cdots + |1-k|^2|1+\gamma A|(W_1 + W_2) + |1-k|(W_1 + W_2) \\ &= |1-k|^n|1+\gamma A|^n|X_1 - x_1| + \frac{|1-k|(W_1 + W_2)(1 - |1-k|^n|1+\gamma A|^n)}{1 - |1-k||1+\gamma A|}, \end{aligned}$$

for $k > \frac{\gamma A}{1+\gamma A}$, and we obtain $(1-k)(1+\gamma A) < 1$. As $n \to \infty$, the limitation of the right side of the above equation is $\frac{|1-k|(W_1+W_2)}{1-|1-k||1+\gamma A|}$. Obviously,

$$|X_{n+1} - x_{n+1}| \to 0\,,\text{ as } k \to 1\,.$$

Similarly, for the model variables y_{n+1} and Y_{n+1}, we substitute the coupling terms into

$$\begin{aligned}
Y_{n+1} &= (1-\gamma)Y_n - \gamma A X_n Z_n + k(((1-\gamma)y_n - \gamma\alpha x_n z_n) - ((1-\gamma)Y_n - \gamma A X_n Z_n)),\\
y_{n+1} &= (1-\gamma)y_n - \gamma\alpha x_n z_n,\\
Y_{n+1} - y_{n+1} &= (1-k)(((1-\gamma)Y_n - \gamma A X_n Z_n) - ((1-\gamma)y_n - \gamma\alpha x_n z_n))\\
&= (1-k)((1-\gamma)(Y_n - y_n) - (\gamma A X_n Z_n - \gamma\alpha x_n z_n)).
\end{aligned}$$

Taking the absolute value, we have

$$|Y_{n+1} - y_{n+1}| \le |1-k||1-\gamma||Y_n - y_n| + |1-k||\gamma A\, X_n Z_n - \gamma\alpha x_n z_n|.$$

According to the previous discussion, there exists $W_3 > 0$ satisfying

$$|\gamma A X_n Z_n - \gamma\alpha x_n z_n| \le W_3.$$

Then,

$$\begin{aligned}
|Y_{n+1} - y_{n+1}| &\le |1-k||1-\gamma||Y_n - y_n| + |1-k|W_3\\
&\le |1-k||1-\gamma|(|1-k||1-\gamma||\,Y_{n-1} - y_{n-1}| + |1-k|W_3) + |1-k|W_3\\
&\le \cdots\\
&\le |1-k|^n|1-\gamma|^n|Y_1 - y_1| + |1-k|^n|1-\gamma|^{n-1}W_3\\
&\quad + \cdots + |1-k|^2|1-\gamma|W_3 + |1-k|W_3\\
&= |1-k|^n|1-\gamma|^n|Y_1 - y_1| + \frac{|1-k|W_3(1-|1-k|^n|1-\gamma|^n)}{1-|1-k||1-\gamma|}.
\end{aligned}$$

As $n \to \infty$, $|Y_{n+1} - y_{n+1}| \to \frac{|1-k|W_3}{1-|1-k||1-\gamma|}$, obviously,

$$|Y_{n+1} - y_{n+1}| \to 0\,,\text{ as } k \to 1\,.$$

Similarly, for the model variables z_{n+1} and Z_{n+1} we have

$$\begin{aligned}
Z_{n+1} &= Z_n + \gamma B + \gamma A X_n Y_n + k((z_n + \gamma\beta + \gamma\alpha x_n y_n) - (Z_n + \gamma B + \gamma A X_n Y_n)),\\
z_{n+1} &= z_n + \gamma\beta + \gamma\alpha x_n y_n,\\
Z_{n+1} - z_{n+1} &= (1-k)((Z_n + \gamma B + \gamma A X_n Y_n) - (z_n + \gamma\beta + \gamma\alpha x_n y_n))\\
&= (1-k)(Z_n - z_n + \gamma B - \gamma\beta + \gamma A X_n Y_n - \gamma\alpha x_n y_n).
\end{aligned}$$

Taking the absolute value

$$|Z_{n+1} - z_{n+1}| \le |1-k||Z_n - z_n| + |1-k||\gamma B - \gamma\beta + \gamma A\, X_n Y_n - \gamma\alpha x_n y_n|,$$

according to the previous discussion there exists $W_4 > 0$ satisfying

$$|\gamma B - \gamma\beta + \gamma A X_n Y_n - \gamma\alpha x_n y_n| \le W_4.$$

Then,

$$\begin{aligned}
|Z_{n+1} - z_{n+1}| &\le |1-k||Z_n - z_n| + |1-k|W_4\\
&\le |1-k|(|1-k||\,Z_{n-1} - z_{n-1}| + |1-k|W_4) + |1-k|W_4\\
&\le \cdots\\
&\le |1-k|^n|Z_1 - z_1| + |1-k|^n W_4 + \cdots + |1-k|^2 W_4 + |1-k|W_4\\
&= |1-k|^n|Z_1 - z_1| + \frac{|1-k|W_4(1-|1-k|^n)}{1-|1-k|}.
\end{aligned}$$

As $n \to \infty, |Z_{n+1} - z_{n+1}| \to \frac{|1-k|W_4}{1-|1-k|}$, obviously,

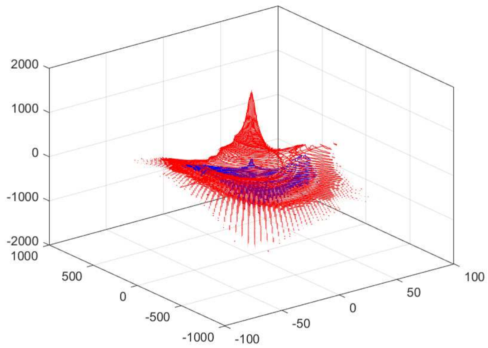
$$|Z_{n+1} - z_{n+1}| \to 0\,, \text{ as } k \to 1\,.$$

Therefore, this paper successfully demonstrates the synchronization of trajectories between system (8) and system (8), leading to the synchronization of Julia sets between the two models. □

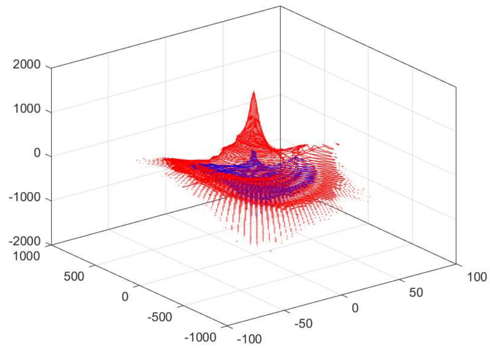
Thus far, we have provided theoretical evidence supporting the notion that the synchronous coupler (9) is capable of achieving the synchronization of Julia sets between the driving system (7) and the response system (8). Additionally, owing to the flexibility in terms of model parameters, the synchronous coupler (I) can facilitate the synchronization of Julia sets between two financial models characterized by differing parameter configurations.

Next, visual representations of the Julia sets are utilized to validate the efficacy of the synchronous coupler. Specifically, the Julia set corresponding to the driver system (7) is depicted in red, while the Julia set associated with the response system (8) is depicted in blue. As the synchronization parameter k is varied, the Julia set of the response system undergoes corresponding changes. In line with the theorem mentioned earlier, as the synchronization parameter k tends to 1, the Julia sets of the two models are expected to achieve synchronization, that is, the blue graph should tend towards consistency with the red graph.

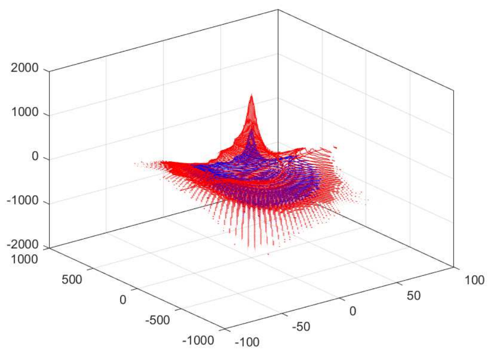
From Figure 2, it can be observed that as k tends from 0 to 1, the blue Julia set of the response system (8) gradually expands to be exactly the same as the red Julia set of the driving system (7). Therefore, the graph demonstrates that the Julia set has achieved synchronization.



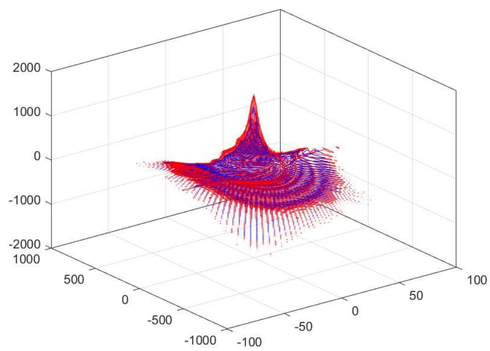
(**a**) $k = 0$



(**b**) $k = 0.1$

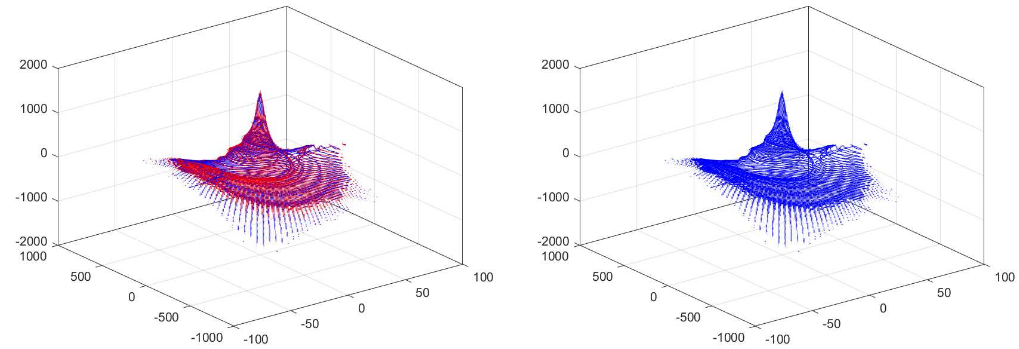


(**c**) $k = 0.3$



(**d**) $k = 0.5$

Figure 2. *Cont.*



(**e**) $k = 0.7$ (**f**) $k = 1$

Figure 2. Changes in the parameter k of the synchronous coupler (I) leading to synchronization of the Julia sets between the financial models.

4.2. Synchronous Coupler (II)

The synchronous coupler (II) is based on the partial nonlinear terms

$$\begin{cases} u_n = h(x_n - X_n) + k(\gamma\alpha(x_n + y_n) - \gamma A(X_n + Y_n)), \\ v_n = k((-\gamma\alpha x_n z_n) - (-\gamma A X_n Z_n)), \\ w_n = h(z_n - Z_n) + k((\gamma\beta + \gamma\alpha x_n y_n) - (\gamma B + \gamma A X_n Y_n)), \end{cases} \tag{10}$$

where k is the synchronization parameter and $h \in (0,1)$ is the arbitrary constant.

Theorem 3. *For the synchronous coupler (10), $|X_{n+1} - x_{n+1}| \to 0, |Y_{n+1} - y_{n+1}| \to 0$ and $|Z_{n+1} - z_{n+1}| \to 0$ when $n \to \infty$ and $k \to 1$, respectively, and h is an arbitrary constant from $(0,1)$, that is, the synchronization of trajectories of system (7) and system (8) is achieved.*

Proof. For model variables x_{n+1} and X_{n+1}, substituting the coupling terms into

$$\begin{aligned} X_{n+1} &= (1+\gamma A)X_n + \gamma A Y_n + h(x_n - X_n) + k(\gamma\alpha(x_n + y_n) - \gamma A(X_n + Y_n)), \\ x_{n+1} &= (1+\gamma\alpha)x_n + \gamma\alpha y_n, \\ X_{n+1} - x_{n+1} &= (1-h)(X_n - x_n) + (1-k)(\gamma A(X_n + Y_n) - \gamma\alpha(x_n + y_n)). \end{aligned}$$

and taking the absolute value, we obtain

$$|X_{n+1} - x_{n+1}| \le |1-h||X_n - x_n| + |1-k||\gamma A\,(X_n + Y_n) - \gamma\alpha(x_n + y_n)|.$$

According to the previous discussion, there exists $W_5 > 0$ satisfying

$$|\gamma A(X_n + Y_n) - \gamma\alpha(x_n + y_n)| \le W_5.$$

Then, we have

$$\begin{aligned} |X_{n+1} - x_{n+1}| &\le |1-h||X_n - x_n| + |1-k|W_5 \\ &\le |1-h|(|1-h||\,X_{n-1} - x_{n-1}| + |1-k|W_5) + |1-k|W_5 \\ &\le \cdots \\ &\le |1-h|^n|X_1 - x_1| + |1-h|^{n-1}|1-k|W_5 + \cdots + |1-h||1-k|W_5 + |1-k|W_5 \\ &= |1-h|^n|X_1 - x_1| + \frac{|1-k|W_5(1-|1-h|^n)}{1-|1-h|}. \end{aligned}$$

For $0 < h < 1$, we obtain $0 < 1-h < 1$. As $n \to \infty$, $|X_{n+1} - x_{n+1}| \to \frac{|1-k|W_5}{1-|1-h|}$, obviously,

$$|X_{n+1} - x_{n+1}| \to 0\,, \text{ as } k \to 1\,.$$

For the model variables y_{n+1} and Y_{n+1}, we have

$$\begin{aligned} Y_{n+1} &= (1-\gamma)Y_n - \gamma A X_n Z_n + k((-\gamma\alpha x_n z_n) - (-\gamma A X_n Z_n)), \\ y_{n+1} &= (1-\gamma)y_n - \gamma\alpha x_n z_n, \\ Y_{n+1} - y_{n+1} &= (1-\gamma)(Y_n - y_n) + (1-k)(\gamma\alpha x_n z_n - \gamma A X_n Z_n). \end{aligned}$$

Taking the absolute value

$$|Y_{n+1} - y_{n+1}| \leq |1-\gamma||Y_n - y_n| + |1-k||\gamma\alpha\, x_n z_n - \gamma A X_n Z_n|,$$

according to the previous discussion, there exists $W_6 > 0$ satisfying

$$|\gamma\alpha x_n z_n - \gamma A X_n Z_n| \leq W_6.$$

Then, we obtain

$$\begin{aligned} |Y_{n+1} - y_{n+1}| &\leq |1-\gamma||Y_n - y_n| + |1-k|W_6 \\ &\leq |1-\gamma|(|1-\gamma||\,Y_{n-1} - y_{n-1}| + |1-k|W_6) + |1-k|W_6 \\ &\leq \cdots \\ &\leq |1-\gamma|^n|Y_1 - y_1| + |1-\gamma|^{n-1}|1-k|W_6 + \cdots + |1-\gamma||1-k|W_6 + |1-k|W_6 \\ &= |1-\gamma|^n|Y_1 - y_1| + \frac{|1-k|W_6(1-|1-\gamma|^n)}{1-|1-\gamma|}. \end{aligned}$$

As $n \to \infty$, $|Y_{n+1} - y_{n+1}| \to \frac{|1-k|W_6}{1-|1-\gamma|}$, obviously,

$$|Y_{n+1} - y_{n+1}| \to 0\,,\ \text{as}\ k \to 1\,.$$

For the model variables z_{n+1} and Z_{n+1}, we have

$$\begin{aligned} Z_{n+1} &= Z_n + \gamma B + \gamma A X_n Y_n + h(z_n - Z_n) + k((\gamma\beta + \gamma\alpha x_n y_n) - (\gamma B + \gamma A X_n Y_n)), \\ z_{n+1} &= z_n + \gamma\beta + \gamma\alpha x_n y_n, \\ Z_{n+1} - z_{n+1} &= (1-h)(Z_n - z_n) + (1-k)((\gamma\beta + \gamma\alpha x_n y_n) - (\gamma B + \gamma A X_n Y_n)). \end{aligned}$$

Taking the absolute value

$$|Z_{n+1} - z_{n+1}| \leq |1-h||Z_n - z_n| + |1-k||\gamma\beta + \gamma\alpha\, x_n y_n - \gamma B - \gamma A X_n Y_n|,$$

according to the previous discussion, there exists $W_7 > 0$ satisfying

$$|\gamma\beta + \gamma\alpha x_n y_n - \gamma B - \gamma A X_n Y_n| \leq W_7.$$

Then, we obtain

$$\begin{aligned} |Z_{n+1} - z_{n+1}| &\leq |1-h||Z_n - z_n| + |1-k|W_7 \\ &\leq |1-h|(|1-h||\,Z_{n-1} - z_{n-1}| + |1-k|W_7) + |1-k|W_7 \\ &\leq \cdots \\ &\leq |1-h|^n|Z_1 - z_1| + |1-h|^{n-1}|1-k|W_7 + \cdots + |1-h||1-k|W_7 + |1-k|W_7 \\ &= |1-h|^n|Z_1 - z_1| + \frac{|1-k|W_7(1-|1-h|^n)}{1-|1-h|}. \end{aligned}$$

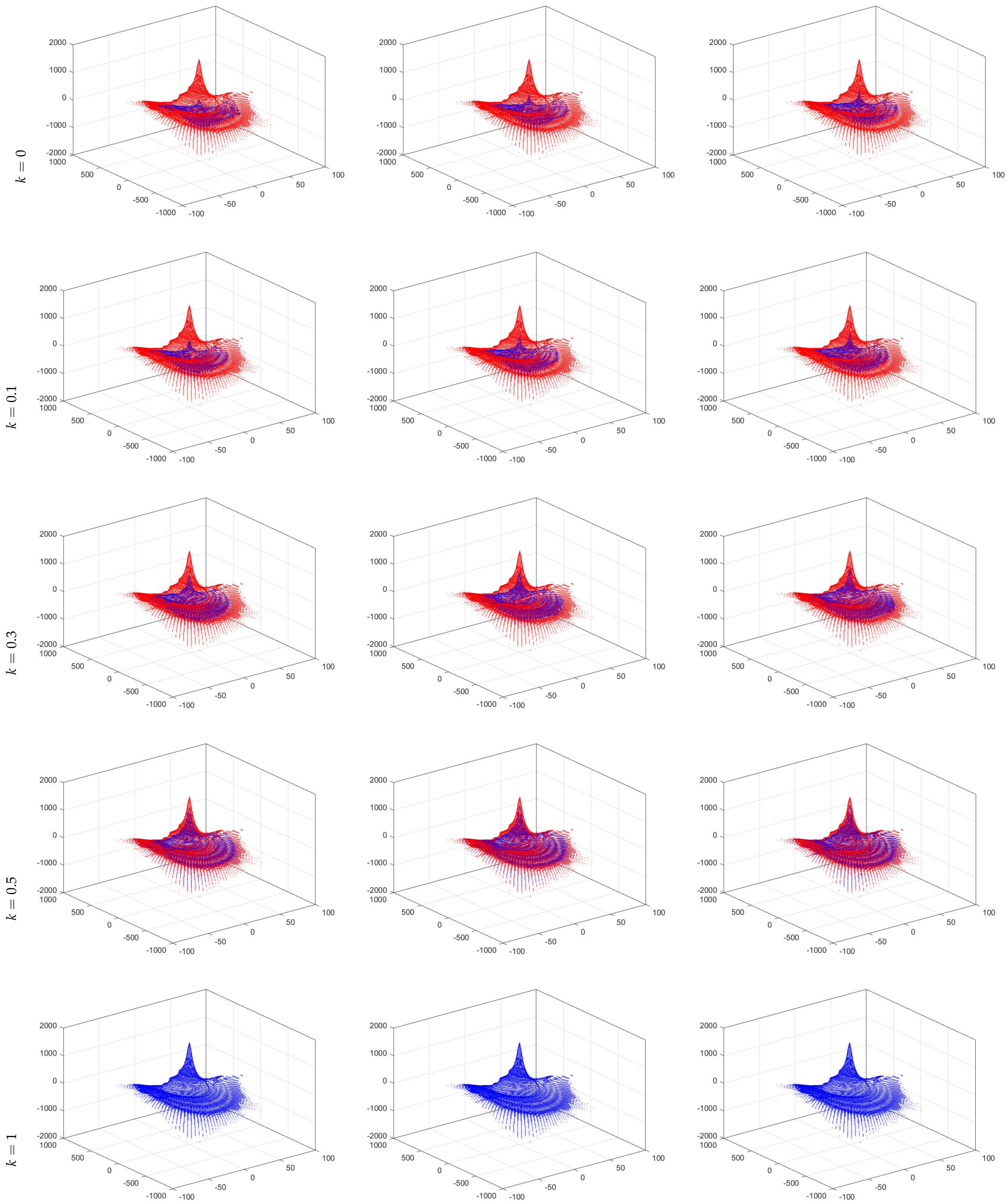
For $0 < h < 1$, we obtain $0 < 1-h < 1$. As $n \to \infty$, $|Z_{n+1} - z_{n+1}| \to \frac{|1-k|W_7}{1-|1-h|}$. Obviously,

$$|Z_{n+1} - z_{n+1}| \to 0\,,\ \text{as}\ k \to 1\,.$$

Consequently, the synchronization of trajectories between system (7) and system (8) is attained, resulting in the synchronization of Julia sets between the two models. □

Thus far, we have successfully demonstrated that the synchronous coupler (II) can achieve the synchronization of Julia sets between two financial models characterized by different model parameters.

According to the above theorem, for arbitrary constant $h \in (0,1)$, when the synchronization parameter k tends to 1, the Julia sets of the two models are synchronized. Due to the presence of arbitrary constant $h \in (0,1)$ uncertainty in the synchronous coupler (II), we continue to discuss the impact of the value of h on the synchronization effect of the Julia set. For the arbitrary constant $h \in (0,1)$, the first column takes a value of $h = 0.2$, the second column takes $h = 0.5$, and the third column takes $h = 0.8$. In the graphics of each column, as k varies from 0 to 1, as shown in Figure 3, the blue Julia set of the response system (8) gradually expands to exactly the same shape as the red Julia set of the driving system (7).



(**a**) $h = 0.2$ (**b**) $h = 0.5$ (**c**) $h = 0.8$

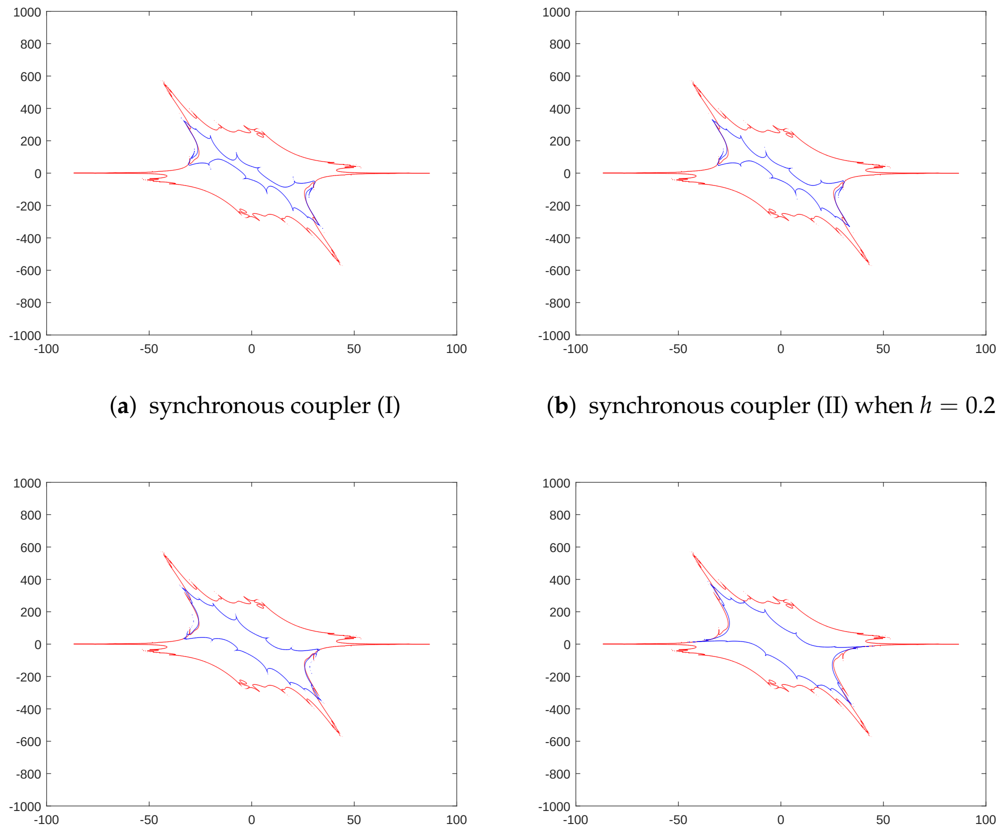
Figure 3. The synchronization of the Julia sets between the financial models when the parameter k of the synchronous coupler (II) changes.

It is difficult to see the impact of the value of the constant h on the synchronization effect of the Julia set from the picture; it can only be seen in the row where the synchronization parameter $k = 0.1$ that the blue Julia set in the right column is closer to the red Julia set at the sharp tip than the blue Julia set in the left column. It can be inferred that the larger the constant h, the faster the synchronization speed. This paper further investigates the impact of the value of the arbitrary constant h in the next section.

4.3. Comparison of Synchronous Couplers (I) and (II)

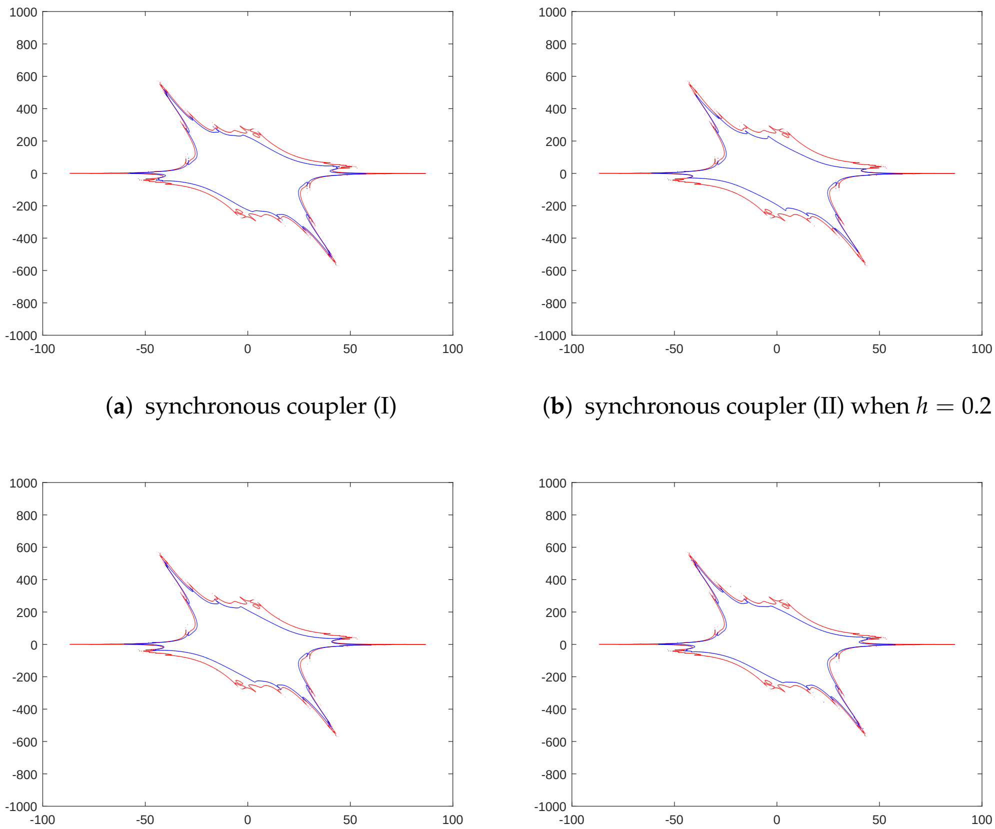
In the above two subsections, we designed two types of synchronous couplers and proved through theoretical derivation that they can achieve synchronization of Julia sets between the financial models. In the graph, it can be seen that the synchronization rates of the two types of couplers are similar, and small differences cannot be better depicted in the three-dimensional graph. Here, we consider taking $z = 0$ to draw the two-dimensional section of the Julia sets to better display the synchronization effect differences between couplers (I) and (II).

As shown in Figure 4, when the synchronization parameter $k = 0.1$, the two-dimensional section of the Julia set of the driving system is red and the blue Julia set section of the response model with synchronous couplers presents different shapes, indicating that the synchronization efficiency of the two types of couplers is different. Comparing the effects of synchronous coupler (II) under different constants of h, it can be seen in in Figure 5b–d that when the constant of $h \in (0, 1)$ is larger, the blue Julia set is more similar to the red Julia set and its synchronization effect is better. The same result appears in Figure 5, where the synchronization parameter $k = 0.5$.



(**a**) synchronous coupler (I)
(**b**) synchronous coupler (II) when $h = 0.2$
(**c**) synchronous coupler (II) when $h = 0.5$
(**d**) synchronous coupler (II) when $h = 0.8$

Figure 4. Comparison of synchronization effects between synchronous couplers (I) and (II) when the synchronization parameter $k = 0.1$.



(**a**) synchronous coupler (I)

(**b**) synchronous coupler (II) when $h = 0.2$

(**c**) synchronous coupler (II) when $h = 0.5$

(**d**) synchronous coupler (II) when $h = 0.8$

Figure 5. Comparison of synchronization effects between synchronous couplers (I) and (II) when the synchronization parameter $k = 0.5$.

This phenomenon is further analyzed in the process of proving Theorem 3. When $n \to \infty$, $|X_{n+1} - x_{n+1}| \to \frac{|1-k|W_5}{1-|1-h|}$; at this time, for any $0 < h < 1$, the larger h is, the smaller the value of $|X_{n+1} - x_{n+1}|$. The same is the case for $|Z_{n+1} - z_{n+1}|$. Thus, when the constant $h \in (0,1)$ is larger, the efficiency of trajectory synchronization is faster. From the graph, it can be seen that when the value of h is larger, the blue Julia set is closer to the red Julia set.

5. Conclusions

In real financial problems, the differences in various factors can lead to different phenomena and characteristics. If it is hoped that problems will develop in the desired direction, it is necessary to generate similar or even identical states between different financial models, that is, to consider the synchronization of Julia sets between the financial models. To achieve the synchronization of financial models with different model parameters, we have two types of synchronization couplers: coupler (I), based on the overall terms, and coupler (II), based on the partial nonlinear terms of the model. In order to better compare the performance of the two synchronous couplers, we provide a two-dimensional section of the Julia sets that effectively reflects the differences between the two couplers.

In summary, this paper focuses on achieving the synchronization of Julia sets in a three-dimensional discrete financial model. Two types of synchronization couplers are designed, and the synchronization parameters are adjusted to achieve the synchronization of Julia sets. The effectiveness of these synchronization couplers is demonstrated through theoretical analysis and graphical representation of Julia sets. The results are innovative and universal, creating certain value for subsequent research on Julia sets.

Author Contributions: Methodology, Z.Z.; investigation, Y.Z.; writing—original draft preparation, Z.Z.; writing—review and editing, Y.Z. and D.T.; supervision, Y.Z. and D.T.; project administration, Y.Z.; funding acquisition, Y.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Natural Science Foundation of Shandong Province (No. ZR2022MA032), the National Natural Science Foundation of China–Shandong Joint Fund (No. U1806203), and the Fundamental Research Funds for the Central Universities (No. 2019ZRJC005).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Beinhocker, E.D. *The Origin of Wealth*; Harvard Business Press: Cambridge, MA, USA, 2006.
2. Gentili, P.L. Why is Complexity Science valuable for reaching the goals of the UN 2030 Agenda? *Rend. Fis. Acc. Lincei* **2021**, *32*, 117–134. [CrossRef] [PubMed]
3. Huang, C.; Wen, F.; Li, J.; Yi, T.; Lin, X. Nonlinear Dynamics in Financial Systems: Advances and Perspectives. *Discret. Dyn. Nat. Soc.* **2014**, *2014*, 275750. [CrossRef]
4. Huang, D.S.; Li, H.Q. *Theories and Methods of Nonlinear Economics*; Sichuan University Press: Chengdu, China, 1993.
5. Ma, J.H.; Chen, Y.S. Study for bifurcation topological structure and the global complicated character of a kind of nonlinear finance system(I). *Appl. Math. Mech.* **2001**, *22*, 1240–1251. [CrossRef]
6. Ma, J.H.; Chen, Y.S. Study for bifurcation topological structure and the global complicated character of a kind of nonlinear finance system(II). *Appl. Math. Mech.* **2001**, *22*, 1375–1382. [CrossRef]
7. Gao, Q.; Ma, J.H. Chaos and Hopf bifurcation of a finance system. *Nonlinear Dyn.* **2009**, *58*, 209–216. [CrossRef]
8. Kai, G.; Zhang, W.; Jin, Z.; Wang, C.Z. Hopf bifurcation and dynamic analysis of an improved financial system with two delays. *Complexity* **2020**, *9*, 3734125. [CrossRef]
9. Zhang, J.Z. *Fractal*; Tsinghua University Press: Beijing, China, 2011.
10. Mandelbrot, B.B. *The Fractal Geometry of Nature*; W. H. Freeman and Company: New York, NY, USA, 1983.
11. Mandelbrot, B.B. *Fractals and Chaos: The Mandelbrot Set and Beyond*; Springer: New York, NY, USA, 2004.
12. Julia, G. Mémoire sur l'itération des fonctions rationnelles. *J. Math. Pures Appl.* **1918**, *8*, 47–246.
13. Mojica, N.S.; Navarro, J.; Marijuan, P.C.; Lahoz-Beltra, R. Cellular "bauplans": Evolving unicellular forms by means of Julia sets and Pickover biomorphs. *BioSystems* **2009**, *98*, 19–30. [CrossRef]
14. Beck, C. Physical meaning for Mandelbrot and Julia sets. *Phys. Nonlinear Phenom.* **1999**, *125*, 171–182. [CrossRef]
15. Sun, Y.Y.; Xu, R.D.; Chen, L.N.; Hu, X.P. Image compression and encryption scheme using fractal dictionary and Julia set. *IET Image Process.* **2015**, *9*, 173–183. [CrossRef]
16. Bhoria, A.; Panwar, A.; Sajid, M. Mandelbrot and Julia Sets of Transcendental Functions Using Picard–Thakur Iteration. *Fractal Fract.* **2023**, *7*, 768. [CrossRef]
17. Kang, J.Y.; Ryoo, C.S. Difference Equations and Julia Sets of Several Functions for Degenerate q-Sigmoid Polynomials. *Fractal Fract.* **2023**, *7*, 791. [CrossRef]
18. Wang, X.; Li, W. Choosing the Best Members of the Optimal Eighth-Order Petković's Family by Its Fractal Behavior. *Fractal Fract.* **2022**, *6*, 749. [CrossRef]
19. Sui, S.G.; Liu, S.T. Control of Julia sets. *Chaos Solitons Fractals* **2005**, *26*, 1135–1147. [CrossRef]
20. Zhang, Y.P.; Sun, W.H.; Liu, C.A. Control and synchronization of second Julia sets. *Chin. Phys. B* **2010**, *19*, 150–157.
21. Sun, W.H.; Zhang, Y.P. Control and synchronization of Julia sets in the forced Brusselator model. *Int. J. Bifurc. Chaos* **2015**, *25*, 1550113. [CrossRef]
22. Sun, W.H.; Liu, S.T. Consensus of Julia Sets. *Fractal Fract.* **2022**, *6*, 43. [CrossRef]
23. Wang, Y.P.; Liu, S.T.; Li, H. Adaptive synchronization of Julia sets generated by Mittag-Leffler function. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *83*, 105115. [CrossRef]
24. Wang, P.; Zhang, H. Adaptive Anti-Synchronization of Julia Sets in Generalized Alternated System. *IEEE Access* **2020**, *8*, 175596–175601. [CrossRef]
25. Falconer, K. *Fractal Geometry: Mathematical Foundations and Applications*; John Wiley and Sons Ltd.: Chichester, UK, 2014.
26. Liu, S.T.; Wang, P. *Fractal Control Theory*; Springer: Singapore, 2018.
27. Wang, D.; Liu, S.T. Synchronization between the spatial Julia sets of complex Lorenz system and complex Henon map. *Nonlinear Dyn.* **2015**, *81*, 1197–1205. [CrossRef]
28. Zhang, R.Y. Bifurcation analysis for a kind of nonlinear finance system with delayed feedback and its application to control of chaos. *J. Appl. Math.* **2012**, *4*, 316390. [CrossRef]