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Convergence Rate of the Diffused Split-Step Truncated Euler–Maruyama Method for Stochastic Pantograph Models with Lévy Leaps

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Abstract: This paper studies the stochastic pantograph model, which is considered a subcategory of stochastic delay differential equations. A more general jump process, which is called the Lévy process, is added to the model for better performance and modeling situations, having sudden changes and extreme events such as market crashes in finance. By utilizing the truncation technique, we propose the diffused split-step truncated Euler–Maruyama method, which is considered as an explicit scheme, and apply it to the addressed model. By applying the Khasminskii-type condition, the convergence rate of the proposed scheme is attained in $\mathcal{L}^p (p \ge 2)$ sense where the non-jump coefficients grow super-linearly while the jump coefficient acts linearly. Also, the rate of convergence of the proposed scheme in $\mathcal{L}^p (0 sense is addressed where all the three coefficients grow beyond linearly. Finally, theoretical findings are manifested via some numerical examples.$

Keywords: stochastic pantograph differential equations; Lévy jumps; diffused split-step truncated Euler–Maruyama method; convergence rate

MSC: 60H10; 65C30; 60H35

1. Introduction

Stochastic differential models are very important, and many researchers have focused their attention on them because they have been widely used in many fields, such as physics, chemistry, engineering, biology, and mathematical finance, to describe dynamical systems affected by uncertain factors. In order to gain more realistic simulations for stochastic systems, it is more desirable and efficient to study stochastic models with delay. Stochastic pantograph models are special kinds of stochastic delay differential equations with unlimited storage and are used in many fields of pure and applied mathematics, such as probability and quantum mechanics. Ockendon and Tayler [1] studied the collection of the electric current via the pantograph of an electric locomotive, from which the name originates.

On the other hand, the Weiner process is not a convenient approach for modeling situations, having sudden changes and extreme events. Therefore, jump models are better for tackling these situations because they play a vital role in describing a sudden change in the system [2,3]. Merton [4] was the first to propose a jump-diffusion model to update the black and Scholes model [5], which did not take into account the jumps that can occur at any time and randomly. Stochastic models interspersed with Poisson jumps have been studied by many scholars [6–8]. However, if the fluctuations are a random process, then the number of points where jumps happen and the magnitude of these jumps are also stochastic. For



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Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). modeling such a kind of these fluctuations, it is more powerful to use a general jump process, arising from Poisson random measures and generated by the Poisson point process instead of using the Poisson process. Furthermore, studying stochastic models with delay and jumps is also preferable for better performance and accuracy. *Accordingly, this paper will focus on the stochastic pantograph model with Lévy jumps*.

Most of stochastic pantograph models do not have analytical solutions, and numerical algorithms are needed to tackle this problem. However, most of these numerical algorithms have been applied under the classical global Lipschitz condition and the linear growth condition [9,10]. In many applications, these conditions are not common to be satisfied, and this in turn leads to violation in the convergence properties of these methods. When the coefficients grow beyond linearly, Hutzenthaler et al. [11] have manifested that the *p*th moments of the Euler–Maruyama method blow up to infinity for all $p \ge 1$. To tackle this problem, Hutzenthaler et al. [12] presented the tamed Euler–Maruyama method, which was a recent approach to deal with this kind of problem. The tamed Euler–Maruyama for stochastic delay models with Lévy bursts whose drift coefficients grow super linearly was investigated in [13]. However, it was mentioned in [14] that the tamed methods can cause significant inaccurate results for even step sizes that are not very small, and this is because of the disorder of the flow caused by modifying the coefficients of the stochastic model.

Recently, Mao [15] introduced the truncated Euler–Maruyama technique for highly nonlinear stochastic models and studied the convergence properties in the presence of local Lipschitz and Khasminskii-type conditions. In 2016, he [16] studied its convergence rate and stability. Guo et al. [17] applied Mao's scheme [15] on stochastic delay differential models. Geng et al. [18] studied the convergence of the truncated Euler–Maruyama method for stochastic differential equations with piecewise continuous arguments. He et al. [19] studied the truncated Euler–Maruyama method for stochastic differential equations driven by fractional Brownian motion with super-linear drift coefficient. An original contribution was made in [20] by introducing the implicit split-step version of the Euler–Maruyama technique for stochastic models. However, the core limitation regarding implicit schemes is the requirement of more computations than explicit ones.

Additionally, as we know, there are not many studies on split-step schemes for stochastic pantograph models with Lévy jumps where coefficients might act super-linearly. *Therefore, motivated by the idea of truncation technique* [15], we propose the diffused split-step truncated Euler–Maruyama method which is explicit for highly nonlinear stochastic pantograph models interspersed with Lévy jumps where all coefficients might exceed linearity and study the convergence *rate in* $\mathcal{L}^p(p \ge 0)$ sense.

The following depicts how this paper is sorted. A collection of notations and model description will be given in Section 2. Section 3 will put the light on the convergence rate in $\mathcal{L}^p(p \ge 2)$ sense. Convergence rate in $\mathcal{L}^p(0 sense will be depicted in Section 4. Numerical examples will be provided in Section 5. Finally, some conclusions will be mentioned in Section 6.$

2. Preliminaries and Model Description

In this section, we are going to present some preliminaries that will help the readers have the necessary background knowledge to understand the subsequent sections of this paper and follow the research methodology, analysis, and results effectively.

Definition 1 ([21]). A stochastic process $\{v(t)\}_{t\geq 0}$ is a collection of random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by time t, where

- For every $t \ge 0$, the function $\omega \to v(t; \omega)$ is a measurable function defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- For each $\omega \in \Omega$, the function $t \to v(t; \omega)$ is named the sample path of the process.

Definition 2 ([22]). A stochastic process $\{v(t)\}_{t\geq 0}$, defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with filtration $\{\mathcal{F}_t\}_{\{t\geq 0\}}$, has the Markov property if for any $t \geq 0$, $\bar{\varsigma} \geq 0$ and $Y \in B$, where B is the set of all Borel sets,

$$\mathbb{P}(v(t+\bar{\varsigma}) \in Y \mid \mathcal{F}_t) = \mathbb{P}(v(t+\bar{\varsigma}) \in Y \mid v(t))$$

Definition 3 ([23]). *The non-anticipating stochastic process* W(t) *satisfies the following attributes:*

- W(0) = 0 and the sample path $t \to W(t; \omega)$ is continuous a.s.
- The increment $W(\varsigma_1) W(\varsigma_2) \sim N(0, \varsigma_1 \varsigma_2)$, where $0 \le \varsigma_2 \le \varsigma_1$.
- The increments $W(\bar{\varsigma_1}) W(\varsigma_1)$ and $W(\bar{\varsigma_2}) W(\varsigma_2)$ are independent for $0 < \varsigma_1 < \bar{\varsigma_1} < \varsigma_2 < \bar{\varsigma_2}$.

is called Brownian motion.

Definition 4 ([24]). The non-anticipating stochastic process N(t) satisfies the following attributes • N(0) = 0 a.s.,

- The increment $N(\varsigma_1) N(\varsigma_2) \sim Po(\lambda^*(\varsigma_1 \varsigma_2))$, where $0 \le \varsigma_2 \le \varsigma_1$ and $\lambda^* > 0$.
- The increments $N(\bar{\varsigma_1}) N(\varsigma_1)$ and $N(\bar{\varsigma_2}) N(\varsigma_2)$ are independent for $0 < \varsigma_1 < \bar{\varsigma_1} < \varsigma_2 < \bar{\varsigma_2}$.

is called Poisson process with intensity λ^* .

Definition 5 ([23]). A right-continuous with left limits and adapted stochastic process L(t), $t \in [0, T]$, defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with filtration $\{\mathcal{F}_t\}_{t\geq 0\}}$, satisfies the following attributes

- $L(0) = 0 \ a.s.,$
- *L*(*t*) *has independent and stationary increments.*
- L(t) is stochastically continuous, which means $\forall \varsigma > 0$ and $\forall \bar{\varsigma} \ge 0$.

$$\lim_{t \to \bar{\varsigma}} \mathbb{P}(\parallel L(t) - L(\bar{\varsigma}) \parallel > \varsigma) = 0$$

is called the Lévy process.

Definition 6 ([25]). A stochastic differential equation (SDE) is a differential equation where one or more of its terms are stochastic processes and therefore the solution of it will be a stochastic process. A typical form is

$$dv(t) = \varphi(v(t))dt + \psi(v(t))dW(t),$$

where W(t) is a Brownian motion.

The functions φ and ψ are called the drift and diffusion coefficients, respectively. Stochastic pantograph differential equations [26] are considered special subcategory of stochastic delay differential equations with the form

$$dv(t) = \varphi(v(t), v(\eta t))dt + \psi(v(t), v(\eta t))dW(t),$$
(1)

with initial data $v(0^-) = v_0$ and $0 < \eta < 1$. Most stochastic pantograph models do not have analytical solutions or are difficult to obtain, and numerical algorithms are needed to tackle this problem. However the classical existence and uniqueness theorems requires the coefficients of the stochastic model to satisfy

• Global Lipschitz condition: There exists a constant C > 0 such that for all $\zeta_1, \zeta_2, \overline{\zeta_1}, \overline{\zeta_2} \in \mathbb{R}^m$,

$$|\varphi(\varsigma_1, \bar{\varsigma_1}) - \varphi(\varsigma_2, \bar{\varsigma_2})|^2 \vee |\psi(\varsigma_1, \bar{\varsigma_1}) - \psi(\varsigma_2, \bar{\varsigma_2})|^2 \le C|\varsigma_1 - \varsigma_2|^2 + |\bar{\varsigma_1} - \bar{\varsigma_2}|^2).$$

• *Linear growth condition*: There exists a constant C > 0 such that for all $\zeta_1, \zeta_2 \in \mathbb{R}^m$,

$$|\varphi(\varsigma_1,\varsigma_2)|^2 \vee |\psi(\varsigma_1,\varsigma_2)|^2 \leq C(1+|\varsigma_1|^2+|\varsigma_2|^2).$$

However, these conditions are very restrictive, and there are many stochastic pantograph models that do not satisfy the linear growth condition, and this in turn leads to some violations in the convergence properties of these numerical algorithms. This is considered one of the motivations behind this paper, where we try to perform some relaxation and replace the linear growth condition with what is known as the Khasminskii-type condition (to be discussed later).

There exist two kinds of convergence of the numerical solutions of stochastic models [25]. The first kind of convergence is strong convergence.

Definition 7. Suppose $\chi(t)$ is a continuous-time approximation of the solution v(t) of Equation (1) with step size $\Delta > 0$. Then, χ converges to v(t) in the strong sense with order $\epsilon \in (0, \infty)$ if there exist positive constants C and Δ^* such that

$$\mathbb{E}|v(t)-\chi(t)|\leq C\Delta^{\epsilon},$$

where $\Delta \in (0, \Delta^*)$.

The other kind of convergence is weak convergence.

Definition 8. Suppose $\chi(t)$ is a continuous-time approximation of the solution v(t) of Equation (1) with step size $\Delta > 0$. Then, χ converges to v(t) in the weak sense with order $\bar{\epsilon} \in (0, \infty)$ if for any function $f : \mathbb{R}^m \to \mathbb{R}$, there exist positive constants C and Δ^* such that

$$|\mathbb{E}f(v(t)) - \mathbb{E}f(\chi(t))| \le C\Delta^{\tilde{\epsilon}},$$

where $\Delta \in (0, \Delta^*)$.

Throughout this paper, let $(\Omega, \mathcal{F}, {\mathcal{F}_t}_{\{t \ge 0\}}, \mathbb{P})$ be a complete probability space with right-continuous and non-decreasing filtration ${\mathcal{F}_t}_{t\ge 0}$ with \mathcal{F}_0 encompassing all \mathbb{P} -null sets. Let $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ indicate the space of random variables Ψ with expectation $\mathbb{E}|\Psi|^p < \infty$ for p > 0. Furthermore, if Z is a vector or matrix, its transpose is represented by Z^T . Let $|\cdot|$ denote the Euclidean vector norm in \mathbb{R}^m , and let $\langle \varsigma, \bar{\varsigma} \rangle$ be the inner product of $\varsigma, \bar{\varsigma}$ in \mathbb{R}^m and $\varsigma \in \mathbb{R}$, $[\varsigma]$ refer to the non-fractional part of ς . Also, $\varsigma \vee \bar{\varsigma}$ and $\varsigma \wedge \bar{\varsigma}$ refer to picking up the bigger and smaller between them, respectively. Let $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))^T$ be d-dimensional Brownian motion and $U \in \mathbb{R}^m \setminus \{0\}$ be the scope of abrupt leaps. Let $N(\cdot, \cdot)$ defined on $\mathbb{R}_+ \times \mathbb{R}^m \setminus \{0\}$ be a \mathcal{F}_t -adapted Poisson random measure and $\widetilde{N}(dt, du) = N(dt, du) - \pi(du)dt$ be its compensated version with Lévy measure π defined on U with $\pi(U) = \lambda$. It is assumed that W(t) is independent of $N(t, \cdot)$.

Let our analysis be focused on *m*-dimensional stochastic pantograph model interspersed with Lévy jumps of the form

$$dv(t) = \varphi(v(t^{-}), v(\eta t^{-}))dt + \psi(v(t^{-}), v(\eta t^{-}))dW(t) + \int_{II} \omega(v(t^{-}), v(\eta t^{-}), u)\widetilde{N}(dt, du),$$
(2)

defined on $0 \le t \le T$ with $0 < \eta < 1$ and initial data $v(0^-) = v_0$, where v_0 is \mathcal{F}_0 measurable, right-continuous, and $\mathbb{E}|v_0|^{\bar{q}} < \infty$ for $\bar{q} > 0$. Here $v(t^-) := \lim_{s \to t^-} v(s)$, $v(\eta t^-) := \lim_{s \to \eta t^-} v(s)$, $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $\psi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ and $\varpi : \mathbb{R}^m \times \mathbb{R}^m \times U \to \mathbb{R}^m$, $m, d \in \mathbb{N}^+$. **Remark 1.** In this paper, v(t) and $v(\eta t)$ are used to express $v(t^-)$ and $v(\eta t^-)$, respectively, and *C* is used to denote a general real positive constant (independent of Δ , *l* later) changing at different positions.

3. Convergence Rate in $\mathcal{L}^p (p \ge 2)$

In some applications, we need to approximate the variance or the higher moment of the solution. In these situations, we need to have the convergence in the $\mathcal{L}^p(p \ge 2)$ sense. Therefore, in this section, the convergence rate of the diffused split-step truncated Euler–Maruyama method for Equation (2) is attained in the $\mathcal{L}^p(p \ge 2)$ sense, where non-jump coefficients behave beyond linearly while the jump coefficient grows linearly. At first, some assumptions and lemmas will be presented as helping tools for proving our main convergence theorem.

Assumption 1. Let $k_1 > 0$, $\zeta \ge 0$ such that

$$\begin{aligned} |\varphi(\varsigma_{1}, \varsigma_{1}) - \varphi(\varsigma_{2}, \varsigma_{2})| &\lor |\psi(\varsigma_{1}, \varsigma_{1}) - \psi(\varsigma_{2}, \varsigma_{2})| \\ &\le k_{1}(1 + |\varsigma_{1}|^{\zeta} + |\varsigma_{1}|^{\zeta} + |\varsigma_{2}|^{\zeta} + |\varsigma_{2}|^{\zeta})(|\varsigma_{1} - \varsigma_{2}| + |\varsigma_{1} - \varsigma_{2}|) \end{aligned}$$

and

$$\int_{U} |\omega(\varsigma_1, \bar{\varsigma_1}, u) - \omega(\varsigma_2, \bar{\varsigma_2}, u)| \pi(du) \le k_1(|\varsigma_1 - \varsigma_2| + |\bar{\varsigma_1} - \bar{\varsigma_2}|)$$

for all $\varsigma_1, \varsigma_2, \overline{\varsigma_1}, \overline{\varsigma_2} \in \mathbb{R}^m$ and $u \in U$.

By utilizing Assumption 1, it can be concluded that

$$|\varphi(\varsigma,\bar{\varsigma})| \vee |\psi(\varsigma,\bar{\varsigma})| \le C(1+|\varsigma|^{\zeta+1}+|\bar{\varsigma}|^{\zeta+1}) \tag{3}$$

and

$$\int_{U} |\omega(\varsigma, \bar{\varsigma}, u)| \pi(du) \le C(1 + |\varsigma| + |\bar{\varsigma}|)$$
(4)

for all $\zeta, \overline{\zeta} \in \mathbb{R}^m$ and $u \in U$.

Assumption 2. *Let* $k_2 > 0$ *,* $\xi > 2$ *such that*

$$\begin{aligned} (\varsigma_1 - \varsigma_2)^T (\varphi(\varsigma_1, \bar{\varsigma_1}) - \varphi(\varsigma_2, \bar{\varsigma_2})) + \frac{\zeta - 1}{2} |\psi(\varsigma_1, \bar{\varsigma_1}) - \psi(\varsigma_2, \bar{\varsigma_2})|^2 \\ &\leq k_2 (|\varsigma_1 - \varsigma_2|^2 + |\bar{\varsigma_1} - \bar{\varsigma_2}|^2) \end{aligned}$$

for all $\varsigma_1, \varsigma_2, \overline{\varsigma_1}, \overline{\varsigma_2} \in \mathbb{R}^m$.

Assumption 3. (*Khasminskii-type condition*) Let $k_3 > 0$, $\bar{p} > \xi$ such that

$$\zeta^{T} \varphi(\zeta, \bar{\zeta}) + \frac{\bar{p} - 1}{2} |\psi(\zeta, \bar{\zeta})|^{2} \le k_{3}(1 + |\zeta|^{2} + |\bar{\zeta}|^{2})$$

for all $\zeta, \overline{\zeta} \in \mathbb{R}^m$.

Lemma 1. Under Assumptions 1 and 3, for any $q \in [2, \bar{p})$

$$\sup_{0 \le t \le T} \mathbb{E}|v(t)|^q \le C, \quad \forall T > 0$$
(5)

Proof. Proving this Lemma can be attained by following the same approach as in [27]. To define the diffused split-step truncated Euler–Maruyama scheme, a strictly non-decreasing continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is selected, where $\beta(\iota) \to \infty$ as $\iota \to \infty$ and

$$\sup_{\varsigma|\vee|\bar{\varsigma}|\leq\iota} (|\varphi(\varsigma,\bar{\varsigma})|\vee|\psi(\varsigma,\bar{\varsigma})|) \leq \beta(\iota),$$
(6)

$$\sup_{0 < |\varsigma_i| \lor |\varsigma_i| \le \iota} \frac{|\varphi(\varsigma_1, \varsigma_1) - \varphi(\varsigma_2, \varsigma_2)|}{(|\varsigma_1 - \varsigma_2| + |\varsigma_1 - \varsigma_2|)} \lor \frac{|\psi(\varsigma_1, \varsigma_1) - \psi(\varsigma_2, \varsigma_2)|}{(|\varsigma_1 - \varsigma_2| + |\varsigma_1 - \varsigma_2|)} \le \beta(\iota),$$
(7)

for all $\iota \ge 1$ and i = 1, 2. Moreover, a strictly non-increasing function $\gamma : (0, 1] \to (0, \infty)$ is chosen such that

$$\lim_{\Delta \to 0} \gamma(\Delta) = \infty \quad \text{and} \quad (\Delta^{1/\bar{p}} \vee \Delta^{1/4}) \gamma(\Delta) \le 1, \quad \forall \Delta \in (0, 1].$$
(8)

For a given $\Delta \in (0, 1]$, a truncated mapping ν_{Δ} from \mathbb{R}^m to the closed ball { $\varsigma \in \mathbb{R}^m : |\varsigma| \le \beta^{-1}(\gamma(\Delta))$ } is defined by

$$\nu_{\Delta}(\varsigma) = (|\varsigma| \wedge \beta^{-1}(\gamma(\Delta))) \frac{\varsigma}{|\varsigma|},\tag{9}$$

where we set $\zeta/|\zeta| = 0$ if $\zeta = 0$. Then, the truncated functions are defined as follows:

$$Y_{\Delta}(\varsigma,\bar{\varsigma}) = Y(\nu_{\Delta}(\varsigma),\nu_{\Delta}(\bar{\varsigma})), \tag{10}$$

for any $\zeta, \overline{\zeta} \in \mathbb{R}^m$, where $\Upsilon = \varphi$ or ψ . It is also obvious that

$$|\varphi_{\Delta}(\varsigma,\bar{\varsigma})| \vee |\psi_{\Delta}(\varsigma,\bar{\varsigma})| \le \beta(\beta^{-1}(\gamma(\Delta))) = \gamma(\Delta), \quad \forall \varsigma, \bar{\varsigma} \in \mathbb{R}^{m}$$
(11)

which indicates that φ_{Δ} , ψ_{Δ} are bounded even though φ , ψ may not. Additionally, it can be concluded

$$|\nu_{\Delta}(\varsigma)| \le |\varsigma|, \quad |\nu_{\Delta}(\varsigma) - \nu_{\Delta}(\bar{\varsigma})| \le |\varsigma - \bar{\varsigma}|, \quad \forall \varsigma, \bar{\varsigma} \in \mathbb{R}^{m}.$$
(12)

Upon utilizing (12) and Assumption 1, it can be concluded that

$$\begin{aligned} |\varphi_{\Delta}(\varsigma_{1}, \bar{\varsigma_{1}}) - \varphi_{\Delta}(\varsigma_{2}, \bar{\varsigma_{2}})| &\lor |\psi_{\Delta}(\varsigma_{1}, \bar{\varsigma_{1}}) - \psi_{\Delta}(\varsigma_{2}, \bar{\varsigma_{2}})| \\ &\le k_{1}(1 + |\varsigma_{1}|^{\zeta} + |\bar{\varsigma_{1}}|^{\zeta} + |\varsigma_{2}|^{\zeta} + |\bar{\varsigma_{2}}|^{\zeta})(|\varsigma_{1} - \varsigma_{2}| + |\bar{\varsigma_{1}} - \bar{\varsigma_{2}}|) \end{aligned}$$
(13)

for all $\varsigma_1, \varsigma_2, \overline{\varsigma_1}, \overline{\varsigma_2} \in \mathbb{R}^m$. \Box

Lemma 2. *Under Assumption 3, for any* $\Delta \in (0, 1]$ *,*

$$\varsigma^{T}\varphi_{\Delta}(\varsigma,\bar{\varsigma}) + \frac{\bar{p}-1}{2}|\psi_{\Delta}(\varsigma,\bar{\varsigma})|^{2} \le C(1+|\varsigma|^{2}+|\bar{\varsigma}|^{2}), \quad \forall \varsigma,\bar{\varsigma} \in \mathbb{R}^{m}$$
(14)

Proof. The verification follows the one discussed in [28]. Now, the diffused split-step truncated Euler–Maruyama scheme for Equation (2) is defined by $Y_0 = v_0$ and Y_{n+1} is computed by

$$Z_n = Y_n + \psi_{\Delta}(Y_n, Y_{[\eta n]}) \Delta W_n \tag{15}$$

$$Y_{n+1} = Z_n + \Delta \varphi_{\Delta}(Z_n, Z_{[\eta n]}) + \int_{t_n}^{t_{n+1}} \int_U \mathcal{O}(Z_n, Z_{[\eta n]}, u) \widetilde{N}(dt, du),$$
(16)

for n = 0, 1, ..., where Y_n approximates $v(t_n)$ at $t_n = n\Delta$, $\Delta W_n := W(t_{n+1}) - W(t_n)$. Wang and Li [29] introduced the fully explicit split-step forward methods for solving Itô stochastic differential models. However, the main limitation of these schemes is that the derivatives of the drift and diffusion coefficients must be calculated at each iteration that is considered computationally intensive. Our proposed scheme is considered as an explicit and derivative-free scheme that does not require the calculation of the derivative at each step with good properties in terms of convergence rate and accuracy. For all $t \in [t_n, t_{n+1})$ and $\Delta \in (0, 1]$, we define

$$\chi_{\Delta}(t) := Y_n + (t - t_n)\varphi_{\Delta}(Z_n, Z_{[\eta n]}) + \psi_{\Delta}(Y_n, Y_{[\eta n]})(W(t) - W(t_n)) + \int_{t_n}^t \int_U \mathcal{O}(Z_n, Z_{[\eta n]}, u)\widetilde{N}(dt, du)$$
(17)

and denote

$$\kappa_1(t) = \sum_{r=0}^{\infty} Y_r I_{[t_r, t_{r+1})}(t), \quad \kappa_2(t) = \sum_{r=0}^{\infty} Y_{[\eta r]} I_{[t_r, t_{r+1})}(t),$$

and

$$\kappa_1^*(t) = \sum_{r=0}^{\infty} Z_r I_{[t_r, t_{r+1})}(t), \quad \kappa_2^*(t) = \sum_{r=0}^{\infty} Z_{[\eta r]} I_{[t_r, t_{r+1})}(t),$$

where $I_Z(\Psi) = 1$ if $\Psi \in Z$. Accordingly, Equation (17) can be rewritten in integral form as

$$\chi_{\Delta}(t) = Y_0 + \int_0^t \varphi_{\Delta}(\kappa_1^*(s), \kappa_2^*(s)) ds + \int_0^t \psi_{\Delta}(\kappa_1(s), \kappa_2(s)) dW(s) + \int_0^t \int_U \varpi(\kappa_1^*(s), \kappa_2^*(s), u) \widetilde{N}(ds, du).$$
(18)

Lemma 3. Under Assumption 1,

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\varrho}} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}^{*}(t)|^{\hat{\varrho}} \\ \leq C((\gamma(\Delta))^{\hat{\varrho}} \Delta^{\hat{\varrho}/2} + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(t)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(t)|^{\hat{\varrho}})), \quad \hat{\varrho} \geq 2,$$
(19)

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\varrho}} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}^{*}(t)|^{\hat{\varrho}} \leq C((\gamma(\Delta))^{\hat{\varrho}}\Delta^{\hat{\varrho}/2} + \Delta^{\hat{\varrho}/2}(1 + \mathbb{E}|\kappa_{1}^{*}(t)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(t)|^{\hat{\varrho}})), \quad 0 < \hat{\varrho} < 2.$$

$$(20)$$

Proof. Select any $\Delta \in (0, 1]$, $\hat{\varrho} \ge 2$. Then, \exists a unique *r* where $r\Delta \le t \le (r + 1)\Delta$. From Equation (18), we have the following:

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\varrho}} = \mathbb{E}|\chi_{\Delta}(t) - \chi_{\Delta}(r\Delta)|^{\hat{\varrho}}$$

$$= \mathbb{E}\left|\int_{r\Delta}^{t} \varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s))ds + \int_{r\Delta}^{t} \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))dW(s) + \int_{r\Delta}^{t} \int_{U} \mathcal{O}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u))\widetilde{N}(ds, du)\right|^{\hat{\varrho}}.$$
(21)

Once utilizing (11), Assumption 1 and the properties of the Itô integral [21], we obtain

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\varrho}} \leq C\Delta^{\hat{\varrho}-1}\mathbb{E}\int_{r\Delta}^{t}|\varphi_{\Delta}(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s))|^{\hat{\varrho}}ds + C\Delta^{(\hat{\varrho}-2)/2}\mathbb{E}\int_{r\Delta}^{t}|\psi_{\Delta}(\kappa_{1}(s),\kappa_{2}(s))|^{\hat{\varrho}}ds + C\mathbb{E}\left(\int_{r\Delta}^{t}\int_{U}|\varpi(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s),u)|^{2}\pi(du)ds\right)^{\hat{\varrho}/2} + C\mathbb{E}\int_{r\Delta}^{t}\int_{U}|\varpi(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s),u)|^{\hat{\varrho}}\pi(du)ds \leq C((\gamma(\Delta))^{\hat{\varrho}}\Delta^{\hat{\varrho}/2} + \Delta(1+\mathbb{E}|\kappa_{1}^{*}(s)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(s)|^{\hat{\varrho}})).$$

$$(22)$$

Therefore

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_1(t)|^{\hat{\varrho}} \le C((\gamma(\Delta))^{\hat{\varrho}}\Delta^{\hat{\varrho}/2} + \Delta(1 + \mathbb{E}|\kappa_1^*(s)|^{\hat{\varrho}} + \mathbb{E}|\kappa_2^*(s)|^{\hat{\varrho}})).$$
(23)

By utilizing (11) and (15), it can be concluded that

$$\mathbb{E}|\kappa_1(t) - \kappa_1^*(t)|^{\hat{\varrho}} \le C\Delta^{(\hat{\varrho}-2)/2} \mathbb{E} \int_{r\Delta}^t |\psi_\Delta(\kappa_1(s),\kappa_2(s))|^{\hat{\varrho}} d(s) \le C(\gamma(\Delta))^{\hat{\varrho}} \Delta^{\hat{\varrho}/2}.$$
(24)

By utilizing (23) and (24), we obtain

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}^{*}(t)|^{\hat{\ell}} \leq C(\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\ell}} + \mathbb{E}|\kappa_{1}(t) - \kappa_{1}^{*}(t)|^{\hat{\ell}})$$

$$\leq C((\gamma(\Delta))^{\hat{\ell}}\Delta^{\hat{\ell}/2} + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(s)|^{\hat{\ell}} + \mathbb{E}|\kappa_{2}^{*}(s)|^{\hat{\ell}})).$$
(25)

By utilizing the Hölder inequality, (23), and (25), we have for any $0 < \hat{\ell} < 2$ the following:

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{\hat{\ell}} \leq (\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}(t)|^{2})^{\hat{\ell}/2} \\
\leq C((\gamma(\Delta))^{2}\Delta + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(s)|^{2} + \mathbb{E}|\kappa_{2}^{*}(s)|^{2}))^{\hat{\ell}/2} \\
\leq C((\gamma(\Delta))^{\hat{\ell}}\Delta^{\hat{\ell}/2} + \Delta^{\hat{\ell}/2}(1 + \mathbb{E}|\kappa_{1}^{*}(s)|^{\hat{\ell}} + \mathbb{E}|\kappa_{2}^{*}(s)|^{\hat{\ell}})),$$
(26)

and

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}^{*}(t)|^{\hat{\varrho}} \leq (\mathbb{E}|\chi_{\Delta}(t) - \kappa_{1}^{*}(t)|^{2})^{\hat{\varrho}/2} \\
\leq C((\gamma(\Delta))^{2}\Delta + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(s)|^{2} + \mathbb{E}|\kappa_{2}^{*}(s)|^{2}))^{\hat{\varrho}/2} \\
\leq C((\gamma(\Delta))^{\hat{\varrho}}\Delta^{\hat{\varrho}/2} + \Delta^{\hat{\varrho}/2}(1 + \mathbb{E}|\kappa_{1}^{*}(s)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(s)|^{\hat{\varrho}})).$$
(27)

Corollary 1. Under Assumption 1,

$$\mathbb{E}|\chi_{\Delta}(\eta t) - \kappa_{2}(t)|^{\hat{\varrho}} \vee \mathbb{E}|\chi_{\Delta}(\eta t) - \kappa_{2}^{*}(t)|^{\hat{\varrho}} \leq C((\gamma(\Delta))^{\hat{\varrho}}\Delta^{\hat{\varrho}/2} + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(t)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(t)|^{\hat{\varrho}})), \quad \hat{\varrho} \geq 2,$$
(28)

$$\mathbb{E}|\chi_{\Delta}(\eta t) - \kappa_{2}(t)|^{\hat{\varrho}} \vee \mathbb{E}|\chi_{\Delta}(\eta t) - \kappa_{2}^{*}(t)|^{\hat{\varrho}} \leq C((\gamma(\Delta))^{\hat{\varrho}} \Delta^{\hat{\varrho}/2} + \Delta^{\hat{\varrho}/2}(1 + \mathbb{E}|\kappa_{1}^{*}(t)|^{\hat{\varrho}} + \mathbb{E}|\kappa_{2}^{*}(t)|^{\hat{\varrho}})), \quad 0 < \hat{\varrho} < 2.$$

$$(29)$$

Proof. The proof of this corollary can be attained by proceeding the same approach as in Lemma 3. \Box

Lemma 4. Under Assumptions 1 and 3, for $q \in [2, \bar{p})$

$$\sup_{0<\Delta\leq 1}\sup_{0\leq t\leq T}\mathbb{E}|\chi_{\Delta}(t)|^{q}\leq C,\quad\forall T>0.$$
(30)

Proof. For fixed $\Delta \in (0, 1]$, we obtain via the Itô formula [30] and (18)

$$\begin{aligned} |\chi_{\Delta}(t)|^{q} &\leq |v_{0}|^{q} + \int_{0}^{t} q |\chi_{\Delta}(s)|^{q-2} \Big(\kappa_{1}^{T}(s) \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)) + \frac{q-1}{2} |\psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2} \Big) ds \\ &+ \int_{0}^{t} q |\chi_{\Delta}(s)|^{q-2} \Big((\chi_{\Delta}(s) - \kappa_{1}(s))^{T} \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)) \Big) ds \\ &+ \int_{0}^{t} q |\chi_{\Delta}(s)|^{q-2} \Big(\chi_{\Delta}^{T}(s) (\varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s)) - \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))) \Big) ds \\ &+ \int_{0}^{t} q |\chi_{\Delta}(s)|^{q-2} \chi_{\Delta}^{T}(s) \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)) dW(s) \\ &+ \int_{0}^{t} \int_{U} q |\chi_{\Delta}(s)|^{q-2} \chi_{\Delta}^{T}(s) \omega(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u) \widetilde{N}(ds, du) \\ &+ \int_{0}^{t} \int_{U} \Big[|\chi_{\Delta}(s) + \omega(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{q} - |\chi_{\Delta}(s)|^{q} \\ &- q |\chi_{\Delta}(s)|^{q-2} \chi_{\Delta}^{T}(s) \omega(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u) \Big] N(ds, du). \end{aligned}$$

Applying Assumption 3, using the Taylor formula [30] and the Young inequality, and then taking the expectation will lead to

$$\mathbb{E}|\chi_{\Delta}(t)|^{q} \le |v_{0}|^{q} + L_{1} + L_{2} + L_{3} + L_{4}, \tag{32}$$

where

$$L_1 = C\mathbb{E}\int_0^t |\chi_{\Delta}(s)|^{q-2} (1+|\kappa_1(s)|^2+|\kappa_2(s)|^2) ds,$$
(33)

$$L_2 = C\mathbb{E}\int_0^t |\chi_{\Delta}(s)|^{q-2} \Big((\chi_{\Delta}(s) - \kappa_1(s))^T \varphi_{\Delta}(\kappa_1(s), \kappa_2(s)) \Big) ds,$$
(34)

$$L_3 = \mathbb{CE} \int_0^t |\chi_{\Delta}(s)|^{q-2} \Big(\chi_{\Delta}^T(s)(\varphi_{\Delta}(\kappa_1^*(s), \kappa_2^*(s)) - \varphi_{\Delta}(\kappa_1(s), \kappa_2(s))) \Big) ds,$$
(35)

and

$$L_4 = C\mathbb{E}\int_0^t \int_U \Big(|\chi_{\Delta}(s)|^{q-2} |\varpi(\kappa_1^*(s), \kappa_2^*(s), u)|^2 + |\varpi(\kappa_1^*(s), \kappa_2^*(s), u)|^q \Big) \pi(du) ds.$$
(36)

From (8), (11) and (15), we obtain

$$\mathbb{E}|\kappa_1^*(t)|^q \le 2^{q-1}(\mathbb{E}|\kappa_1(t)|^q + (\gamma(\Delta))^q \mathbb{E}|\Delta W_r|^q) \le C + C\mathbb{E}|\kappa_1(t)|^q.$$
(37)

By the same analogy, we obtain

$$\mathbb{E}|\kappa_2^*(t)|^q \le C + C\mathbb{E}|\kappa_2(t)|^q.$$
(38)

Utilizing the Young inequality $a^{q-2}b \leq \frac{q-2}{q}a^q + \frac{2}{q}b^{q/2}$ leads to

$$L_{1} \leq C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds + C\mathbb{E} \int_{0}^{t} (1 + |\kappa_{1}(s)|^{q} + |\kappa_{2}(s)|^{q}) ds$$

$$\leq C + C \int_{0}^{t} (\mathbb{E}|\chi_{\Delta}(s)|^{q} + \mathbb{E}|\kappa_{1}(s)|^{q} + \mathbb{E}|\kappa_{2}(s)|^{p}) ds$$

$$\leq C + C \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E}|\chi_{\Delta}(r)|^{q} ds$$
(39)

By applying the Young inequality, Lemma 3, (10), (11), (37), and (38), we obtain

$$L_{2} \leq C + C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds + C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s) - \kappa_{1}(s)|^{q} |\varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{q} ds$$

$$\leq C + C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds$$

$$+ C(\gamma(\Delta))^{q} \int_{0}^{t} ((\gamma(\Delta))^{q} \Delta^{q/2} + \Delta(1 + \mathbb{E}|\kappa_{1}^{*}(t)|^{q} + \mathbb{E}|\kappa_{2}^{*}(t)|^{q})) ds$$

$$\leq C + C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds$$

$$+ C(\gamma(\Delta))^{q} \int_{0}^{t} ((\gamma(\Delta))^{q} \Delta^{q/2} + \Delta(1 + \mathbb{E}|\kappa_{1}(t)|^{q} + \mathbb{E}|\kappa_{2}(t)|^{q})) ds$$

$$\leq C + C \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E} |\chi_{\Delta}(r)|^{q} ds$$
(40)

By the Young inequality, (7), (10), (11), (12), and (24), we have

$$\begin{split} L_{3} &\leq C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds + C\mathbb{E} \int_{0}^{T} |\varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s)) - \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{q} ds \\ &\leq C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds \\ &+ C\mathbb{E} \int_{0}^{T} |\varphi(\nu_{\Delta}(\kappa_{1}^{*}(s)), \nu_{\Delta}(\kappa_{2}^{*}(s))) - \varphi(\nu_{\Delta}(\kappa_{1}(s)), \nu_{\Delta}(\kappa_{2}(s)))|^{q} ds \\ &\leq C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds \\ &+ C(\gamma(\Delta))^{q} \int_{0}^{T} (\mathbb{E} |\nu_{\Delta}(\kappa_{1}^{*}(s)) - \nu_{\Delta}(\kappa_{1}(s))|^{q} + \mathbb{E} |\nu_{\Delta}(\kappa_{2}^{*}(s)) - \nu_{\Delta}(\kappa_{2}(s))|^{q} ds \\ &\leq C\mathbb{E} \int_{0}^{t} |\chi_{\Delta}(s)|^{q} ds + C \int_{0}^{T} (\gamma(\Delta))^{2q} \Delta^{q/2} ds \\ &\leq C + C \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E} |\chi_{\Delta}(r)|^{q} ds. \end{split}$$

$$(41)$$

By utilizing the Young inequality and Assumption 1, then proceeding the same as before, we obtain

$$L_4 \le C + C \int_0^t \sup_{0 \le r \le s} \mathbb{E} |\chi_{\Delta}(r)|^q ds.$$
(42)

By plugging (39), (40), (41), and (42) into (32), we obtain

$$\mathbb{E}|\chi_{\Delta}(t)|^{q} \leq C + C \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E}|\chi_{\Delta}(r)|^{q} ds,$$
(43)

where the R.H.S of (43) is increasing in *t*. Then,

$$\sup_{0 \le r \le t} \mathbb{E} |\chi_{\Delta}(r)|^q \le C + C \int_0^t \sup_{0 \le r \le s} \mathbb{E} |\chi_{\Delta}(r)|^q ds.$$
(44)

By the Gronwall inequality,

$$\sup_{0 \le r \le t} \mathbb{E} |\chi_{\Delta}(r)|^q \le C.$$
(45)

Because this is valid regardless, the value of Δ , (30) is obtained. \Box

Lemma 5. Under Assumptions 1 and 3,

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_1(t)|^{\varrho^*} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_1^*(t)|^{\varrho^*} \le C((\gamma(\Delta))^{\varrho^*} \Delta^{\varrho^*/2} + \Delta), \quad 2 \le \varrho^* \le \bar{p},$$
(46)

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_1(t)|^{\varrho^*} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_1^*(t)|^{\varrho^*} \le C((\gamma(\Delta))^{\varrho^*} \Delta^{\varrho^*/2} + \Delta^{\varrho^*/2}), \ 0 < \varrho^* < 2.$$
(47)

Proof. By utilizing Lemma 4, (19), and (37), the required assertion (46) is directly attained. For any $0 < \varrho^* < 2$, by utilizing Hölder's inequality, we obtain

$$egin{aligned} \mathbb{E} |\chi_\Delta(t) - \kappa_1(t)|^{arrho^*} &\leq (\mathbb{E} |\chi_\Delta(t) - \kappa_1(t)|^2)^{arrho^*/2} \ &\leq (C((\gamma(\Delta))^2\Delta + \Delta))^{arrho^*/2} \ &\leq C((\gamma(\Delta))^{arrho^*}\Delta^{arrho^*/2} + \Delta^{arrho^*/2}) \end{aligned}$$

Similarly,

$$\mathbb{E}|\chi_{\Delta}(t)-\kappa_1^*(t)|^{\varrho^*} \leq C((\gamma(\Delta))^{\varrho^*}\Delta^{\varrho^*/2}+\Delta^{\varrho^*/2}).$$

The proof is complete. \Box

Lemma 6. Suppose that Assumptions 1 and 3 hold. Then, for any real number l > |v(0)| and $\Delta \in (0,1]$, we define the stopping time $\rho_l = \inf\{t \ge 0 : |v(t)| \ge l\}$ such that

$$\mathbb{P}(\rho_l \le T) \le \frac{C}{l^q} \tag{48}$$

Proof. By utilizing (5), we have

$$\sup_{0\leq u\leq T}\mathbb{E}|v(u\wedge\rho_l)|^q\leq C.$$

Then, by applying Chebyshev's inequality, we have

$$l^q \mathbb{P}(\rho_l \le T) \le C. \tag{49}$$

The proof is complete. \Box

Lemma 7. Suppose that Assumptions 1 and 3 hold. Then, for any real number l > |v(0)| and $\Delta \in (0,1]$, we define stopping times $\vartheta_{\Delta,l} = \inf\{t \ge 0 : |\chi_{\Delta}(t)| \ge l\}$ and $\bar{\vartheta}_{\Delta,l} = \inf\{t \ge 0 : |\kappa_1^*(t)| \ge l\}$ such that

$$\mathbb{P}(\vartheta_{\Delta,l} \le T) \lor \mathbb{P}(\bar{\vartheta}_{\Delta,l} \le T) \le \frac{C}{19}$$
(50)

Proof. Upon proceeding in the same manner as in in Lemma 4, it can be shown that

$$\sup_{0\leq u\leq T}\mathbb{E}|\chi_{\Delta}(u\wedge\vartheta_{\Delta,l})|^{q}\leq C.$$

Then, by applying Chebyshev's inequality, we obtain

$$l^{q}\mathbb{P}(\vartheta_{\Delta,l} \le T) \le C.$$
(51)

Then, by utilizing (37) and Chebyshev's inequality, we can obtain

$$l^{q}\mathbb{P}(\bar{\vartheta}_{\Delta,l} \le T) \le C.$$
(52)

Theorem 1. Let Assumptions 1–3 hold, $q \in (2, \bar{p})$ such that $q > (1 + \zeta)\xi$. Then, for $p \in [2, \xi)$ and $\Delta \in (0, 1]$

$$\mathbb{E}|e_{\Delta}(T \wedge \tau_{\Delta,l})|^{p} \leq C((\beta^{-1}(\gamma(\Delta)))^{p\zeta+p-q} + (\gamma(\Delta))^{p}\Delta^{p/2} + \Delta^{(q-p\zeta)/q}),$$
(53)

where $e_{\Delta}(t) = v(t) - \chi_{\Delta}(t)$ and $\tau_{\Delta,l} = \rho_l \wedge \vartheta_{\Delta,l} \wedge \bar{\vartheta}_{\Delta,l}$.

Proof. Let $\tau_{\Delta,l} = \tau$ be a sort of simplicity, and note that $q > (1 + \zeta)\omega$ if $\omega \in (p, \xi)$. Upon applying the Itô formula, using the Taylor formula, and taking the expectation, we have

$$\mathbb{E}|e_{\Delta}(t \wedge \tau)|^{p} \leq \mathbb{E}\int_{0}^{t \wedge \tau} p|e_{\Delta}(s)|^{p-2} \Big(e_{\Delta}^{T}(s)[\varphi(v(s), v(\eta s)) - \varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s))] \\ + \frac{p-1}{2}|\psi(v(s), v(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2}\Big)ds \qquad (54)$$

$$+ C\mathbb{E}\int_{0}^{t \wedge \tau}\int_{U}|e_{\Delta}(s)|^{p-2}|\varpi(v(s), v(\eta s), u) - \varpi(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{2}\pi(du)ds \\ + C\mathbb{E}\int_{0}^{t \wedge \tau}\int_{U}|\varpi(v(s), v(\eta s), u) - \varpi(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{p}\pi(du)ds.$$

Applying the Young inequality leads to

$$\frac{p-1}{2} |\psi(v(s), v(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2}
\leq \frac{p-1}{2} \times \left(\left(1 + \frac{\omega - p}{p-1} \right) |\psi(v(s), v(\eta s)) - \psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{2}
+ \left(1 + \frac{p-1}{\omega - p} \right) |\psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2} \right)$$

$$= \frac{\omega - 1}{2} |\psi(v(s), v(\eta s)) - \psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{2}
+ \frac{(\omega - 1)(p-1)}{2(\omega - p)} |\psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2}.$$
(55)

Plugging (55) into (54) yields

$$\mathbb{E}|e_{\Delta}(t\wedge\tau)|^{p} \leq J_{1}+J_{2}+J_{3},$$
(56)

where

$$J_{1} = \mathbb{E} \int_{0}^{t\wedge\tau} p|e_{\Delta}(s)|^{p-2} \left(e_{\Delta}^{T}(s) [\varphi(\upsilon(s), \upsilon(\eta s)) - \varphi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))] \right.$$

$$\left. + \frac{\omega - 1}{2} |\psi(\upsilon(s), \upsilon(\eta s)) - \psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{2} \right) ds,$$

$$J_{2} = \mathbb{E} \int_{0}^{t\wedge\tau} p|e_{\Delta}(s)|^{p-2} \left(e_{\Delta}^{T}(s) [\varphi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s))] \right.$$

$$\left. + \frac{(\omega - 1)(p - 1)}{2(\omega - p)} |\psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2} \right) ds,$$

$$(57)$$

and

$$J_{3} = C\mathbb{E} \int_{0}^{t\wedge\tau} \int_{U} |e_{\Delta}(s)|^{p-2} |\omega(v(s), v(\eta s), u) - \omega(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{2} \pi(du) ds + C\mathbb{E} \int_{0}^{t\wedge\tau} \int_{U} |\omega(v(s), v(\eta s), u) - \omega(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{p} \pi(du) ds.$$
(59)

By utilizing Assumption 2, it can be directly concluded that

$$J_1 \le C \int_0^t \mathbb{E} |e_{\Delta}(s \wedge \tau)|^p ds.$$
(60)

$$J_{2} \leq \mathbb{E} \int_{0}^{t \wedge \tau} p |e_{\Delta}(s)|^{p-2} \Big(e_{\Delta}^{T}(s) [\varphi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \varphi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))] \\ + \frac{(\omega - 1)(p - 1)}{(\omega - p)} |\psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{2} \Big) ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} p |e_{\Delta}(s)|^{p-2} \Big(e_{\Delta}^{T}(s) [\varphi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \varphi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s))] \\ + \frac{(\omega - 1)(p - 1)}{(\omega - p)} |\psi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2} \Big) ds \\ = J_{21} + J_{22}.$$

$$(61)$$

By utilizing the Young inequality, Hölder's inequality, (10), and Assumption 1 and (12), we obtain

$$J_{21} \leq C\mathbb{E} \int_{0}^{t} |e_{\Delta}(s \wedge \tau)|^{p} ds + C\mathbb{E} \int_{0}^{T} \left(|\varphi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \varphi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{p} + |\psi(\chi_{\Delta}(s), \chi_{\Delta}(\eta s)) - \psi_{\Delta}(\chi_{\Delta}(s), \chi_{\Delta}(\eta s))|^{p} \right) ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C\mathbb{E} \int_{0}^{T} \left[(1 + |\chi_{\Delta}(s)|^{p\zeta} + |\chi_{\Delta}(\eta s)|^{p\zeta} + |\nu_{\Delta}(\chi_{\Delta}(s))|^{p\zeta} + |\nu_{\Delta}(\chi_{\Delta}(s))|^{p\zeta} + |\nu_{\Delta}(\chi_{\Delta}(s))|^{p\zeta} + |\nu_{\Delta}(\chi_{\Delta}(s))|^{p\zeta} \right] ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \left[\mathbb{E} (1 + |\chi_{\Delta}(s)|^{q} + |\chi_{\Delta}(\eta s)|^{q}) \right]^{p\zeta/q} \times \left[\mathbb{E} |\chi_{\Delta}(s) - \nu_{\Delta}(\chi_{\Delta}(s))|^{pq/(q-p\zeta)} + \mathbb{E} |\chi_{\Delta}(\eta s) - \nu_{\Delta}(\chi_{\Delta}(\eta s))|^{pq/(q-p\zeta)} \right]^{(q-p\zeta)/q} ds.$$
(62)

Utilizing Lemma 4 leads to

$$J_{21} \leq C \int_0^t \mathbb{E} |e_{\Delta}(s \wedge \tau)|^p ds + C \int_0^T [\mathbb{E} |\chi_{\Delta}(s) - \nu_{\Delta}(\chi_{\Delta}(s))|^{pq/(q-p\zeta)}]^{(q-p\zeta)/q} ds$$

$$\leq C \int_0^t \mathbb{E} |e_{\Delta}(s \wedge \tau)|^p ds + C \int_0^T [\mathbb{E} (\mathbf{I}_{|\chi_{\Delta}(s)| > \beta^{-1}(\gamma(\Delta))} |\chi_{\Delta}(s)|^{pq/(q-p\zeta)})]^{(q-p\zeta)/q} ds.$$
(63)

By exploiting the fundamental bridge and Chebyshev's inequality, we reach

$$J_{21} \leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \left([\mathbb{P}(|\chi_{\Delta}(s)| > \beta^{-1}(\gamma(\Delta)))]^{(q-p\zeta-p)/(q-p\zeta)} \times [\mathbb{E} |\chi_{\Delta}(s)|]^{p/(q-p\zeta)} \right)^{(q-p\zeta)/q} ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \left(\frac{\mathbb{E} |\chi_{\Delta}(s)|^{q}}{(\beta^{-1}(\gamma(\Delta)))^{q}} \right)^{(q-p\zeta-p)/q} ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C (\beta^{-1}(\gamma(\Delta)))^{p\zeta+p-q}.$$
(64)

Applying the Young inequality and (13) yields

$$J_{22} \leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \mathbb{E} \int_{0}^{t \wedge \tau} |e_{\Delta}(s)|^{p-2} (1 + |\chi_{\Delta}(s)|^{2\zeta} + |\chi_{\Delta}(\eta s)|^{2\zeta} + |\kappa_{1}^{*}(s)|^{2\zeta} + |\kappa_{2}^{*}(s)|^{2\zeta}) \times (|\chi_{\Delta}(s) - \kappa_{1}^{*}(s)|^{2} + |\chi_{\Delta}(\eta s) - \kappa_{2}^{*}(s)|^{2}) ds + C \mathbb{E} \int_{0}^{s \wedge \tau} |e_{\Delta}(s)|^{p-2} (1 + |\chi_{\Delta}(s)|^{2\zeta} + |\chi_{\Delta}(\eta s)|^{2\zeta} + |\kappa_{1}(s)|^{2\zeta} + |\kappa_{2}(s)|^{2\zeta})$$
(65)
$$\times (|\chi_{\Delta}(s) - \kappa_{1}(s)|^{2} + |\chi_{\Delta}(\eta s) - \kappa_{2}(s)|^{2}) ds = C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + J_{221} + J_{222},$$

where

$$J_{221} = C\mathbb{E} \int_{0}^{t\wedge\tau} |e_{\Delta}(s)|^{p-2} (1+|\chi_{\Delta}(s)|^{2\zeta}+|\chi_{\Delta}(\eta s)|^{2\zeta}+|\kappa_{1}^{*}(s)|^{2\zeta}+|\kappa_{2}^{*}(s)|^{2\zeta}) \times (|\chi_{\Delta}(s)-\kappa_{1}^{*}(s)|^{2}+|\chi_{\Delta}(\eta s)-\kappa_{2}^{*}(s)|^{2}) ds,$$
(66)

and

$$J_{222} = C\mathbb{E} \int_{0}^{t\wedge\tau} |e_{\Delta}(s)|^{p-2} (1+|\chi_{\Delta}(s)|^{2\zeta}+|\chi_{\Delta}(\eta s)|^{2\zeta}+|\kappa_{1}(s)|^{2\zeta}+|\kappa_{2}(s)|^{2\zeta}) \times (|\chi_{\Delta}(s)-\kappa_{1}(s)|^{2}+|\chi_{\Delta}(\eta s)-\kappa_{2}(s)|^{2}) ds.$$
(67)

Upon applying the Young inequality, Hölder's inequality, and Lemmas 4 and 5, and utilizing Inequalities (37) and (38) and $pq/(q - p\zeta) \ge 2$, we obtain

$$J_{221} \leq C\mathbb{E} \int_{0}^{t} |e_{\Delta}(s \wedge \tau)|^{p} ds + C\mathbb{E} \int_{0}^{T} \left(1 + |\chi_{\Delta}(s)|^{p\zeta} + |\chi_{\Delta}(\eta s)|^{p\zeta} + |\chi_{\Delta}(\eta s)|^{p\zeta} + |\kappa_{1}^{*}(s)|^{p\zeta} + |\kappa_{2}^{*}(s)|^{p\zeta}\right) \times \left(|\chi_{\Delta}(s) - \kappa_{1}^{*}(s)|^{p} + |\chi_{\Delta}(\eta s) - \kappa_{2}^{*}(s)|^{p}\right) ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds$$

$$+ C \int_{0}^{T} \left[\mathbb{E} (1 + |\chi_{\Delta}(s)|^{q} + |\chi_{\Delta}(\eta s)|^{q} + |\kappa_{1}^{*}(s)|^{q} + |\kappa_{2}^{*}(s)|^{q})\right]^{p\zeta/q}$$

$$\times \left[\mathbb{E} |\chi_{\Delta}(s) - \kappa_{1}^{*}(s)|^{pq/(q-p\zeta)} + \mathbb{E} |\chi_{\Delta}(\eta s) - \kappa_{2}^{*}(s)|^{pq/(q-p\zeta)}\right]^{(q-p\zeta)/q} ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \left((\gamma(\Delta))^{pq/(q-p\zeta)} \Delta^{pq/2(q-p\zeta)} + \Delta\right)^{(q-p\zeta)/q} ds$$

$$\leq C \int_{0}^{t} \mathbb{E} |e_{\Delta}(s \wedge \tau)|^{p} ds + C((\gamma(\Delta))^{p} \Delta^{p/2} + \Delta^{(q-p\zeta)/q}).$$

By following the same approach as for J_{221} , it can be concluded that

$$J_{222} \le C \int_0^t \mathbb{E} |e_\Delta(s \wedge \tau)|^p ds + C((\gamma(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\zeta)/q}).$$
(69)

Therefore by plugging (68) and (69) into (65) and substituting with (64) and (65) into (61), we obtain

$$J_2 \le C \int_0^t \mathbb{E} |e_{\Delta}(s \wedge \tau)|^p ds + C((\beta^{-1}(\gamma(\Delta)))^{p\zeta + p - q} + (\gamma(\Delta))^p \Delta^{p/2} + \Delta^{(q - p\zeta)/q}).$$
(70)

By applying Assumption 1, the Young inequality, and Lemmas 4 and 5,

$$J_{3} \leq C \int_{0}^{t} \mathbb{E}|e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \mathbb{E}(|v(s) - \kappa_{1}^{*}(s)|^{p} + |v(\eta s) - \kappa_{2}^{*}(s))|^{p}) ds$$

$$\leq C \mathbb{E} \int_{0}^{t} |e_{\Delta}(s \wedge \tau)|^{p} ds + C \int_{0}^{T} \mathbb{E}(|\chi_{\Delta}(s) - \kappa_{1}^{*}(s)|^{p} + |\chi_{\Delta}(\eta s) - \kappa_{2}^{*}(s)|^{p}) ds \qquad (71)$$

$$\leq C \mathbb{E} \int_{0}^{t} |e_{\Delta}(s \wedge \tau)|^{p} ds + C((\gamma(\Delta))^{p} \Delta^{p/2} + \Delta).$$

Then, by plugging (60), (70) and (71) into (56), we reach

$$\mathbb{E}|e_{\Delta}(t\wedge\tau)|^{p} \leq C \int_{0}^{t} \mathbb{E}|e_{\Delta}(s\wedge\tau)|^{p} ds + C((\beta^{-1}(\gamma(\Delta)))^{p\zeta+p-q} + (\gamma(\Delta))^{p} \Delta^{p/2} + \Delta^{(q-p\zeta)/q}).$$
(72)

Then, the Gronwall inequality leads to

$$\mathbb{E}|e_{\Delta}(T\wedge\tau)|^{p} \leq C((\beta^{-1}(\gamma(\Delta)))^{p\zeta+p-q} + (\gamma(\Delta))^{p}\Delta^{p/2} + \Delta^{(q-p\zeta)/q}).$$
(73)

Corollary 2. Let Assumptions 1 and 2 hold and Assumption 3 holds for all $\bar{p} \in (\xi, \infty)$. Define

$$\beta(x) = Cx^{1+\zeta}, \quad x \ge 0, \quad and \quad \gamma(\Delta) = \Delta^{-\varepsilon}, \quad \varepsilon \in (0, 1/4 \land 1/q].$$
(74)

Then, for any

$$p \in [2,\xi), \quad q \in ((1+\zeta)p \lor \xi, \bar{p}) \quad and \quad \varepsilon \in (0,1/4 \land 1/q],$$
(75)

we have

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq C\Delta^{\varepsilon(q-(1+\zeta)p)/(1+\zeta)\wedge(q-\zeta p)/q}.$$
(76)

Proof. By utilizing (75), it can be concluded that

$$\varepsilon \le \frac{1}{q} < \frac{p(1+\zeta)}{2q}$$

which implies

$$\varepsilon(q - (1 + \zeta)p) / (1 + \zeta) < p(1 - 2\varepsilon) / 2.$$

Then, by applying Theorem 1 and (74), the required assertion (76) can be easily obtained.

4. Convergence Rate in $\mathcal{L}^p(0$

In some applications, we need to approximate the mean value of the solution or the European call option value. In these situations, we need to have the convergence in $\mathcal{L}^p(0 sense. Therefore, in this section the convergence rate of the diffused split-step truncated Euler–Maruyama method for Equation (2) is attained in <math>\mathcal{L}^p(0 sense where all the coefficients behave beyond linearly. Also, we first will present some assumptions and lemmas for helping us in proving the convergence theorem.$

Assumption 4. *Let* $k_R > 0$ *such that*

$$|\varphi(\varsigma_1, \bar{\varsigma_1}) - \varphi(\varsigma_2, \bar{\varsigma_2})| \lor |\psi(\varsigma_1, \bar{\varsigma_1}) - \psi(\varsigma_2, \bar{\varsigma_2})| \le k_R(|\varsigma_1 - \varsigma_2| + |\bar{\varsigma_1} - \bar{\varsigma_2}|)$$

and

$$\int_{U} |\varpi(\varsigma_1, \bar{\varsigma_1}, u) - \varpi(\varsigma_2, \bar{\varsigma_2}, u)| \pi(du) \le k_R(|\varsigma_1 - \varsigma_2| + |\bar{\varsigma_1} - \bar{\varsigma_2}|),$$

for all $\varsigma_1, \varsigma_2, \bar{\varsigma_1}, \bar{\varsigma_2} \in \mathbb{R}^m$ with $|\varsigma_1| \lor |\varsigma_2| \lor |\bar{\varsigma_1}| \lor |\bar{\varsigma_2}| \le R$ and $u \in U$.

Assumption 5. Let k > 0 such that

$$2\langle\varsigma,\varphi(\varsigma,\bar{\varsigma}) + \int_{U} \mathcal{O}(\varsigma,\bar{\varsigma},u)\pi(du)\rangle + |\psi(\varsigma,\bar{\varsigma})|^{2} + \int_{U} |\mathcal{O}(\varsigma,\bar{\varsigma},u)|^{2}\pi(du) \le k(1+|\varsigma|^{2}+|\bar{\varsigma}|^{2})$$

$$for all \ \varsigma,\bar{\varsigma} \in \mathbb{R}^{m} \ and \ u \in U.$$

$$(77)$$

By following the same approach and procedures as for proving Lemma 1, we have the following lemma.

Lemma 8. Under Assumptions 4 and 5,

$$\sup_{0 \le t \le T} \mathbb{E} |v(t)|^2 < \infty, \quad \forall T > 0.$$
(78)

In Section 3, the jump term was acting linearly, but in this section, according to Assumptions 4 and 5, the jump term is permitted to grow super-linearly; therefore drift, diffusion, and jump coefficients will be truncated. By proceeding the same as in in Section 3, β is selected such that $\beta(\iota) \rightarrow \infty$ as $\iota \rightarrow \infty$ and

$$\sup_{|\varsigma| \lor |\bar{\varsigma}| \le \iota} (|\varphi(\varsigma, \bar{\varsigma})| \lor |\psi(\varsigma, \bar{\varsigma})| \lor |\varpi(\varsigma, \bar{\varsigma}, u)|) \le \beta(\iota), \quad \forall \iota \ge 1.$$
(79)

Moreover, a strictly non-increasing function $\gamma : (0, 1] \rightarrow (0, \infty)$ is chosen such that

$$\lim_{\Delta \to 0} \gamma(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} \gamma(\Delta) \le 1, \quad \forall \Delta \in (0, 1].$$
(80)

For a given $\Delta \in (0, 1]$, ν_{Δ} is the same as (9) and

$$\phi_{\Delta}(\varsigma,\bar{\varsigma}) = \phi(\nu_{\Delta}(\varsigma),\nu_{\Delta}(\bar{\varsigma})) \quad \text{and} \quad \varpi_{\Delta}(\varsigma,\bar{\varsigma},u) = \varpi(\nu_{\Delta}(\varsigma),\nu_{\Delta}(\bar{\varsigma}),u)$$
(81)

for all $\zeta, \overline{\zeta} \in \mathbb{R}^m$ and $u \in U$ where $\phi = f$ or g. It is also obvious that

$$|\varphi_{\Delta}(\varsigma,\bar{\varsigma})| \vee |\psi_{\Delta}(\varsigma,\bar{\varsigma})| \vee |\omega_{\Delta}(\varsigma,\bar{\varsigma},u)| \le \beta(\beta^{-1}(\gamma(\Delta))) = \gamma(\Delta)$$
(82)

for all $\zeta, \overline{\zeta} \in \mathbb{R}^m$ and $u \in U$. Additionally, by utilizing (12), (82), and Assumption 5, it can be concluded that for any $\zeta, \overline{\zeta} \in \mathbb{R}^m$,

$$2\langle \varsigma, \varphi_{\Delta}(\varsigma, \bar{\varsigma}) + \int_{U} \varpi_{\Delta}(\varsigma, \bar{\varsigma}, u) \pi(du) \rangle + |\psi_{\Delta}(\varsigma, \bar{\varsigma})|^{2} + \int_{U} |\varpi_{\Delta}(\varsigma, \bar{\varsigma}, u)|^{2} \pi(du)$$

$$\leq C(1 + |\varsigma|^{2} + |\bar{\varsigma}|^{2}).$$
(83)

Now, the diffused split-step truncated Euler–Maruyama scheme for Equation (2) is established by the initial value $Y_0 = v_0$, and Y_{n+1} is computed by

$$Z_n = Y_n + \psi_{\Delta}(Y_n, Y_{[nn]}) \Delta W_n \tag{84}$$

$$Y_{n+1} = Z_n + \Delta \varphi_{\Delta}(Z_n, Z_{[\eta n]}) + \int_{t_n}^{t_{n+1}} \int_U \omega_{\Delta}(Z_n, Z_{[\eta n]}, u) \widetilde{N}(dt, du)$$
(85)

for n = 0, 1, ... and $\chi_{\Delta}(t)$ is defined by

$$\chi_{\Delta}(t) = Y_0 + \int_0^t \varphi_{\Delta}(\kappa_1^*(s), \kappa_2^*(s)) ds + \int_0^t \psi_{\Delta}(\kappa_1(s), \kappa_2(s)) dW(s) + \int_0^t \int_U \varphi_{\Delta}(\kappa_1^*(s), \kappa_2^*(s), u) \widetilde{N}(ds, du),$$
(86)

where $\kappa_1(t)$, $\kappa_2(t)$, $\kappa_1^*(t)$ and $\kappa_2^*(t)$ are the same as defined before.

Lemma 9. Under Assumptions 4 and 5,

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_1(t)|^{\varrho^*} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_1^*(t)|^{\varrho^*} \le C(\gamma(\Delta))^{\varrho^*}\Delta, \quad \varrho^* \ge 2$$
(87)

$$\mathbb{E}|\chi_{\Delta}(t) - \kappa_1(t)|^{\varrho^*} \vee \mathbb{E}|\chi_{\Delta}(t) - \kappa_1^*(t)|^{\varrho^*} \le C(\gamma(\Delta))^{\varrho^*} \Delta^{\varrho^*/2}, \quad 0 < \varrho^* < 2$$
(88)

Proof. By utilizing $|\omega_{\Delta}(\varsigma, \overline{\varsigma}, u)| \leq \gamma(\Delta)$ for all $u \in U$ and following the same approach and procedures performed in Lemma 3, the required assertions (87) and (88) can be easily attained. \Box

Lemma 10. Under Assumptions 4 and 5, we have

$$\sup_{0<\Delta\leq 1}\sup_{0\leq t\leq T}\mathbb{E}|\chi_{\Delta}(t)|^{2}\leq C,\quad\forall T>0$$
(89)

Proof. For fixed $\Delta \in (0, 1]$, we obtain via the Itô formula and Equation (86)

,

$$\mathbb{E}|\chi_{\Delta}(t)|^{2} \leq \mathbb{E}|x_{0}|^{2} + \mathbb{E}\int_{0}^{t} (2\kappa_{1}^{T}(s)\varphi_{\Delta}(\kappa_{1}(s),\kappa_{2}(s)) + |\psi_{\Delta}(\kappa_{1}(s),\kappa_{2}(s))|^{2})ds + \mathbb{E}\int_{0}^{t} 2(\chi_{\Delta}(s) - \kappa_{1}(s))^{T}\varphi_{\Delta}(\kappa_{1}(s),\kappa_{2}(s))ds + \mathbb{E}\int_{0}^{t} 2\chi_{\Delta}^{T}(s)(\varphi_{\Delta}(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s)) - \varphi_{\Delta}(\kappa_{1}(s),\kappa_{2}(s)))ds + \mathbb{E}\int_{0}^{t}\int_{U} 2\kappa_{1}^{T}(s)\omega_{\Delta}(\kappa_{1}(s),\kappa_{2}(s),u)\pi(du)ds + \mathbb{E}\int_{0}^{t}\int_{U} 2(\chi_{\Delta}(s) - \kappa_{1}(s))^{T}\omega_{\Delta}(\kappa_{1}(s),\kappa_{2}(s),u)\pi(du)ds + \mathbb{E}\int_{0}^{t}\int_{U} 2\chi_{\Delta}^{T}(s)(\omega_{\Delta}(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s),u) - \omega_{\Delta}(\kappa_{1}(s),\kappa_{2}(s),u))\pi(du)ds + 2\mathbb{E}\int_{0}^{t}\int_{U}|\omega_{\Delta}(\kappa_{1}(s),\kappa_{2}(s),u)|^{2}\pi(du)ds + 2\mathbb{E}\int_{0}^{t}\int_{U}|\omega_{\Delta}(\kappa_{1}^{*}(s),\kappa_{2}^{*}(s),u) - \omega_{\Delta}(\kappa_{1}(s),\kappa_{2}(s),u)|^{2}\pi(du)ds.$$
(90)

Applying (81), (82), and (83), Assumption 4 leads to

$$\mathbb{E}|\chi_{\Delta}(t)|^{2} \leq C + C\mathbb{E}\int_{0}^{t} (|\chi_{\Delta}(s)|^{2} + |\kappa_{1}(s)|^{2} + |\kappa_{2}(s)|^{2})ds + C\gamma(\Delta)\int_{0}^{t} \mathbb{E}|\chi_{\Delta}(s) - \kappa_{1}(s)|ds + C\int_{0}^{t} \mathbb{E}(|\kappa_{1}^{*}(s) - \kappa_{1}(s)|^{2} + |\kappa_{2}^{*}(s) - \kappa_{2}(s)|^{2})ds.$$
(91)

Then, by using Lemma 9 and noting from (80) that $((\gamma(\Delta))^2 \Delta^{1/2} \leq 1)$, we could obtain

$$\mathbb{E}|\chi_{\Delta}(t)|^{2} \leq C \left(1 + \int_{0}^{t} \mathbb{E}|\chi_{\Delta}(s)|^{2} + \mathbb{E}|\kappa_{1}(s)|^{2} + \mathbb{E}|\kappa_{2}(s)|^{2}\right) ds$$

$$\leq C \left(1 + \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E}|\chi_{\Delta}(r)|^{2} ds\right).$$
(92)

Upon proceeding in a similar fashion as for Lemma 4, (89) is obtained. \Box

The following Lemma can be obtained by the same approach in Lemmas 6 and 7.

Lemma 11. Under Assumptions 4 and 5, for any real number l > |v(0)| and $\Delta \in (0, 1]$,

$$\mathbb{P}(\rho_l \le T) \le \frac{C}{l^2} \quad and \quad \mathbb{P}(\vartheta_{\Delta,l} \le T) \lor \mathbb{P}(\bar{\vartheta}_{\Delta,l} \le T) \le \frac{C}{l^2}, \tag{93}$$

where ρ_l , $\vartheta_{\Delta,l}$ and $\bar{\vartheta}_{\Delta,l}$ are the same as defined before.

Assumption 6. *Let* $c_1 > 0$ *such that*

$$2\langle\varsigma_{1}-\varsigma_{2},(\varphi(\varsigma_{1},\bar{\varsigma_{1}})-\varphi(\varsigma_{2},\bar{\varsigma_{2}}))+\int_{U}(\varpi(\varsigma_{1},\bar{\varsigma_{1}},u)-\varpi(\varsigma_{2},\bar{\varsigma_{2}},u))\pi(du)\rangle +|\psi(\varsigma_{1},\bar{\varsigma_{1}})-\psi(\varsigma_{2},\bar{\varsigma_{2}})|^{2}+\int_{U}|\varpi(\varsigma_{1},\bar{\varsigma_{1}},u)-\varpi(\varsigma_{2},\bar{\varsigma_{2}},u)|^{2}\pi(du) \qquad (94)$$
$$\leq c_{1}(|\varsigma_{1}-\varsigma_{2}|^{2}+|\bar{\varsigma_{1}}-\bar{\varsigma_{2}}|^{2})$$

for all $\zeta_1, \zeta_2, \overline{\zeta_1}, \overline{\zeta_2} \in \mathbb{R}^m$ and $u \in U$.

Assumption 7. Let $c_2 > 0$, $\bar{\zeta} \ge 0$ such that

$$\begin{aligned} |\varphi(\varsigma_{1},\bar{\varsigma_{1}}) - \varphi(\varsigma_{2},\bar{\varsigma_{2}})| &\lor \int_{U} |\varpi(\varsigma_{1},\bar{\varsigma_{1}},u) - \varpi(\varsigma_{2},\bar{\varsigma_{2}},u)|\pi(du) \\ &\le c_{2}(1+|\varsigma_{1}|^{\bar{\zeta}}+|\bar{\varsigma_{1}}|^{\bar{\zeta}}+|\varsigma_{2}|^{\bar{\zeta}}+|\bar{\varsigma_{2}}|^{\bar{\zeta}})(|\varsigma_{1}-\varsigma_{2}|+|\bar{\varsigma_{1}}-\bar{\varsigma_{2}}|) \end{aligned}$$
(95)

for all $\varsigma_1, \varsigma_2, \bar{\varsigma_1}, \bar{\varsigma_2} \in \mathbb{R}^m$ and $u \in U$.

Lemma 12. Under Assumptions 4, 5, 6, and 7, let $l > |v_0|$ be a real number and Δ be small enough such that $\beta^{-1}(\gamma(\Delta)) \ge l$. Then,

$$\mathbb{E}|e_{\Delta}(T \wedge \tau_{\Delta,l})|^2 \le C(\gamma(\Delta))^2 \Delta, \tag{96}$$

where $e_{\Delta}(t)$, $\tau_{\Delta,l} = \rho_l \wedge \vartheta_{\Delta,l} \wedge \overline{\vartheta}_{\Delta,l}$ are the same as defined before.

Proof. For simplification, we denote $\tau_{\Delta,l} = \tau$. By the Itô formula,

$$\begin{split} \mathbb{E}|e_{\Delta}(t \wedge \tau)|^{2} \\ \leq \mathbb{E} \int_{0}^{t \wedge \tau} 2(v(s) - \kappa_{1}(s))^{T}(\varphi(v(s), v(\eta s)) - \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))) \\ + |\psi(v(s), v(\eta s)) - \psi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s))|^{2})ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} 2(v(s) - \kappa_{1}(s))^{T}(\varpi(v(s), v(\eta s), u) - \varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s), u))\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} |\varpi(v(s), v(\eta s), u) - \varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s), u)|^{2}\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} 2(\kappa_{1}(s) - \kappa_{\Delta}(s))^{T}(\varphi(v(s), v(\eta s)) - \varphi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)))ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} 2(v(s) - \chi_{\Delta}(s))^{T}(\varpi(v(s), v(\eta s), u) - \varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)), u)\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} 2(v(s) - \chi_{\Delta}(s))^{T}(\varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(\eta s), u) - \varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(s)), u)\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} 2(v(s) - \chi_{\Delta}(s))^{T}(\varpi_{\Delta}(\kappa_{1}(s), \kappa_{2}(\eta s), u) - \varpi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u))\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} |\omega_{\Delta}(\kappa_{1}(s), \kappa_{2}(\eta s), u) - \varpi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u))\pi(du)ds \\ + \mathbb{E} \int_{0}^{t \wedge \tau} \int_{U} |\omega_{\Delta}(\kappa_{1}(s), \kappa_{2}(\eta s), u) - \varpi_{\Delta}(\kappa_{1}^{*}(s), \kappa_{2}^{*}(s), u)|^{2}\pi(du)ds. \end{split}$$

$$|v(s)| \vee |v(\eta s)| \vee |\kappa_1(s)| \vee |\kappa_2(s)| \vee |\kappa_1^*(s)| \vee |\kappa_2^*(s)| \le l.$$

But due to $\beta^{-1}(\gamma(\Delta)) \ge l$,

$$|v(s)| \lor |v(\eta s)| \lor |\kappa_1(s)| \lor |\kappa_2(s)| \lor |\kappa_1^*(s)| \lor |\kappa_2^*(s)| \le \beta^{-1}(\gamma(\Delta)).$$

Due to (81), we have for $s \in [0, t \land \tau]$

$$\phi_{\Delta}(i,j) = \phi(i,j), \quad \varpi_{\Delta}(i,j,u) = \varpi(i,j,u), \quad \forall u \in U,$$
(98)

where $\phi = \phi$ or ψ whereas $i = \kappa_1(s)$ or $\kappa_1^*(s)$ and $j = \kappa_2(s)$ or $\kappa_2^*(s)$. Therefore, applying (98), Assumptions 6 and 7 to (97) yields

$$\begin{split} \mathbb{E}|e_{\Delta}(t\wedge\tau)|^{2} \\ &\leq C\mathbb{E}\int_{0}^{t\wedge\tau}(|v(s)-\kappa_{1}(s)|^{2}+|v(\eta s)-\kappa_{2}(s)|^{2})ds \\ &+C\mathbb{E}\int_{0}^{t\wedge\tau}(\kappa_{1}(s)-\chi_{\Delta}(s))^{T}(1+|v(s)|^{\bar{\zeta}}+|v(\eta s)|^{\bar{\zeta}}+|\kappa_{1}(s)|^{\bar{\zeta}}+|\kappa_{2}(s)|^{\bar{\zeta}}) \\ &\times(|v(s)-\kappa_{1}(s)|+|v(\eta s)-\kappa_{2}(s)|)ds \\ &+C\mathbb{E}\int_{0}^{t\wedge\tau}(v(s)-\chi_{\Delta}(s))^{T}(1+|\kappa_{1}(s)|^{\bar{\zeta}}+|\kappa_{2}(s)|^{\bar{\zeta}}+|\kappa_{1}^{*}(s)|^{\bar{\zeta}}+|\kappa_{2}^{*}(s)|^{\bar{\zeta}}) \\ &\times(|\kappa_{1}(s)-\kappa_{1}^{*}(s)|+|\kappa_{2}(s)-\kappa_{2}^{*}(s)|)ds \\ &+C\mathbb{E}\int_{0}^{t\wedge\tau}(1+|\kappa_{1}(s)|^{2\bar{\zeta}}+|\kappa_{2}(s)|^{2\bar{\zeta}}+|\kappa_{1}^{*}(s)|^{2\bar{\zeta}}+|\kappa_{2}^{*}(s)|^{2\bar{\zeta}}) \\ &\times(|\kappa_{1}(s)-\kappa_{1}^{*}(s)|^{2}+|\kappa_{2}(s)-\kappa_{2}^{*}(s)|^{2})ds \end{split}$$
(99)

Utilizing the Young inequality, Hölder's inequality, Lemmas 8, 9, and 10 cause

$$\begin{split} \mathbb{E}|e_{\Delta}(t\wedge\tau)|^{2} \\ &\leq C\int_{0}^{t} \mathbb{E}|e_{\Delta}(s\wedge\tau)|^{2}ds + C\int_{0}^{T} \mathbb{E}(|\chi_{\Delta}(s) - \kappa_{1}(s)|^{2} + |\chi_{\Delta}(\eta s) - \kappa_{2}(s)|^{2})ds \\ &+ C\int_{0}^{T} \left(1 + \mathbb{E}|v(s)|^{2} + \mathbb{E}|v(\eta s)|^{2} + \mathbb{E}|\kappa_{1}(s)|^{2} + \mathbb{E}|\kappa_{2}(s)|^{2}\right)^{\bar{\zeta}} \\ &\times \left(\mathbb{E}|\chi_{\Delta}(s) - \kappa_{1}(s)|^{2/(1-\bar{\zeta})} + \mathbb{E}|\chi_{\Delta}(\eta s) - \kappa_{2}(s)|^{2/(1-\bar{\zeta})}\right)^{(1-\bar{\zeta})}ds \\ &+ C\int_{0}^{T} \left(1 + \mathbb{E}|\kappa_{1}(s)|^{2} + \mathbb{E}|\kappa_{2}(s)|^{2} + \mathbb{E}|\kappa_{1}^{*}(s)|^{2} + \mathbb{E}|\kappa_{2}^{*}(s)|^{2}\right)^{\bar{\zeta}} \\ &\times \left(\mathbb{E}|\kappa_{1}(s) - \kappa_{1}^{*}(s)|^{2/(1-\bar{\zeta})} + \mathbb{E}|\chi_{2}(\eta s) - \kappa_{2}^{*}(s)|^{2/(1-\bar{\zeta})}\right)^{(1-\bar{\zeta})}ds \\ &\leq C\int_{0}^{t} \mathbb{E}|e_{\Delta}(s\wedge\tau)|^{2}ds + C(\gamma(\Delta))^{2}\Delta. \end{split}$$
(100)

By the Gronwall inequality,

$$\mathbb{E}|e_{\Delta}(T \wedge \tau)|^2 \le C(\gamma(\Delta))^2 \Delta.$$
(101)

The proof is complete. \Box

Theorem 2. Under Assumptions 4, 5, 6 and 7. Let $p \in (0, 2)$ and constant c > 0 such that

$$\gamma(\Delta) \ge \beta \left(c^{-(1+\zeta)} ((\gamma(\Delta))^p \Delta^{p/2})^{-1/(2-p)} \right)$$
(102)

holds for small values of $\Delta \in (0, 1]$. Then, for these small values of Δ

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq C(\gamma(\Delta))^{p} \Delta^{p/2}.$$
(103)

Proof. Let $e_{\Delta}(t)$, ρ_l , $\vartheta_{\Delta,l}$, $\bar{\vartheta}_{\Delta,l}$, and τ be the same as defined before. By [20], for any $\varrho > 0$ and $p \in (0, 2)$,

$$\mathbb{E}|e_{\Delta}(T)|^{p} = \mathbb{E}(|e_{\Delta}(T)|^{p}I_{\{\tau>T\}}) + \mathbb{E}(|e_{\Delta}(T)|^{p}I_{\{\tau\leq T\}}) \\ \leq \mathbb{E}(|e_{\Delta}(T)|^{p}I_{\{\tau>T\}}) + \frac{p\varrho}{2}\mathbb{E}|e_{\Delta}(T)|^{2} + \frac{2-p}{2\rho^{\frac{p}{2-p}}}\mathbb{P}(\tau\leq T).$$
(104)

By Lemmas 8 and 10,

$$\mathbb{E}|e_{\Delta}(T)|^2 \le 2\mathbb{E}(|v(T)|^2 + |\chi_{\Delta}(T)|^2) \le C.$$
(105)

By Lemma 11, we obtain

$$\mathbb{P}(\tau \le T) \le \mathbb{P}(\rho_l \le T) + \mathbb{P}(\vartheta_{\Delta,l} \le T) + \mathbb{P}(\bar{\vartheta}_{\Delta,l} \le T) \le \frac{C}{l^2}$$
(106)

By plugging (105) and (106) into (104), we obtain

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq \mathbb{E}|e_{\Delta}(T \wedge \tau)|^{p} + \frac{Cp\varrho}{2} + \frac{C(2-p)}{2l^{2}\varrho^{\frac{p}{2-p}}}.$$
(107)

holds for any $\Delta \in (0, 1)$, $l > |x_0|$ and $\varrho > 0$. Then, by selecting

$$\varrho = (\gamma(\Delta))^p \Delta^{p/2}, \quad l = c^{-(1+\bar{\zeta})} \varrho^{-1/(2-p)},$$

and substituting in (107), we obtain

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq \mathbb{E}|e_{\Delta}(T \wedge \tau)|^{p} + C(\gamma(\Delta))^{p} \Delta^{p/2}.$$
(108)

Furthermore, by Condition (102), we obtain

$$\beta^{-1}(\gamma(\Delta)) \ge c^{-(1+\zeta)}((\gamma(\Delta))^p \Delta^{p/2})^{-1/(2-p)} = l.$$
(109)

Therefore, by applying Lemma 12, we obtain

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq (\mathbb{E}|e_{\Delta}(T)|^{2})^{p/2} \leq C((\gamma(\Delta))^{2}\Delta)^{p/2} = C(\gamma(\Delta))^{p}\Delta^{p/2}.$$
(110)

Corollary 3. Under Assumptions 4, 5, 6 and 7. Define

$$\beta(x) = cx^{1+\bar{\zeta}}, \quad x \ge 0, \quad and \quad \gamma(\Delta) = \Delta^{-\varepsilon}, \quad \varepsilon \in \left[\frac{p(1+\zeta)}{4+2p\bar{\zeta}}, \frac{1}{4}\right], \tag{111}$$

where $0 . Assume also that (102) holds for small values of <math>\Delta \in (0, 1]$. Then, for these small values of Δ

$$\mathbb{E}|e_{\Delta}(T)|^{p} \le C\Delta^{p(1-2\varepsilon)/2}.$$
(112)

Proof. By utilizing Theorem 2 and (111), the required assertion (112) can be easily obtained. □

5. Numerical Examples

In this section, we will present two examples to verify our theoretical results that were obtained in the previous sections, and to open up new avenues as a future objective (to be taken into consideration) in our upcoming papers to mention that stochastic pantograph models with Lévy jumps can be applied in real-life applications, such as financial markets, where the proposed diffused split-step truncated Euler–Maruyama method can be applied for capturing the stock price behavior with nonlinear drift, diffusion, and Lévy jumps, allowing for better pricing and risk management in financial markets. Also, stochastic pantograph models can be employed to study the spread of infectious diseases and analyze the effectiveness of control strategies where the applicability of the proposed scheme can be utilized for simulating the epidemic's progression accurately, capturing the impact of delays and sudden changes in the infection rate, and aiding in designing effective intervention strategies.

Example 1. Consider a stochastic pantograph model for modeling stock prices with Lévy jumps

$$dv(t) = -2(v^{5}(t) + v^{5}(\eta t))dt + (v^{2}(t) + v(\eta t))dW(t) + \int_{U} (v^{2}(t) + v^{2}(\eta t))u^{2}\widetilde{N}(dt, du),$$
(113)

with initial data v(0) = 1, $\eta = 0.5$ and the compensator given by $\pi(du)dt = \lambda f(u)dudt$, where $\lambda = 1$ and f(u) is the pdf of the standard normal random variable

$$f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \quad -\infty < u < \infty.$$

Therefore, we deduce that $\varphi(v(t), v(\eta t)) = -2(v^5(t) + v^5(\eta t)), \psi(v(t), v(\eta t)) = v^2(t) + v(\eta t)$ and $\varpi(v(t), v(\eta t), u) = (v^2(t) + v^2(\eta t))u^2$. Then, it can be easily noticed that Assumption Assumption of the set of the

tions 4 and 7 are satisfied. For Assumption 6, by utilizing $2\varsigma\bar{\varsigma} \leq \varsigma^2 + \bar{\varsigma}^2$ and noting that $-(\varsigma^3\bar{\varsigma} + \varsigma^2\bar{\varsigma}^2 + \varsigma\bar{\varsigma}^3) \leq 0.5(\varsigma^4 + \bar{\varsigma}^4)$, we have

$$2\langle \varsigma_{1} - \varsigma_{2}, (f(\varsigma_{1}, \bar{\varsigma_{1}}) - f(\varsigma_{2}, \bar{\varsigma_{2}})) \rangle$$

$$= 2(\varsigma_{1} - \varsigma_{2})(-2(\varsigma_{1}^{5} + \bar{\varsigma_{1}}^{5}) + 2(\varsigma_{2}^{5} + \bar{\varsigma_{2}}^{5}))$$

$$= -4(\varsigma_{1} - \varsigma_{2})(\varsigma_{1}^{5} - \varsigma_{2}^{5}) - 4(\varsigma_{1} - \varsigma_{2})(\bar{\varsigma_{1}}^{5} - \bar{\varsigma_{2}}^{5})$$

$$= -4(\varsigma_{1} - \varsigma_{2})(\varsigma_{1} - a_{2})(\varsigma_{1}^{4} + \varsigma_{1}^{3}a_{2} + a_{1}^{2}a_{2}^{2} + \varsigma_{1}a_{2}^{3} + \varsigma_{2}^{4})$$

$$- 4(\varsigma_{1} - \varsigma_{2})(\bar{\varsigma_{1}} - \bar{\varsigma_{2}})(\bar{\varsigma_{1}}^{4} + \bar{\varsigma_{1}}^{3}\bar{\varsigma_{2}} + \bar{\varsigma_{1}}^{2}\bar{\varsigma_{2}}^{2} + \bar{\varsigma_{1}}\bar{\varsigma_{2}}^{3} + \bar{\varsigma_{2}}^{4})$$

$$\leq [-2(\varsigma_{1}^{4} + \varsigma_{2}^{4}) - 2(\bar{\varsigma_{1}}^{4} + \bar{\varsigma_{2}}^{4})](|\varsigma_{1} - \varsigma_{2}|^{2} + |\bar{\varsigma_{1}} - \bar{\varsigma_{2}}|^{2}).$$
(114)

Also,

$$2\langle \varsigma_{1} - \varsigma_{2}, \int_{U} (\omega(\varsigma_{1}, \bar{\varsigma_{1}}, u) - \omega(\varsigma_{2}, \bar{\varsigma_{2}}, u)) \pi(du) \rangle$$

$$= 2(\varsigma_{1} - \varsigma_{2}) \int_{U} (\varsigma_{1}^{2} + \bar{\varsigma_{1}}^{2} - \varsigma_{2}^{2} - \bar{\varsigma_{2}}^{2}) u^{2} f(u) du \qquad (115)$$

$$\leq 2(\varsigma_{1} + \varsigma_{2})(\varsigma_{1} - \varsigma_{2})^{2} + 2(\varsigma_{1} - \varsigma_{2})(\bar{\varsigma_{1}} + \bar{\varsigma_{2}})(\bar{\varsigma_{1}} - \bar{\varsigma_{2}})$$

$$\leq [2(\varsigma_{1}^{2} + \varsigma_{2}^{2}) + 2(\bar{\varsigma_{1}}^{2} + \bar{\varsigma_{2}}^{2})](|\varsigma_{1} - \varsigma_{2}|^{2} + |\bar{\varsigma_{1}} - \bar{\varsigma_{2}}|^{2}).$$

$$|\psi(\varsigma_{1}, \bar{\varsigma_{1}}) - \psi(\varsigma_{2}, \bar{\varsigma_{2}})|^{2} + \int_{U} |\omega(\varsigma_{1}, \bar{\varsigma_{1}}, u) - \omega(\varsigma_{2}, \bar{\varsigma_{2}}, u)|^{2} \pi(du)$$

$$= |\varsigma_{1}^{2} + \bar{\varsigma_{1}} - \varsigma_{2}^{2} - \bar{\varsigma_{2}}|^{2} + \int_{U} |\varsigma_{1}^{2} + \bar{\varsigma_{1}}^{2} - \varsigma_{2}^{2} - \bar{\varsigma_{2}}^{2}|^{2} u^{4} f(u) du \qquad (116)$$

$$\leq 2|\varsigma_{1}^{2} - \varsigma_{2}^{2}|^{2} + 2|\varsigma_{1} - \varsigma_{2}|^{2} + 6|\varsigma_{1}^{2} - \varsigma_{2}^{2}|^{2} + 6|\varsigma_{1}^{2} - \bar{\varsigma_{2}}^{2}|^{2}$$

$$\leq [16(\varsigma_{1}^{2} + \varsigma_{2}^{2}) + 12(\bar{\varsigma_{1}}^{2} + \bar{\varsigma_{2}}^{2}) + 2](|\varsigma_{1} - \varsigma_{2}|^{2} + |\bar{\varsigma_{1}} - \bar{\varsigma_{2}}|^{2}).$$

Then, combining (114), (115), and (116) and utilizing the inequality $\zeta \overline{\zeta} \leq \zeta^2 + \overline{\zeta}^2/4$ yield

$$2\langle \varsigma_{1}-\varsigma_{2}, (\varphi(\varsigma_{1},\bar{\varsigma_{1}})-\varphi(\varsigma_{2},\bar{\varsigma_{2}})) + \int_{U} (\varpi(\varsigma_{1},\bar{\varsigma_{1}},u)-\varpi(\varsigma_{2},\bar{\varsigma_{2}},u))\pi(du)\rangle + |\psi(\varsigma_{1},\bar{\varsigma_{1}})-\psi(\varsigma_{2},\bar{\varsigma_{2}})|^{2} + \int_{U} |\varpi(\varsigma_{1},\bar{\varsigma_{1}},u)-\varpi(\varsigma_{2},\bar{\varsigma_{2}},u)|^{2}\pi(du) \leq [-2(\varsigma_{1}^{4}+\varsigma_{2}^{4})-2(\bar{\varsigma_{1}}^{4}+\bar{\varsigma_{2}}^{4})+2(\varsigma_{1}^{2}+\varsigma_{2}^{2})+2(\bar{\varsigma_{1}}^{2}+\bar{\varsigma_{2}}^{2})+16(\varsigma_{1}^{2}+\varsigma_{2}^{2}) +12(\bar{\varsigma_{1}}^{2}+\bar{\varsigma_{2}}^{2})+2](|\varsigma_{1}-\varsigma_{2}|^{2}+|\bar{\varsigma_{1}}-\bar{\varsigma_{2}}|^{2}) \leq c_{1}(|\varsigma_{1}-\varsigma_{2}|^{2}+|\bar{\varsigma_{1}}-\bar{\varsigma_{2}}|^{2}).$$

$$(117)$$

Therefore, Assumption 6 is satisfied. Furthermore,

$$\begin{aligned} 2\langle\varsigma,\varphi(\varsigma,\bar{\varsigma}) + \int_{U} \varpi(\varsigma,\bar{\varsigma},u)\pi(du)\rangle + |\psi(\varsigma,\bar{\varsigma})|^{2} + \int_{U} |\varpi(\varsigma,\bar{\varsigma},u)|^{2}\pi(du) \\ &= -4\varsigma^{6} - 4\varsigma\bar{\varsigma}^{5} + \int_{U} (2\varsigma^{3} + 2\varsigma\bar{\varsigma}^{2})u^{2}f(u)du + |\varsigma^{2} + \bar{\varsigma}|^{2} + \int_{U} |\varsigma^{2} + \bar{\varsigma}^{2}|^{2}u^{4}f(u)du \\ &\leq \frac{-14}{3}\varsigma^{6} - \frac{10}{3}\bar{\varsigma}^{6} + 2\varsigma^{3} + \varsigma^{2} + \bar{\varsigma}^{4} + 2\varsigma^{4} + 2\bar{\varsigma}^{2} + 6\varsigma^{4} + 6\bar{\varsigma}^{4} \\ &\leq \frac{-14}{6}\varsigma^{6} + 8\varsigma^{4} + 2\varsigma^{3} - \frac{14}{6}\varsigma^{6} - \frac{10}{3}\bar{\varsigma}^{6} + 7\bar{\varsigma}^{4} + 2\bar{\varsigma}^{2} + \varsigma^{2} \\ &\leq \frac{-14}{6}\varsigma^{2}(\varsigma^{4} - \frac{24}{7}\varsigma^{4}) - \frac{1}{6}\varsigma^{6} + 1 - \frac{10}{3}\bar{\varsigma}^{2}(\bar{\varsigma}^{4} - \frac{21}{10}\bar{\varsigma}^{2}) + 2\bar{\varsigma}^{2} + \varsigma^{2} \\ &\leq \frac{-14}{6}\varsigma^{2}\left[\left(\varsigma^{2} - \frac{12}{7}\right)^{2} - \frac{144}{49}\right] - \frac{1}{6}\varsigma^{6} - \frac{10}{3}\bar{\varsigma}^{2}\left[\left(\bar{\varsigma}^{2} - \frac{21}{20}\right)^{2} - \frac{441}{400}\right] + 1 + \varsigma^{2} + 2\bar{\varsigma}^{2} \\ &\leq c(1 + |\varsigma|^{2} + |\bar{\varsigma}|^{2}). \end{aligned}$$

Hence, Assumption 5 is also satisfied. It can be noticed that

$$\sup_{|\varsigma| \lor |\bar{\varsigma}| \leq \iota} (|\varphi(\varsigma,\bar{\varsigma})| \lor |\psi(\varsigma,\bar{\varsigma})| \lor |\varpi(\varsigma,\bar{\varsigma},u)|) \leq 2\iota^5, \quad \forall \iota \geq 1.$$

Therefore, we can select $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ by $\beta(\iota) = 2\iota^5$, $r \ge 0$ with c = 2 and $\overline{\zeta} = 4$. $\beta^{-1}(\iota) = (\frac{\iota}{2})^{1/5}$, $\iota \ge 0$. Also, let $0 and define <math>\gamma(\Delta) = \Delta^{-\varepsilon}$, $\varepsilon \in \left[\frac{p(1+\overline{\zeta})}{4+2p\overline{\zeta}}, \frac{1}{4}\right]$, then all conditions in (80) and (102) are satisfied for all $\Delta \in (0, 1]$. Therefore, with these selected functions β and γ , the diffused split-step truncated Euler–Maruyama scheme (85) can be utilized to gain the numerical solution of Equation (113), and by utilizing Corollary 3, we obtain

$$\mathbb{E}|e_{\Lambda}(T)|^{p} \le C\Delta^{p(1-2\varepsilon)/2},\tag{119}$$

Example 2. Consider a stochastic pantograph model for modeling the transmission dynamics of a viral outbreak with delays and Lévy jumps.

$$dv(t) = -2v^{5}(t)dt + (v^{2}(t) + v(t)\sin^{2}(v(\eta t)))dW(t) + \int_{U} (v(t) + v(\eta t)u\widetilde{N}(dt, du),$$
(120)

with initial data v(0) = 2, $\eta = 0.7$ and compensator given by $\pi(du)dt = \lambda f(u)dudt$, where $\lambda = 2$ and f(u)

$$f(u) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{(\ln u)^2}{2}}, \quad 0 \le u < \infty.$$

Here, it is noticed that $\varphi(v(t), v(\eta t)) = -2v^5(t), \psi(v(t), v(\eta t)) = v^2(t) + v(t) \sin^2(v(\eta t))$ and $\omega(v(t), v(\eta t), u) = (v(t) + v(\eta t))u$. Then, it can be easily checked that Assumption 1 is satisfied. For Assumption 2, it can be seen that

$$\begin{split} (\varsigma_1 - \varsigma_2)(\varphi(\varsigma_1, \bar{\varsigma_1}) - \varphi(\varsigma_2, \bar{\varsigma_2})) + \frac{\xi - 1}{2} |\psi(\varsigma_1, \bar{\varsigma_1}) - \psi(\varsigma_2, \bar{\varsigma_2})|^2 \\ &= (\varsigma_1 - \varsigma_2)(-2(\varsigma_1^5 - \varsigma_2^5)) + \frac{\xi - 1}{2} |\varsigma_1^2 + \varsigma_1 \sin^2 \bar{\varsigma_1} - \varsigma_2^2 - \varsigma_2 \sin^2 \bar{\varsigma_2}|^2 \\ &\leq (\varsigma_1 - \varsigma_2)(-2(\varsigma_1 - \varsigma_2)(\varsigma_1^4 + \varsigma_1^3 \varsigma_2 + \varsigma_1^2 \varsigma_2^2 + \varsigma_1 \varsigma_2^3 + \varsigma_2^4)) \\ &+ (\xi - 1)(|\varsigma_1^2 - \varsigma_2^2|^2 + |(\varsigma_1 - \varsigma_2) \sin^2 \bar{\varsigma_1} + \varsigma_2(\sin^2 \bar{\varsigma_1} - \sin^2 \bar{\varsigma_2})|^2). \end{split}$$

Then, by performing a little bit of simplification and utilizing the elementary inequalities $\varsigma \overline{\varsigma} \leq \varsigma^2 + \overline{\varsigma}^2/4$ and $-(\varsigma_1^3 \varsigma_2 + \varsigma_1^2 \varsigma_2^2 + \varsigma_1 \varsigma_2^3) \leq 0.5(\varsigma_1^4 + \varsigma_2^4)$, we obtain

$$\begin{split} (\varsigma_1 - \varsigma_2)(\varphi(\varsigma_1, \bar{\varsigma_1}) - \varphi(\varsigma_2, \bar{\varsigma_2})) + \frac{\xi - 1}{2} |\psi(\varsigma_1, \bar{\varsigma_1}) - \psi(\varsigma_2, \bar{\varsigma_2})|^2 \\ &\leq \left[-(\varsigma_1^4 + \varsigma_2^4) + 2(\xi - 1)(\varsigma_1^2 + \varsigma_2^2) + (\xi - 1)(1 + \varsigma_1^2 + \varsigma_2^2) \right] \\ &\times (|\varsigma_1 - \varsigma_2|^2 + |\bar{\varsigma_1} - \bar{\varsigma_2}|^2) \\ &\leq \left[\frac{1}{4} + 6(\xi - 1)^2 \right] (|\varsigma_1 - \varsigma_2|^2 + |\bar{\varsigma_1} - \bar{\varsigma_2}|^2) \\ &\leq k_2 (|\varsigma_1 - \varsigma_2|^2 + |\bar{\varsigma_1} - \bar{\varsigma_2}|^2) \end{split}$$

Therefore, Assumption 2 is satisfied. Furthermore,

$$\begin{split} \varsigma\varphi(\varsigma,\bar{\varsigma}) &+ \frac{\bar{p}-1}{2} |\psi(\varsigma,\bar{\varsigma})|^2 = -2\varsigma^6 + \frac{\bar{p}-1}{2} |\varsigma^2 + \varsigma \sin^2 \bar{\varsigma}|^2 \\ &\leq -2\varsigma^6 + (\bar{p}-1)(|\varsigma|^4 + |\varsigma|^2) \\ &\leq -2\varsigma^2 \bigg[\left(\varsigma^2 - \frac{\bar{p}-1}{4} \right)^2 - \left(\frac{\bar{p}-1}{4} \right)^2 \bigg] + (\bar{p}-1)|\varsigma|^2 \end{split}$$
(121)
$$&\leq \frac{9}{8} (\bar{p}-1)^2 (1+|\varsigma|^2 + |\bar{\varsigma}|^2) \\ &\leq k_3 (1+|\varsigma|^2 + |\bar{\varsigma}|^2). \end{split}$$

Hence, Assumption 3 is also satisfied for all $\bar{p} \in (\xi, \infty)$ *. It should be also noted that*

$$\sup_{|\varsigma| \lor |\varsigma| \leq \iota} (|\varphi(\varsigma, \bar{\varsigma})| \lor |\psi(\varsigma, \bar{\varsigma})|) \leq 2\iota^5, \quad \forall \iota \geq 1.$$

Therefore, we select $\beta(\iota) = 2\iota^5$, $\iota \ge 0$ with C = 2 and $\zeta = 4$. Then, by selecting $\xi > 2$, letting $2 \le p < \xi$, choosing $\bar{p} \in (\xi, \infty)$ large enough such that $q \in ((1+\zeta)p \lor \xi, \bar{p})$ and defining $\gamma(\Delta) = \Delta^{-\varepsilon}$, $\varepsilon \in (0, \frac{1}{4} \land 1/q]$ such that all conditions in (8) hold for all $\Delta \in (0, 1]$, it can be concluded by utilizing Corollary 2 that

$$\mathbb{E}|e_{\Delta}(T)|^{p} \leq C\Delta^{\varepsilon(q-(1+\zeta)p)/(1+\zeta)\wedge(q-\zeta p)/q}.$$
(122)

6. Conclusions

This paper studied the stochastic pantograph model with Lévy jumps, which can be applied in real-life applications such as financial markets and biology. This paper also contributed to the field of stochastic modeling by providing a robust and efficient numerical method, which is called the diffused split-step truncated Euler–Maruyama method, for analyzing stochastic pantograph models with Lévy jumps. The finite time $\mathcal{L}^p (p \ge 2)$ convergence rate was obtained where non-jump coefficients behaved beyond linearly, while the jump coefficient increased linearly and this can be utilized to approximate the variance or the higher moment of the solution. Also, when $0 , the <math>\mathcal{L}^p$ convergence rate was addressed with drift, diffusion, and jump coefficients exceeding linearity, and this can be used to approximate the mean value of the solution or the European call option value in financial mathematics. The obtained convergence rates and numerical examples demonstrated the effectiveness and practical relevance of the proposed approach, which in turn opened up new avenues for studying and understanding complex dynamical systems influenced by random factors.

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