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The Impulsive Coupled Langevin ψ-Caputo Fractional Problem with Slit-Strip-Generalized-Type Boundary Conditions

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Abstract: In this paper, the existence of a unique solution is established for a coupled system of Langevin fractional problems of ψ-Caputo fractional derivatives with generalized slit-strip-type integral boundary conditions and impulses using the Banach contraction principle. We also find at least one solution to the aforementioned system using some assumptions and Schaefer's fixed point theorem. After that, Ulam–Hyers stability is discussed. Finally, to provide additional support for the main results, pertinent examples are presented.

Keywords: coupled system; integro-multipoint–multistrip boundary conditions; ψ-Caputo fractional derivative; Schaefer's fixed point theorem; Ulam–Hyers stability

MSC: 26A33; 34B27; 39B82; 45M10

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1. Introduction

Higher-order derivatives and n-fold integrals in ordinary calculus are considered only for the particular case when the order belongs to a set of natural numbers. Integrations and derivatives of any real number $\alpha > 0$ are discussed in fractional calculus, in which the ordinary definitions of derivatives and integrals are considered as special cases, i.e., for when α belongs to a set of natural numbers. Fractional calculus has become increasingly significant owing to its wide-ranging uses in science. In the literature, there are more precise and mathematical representations of a range of phenomena modeled with the help of fractional derivatives [1–3]. Among the different qualitative properties of Fractional Differential Equations (FDEs), researchers investigated the existence of unique solutions to and stability analyses of FDEs under different conditions [4–8].

The Langevin equation is a great method for describing phenomena related to Brownian motion, and it may be used to successfully describe processes by economists, engineers, doctors, and other professionals. It was discovered that the Langevin equation, which was initially formulated by Langevin in 1908, is a useful tool for accurately describing the evolution of physical processes, including stochastic difficulties in various disciplines, including mechanical and electrical engineering, chemistry, physics, military systems, image processing, and astronomy. When the random oscillation force is assumed to be Gaussian, it is also utilized to characterize Brownian motion. For more details, see other studies [9–11]. Coupled system of FDEs are used in a variety of physical and practical models, including those that simulate diseases [12,13], the environment [14], chaotic systems [15], and many more. Ahmad et al. discussed basic applications of slit strip conditions in imaging and acoustics using strip detectors [16,17].

The authors of [18] investigated the existence of a unique solution to the following problem:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0+}\wp(\sigma) = f(\sigma, \wp(\sigma), {}^{c}\mathcal{D}^{\beta}_{0+}\wp(\sigma)), \ \sigma \in [0,1], \\ \wp''(0) = 0, \wp'''(0) = 0, ..., \wp^{n-2}(0) = 0, \\ \wp(0) + \wp'(0) = \chi(\wp), \displaystyle\int_0^1 \wp(\sigma)d\sigma = m, \end{cases}$$

where ${}^{c}\mathcal{D}^{\alpha}_{0+}$ and ${}^{c}\mathcal{D}^{\beta}_{0+}$ are Caputo Fractional Derivatives (CFDs) of order α with $n-1 < \alpha < n$ $(n \geqslant 2)$ and $0 < \beta < 1$, respectively. $m \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\chi : C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions.

A coupled system of nonlinear FDEs was studied by Ahmad et al. [19],

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0}\wp_1(\sigma) &= h_1(\sigma, \wp_1(\sigma), \wp_2(\sigma)), \ \sigma \in [0,1], \ 1 < \alpha \leqslant 2, \\ {}^{c}\mathcal{D}^{\beta}_{0}\wp_2(\sigma) &= h_2(\sigma, \wp_1(\sigma), \wp_2(\sigma)), \ \sigma \in [0,1], \ 1 < \beta \leqslant 2, \end{cases}$$

with the coupled and uncoupled boundary conditions of the type:

$$\begin{aligned} &\wp_1(0) = 0, \quad \wp_1(\nu) = d_1 \textstyle\int_0^{\mu} \wp_2(s)ds + d_2 \int_{\nu_1}^{1} \wp_2(s)ds, \ 0 < \mu < \nu < \nu_1 < 1, \\ &\wp_2(0) = 0, \quad \wp_2(\nu) = d_1 \textstyle\int_0^{\mu} \wp_1(s)ds + d_2 \int_{\nu_1}^{1} \wp_1(s)ds, \ 0 < \mu < \nu < \nu_1 < 1, \\ &\wp_1(0) = 0, \quad \wp_1(\nu) = d_1 \textstyle\int_0^{\mu} \wp_1(s)ds + d_2 \int_{\nu_1}^{1} \wp_1(s)ds, \ 0 < \mu < \nu < \nu_1 < 1, \\ &\wp_2(0) = 0, \quad \wp_2(\nu) = d_1 \textstyle\int_0^{\mu} \wp_2(s)ds + d_2 \int_{\nu_1}^{1} \wp_2(s)ds, \ 0 < \mu < \nu < \nu_1 < 1, \end{aligned}$$

where ${}^{c}\mathcal{D}^{\alpha}_{0}$ and ${}^{c}\mathcal{D}^{\beta}_{0}$ are CFDs of order α, β, respectively. $h_1, h_2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given continuous functions and $d_1, d_2 \in \mathbb{R}$.

A coupled system of hybrid nonlinear FDEs was investigated by Ahmad et al. [20].

$$\begin{aligned} {}^{c}\mathcal{D}^{\alpha}_{0}[\wp_1(\sigma) - h_1(\sigma, \wp_1(\sigma), \wp_2(\sigma))] &= \theta_1(\sigma, \wp_1(\sigma), \wp_2(\sigma)), \quad \sigma \in [0,1], \quad 1 < \alpha \leqslant 2, \\ {}^{c}\mathcal{D}^{\beta}_{0}[\wp_2(\sigma) - h_2(\sigma, \wp_1(\sigma), \wp_2(\sigma))] &= \theta_2(\sigma, \wp_1(\sigma), \wp_2(\sigma)), \quad \sigma \in [0,1], \quad 1 < \beta \leqslant 2, \end{aligned}$$

with coupled slit-strip-type integral boundary conditions:

$$\begin{aligned} &\wp_1(0) = 0, \ \ \wp_1(\mu) = d_1 \textstyle\int_0^{\nu_1} \wp_2(s)ds + d_2 \int_{\nu_2}^{1} \wp_2(s)ds, \quad 0 < \nu_1 < \mu < \nu_2 < 1, \\ &\wp_2(0) = 0, \ \ \wp_2(\mu) = d_1 \textstyle\int_0^{\nu_1} \wp_1(s)ds + d_2 \int_{\nu_2}^{1} \wp_1(s)ds, \quad 0 < \nu_1 < \mu < \nu_2 < 1, \end{aligned}$$

where ${}^{c}\mathcal{D}^{\alpha}_{0}$ with order α and ${}^{c}\mathcal{D}^{\beta}_{0}$ with order β are CFDs. $\theta_i, h_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $h_i(0, \Im_1(0), \Im_2(0)) = 0$, $i = 1,2$ and $d_1, d_2 \in \mathbb{R}$.

In 2022, Zhiwei et al. [21] studied the following coupled system:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha;\psi}_{\sigma_k,\sigma}[\wp_1(\sigma) - h_1(\sigma, \wp_1(\sigma), {}^{c}\mathcal{D}^{\alpha;\psi}_{\sigma_k,\sigma}\wp_1(\sigma))] = j_1(\sigma, \wp_1(\sigma), {}^{c}\mathcal{D}^{\alpha;\psi}_{\sigma_k,\sigma}\wp_1(\sigma)), \quad \sigma \in (\sigma_k, \sigma_{k+1}], \quad k = 0,1,\ldots,p, \\ {}^{c}\mathcal{D}^{\beta;\psi}_{\sigma_k,\sigma}[\wp_2(\sigma) - h_2(\sigma, \wp_2(\sigma), {}^{c}\mathcal{D}^{\beta;\psi}_{\sigma_k,\sigma}\wp_2(\sigma))] = j_2(\sigma, \wp_2(\sigma), {}^{c}\mathcal{D}^{\beta;\psi}_{\sigma_k,\sigma}\wp_2(\sigma)), \quad \sigma \in (\sigma_k, \sigma_{k+1}], \quad k = 0,1,\ldots,p, \\ \wp_1(0) = 0, \ \ \wp_1(\eta) = a_1 \int_{\sigma_k}^{\delta_{2k}} \wp_1(\sigma)d\sigma + a_2 \int_{\delta_{2k+1}}^{\sigma_{k+1}} \wp_1(\sigma)d\sigma, \quad \sigma_k < \delta_{2k} < \eta < \delta_{2k+1} < \sigma_{k+1}, \\ \wp_2(0) = 0, \ \ \wp_2(\eta) = a_1 \int_{\sigma_k}^{\delta_{2k}} \wp_2(\sigma)d\sigma + a_2 \int_{\delta_{2k+1}}^{\sigma_{k+1}} \wp_2(\sigma)d\sigma, \quad \sigma_k < \delta_{2k} < \eta < \delta_{2k+1} < \sigma_{k+1}, \\ \Delta\wp_1(\sigma_k) = \wp_1(\sigma_k^+) - \wp_1(\sigma_k^-) = I_k(\wp_1(\sigma_k)), \quad \Delta\wp_1'(\sigma_k) = \wp_1'(\sigma_k^+) - \wp_1'(\sigma_k^-) = J_k(\wp_1(\sigma_k)), \quad k = 1,2,\ldots,p, \\ \Delta\wp_2(\sigma_k) = \wp_2(\sigma_k^+) - \wp_2(\sigma_k^-) = I_k^*(\wp_2(\sigma_k)), \quad \Delta\wp_2'(\sigma_k) = \wp_2'(\sigma_k^+) - \wp_2'(\sigma_k^-) = J_k^*(\wp_2(\sigma_k)), \quad k = 1,2,\ldots,p, \end{cases}$$

where ${}^{c}\mathcal{D}^{\alpha;\psi}_{\sigma_k,\sigma}$ and ${}^{c}\mathcal{D}^{\beta;\psi}_{\sigma_k,\sigma}$ denote the ψ-CFD of order $\alpha, \beta \in (1,2]$, and $\mathcal{J} = [0, R]$ with $R > 0$, $h_1, j_1, h_2, j_2 : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions with $h_1(0, \wp_1(0), {}^{c}\mathcal{D}^{\alpha;\psi}_{\sigma_k,\sigma}\wp_1(0)) = 0 = h_2(0, \wp_2(0), {}^{c}\mathcal{D}^{\beta;\psi}_{\sigma_k,\sigma}\wp_2(0))$ and a_i are real constants for $i = 1, 2$.

In [22], Almaghamsi et al. investigated the following system:

$$ {}^{c}\mathcal{D}^{\gamma_i,\mu}({}^{c}\mathcal{D}^{\sigma_i,\mu} + \alpha_i)\varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b],\ i = 1, 2. $$

with boundary conditions

$$ \varphi_i(a) = 0, \quad I^{\vartheta_i,\mu}\varphi_i(b) = 0, $$
$$ {}^{c}\mathcal{D}^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^{\zeta} \varphi_2(s)ds. $$

where for $i = 1, 2$, ${}^{c}\mathcal{D}^{\gamma_i,\mu}$ and ${}^{c}\mathcal{D}^{\sigma_i,\mu}$ are μ- CFD, $0 < \sigma_i, \gamma_2 < 1$, $1 < \gamma_1 \leqslant 2$, $\alpha_i, \kappa \in \mathbb{R}$.

In this paper, we investigate the coupled system of Langevin fractional problems of ψ-CFDs with generalized slit-strip-type integral boundary conditions and impulses:

$$ \begin{cases} {}^{c}\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma} + \alpha_1)\wp(\sigma) = F_1(\sigma, \wp(\sigma), \Im(\sigma)), \quad \sigma \in (\sigma_k, \sigma_{k+1}], \quad k = 0, 1, \ldots, p, \\ {}^{c}\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma} + \alpha_2)\Im(t) = F_2(\sigma, \wp(\sigma), \Im(\sigma)), \quad \sigma \in (\sigma_k, \sigma_{k+1}], \quad k = 0, 1, \ldots, p, \\ \wp(0) = 0, \ \ \wp(\eta) = a_1 \int_{\sigma_k}^{\delta_{2k}} \wp(\tau)d\tau + a_2 \int_{\delta_{2k+1}}^{\sigma_{k+1}} \wp(\tau)d\tau, \quad \sigma_k < \delta_{2k} < \eta < \delta_{2k+1} < \sigma_{k+1}, \\ \Im(0) = 0, \ \ \Im(\eta) = a_1 \int_{\sigma_k}^{\delta_{2k}} \Im(\tau)d\tau + a_2 \int_{\delta_{2k+1}}^{\sigma_{k+1}} \Im(\tau)d\tau, \quad \sigma_k < \delta_{2k} < \eta < \delta_{2k+1} < \sigma_{k+1}, \\ \Delta\wp(\sigma_k) = \wp(\sigma_k^+) - \wp(\sigma_k^-) = I_k(\wp(\sigma_k)), \quad \Delta\wp'(\sigma_k) = \wp'(\sigma_k^+) - \wp'(\sigma_k^-) = J_k(\wp(\sigma_k)), \quad k = 1, 2, \ldots, p, \\ \Delta\Im(\sigma_k) = \Im(\sigma_k^+) - \Im(\sigma_k^-) = I_k^*(\Im(\sigma_k)), \quad \Delta\Im'(\sigma_k) = \Im'(\sigma_k^+) - \Im'(\sigma_k^-) = J_k^*(\Im(\sigma_k)), \quad k = 1, 2, \ldots, p, \end{cases} \tag{1} $$

where ${}^{c}\mathcal{D}^{\gamma_i;\psi}_{\sigma_k,\sigma}$ and ${}^{c}\mathcal{D}^{\beta_i;\psi}_{\sigma_k,\sigma}$ denote the ψ-CFD with $\beta_i \in (0, 1]$, $\gamma_i \in (1, 2]$. $J = [0, T]$ with $T > 0$, $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and a_1, a_2 are real constants.

The subsequent sections of the paper are structured as follows: Section 2 presents some basic materials relevant to our results. The proof of a lemma that characterizes the solution of our problem is found in Section 3. In Section 4, the Shaefer fixed point theorem and the Banach contraction principle are applied to prove the existence and uniqueness of the problem, while Section 5 presents Ulam–Hyers stability for problem (1) and Section 6 provide examples, demonstrating our findings.

2. Preliminaries and Notations

Several definitions and results from this section are required later.

Definition 1 ([23])**.** *Let* $F : [0, 1] \to \mathbb{R}$ *be an integrable function and* $\psi : [a_0, b_0] \to \mathbb{R}$ *be an increasing and differentiable function such that* $\psi'(\sigma) \neq 0$ *for all* $\sigma \in [a_0, b_0]$*. Then, the left-sided* ψ*–Riemann–Liouville (RL) fractional integral of order* $(\alpha > 0)$ *is defined by*

$$ \mathcal{J}^{\alpha;\psi}_{a_0+}F(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\sigma} \psi'(r)(\psi(\sigma) - \psi(r))^{\alpha-1}F(r)dr, $$

where Γ *denotes the Euler Gamma function.*

Definition 2 ([23])**.** *Let* F *and* $\psi \in C^n([a_0, b_0], \mathbb{R})$ $(n \in \mathbb{N})$ *be functions where* $\psi(\sigma)$ *is increasing and* $\psi'(\sigma) \neq 0$ *for all* $\sigma \in [a_0, b_0]$*. Then, the left-sided* ψ*-RL fractional derivative of order* α *of a function* F *is defined by*

$$ \begin{aligned} \mathcal{D}^{\alpha;\psi}_{a_0+}F(\sigma) &= \left(\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}\right)^n \mathcal{J}^{n-\alpha;\psi}_{a_0+}F(\sigma) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}\right)^n \int_{a_0}^{\sigma} \psi'(r)(\psi(\sigma) - \psi(r))^{n-\alpha-1}F(r)dr, \end{aligned} $$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Definition 3 ([23]). *Let* F *and* $\psi \in C^n([a_0, b_0], \mathbb{R})$ $(n \in \mathbb{N})$ *be functions where* $\psi(\sigma)$ *is increasing and* $\psi'(\sigma) \neq 0$ *for all* $\sigma \in [a_0, b_0]$. *Then, the left-sided* ψ*-CFD of order* α $(\alpha \in (n-1, n))$ *of a function* F *is defined by*

$$ {}^{c}\mathcal{D}^{\alpha;\psi}_{a_0+}F(\sigma) = \mathcal{D}^{\alpha;\psi}_{a_0+}[F(\sigma) - \sum_{l=0}^{n-1} \frac{F^{[l](a_0)}_{\psi}}{l!}(\psi(\sigma) - \psi(a_0))^l], $$

where $F^{[l]}_{\psi}(\sigma) = (\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma})^l F(\sigma)$ *and* $n = [\alpha] + 1$ *for* $\alpha \notin N$, $n = \alpha$ *for* $\alpha \in N$.

Further, if $F \in C^n([a_0, b_0], \mathbb{R})$ *and* $\alpha \notin N$, *then*

$$ \begin{aligned} {}^{c}\mathcal{D}^{\alpha;\psi}_{a_0+}F(\sigma) &= \mathcal{I}^{n-\alpha;\psi}_{a_0+}\left(\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}\right)^n F(\sigma) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{a_0}^{\sigma} \psi'(r)(\psi(\sigma) - \psi(r))^{n-\alpha-1}F^{[n]}_{\psi}(r)dr. \end{aligned} $$

Thus, if $\alpha = n \in N$, *then* ${}^{c}\mathcal{D}^{\alpha;\psi}_{a_0+}F(\sigma) = F^{[n]}_{\psi}(\sigma)$.

Lemma 1 ([23]). *For* $\alpha > 0$,

If $F \in C([a_0, b_0], \mathbb{R})$, *then* ${}^{c}\mathcal{D}^{\alpha;\psi}_{a_0+}\mathcal{I}^{\alpha;\psi}_{a_0+}F(\sigma) = F(\sigma), \quad \sigma \in [a_0, b_0]$.

If $F \in C^n([a_0, b_0], \mathbb{R})$, $n - 1 < \alpha < n$, *then*

$$ \mathcal{I}^{\alpha;\psi}_{a_0+}{}^{c}\mathcal{D}^{\alpha;\psi}_{a_0+}F(\sigma) = F(\sigma) - \sum_{l=0}^{n-1} c_l(\psi(\sigma) - \psi(a_0))^l, \quad \sigma \in [a_0, b_0], $$

where $c_l = \frac{F^{[l](a_0)}_{\psi}}{l!}$.

Lemma 2 (Contraction theorem [24]). *Let a metric space* X *be complete and* $\mathcal{P} : X \to X$ *be a contraction on* X. *Then,* $\mathcal{P}$ *has unique fixed point.*

Lemma 3 (Arzela–Ascoli theorem [24]). *Assume that* X *is a compact set in* $\mathbb{R}^n$, $n \geqslant 1$. *Then, a set* $S \subset \mathbf{C}(X)$ *is relatively compact in* $\mathbf{C}(X)$ *if and only if the functions in* S *are uniformly bounded and equicontinuous on* X.

Lemma 4 (Schaefer's fixed point theorem [24]). *Let* $\mathcal{P} : X' \times Y' \to X' \times Y'$ *be a completely continuous operator. Consider a set* $G(\mathcal{P}) = \{(\wp, \Im) \in X' \times Y'; (\wp, \Im) = \lambda\mathcal{P}(\wp, \Im); 0 < \lambda < 1\}$. *Then, either* $\mathcal{P}$ *has at least one fixed point or the set* $G(\mathcal{P})$ *is unbounded.*

3. Main Results

For $\sigma_k \in \mathcal{J}_k$ such that $0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_p = T$ and $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_p$, where $\mathcal{J}_0 = (0, \sigma_1]$, $\mathcal{J}_1 = (\sigma_1, \sigma_2], \ldots, \mathcal{J}_p = (\sigma_p, \sigma_{p+1}]$ and $\mathcal{J}' = \mathcal{J} - \{\sigma_0, \sigma_1, \ldots, \sigma_p\}$, we define the space $X' = \{\wp : \mathcal{J} \to \mathbb{R} \mid \wp \in PC([\mathcal{J}, \mathbb{R}])$, such that the right limits $\wp(\sigma_k^+)$, $\wp'(\sigma_k^+)$ and left limits $\wp(\sigma_k^-)$, $\wp'(\sigma_k^-)$ exist and $\Delta\wp(\sigma_k) = \wp(\sigma_k^+) - \wp(\sigma_k^-)$, $\Delta\wp'(\sigma_k) = \wp'(\sigma_k^+) - \wp'(\sigma_k^-)$, $k = 1, 2, \ldots, p\}$. Then, clearly, X' is a Banach space equipped with the norm $||\wp(\sigma)|| = \max_{\sigma \in \mathcal{J}} |\wp(\sigma)|$. Similarly, define the space $Y' = \{\Im : \mathcal{J} \to \mathbb{R} \mid \Im \in PC([\mathcal{J}, \mathbb{R}])$, the right limits $\Im(\sigma_k^+)$, $\Im'(\sigma_k^+)$ and left limits $\Im(\sigma_k^-)$, $\Im'(\sigma_k^-)$ exist and $\Delta\Im(\sigma_k) = \Im(\sigma_k^+) - \Im(\sigma_k^-)$, $\Delta\Im'(\sigma_k) = \Im'(\sigma_k^+) - \Im'(\sigma_k^-)$, $k = 1, 2, \ldots, p\}$. Then, clearly, Y' is a Banach space equipped with the norm $||\Im(t)|| = \max_{\sigma \in \mathcal{J}} |\Im(\sigma)|$.

Lemma 5. *Let* F_1, F_2 *be real-valued continuous functions on* $\mathfrak{J}$. *Then, the coupled system:*

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma}+\alpha_1)\wp(\sigma)=F_1(\sigma), \quad \sigma\in J,\ \sigma\neq\sigma_k, \quad k=0,1,\ldots,p, \\ {}^{c}\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma}+\alpha_2)\Im(\sigma)=F_2(\sigma), \quad \sigma\in J,\ \sigma\neq\sigma_k, \quad k=0,1,\ldots,p, \end{cases} \tag{2}$$

equipped with the boundary conditions given in (1) only has one solution, which is given by

$$\begin{aligned}
&\wp(\sigma)= \\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}\wp(r)dr+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right) \\
&\times\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\
&-\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr+I_i(\wp(\sigma_i))+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg) \\
&-a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+I_i(\wp(\sigma_i))+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg)\Bigg\}d\tau-a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr+I_i(\wp(\sigma_i))+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg)\Bigg\}d\tau\Bigg] \\
&+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr-\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr+I_i(\wp(\sigma_i))+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg),
\end{aligned} \tag{3}$$

$$
\begin{aligned}
&\mathfrak{I}(\sigma) = \\
&\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + \Big((\psi(\sigma) \\
&-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Big).\frac{1}{\Delta}\Big[\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r) \\
&(\psi(\eta)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + \sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r)dr \\
&-\frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\mathfrak{I}(r)dr + I_i^*(\mathfrak{I}(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\mathfrak{I}(\sigma_i))\Big) \\
&-a_1\int_{\sigma_k}^{\delta_{2k}}\Big\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr \\
&+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr \\
&+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r)dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr \\
&+I_i^*(\mathfrak{I}(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\mathfrak{I}(\sigma_i))\Big)\Big\}d\tau - a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Big\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr \\
&-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + \sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr \\
&+\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r)dr \\
&-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr + I_i^*(\mathfrak{I}(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\mathfrak{I}(\sigma_i))\Big)\Big\}d\tau\Big] \\
&+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r)dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\mathfrak{I}(r)dr \\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r)dr - \frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\mathfrak{I}(r)dr + I_i^*(\mathfrak{I}(\sigma_i)) + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i^*(\mathfrak{I}(\sigma_i))\Big),
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
\Delta = {} & a_1\int_{\sigma_k}^{\delta_{2k}}((\psi(\tau)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1})))d\tau + a_2\int_{\delta_{2k+1}}^{\sigma_{k+1}}((\psi(\tau)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1})))d\tau \\
&-((\psi(\eta)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))),
\end{aligned}
$$

and it is assumed that

$$\Delta \neq 0.$$

Proof. Let

$$ {}^{c}\mathcal{D}_{\sigma_k,\sigma}^{\gamma_1;\psi}({}^{c}\mathcal{D}_{\sigma_k,\sigma}^{\beta_1;\psi} + \alpha_1)\wp(\sigma) = F_1(\sigma).$$

Then, using Lemma 1 in the differential Equation (2), for any $\sigma \in \mathcal{J}_0$, there exist constants $c_0, c_1 \in \mathbb{R}$, such that:

$$\wp(\sigma) = I^{\gamma_1+\beta_1;\psi} F_1(\sigma) - \alpha_1 I^{\beta_1;\psi} \wp(\sigma) + c_0 + c_1(\psi(\sigma) - \psi(\sigma_0)).$$

$$\begin{aligned} \wp(\sigma) &= \frac{1}{\Gamma(\gamma_1+\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1} \wp(r)dr + c_0 + c_1(\psi(\sigma)-\psi(\sigma_0)), \end{aligned} \quad (5)$$

Using the initial condition $\wp(0) = 0$, we get $c_0 = 0$, so

$$\begin{aligned} \wp(\sigma) &= \frac{1}{\Gamma(\gamma_1+\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1} \wp(r)dr + c_1(\psi(\sigma)-\psi(\sigma_0)). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} \wp'(\sigma) &= \frac{1}{\Gamma(\gamma_1+\beta_1-1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-2} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1-1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-2} \wp(r)dr + c_1\psi'(\sigma). \end{aligned}$$

For $\sigma \in \mathcal{J}_1$, there are $d_0,\ d_1 \in \mathbb{R}$ such that

$$\begin{aligned} \wp(\sigma) &= \frac{1}{\Gamma(\gamma_1+\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1} \wp(r)dr + d_0 + d_1(\psi(\sigma)-\psi(\sigma_0)). \end{aligned}$$

$$\begin{aligned} \wp'(\sigma) &= \frac{1}{\Gamma(\gamma_1+\beta_1-1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-2} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1-1)} \int_{\sigma_0}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-2} \wp(r)dr + d_1\psi'(\sigma). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \wp(\sigma_1^-) &= \frac{1}{\Gamma(\gamma_1+\beta_1)} \int_{\sigma_0}^{\sigma_1} \psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-1} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1)} \int_{\sigma_0}^{\sigma_1} \psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-1} \wp(r)dr + c_1(\psi(\sigma_1)-\psi(\sigma_0)), \end{aligned}$$

$$\wp(\sigma_1^+) = d_0,$$

$$\begin{aligned} \wp'(\sigma_1^-) &= \frac{1}{\Gamma(\gamma_1+\beta_1-1)} \int_{\sigma_0}^{\sigma_1} \psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-2} F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1-1)} \int_{\sigma_0}^{\sigma_1} \psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-2} \wp(r)dr + c_1\psi'(\sigma_1), \end{aligned}$$

$$\wp'(\sigma_1^+) = d_1\psi'(\sigma_1).$$

Using

$$\begin{cases} \Delta\wp(\sigma_1) = \wp(\sigma_1^+) - \wp(\sigma_1^-) = I_1(\wp(\sigma_1)), \\ \Delta\wp'(\sigma_1) = \wp'(\sigma_1^+) - \wp'(\sigma_1^-) = J_1(\wp(\sigma_1)). \end{cases}$$

we obtain

$$\begin{aligned} d_0 &= \frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\ &-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-1}\wp(r)dr + c_1(\psi(\sigma_1)-\psi(\sigma_0)) + I_1(\wp(\sigma_1)), \\ d_1 &= \frac{1}{\psi'(\sigma_1)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\ &-\frac{\alpha_1}{\psi'(\sigma_1)\Gamma(\beta_1-1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-2}\wp(r)dr + c_1 + \frac{1}{\psi'(\sigma_1)}J_1(\wp(\sigma_1)). \end{aligned}$$

Thus,

$$\begin{aligned} &\wp(\sigma) = \\ &\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_1}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_1}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{1}{\Gamma(\gamma_1+\beta_1)} \\ &\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-1}\wp(r)dr + c_1(\psi(\sigma_1)-\psi(\sigma_0)) \\ &+I_1(\wp(\sigma_1)) + c_1(\psi(\sigma)-\psi(\sigma_1)) + \frac{(\psi(\sigma)-\psi(\sigma_1))}{\psi'(\sigma_1)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\ &-\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_1))}{\psi'(\sigma_1)\Gamma(\beta_1-1)}\int_{\sigma_0}^{\sigma_1}\psi'(r)(\psi(\sigma_1)-\psi(r))^{\beta_1-2}\wp(r)dr + \frac{(\psi(\sigma)-\psi(\sigma_1))}{\psi'(\sigma_1)}J_1(\wp(\sigma_1)), \quad \sigma \in J_1. \end{aligned}$$

Similarly, for $\sigma \in \mathcal{J}_k$, $k = 1, 2, \ldots, p$, we have

$$\begin{aligned} &\wp(\sigma) = \\ &\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}\wp(r)dr + c_1\Bigg((\psi(\sigma) \\ &-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg) + \sum_{i=1}^{p}\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr + \sum_{i=1}^{p}\frac{\alpha_1}{\Gamma(\beta_1)} \\ &\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\ &-\sum_{i=1}^{p}\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + \sum_{i=1}^{p}I_i(\wp(\sigma_i)) + \sum_{i=1}^{p}\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i)). \end{aligned} \tag{6}$$

Finally, applying $\wp(\eta) = a_1\int_{\sigma_k}^{\delta_{2k}}\wp(\tau)d\tau + a_2\int_{\delta_{2k+1}}^{\sigma_{k+1}}\wp(\tau)d\tau$, we get the value of c_1 as

$$
\begin{aligned}
c_1 ={}& \\
&\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\Bigg(\\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr - \frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + I_i(\wp(\sigma_i)) + \frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg) - a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau) \\
&-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr - \frac{\alpha_1(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + I_i(\wp(\sigma_i)) \\
&+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg)\Bigg\}d\tau - a_2\int_{\delta_{2k+1}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r)dr \\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r)dr \\
&-\frac{\alpha_1(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + I_i(\wp(\sigma_i)) + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\Bigg)\Bigg\}d\tau\Bigg].
\end{aligned} \tag{7}
$$

Putting (7) in (6), we obtain our required result (3) and similarly, we get (4). The converse of the lemma follows by direct computation. This concludes the proof. □

4. Existence Results for the Problem (1)

Here, we consider some hypotheses.

Hypothesis 1. *For each* $\sigma \in \mathcal{J}$ *and* $\wp_1, \wp_2 \in X'$, $\Im_1, \Im_2 \in Y'$, *there exist positive constants* $M_{f_1} > 0$, $N_{f_1} > 0$, *such that*

$$|F_1(\sigma, \wp_1(\sigma), \Im_1(\sigma)) - F_1(\sigma, \wp_2(\sigma), \Im_2(\sigma))| \leqslant M_{f_1}|\wp_1(\sigma) - \wp_2(\sigma)| + N_{f_1}|\Im_1(\sigma) - \Im_2(\sigma)|.$$

Hypothesis 2. *For each* $\sigma \in \mathcal{J}$ *and* $\wp_1, \wp_2 \in X'$, $\Im_1, \Im_2 \in Y'$, *there exist positive constants* $M_{f_2} > 0$, $N_{f_2} > 0$, *such that*

$$|F_2(\sigma, \wp_1(\sigma), \Im_1(\sigma)) - F_2(\sigma, \wp_2(\sigma), \Im_2(\sigma))| \leqslant M_{f_2}|\wp_1(\sigma) - \wp_2(\sigma)| + N_{f_2}|\Im_1(\sigma) - \Im_2(\sigma)|.$$

Hypothesis 3. *For every* $\wp_1, \wp_2 \in X'$ *and* $\Im_1, \Im_2 \in Y'$*, there exist constants* $A_1, A_2, A_3, A_4 > 0$ *such that*

$$|I_k(\wp_1(\sigma_k)) - I_k(\wp_2(\sigma_k))| \leqslant A_1 |\wp_1(\sigma_k) - \wp_2(\sigma_k)|,$$
$$|J_k(\wp_1(\sigma_k)) - J_k(\wp_2(\sigma_k))| \leqslant A_2 |\wp_1(\sigma_k) - \wp_2(\sigma_k)|,$$
$$|I_k^*(\Im_1(\sigma_k)) - I_k^*(\Im_2(\sigma_k))| \leqslant A_3 |\Im_1(\sigma_k) - \Im_2(\sigma_k)|,$$
$$|J_k^*(\Im_1(\sigma_k)) - J_k^*(\Im_2(\sigma_k))| \leqslant A_4 |\Im_1(\sigma_k) - \Im_2(\sigma_k)|.$$

Hypothesis 4. *There exist constants* θ_0*,* θ_1 *and* θ_2 *such that*

$$|F_1(\sigma, \wp(\sigma), \Im(\sigma))| < \theta_0(\sigma) + \theta_1(\sigma)|\wp(\sigma)| + \theta_2(\sigma)|\Im(\sigma)|,$$

with $\sup_{\sigma\in J} \theta_0(\sigma) = \theta_0^*$, $\sup_{\sigma\in J} \theta_1(\sigma) = \theta_1^*$, $\sup_{\sigma\in J} \theta_2(\sigma) = \theta_2^*$.

There exist constants θ_3*,* θ_4 *and* θ_5 *such that*

$$|F_2(\sigma, \wp(\sigma), \Im(\sigma))| < \theta_3(\sigma) + \theta_4(\sigma)|\wp(\sigma)| + \theta_5(\sigma)|\Im(\sigma)|,$$

with $\sup_{\sigma\in J} \theta_3(\sigma) = \theta_3^*$, $\sup_{\sigma\in J} \theta_4(\sigma) = \theta_4^*$, $\sup_{\sigma\in J} \theta_5(\sigma) = \theta_5^*$.

Hypothesis 5. *For each* $\wp(\sigma) \in X'$, $\Im(\sigma) \in Y'$*, there exist constants* $A_5, A_6, A_7, A_8, N_1, N_2, N_3, N_4 > 0$ *such that the functions* $I_k, J_k, I_k^*, J_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and satisfy the inequalities:*

$$|I_k(\wp(\sigma_k))| \leqslant A_5|\wp(\sigma)| + N_1, \quad |J_k(\wp(\sigma_k))| \leqslant A_6|\wp(\sigma)| + N_2,$$
$$|I_k^*(\Im(\sigma_k))| \leqslant A_7|\Im(\sigma)| + N_3, \quad |J_k^*(\Im(\sigma_k))| \leqslant A_8|\Im(\sigma)| + N_4,$$

for k = 1, 2, . . . , p.

Let us define an operator $\mathcal{P} : X' \times Y' \longrightarrow X' \times Y'$ such that

$$\mathcal{P}(\wp, \Im)(\sigma) = (\mathcal{P}_1(\wp, \Im)(\sigma), \mathcal{P}_2(\wp, \Im)(\sigma)),$$

where

$$
\begin{aligned}
&\mathcal{P}_1(\wp,\Im)(\sigma) = \\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right).\frac{1}{\Delta}\left[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\right. \\
&\Im(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}\right. \\
&F_1(r,\wp(r),\Im(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r,\wp(r),\Im(r))dr - \frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr \\
&\left.+I_i(\wp(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\right) - a_1\int_{\sigma_k}^{\delta_{2k}}\left\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\right. \\
&\Im(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}\right. \\
&F_1(r,\wp(r),\Im(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r,\wp(r),\Im(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + I_i(\wp(\sigma_i)) \\
&\left.\left.+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\right)\right\}d\tau - a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\left\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr\right. \\
&-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\right. \\
&\Im(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2} \\
&\left.\left.\left.F_1(r,\wp(r),\Im(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + I_i(\wp(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\right)\right\}d\tau\right] \\
&+\sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)\right. \\
&-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r,\wp(r),\Im(r))dr \\
&\left.-\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + I_i(\wp(\sigma_i)) + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{P}_2(\wp,\Im)(\sigma) = \\
&\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+\Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg).\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r,\wp(r), \\
&\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_2-1}\Im(r)dr + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(\sigma))^{\gamma_2+\beta_2-1} \\
&F_2(r,\wp(r),\Im(r))dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\Im(\sigma_i))\Bigg) - a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_2+\beta_2-1}F_2(r, \\
&\wp(r),\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\Im(r)dr + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&-\psi(r))^{\gamma_2+\beta_2-1}F_2(r,\wp(r),\Im(r))dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\Im(\sigma_i))\Bigg)\Bigg\}d\tau - a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_2+\beta_2-1} \\
&F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\Im(r)dr + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&-\psi(r))^{\gamma_2+\beta_2-1}F_2(r,\wp(r),\Im(r))dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i)) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}J_i^*(\Im(\sigma_i))\Bigg)\Bigg\}d\tau\Bigg] + \sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1} \\
&F_2(r,\wp(r),\Im(r))dr + \frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}F_2(r,\wp(r),\Im(r))dr - \frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i)) + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i^*(\Im(\sigma_i))\Bigg).
\end{aligned}
$$

Our first result is stated as follows.

Theorem 1. *Assume that the conditions* (H_1) *and* (H_2) *are satisfied, and*

$$\Omega = \max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} < 1, \tag{8}$$

where the proof includes Ω_1, Ω_2, Ω_3, *and* Ω_4. *Then, (1) has a unique solution.*

Proof. Let $(\wp,\Im),(\bar{\wp},\bar{\Im})\in X^{'}\times Y^{'}$. Then,

$$
\begin{aligned}
&|\mathcal{P}_1(\wp,\Im)(\sigma)-\mathcal{P}_1(\bar{\wp},\bar{\Im})(\sigma)|\\
\leqslant\;& \frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma}\psi^{'}(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi^{'}(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)\\
&-\psi(\sigma_{i-1}))\Bigg)\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi^{'}(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))\\
&-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi^{'}(r)(\psi(\eta)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\sum_{i=1}^{p}\Bigg(\\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi^{'}(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)\\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi^{'}(\sigma_i)\Gamma(\beta_1-1)}\\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}|\wp(r)-\bar{\wp}(r)|dr+|I_i(\wp(\sigma_i))-I_i(\bar{\wp}(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi^{'}(\sigma_i)}\\
&|J_i(\wp(\sigma_i))-J_i(\bar{\wp}(\sigma_i))|\Bigg)+a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi^{'}(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\\
&\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi^{'}(r)(\psi(\tau)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\\
&\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi^{'}(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)\\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)\\
&-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+|I_i(\wp(\sigma_i))-I_i(\bar{\wp}(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi^{'}(\sigma_i)}|J_i(\wp(\sigma_i))-\\
&J_i(\bar{\wp}(\sigma_i))|\Bigg)\Bigg\}d\tau+a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi^{'}(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))\\
&-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi^{'}(r)(\psi(\tau)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\sum_{i=1}^{p}\Bigg(\\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi^{'}(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)\\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))-F_1(r,\bar{\wp}(r),\bar{\Im}(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi^{'}(r)\\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr+|I_i(\wp(\sigma_i))-I_i(\bar{\wp}(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi^{'}(\sigma_i)}|J_i(\wp(\sigma_i))-J_i(\bar{\wp}(\sigma_i))|\Bigg)\Bigg\}d\tau\Bigg]
\end{aligned}
$$

$$+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\mathfrak{I}(r))$$
$$-F_1(r,\bar{\wp}(r),\bar{\mathfrak{I}}(r))|ds+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)-\bar{\wp}(r)|dr$$
$$+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\mathfrak{I}(r))$$
$$-F_1(r,\bar{\wp}(r),\bar{\mathfrak{I}}(r))|dr+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}|\wp(r)-\bar{\wp}(r)|dr$$
$$+|I_i(\wp(\sigma_i))-I_i(\bar{\wp}(\sigma_i))|+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))-J_i(\bar{\wp}(\sigma_i))|\Bigg).$$

$$\begin{aligned}
&|\mathcal{P}_1(\wp,\mathfrak{I})(\sigma)-\mathcal{P}_1(\bar{\wp},\bar{\mathfrak{I}})(\sigma)|\\
\leqslant\;&\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||\\
&+\Big((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Big)\frac{1}{|\Delta|}\Bigg[\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\Big(M_{f_1}||\wp-\bar{\wp}||\\
&+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||\\
&+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)\\
&+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}||\wp-\bar{\wp}|+A_1||\wp-\bar{\wp}||+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\\
&||\wp-\bar{\wp}||\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)\\
&+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)\\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)||\wp-\bar{\wp}||\\
&+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)\\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}||\wp-\bar{\wp}||+A_1||\wp-\bar{\wp}||(\delta_{2k}-\sigma_k)+A_2\\
&\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}||\wp-\bar{\wp}||\Bigg)+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}\\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}\\
&||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\Big)
\end{aligned}$$

$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})||\wp-\bar{\wp}||+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}$$
$$\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\left(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\right)$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}||\wp-\bar{\wp}||$$
$$+A_1||\wp-\bar{\wp}||(\sigma_{k+1}-\delta_{2k+1})+\frac{A_2||\wp-\bar{\wp}||}{\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\bigg)\bigg]$$
$$+\sum_{i=1}^{p}\left(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\left(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\right)\right.$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}$$
$$\left(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\mathfrak{I}-\bar{\mathfrak{I}}||\right)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}||\wp-\bar{\wp}||$$
$$\left.+A_1||\wp-\bar{\wp}||+A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}||\wp-\bar{\wp}||\right).$$

$$|\mathcal{P}_1(\wp,\mathfrak{I})(\sigma)-\mathcal{P}_1(\bar{\wp},\bar{\mathfrak{I}})(\sigma)|$$
$$\leqslant\ \Bigg[M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)$$
$$-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{M_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\sum_{i=1}^{p}\Bigg(M_{f_1}$$
$$\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+M_{f_1}$$
$$\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}$$
$$+A_1+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)+a_1M_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}$$
$$+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)$$
$$+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}+A_1(\delta_{2k}-\sigma_k)+A_2\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)$$

$$
\begin{aligned}
&+a_2 M_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}\\
&+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\Bigg(M_{f_1}\\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})\\
&+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\\
&+A_1(\sigma_{k+1}-\delta_{2k+1})+\frac{A_2}{\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg\}\\
&+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}\\
&+M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}\\
&+A_1+A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)\Bigg]||\wp-\bar{\wp}||+\Bigg[N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))\\
&+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\\
&+N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)+a_1 N_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}\\
&+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)\\
&+N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Bigg)+a_2 N_{f_1}\\
&\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}+\sum_{i=1}^{p}\Bigg(N_{f_1}\\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg\}+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\\
&+N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)\Bigg]||\Im-\bar{\Im}||.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\Omega_1 = \\
&\Bigg[M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)} + \Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{M_{f_1} \\
&\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)} + \sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)} \\
&+M_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)} + \frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)} + A_1 \\
&+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg) + a_1 M_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)} + \frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)} + \sum_{i=1}^{p}\Bigg(M_{f_1} \\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k) \\
&+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)} + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)} \\
&+A_1(\delta_{2k}-\sigma_k) + A_2\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg) + a_2 M_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)} \\
&+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)} + \sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1}) \\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1}) + M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} \\
&-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\} + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\} \\
&+A_1(\sigma_{k+1}-\delta_{2k+1}) + \frac{A_2}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\} + \sum_{i=1}^{p}\Bigg(M_{f_1} \\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)} + M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)} \\
&+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)} + A_1 + A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_2 = \\
&\Bigg[N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \sum_{i=1}^{p}\Bigg(N_{f_1} \\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg) + a_1 N_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)} \\
&+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k) + N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Bigg) \\
&+a_2 N_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)} + \sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} \\
&(\sigma_{k+1}-\delta_{2k+1}) + N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\} \\
&+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)\Bigg].
\end{aligned}
$$

Then,

$$|\mathcal{P}_1(\wp,\mathfrak{I})(\sigma)-\mathcal{P}_1(\bar{\wp},\bar{\mathfrak{I}})(\sigma)|\leqslant\Omega_1||\wp-\bar{\wp}||+\Omega_2||\mathfrak{I}-\bar{\mathfrak{I}}||.$$

Similarly,

$$|\mathcal{P}_2(\wp,\mathfrak{I})(\sigma)-\mathcal{P}_2(\bar{\wp},\bar{\mathfrak{I}})(\sigma)|\leqslant\Omega_3||\wp-\bar{\wp}||+\Omega_4||\mathfrak{I}-\bar{\mathfrak{I}}||,$$

where

$$\begin{aligned}
&\Omega_3=\\
&\Bigg[M_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma)-\psi(\sigma_k))^{\beta_2}}{\Gamma(\beta_2+1)}+\bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\bigg)\frac{1}{|\Delta|}\bigg\{M_{f_2}\\
&\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_2+1)}+\sum_{i=1}^{p}\bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}\\
&+M_{f_2}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}+\frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)}+A_3\\
&+A_4\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\bigg)+a_1M_{f_2}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)}+\frac{\alpha_2(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\beta_2+2)}+\sum_{i=1}^{p}\bigg(M_{f_2}(\delta_{2k}-\sigma_k)\\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\delta_{2k}-\sigma_k)+M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}\\
&+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_2)}+A_3(\delta_{2k}-\sigma_k)+A_4\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\bigg)+a_2M_{f_2}\\
&\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)}\\
&+\frac{\alpha_2((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_2+1})}{\Gamma(\beta_2+2)}\\
&+\sum_{i=1}^{p}\bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})+M_{f_2}\\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\Gamma(\beta_2)\psi'(\sigma_i)}\\
&\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}+A_3(\sigma_{k+1}-\delta_{2k+1})+\frac{A_4}{\psi(\sigma_i)}\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}\\
&-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}\bigg)\bigg\}+\sum_{i=1}^{p}\bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}+M_{f_2}\\
&\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}+\frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)}\\
&+A_3+A_4\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\bigg)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
&\Omega_4 = \\
&\Bigg[N_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg) \\
&\times\frac{1}{|\Delta|}\Bigg\{N_{f_2}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \sum_{i=1}^{p}\Bigg(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} \\
&+N_{f_2}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\Bigg) + a_1 N_{f_2}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)} + \\
&\sum_{i=1}^{p}\Bigg(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\delta_{2k}-\sigma_k) + N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}\Bigg) + a_2 N_{f_2} \\
&\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)} + \sum_{i=1}^{p}\Bigg(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1}) \\
&+N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\} + \sum_{i=1}^{p}\Bigg(N_{f_2} \\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + N_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\Bigg)\Bigg].
\end{aligned}
$$

As it is assumed that

$$\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} = \Omega < 1.$$

we have

$$|\mathcal{P}(\wp, \mathfrak{I})(\sigma) - \mathcal{P}(\bar{\wp}, \bar{\mathfrak{I}})(\sigma)| \leqslant \Omega(||\wp - \bar{\wp}|| + ||\mathfrak{I} - \bar{\mathfrak{I}}||).$$

Then, from the above inequality we get that $\mathcal{P}$ is a contraction mapping and by the contraction principle $\mathcal{P}$ it has a unique fixed point. □

Theorem 2. *Assume that the conditions* (H_1)–(H_5) *are satisfied, then the coupled system (1) has at least one solution.*

Proof. To prove that the coupled system (1) has at least one solution, we use Schaefer's fixed point theorem. As F_1, I and J are continuous functions, so $\mathcal{P}_1$ is continuous. Furthermore, from the continuity of F_2, I^*, J^*, the operator $\mathcal{P}_2$ is continuous. This shows that $\mathcal{P}$ is continuous. Consider a set:

$$Q_s = \{(\wp, \mathfrak{I}) \in X' \times Y' : ||(\wp, \mathfrak{I})|| \leqslant s\}.$$

For any $\sigma \in [0, T]$, we have

$$
\begin{aligned}
&|\mathcal{P}_1(\wp,\Im)(\sigma)|\\
\leqslant\;& \frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1}\\
&|\wp(r)|dr+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right)\frac{1}{\Delta}\left[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}\right.\\
&|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)\right.\\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr\\
&+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}|\wp(r)|dr+|I_i(\wp(\sigma_i))|+\left.\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\right)+a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\\
&|\wp(r)|dr+\sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)\right.\\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+|I_i(\wp(\sigma_i))|+\left.\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\right)\Bigg\}d\tau+a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\\
&\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\\
&|\wp(r)|dr+\sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)\right.\\
&(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr\\
&+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+|I_i(\wp(\sigma_i))|+\left.\left.\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\right)\Bigg\}d\tau\right]+\sum_{i=1}^{p}\Bigg(\\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\\
&|\wp(r)|dr+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|ds\\
&+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}|\wp(r)|dr+|I_i(\wp(\sigma_i))|+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{P}_1(\wp,\Im)(\sigma)|\\
\leqslant\ & \frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|+\Bigg((\psi(\sigma)-\psi(\sigma_k))\\
&+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg[\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)\\
&+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)\\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0(r)+\theta_1(r)|\wp(r)|\\
&+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}|\wp(r)|+A_5|\wp(\sigma)|+N_1+\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\\
&(A_6|\wp(\sigma)|+N_2)\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)\\
&+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}|\wp(r)|+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)(\theta_0(r)+\theta_1(r)|\wp(r)|\\
&+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)|\wp(r)|+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\\
&(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}|\wp(r)|+(A_5|\wp(r)|\\
&+N_1)(\delta_{2k}-\sigma_k)+\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}(A_6|\wp(r)|+N_2)\Bigg)+a_2(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)\\
&\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}+\\
&\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}|\wp(r)|+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}\\
&-\delta_{2k+1})(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})|\wp(r)|\\
&+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)\\
&|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}|\wp(r)|+(A_5|\wp(r)|\\
&+N_1)(\sigma_{k+1}-\delta_{2k+1})+\frac{(A_6|\wp(r)|+N_2)}{(\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg]\\
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|\\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0(r)+\theta_1(r)|\wp(r)|+\theta_2(r)|\Im(r)|)\\
&+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}|\wp(r)|+(A_5|\wp(r)|+N_1)+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}(A_6|\wp(r)|+N_2)\Bigg).
\end{aligned}
$$

Let $|\wp(r)|\leqslant s_1$ and $|\Im(r)|\leqslant s_2$; then, we get

$$
\begin{aligned}
&|\mathcal{P}_1(\wp,\mathfrak{I})(\sigma)| \\
\leqslant\ & \frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2)+s_1\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i) \\
&-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg[\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2)+s_1\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)} \\
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2)+s_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+s_1\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_5 s_1+N_1 \\
&+(A_6 s_1+N_2)\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+r_1\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+r_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+r_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}+(A_5 s_1+N_1)(\delta_{2k}-\sigma_k)+(A_6 s_1+N_2) \\
&\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+r_1\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1} \\
&-\delta_{2k+1})(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2)+s_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2)+s_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}+(A_5 s_1+N_1)(\sigma_{k+1}-\delta_{2k+1})+\frac{(A_6 s_1+N_2)}{\psi(\sigma_i)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg]+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+s_1\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_1^* s_1+\theta_2^* s_2) \\
&+s_1\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+(A_5 s_1+N_1)+(A_6 s_1+N_2)\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)=\mathcal{F}_1.
\end{aligned}
$$

$$|\mathcal{P}_1(\wp,\mathfrak{I})(\sigma)| \leqslant \mathcal{F}_1.$$

Similarly,

$$|\mathcal{P}_2(\wp,\mathfrak{I})(\sigma)| \leqslant \mathcal{F}_2,$$

where

$$
\begin{aligned}
&\mathcal{F}_2 = \\
&\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\bigg((\psi(\sigma)-\psi(\sigma_k)) \\
&+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\bigg) \\
&\frac{1}{|\Delta|}\Bigg[\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} \\
&(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2) \\
&+s_2\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_7s_2+N_3+(A_8s_2+N_4)\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\bigg) \\
&+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)} \\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2) \\
&+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)} \\
&(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}+(A_7s_2+N_3)(\delta_{2k}-\sigma_k)+(A_8s_2+N_4) \\
&\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\bigg)+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2) \\
&+s_2\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1}) \\
&(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2)+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}+(A_7s_2+N_3)(\sigma_{k+1}-\delta_{2k+1})+\frac{(A_8s_2+N_4)}{\psi(\sigma_i)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\bigg)\Bigg]+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2) \\
&+s_2\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_3^*+\theta_4^*s_1+\theta_5^*s_2) \\
&+s_2\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+(A_7s_2+N_3)+(A_8s_2+N_4)\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\bigg).
\end{aligned}
$$

Now, let $\max\{\mathcal{F}_1,\mathcal{F}_2\}=\mathcal{F}$. Then, we have

$$||\mathcal{P}(\wp,\mathfrak{I})||_{X'\times Y'}\leqslant\mathcal{F}.$$

The aforementioned inequality indicates the boundedness of the operator $\mathcal{P}$. The operator $\mathcal{P}$ must then be demonstrated to be equicontinuous. For this, let ω_1, $\omega_2\in J_k$ such that $\omega_1<\omega_2$, where $k=0,1,\dots,p$.

Let $(\wp,\mathfrak{I})\in Q_s$; then, we have

$$
\begin{aligned}
& |\mathcal{P}_1(\wp,\Im)(\omega_2)-\mathcal{P}_1(\wp,\Im)(\omega_1)| \\
\leqslant\ & \frac{1}{\Gamma(\gamma_1+\beta_1)}\Big|\int_{\sigma_k}^{\omega_2}\psi'(r)(\psi(\omega_2)-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr-\int_{\sigma_k}^{\omega_1}\psi'(r)(\psi(\omega_1) \\
& -\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr\Big|+\frac{\alpha_1}{\Gamma(\beta_1)}\Big|\int_{\sigma_k}^{\omega_2}\psi'(r)(\psi(\omega_2)-\psi(r))^{\beta_1-1}\wp(r)dr- \\
& \int_{\sigma_k}^{\omega_1}\psi'(r)(\psi(\omega_1)-\psi(r))^{\beta_1-1}\wp(r)dr\Big|+\Big\{\Big((\psi(\omega_2)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Big) \\
& -\Big((\psi(\omega_1)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Big)\Big\}\frac{1}{|\Delta|}\Big[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta) \\
& -\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}|\wp(r)|dr \\
& +\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)} \\
& \int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
& -\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2} \\
& |\wp(r)|dr+|I_i(\wp(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\Big)+a_1\int_{\sigma_k}^{\delta_{2k}}\Big\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r) \\
& (\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1} \\
& |\wp(r)|dr+\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr \\
& +\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
& -\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr \\
& +|I_i(\wp(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\Big)\Big\}d\tau+a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Big\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau) \\
& -\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}|\wp(r)|dr \\
& +\sum_{i=1}^{p}\Big(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)} \\
& \int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
& -\psi(r))^{\gamma_1+\beta_1-2}|F_1(r,\wp(r),\Im(r))|dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}|\wp(r)|dr
\end{aligned}
$$

$$+|I_i(\wp(\sigma_i))|+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}|J_i(\wp(\sigma_i))|\Big)\Big\}d\tau\Big]+\Bigg|\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)$$
$$-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr-\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)$$
$$-\psi(r))^{\gamma_1+\beta_1-1}F_1(r,\wp(r),\Im(r))dr+\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr$$
$$-\frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr+\left(\frac{(\psi(\omega_2)-\psi(\sigma_i))-(\psi(\omega_1)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\right)$$
$$\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}F_1(r,\wp(r),\Im(r))dr+\frac{\alpha_1}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\Big((\psi(\omega_2)$$
$$-\psi(\sigma_i))-(\psi(\omega_1)-\psi(\sigma_i))\Big)\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr+I_i(\wp(\sigma_i))$$
$$-I_i(\wp(\sigma_i))+\left(\frac{(\psi(\omega_2)-\psi(\sigma_i))-(\psi(\omega_1)-\psi(\sigma_i))}{\psi'(\sigma_i)}J_i(\wp(\sigma_i))\right)\Bigg|.$$

From above inequality, if $\omega_1\to\omega_2$, we can deduce that

$$|\mathcal{P}_1(\wp,\Im)(\omega_2)-\mathcal{P}_1(\wp,\Im)(\omega_1)|\longrightarrow 0.$$

In the same way, we can prove that

$$|\mathcal{P}_2(\wp,\Im)(\omega_2)-\mathcal{P}_2(\wp,\Im)(\omega_1)|\longrightarrow 0.$$

Therefore, $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely continuous according to the Arzila–Ascoli theorem. Thus, $\mathcal{P}$ is completely continuous.

Now, let us define a set:

$$G=\{(\wp,\Im)\in X'\times Y';(\wp,\Im)=\lambda\mathcal{P}(\wp,\Im);0<\lambda<1\}.$$

We need to prove that the set G is bounded. For $\sigma\in\mathcal{J}$ and $(\wp,\Im)\in G$, then $(\wp,\Im)=\lambda\mathcal{P}(\wp,\Im)$, i.e.,

$\wp(\sigma)=\lambda\mathcal{P}_1(\wp,\Im)$ and $\Im(\sigma)=\lambda\mathcal{P}_2(\wp,\Im)$. Now,

$$|\wp(\sigma)|=|\lambda\mathcal{P}_1(\wp,\Im)|.$$

$$\begin{aligned}
&|\wp(\sigma)|\\
\leqslant\ &\lambda\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^*|\wp(r)|+\theta_2^*|\Im(r)|)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|+\Bigg((\psi(\sigma)-\psi(\sigma_k))\\
&+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^*|\wp(r)|+\theta_2^*|\Im(r)|)+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|\\
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_1^*|\wp(r)|+\theta_2^*|\Im(r)|)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}|\wp(r)|\\
&+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_1^*|\wp(r)|+\theta_2^*|\Im(r)|)+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}\\
&|\wp(r)|+A_5|\wp(\sigma)|+N_1+\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}(A_6|\wp(\sigma)|+N_2)\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}(\theta_0^*+\theta_1^*|\wp(r)|\\
&+\theta_2^*|\Im(r)|)+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}|\wp(r)|+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)(\theta_0^*+\theta_1^*|\wp(r)|
\end{aligned}$$

$$
\begin{aligned}
&+\theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)\,|\,\wp(r)\,|\, + \frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}(\theta_0^* \\
&+\theta_1^* \,|\,\wp(r)\,|\, + \theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}\,|\,\wp(r)\,|\, + (A_5\,|\,\wp(r)\,|\, + N_1)(\delta_{2k}-\sigma_k) + \\
&\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}(A_6\,|\,\wp(r)\,|\, + N_2)\bigg) + a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)} \\
&(\theta_0^* + \theta_1^* \,|\,\wp(r)\,|\, + \theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}\,|\,\wp(r)\,| \\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})(\theta_0^* + \theta_1^* \,|\,\wp(r)\,|\, + \theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1} \\
&-\delta_{2k+1})\,|\,\wp(r)\,|\, + \frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}(\theta_0^* + \theta_1^* \,|\,\wp(r)\,| \\
&+\theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}\,|\,\wp(r)\,|\, + (A_5\,|\,\wp(r)\,| \\
&+N_1)(\sigma_{k+1}-\delta_{2k+1}) + \frac{(A_6\,|\,\wp(r)\,|\, + N_2)}{\psi(\sigma_i)}\bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\bigg\}\bigg)\bigg\} \\
&+\sum_{i=1}^{p}\bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^* + \theta_1^* \,|\,\wp(r)\,|\, + \theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}\,|\,\wp(r)\,|\, + \\
&\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0^* + \theta_1^* \,|\,\wp(r)\,|\, + \theta_2^* \,|\,\Im(r)\,|\,) + \frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)} \\
&|\,\wp(r)\,|\, + (A_5\,|\,\wp(r)\,|\, + N_1) + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}(A_6\,|\,\wp(r)\,|\, + N_2)\bigg).
\end{aligned}
$$

Let

$$
\begin{aligned}
&\mathcal{M}^* = \\
&\lambda\bigg[\theta_1^*\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)} + \bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\bigg) \\
&\frac{1}{|\Delta|}\bigg\{\theta_1^*\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)} + \sum_{i=1}^{p}\bigg(\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} \\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)} + \theta_1^*\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)} \\
&+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)} + A_5 + A_6\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\bigg) + a_1\theta_1^*\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)} \\
&+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)} + \sum_{i=1}^{p}\bigg(\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k) + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k) \\
&+\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)} + \frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)} + A_5(\delta_{2k}-\sigma_k) \\
&+A_6\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\bigg) + a_2\theta_1^*\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1} - (\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}
\end{aligned}
$$

$$+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\Bigg(\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}$$
$$-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}$$
$$+A_5(\sigma_{k+1}-\delta_{2k+1})+\frac{A_6}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}$$
$$+\sum_{i=1}^{p}\Bigg(\theta_1^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+\theta_1^*\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}$$
$$+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_5+A_6\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg).$$

$$|\wp(\sigma)|$$
$$\leqslant\frac{\lambda}{1-\mathcal{M}^*}\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_2^*|\Im(r)|)+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)$$
$$\frac{1}{|\Delta|}\Bigg\{\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_2^*|\Im(r)|)+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_2^*|\Im(r)|)$$
$$+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_2^*|\Im(r)|)+N_1+N_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)$$
$$+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}(\theta_0^*+\theta_2^*|\Im(r)|)+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)(\theta_0^*+\theta_2^*|\Im(r)|)$$
$$+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_2^*|\Im(r)|)+N_1(\delta_{2k}-\sigma_k)+N_2\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)$$
$$+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}(\theta_0^*+\theta_2^*|\Im(r)|)+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}$$
$$(\sigma_{k+1}-\delta_{2k+1})(\theta_0^*+\theta_2^*|\Im(r)|)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}$$
$$(\theta_0^*+\theta_2^*|\Im(r)|)+N_1(\sigma_{k+1}-\delta_{2k+1})+\frac{N_2}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\theta_0^*+\theta_2^*|\Im(r)|)+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}(\theta_0^*+\theta_2^*|\Im(r)|)$$
$$+N_1+N_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)=\mathcal{Z}_1.$$

Thus, there exists a positive constant $\mathcal{Z}_1$, such that $||\wp||_{\chi'}\leqslant\mathcal{Z}_1$.

Similarly, there exists $\mathcal{Z}_2$ such that $||\Im||_{Y'} \leqslant \mathcal{Z}_2$, where

$$\mathcal{Z}_2 =$$
$$\frac{\lambda}{1-\mathcal{N}^*}\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\theta_3^*+\theta_4^*|\wp(r)|)+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)$$
$$\times\frac{1}{|\Delta|}\Bigg\{(\theta_3^*+\theta_4^*|\wp(r)|)\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\theta_3^*+\theta_4^*|\wp(r)|)+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}$$
$$(\theta_3^*+\theta_4^*|\wp(r)|)+N_3+N_4\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)}(\theta_3^*+\theta_4^*|\wp(r)|)$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\delta_{2k}-\sigma_k)(\theta_3^*+\theta_4^*|\wp(r)|)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}(\theta_3^*+\theta_4^*$$
$$|\wp(r)|)+N_3(\delta_{2k}-\sigma_k)+N_4\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)}$$
$$(\theta_3^*+\theta_4^*|\wp(r)|)+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})(\theta_3^*+\theta_4^*|\wp(r)|)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}$$
$$\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}(\theta_3^*+\theta_4^*|\wp(r)|)+N_3(\sigma_{k+1}-\delta_{2k+1})+\frac{N_4}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}$$
$$-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\theta_3^*+\theta_4^*|\wp(r)|)+(\theta_3^*+\theta_4^*|\wp(r)|)$$
$$\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}+N_3+N_4\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg).$$

and

$$\mathcal{N}^* =$$
$$\lambda\Bigg[\theta_5^*\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma)-\psi(\sigma_k))^{\beta_2}}{\Gamma(\beta_2+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{\theta_5^*$$
$$\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\eta)-\psi(\sigma_k))^{\beta_2}}{\Gamma(\beta_2+1)}+\sum_{i=1}^{p}\Bigg(\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}$$
$$+\theta_5^*\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}+\frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)}+A_7+A_8$$
$$\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)+a_1\theta_5^*\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)}+\frac{\alpha_2(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\beta_2+2)}+\sum_{i=1}^{p}\Bigg(\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}$$
$$(\delta_{2k}-\sigma_k)+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\delta_{2k}-\sigma_k)+\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}$$

$$+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_2)}+A_7(\delta_{2k}-\sigma_k)+A_8\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)$$
$$+a_2\theta_5^*\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)}$$
$$+\frac{\alpha_2((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_2+1})}{\Gamma(\beta_2+2)}+\sum_{i=1}^{p}\Bigg(\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})$$
$$+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})+\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}$$
$$-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\Gamma(\beta_2)\psi'(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}$$
$$+A_7(\sigma_{k+1}-\delta_{2k+1})+\frac{A_8}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}$$
$$+\sum_{i=1}^{p}\Bigg(\theta_5^*\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}+\theta_5^*\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}$$
$$+\frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)}+A_7+A_8\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg).$$

Let $\max\{\mathcal{Z}_1,\mathcal{Z}_2\}=\mathcal{Z}$. Then, we have $||(\wp,\mathfrak{I})||_{X'\times Y'}\leqslant\mathcal{Z}$.

Thus, the set G is bounded, and the operator $\mathcal{P}$ has at least one fixed point according to Schaefer's fixed point theorem., i.e., the problem (1) has at least one solution. □

5. Ulam's Stability Results

This section is concerned with the Ulam–Hyers stability of problem (1). First, we present some definitions introduced in [25].

Definition 4. *For* $\epsilon=\max\{\epsilon_1,\epsilon_2\}$, *consider the system of inequalities*

$$\begin{cases} |{}^c\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^c\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma}+\alpha_1)\wp(\sigma)-F_1(\sigma,\wp(\sigma),\mathfrak{I}(\sigma))|<\epsilon_1,\ \sigma\in(\sigma_k,\sigma_{k+1}],\\ |\Delta\wp(\sigma_k)-I_k(\wp(\sigma_k))|<\epsilon_1,\quad k=1,2,\ldots,p,\\ |\Delta\wp'(\sigma_k)-J_k(\wp(\sigma_k))|<\epsilon_1.\\ |{}^c\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^c\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma}+\alpha_2)\mathfrak{I}(\sigma)-F_2(\sigma,\wp(\sigma),\mathfrak{I}(\sigma))|<\epsilon_2,\ \sigma\in(\sigma_k,\sigma_{k+1}],\\ |\Delta\mathfrak{I}(\sigma_k)-I_k^*(\mathfrak{I}(\sigma_k))|<\epsilon_1,\quad k=1,2,\ldots,p,\\ |\Delta\mathfrak{I}'(\sigma_k)-J_k^*(\mathfrak{I}(\sigma_k))|<\epsilon_1. \end{cases}\tag{9}$$

The system (1) is called Ulam–Hyers stable if we can find $\vartheta>0$ *such that, for each solution* $(\bar{\wp},\bar{\mathfrak{I}})\in X'\times Y'$ *of (9), there exists a solution* $(\wp,\mathfrak{I})\in X'\times Y'$ *of the system (1) satisfying* $||(\bar{\wp},\bar{\mathfrak{I}})-(\wp,\mathfrak{I})||\leqslant\vartheta\epsilon$.

Remark 1. $(\bar{\wp},\bar{\mathfrak{I}})\in X'\times Y'$ *is a solution of the system of inequalities (9) if and only if there exist functions* $\phi,\phi^*\in C((\sigma_k,\sigma_{k+1}],\mathbb{R})$ *such that* $|\phi|\leqslant\epsilon_1$ *and* $|\phi^*|\leqslant\epsilon_2$, $\sigma\in(\sigma_k,\sigma_{k+1}]$ *and*

$$\begin{cases} {}^c\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^c\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma}+\alpha_1)\wp(\sigma)=F_1(\sigma,\wp(\sigma),\mathfrak{I}(\sigma))+\phi(\sigma),\ \sigma\in(\sigma_k,\sigma_{k+1}],k=0,1,\ldots,p,\\ \Delta\wp(\sigma_k)=I_k(\wp(\sigma_k))+\phi_k(\sigma),\quad k=1,2,\ldots,p,\\ \Delta\wp'(\sigma_k)=J_k(\wp(\sigma_k))+\phi_k(\sigma).\\ {}^c\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^c\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma}+\alpha_2)\mathfrak{I}(\sigma)=F_2(\sigma,\wp(\sigma),\mathfrak{I}(\sigma))+\phi^*(\sigma),\ \sigma\in(\sigma_k,\sigma_{k+1}],k=0,1,\ldots,p,\\ \Delta\mathfrak{I}(\sigma_k)=I_k^*(\mathfrak{I}(\sigma_k))+\phi_k^*(\sigma),\quad k=1,2,\ldots,p,\\ \Delta y'(\sigma_k)=J_k^*(\mathfrak{I}(\sigma_k))+\phi_k^*(\sigma). \end{cases}\tag{10}$$

Theorem 3. *System ((1)) is Ulam–Hyers stable if* (H_1) *and* (H_2) *are met.*

Proof. Suppose that $(\bar{\wp}, \bar{\Im}) \in X' \times Y'$ is the solution of the following inequality:

$$\begin{cases} |{}^{c}\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma} + \alpha_1)\wp(\sigma) - F_1(\sigma, \wp(\sigma), \Im(\sigma))| < \epsilon_1, \ \sigma \in (\sigma_k, \sigma_{k+1}], \\ |\Delta\wp(\sigma_k) - I_k(\wp(\sigma_k))| < \epsilon_1, \quad k = 1,2,\dots,p, \\ |\Delta\wp'(\sigma_k) - J_k(\wp(\sigma_k))| < \epsilon_1. \\ |{}^{c}\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma} + \alpha_2)\Im(\sigma) - F_2(\sigma, \wp(\sigma), \Im(\sigma))| < \epsilon_2, \ \sigma \in (\sigma_k, \sigma_{k+1}], \\ |\Delta\Im(\sigma_k) - I^*_k(\Im(\sigma_k))| < \epsilon_1, \quad k = 1,2,\dots,p, \\ |\Delta\Im'(\sigma_k) - J^*_k(\Im(\sigma_k))| < \epsilon_1. \end{cases} \tag{11}$$

From inequality (11)

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma_1;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_1;\psi}_{\sigma_k,\sigma} + \alpha_1)\wp(\sigma) = F_1(\sigma, \wp(\sigma), \Im(\sigma)) + \phi(\sigma), \ \sigma \in (\sigma_k, \sigma_{k+1}], k = 0,1,\dots,p, \\ \Delta\wp(\sigma_k) = I_k(\wp(\sigma_k)) + \phi_k(\sigma), \quad k = 1,2,\dots,p, \\ \Delta\wp'(\sigma_k) = J_k(\wp(\sigma_k)) + \phi_k(\sigma). \\ {}^{c}\mathcal{D}^{\gamma_2;\psi}_{\sigma_k,\sigma}({}^{c}\mathcal{D}^{\beta_2;\psi}_{\sigma_k,\sigma} + \alpha_2)\Im(\sigma) = F_2(\sigma, \wp(\sigma), \Im(\sigma)) + \phi^*(\sigma), \ \sigma \in (\sigma_k, \sigma_{k+1}], k = 0,1,\dots,p, \\ \Delta\Im(\sigma_k) = I^*_k(\Im(\sigma_k)) + \phi^*_k(\sigma), \quad k = 1,2,\dots,p, \\ \Delta\Im'(\sigma_k) = J^*_k(\Im(\sigma_k)) + \phi^*_k(\sigma). \end{cases} \tag{12}$$

Using Lemma 1, we get:

$$\begin{aligned}
&\bar{\wp}(\sigma) = \\
&\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\sigma} \psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_1-1} \\
&\wp(r)dr + \left((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right).\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r), \\
&\Im(r)) + \phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}\right. \\
&(F_1(r,\wp(r),\Im(r))+\phi(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr \\
&\left.+I_i(\wp(\sigma_i)) + \phi_i(\sigma_i) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i(\wp(\sigma_i))+\phi_i(\sigma_i))\right) - a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_1+\beta_1-1} \\
&(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)\right. \\
&-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r),\Im(r))+\phi(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr + \frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&\left.+I_i(\wp(\sigma_i)) + \phi_i(\sigma_i) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i(\wp(\sigma_i))+\phi_i(\sigma_i))\right)\Bigg\}d\tau - a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_1+\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau) \\
&-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_1-1}\wp(r)dr + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\right. \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r),\Im(r))+\phi(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&\left.-\psi(r))^{\beta_1-1}\wp(r)dr + I_i(\wp(\sigma_i)) + \phi_i(\sigma_i) + \frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i(\wp(\sigma_i))+\phi_i(\sigma_i))\right)\Bigg\}d\tau\Bigg] + \sum_{i=1}^{p}\left(\frac{1}{\Gamma(\gamma_1+\beta_1)}\right. \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-1}(F_1(r,\wp(r),\Im(r))+\phi(r))dr + \frac{\alpha_1}{\Gamma(\beta_1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-1}\wp(r)dr \\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_1+\beta_1-2}(F_1(r,\wp(r),\Im(r))+\phi(r))dr - \frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_1-1)} \\
&\left.\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_1-2}\wp(r)dr + I_i(\wp(\sigma_i)) + \phi_i(\sigma_i) + \frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}(J_i(\wp(\sigma_i))+\phi_i(\sigma_i))\right),
\end{aligned}$$

$$
\begin{aligned}
&\bar{\Im}(\sigma) = \\
&\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\sigma}\psi'(r)(\psi(\sigma)-\psi(r))^{\beta_2-1} \\
&\Im(r)dr+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right).\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r), \\
&\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\eta}\psi'(r)(\psi(\eta)-\psi(r))^{\beta_2-1}\Im(r)dr+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1} \\
&(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr+\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r) \\
&(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i)+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i))\Bigg)-a_1\int_{\sigma_k}^{\delta_{2k}}\Bigg\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\gamma_2+\beta_2-1} \\
&(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\Im(r)dr+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr+\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+I_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i)+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i))\Bigg)\Bigg\}d\tau-a_2\int_{\delta_{2k+}}^{\sigma_{k+1}}\Bigg\{\frac{1}{\Gamma(\gamma_2+\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau) \\
&-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_k}^{\tau}\psi'(r)(\psi(\tau)-\psi(r))^{\beta_2-1}\Im(r)dr+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr+\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+\frac{(\psi(\tau)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i) \\
&-\psi(r))^{\beta_2-1}\Im(r)dr+I_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i)+\frac{\psi(\eta)-\psi(\sigma_i)}{\psi'(\sigma_i)}(J_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i))\Bigg)\Bigg\}d\tau\Bigg]+\sum_{i=1}^{p}\Bigg(\frac{1}{\Gamma(\gamma_2+\beta_2)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-1}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr+\frac{\alpha_2}{\Gamma(\beta_2)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-1}\Im(r)dr \\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2-1)}\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\gamma_2+\beta_2-2}(F_2(r,\wp(r),\Im(r))+\phi^*(r))dr-\frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)\Gamma(\beta_2-1)} \\
&\int_{\sigma_{i-1}}^{\sigma_i}\psi'(r)(\psi(\sigma_i)-\psi(r))^{\beta_2-2}\Im(r)dr+I_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i)+\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}(J_i^*(\Im(\sigma_i))+\phi_i^*(\sigma_i))\Bigg).
\end{aligned}
$$

Now, as $(\wp, \Im)$ is the solution of (1) and $(\bar{\wp}, \bar{\Im})$ is the solution of (11). Then,

$$
\begin{aligned}
&|\wp(\sigma)-\bar{\wp}(\sigma)| \\
\leqslant\ & \frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}|| \\
&+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg[\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\Big(M_{f_1}||\wp-\bar{\wp}|| \\
&+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} \\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}|| \\
&+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big) \\
&+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}||\wp-\bar{\wp}|+A_1||\wp-\bar{\wp}||+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)} \\
&||\wp-\bar{\wp}||\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big) \\
&+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k) \\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)||\wp-\bar{\wp}|| \\
&+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big) \\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}||\wp-\bar{\wp}||+A_1||\wp-\bar{\wp}||(\delta_{2k}-\sigma_k)+A_2 \\
&\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}||\wp-\bar{\wp}||\Bigg)+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)} \\
&\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)} \\
&||\wp-\bar{\wp}||+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big) \\
&+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})||\wp-\bar{\wp}||+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)} \\
&\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)
\end{aligned}
$$

$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}||\wp-\bar{\wp}||$$
$$+A_1||\wp-\bar{\wp}||(\sigma_{k+1}-\delta_{2k+1})+\frac{A_2||\wp-\bar{\wp}||}{\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg]$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}||\wp-\bar{\wp}||+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}$$
$$\Big(M_{f_1}||\wp-\bar{\wp}||+N_{f_1}||\Im-\bar{\Im}||+|\phi(r)|\Big)+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}||\wp-\bar{\wp}||$$
$$+A_1||\wp-\bar{\wp}||+A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}||\wp-\bar{\wp}||\Bigg).$$

$$|\wp(\sigma)-\bar{\wp}(\sigma)|$$
$$\leqslant\Bigg[M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{$$
$$M_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}$$
$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+M_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+A_1+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}$$
$$+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}\Bigg)+a_1M_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}$$
$$+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)$$
$$+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}$$
$$+A_1(\delta_{2k}-\sigma_k)+A_2\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg)+a_2M_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}$$
$$+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}$$
$$-\delta_{2k+1})+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}$$
$$-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}$$
$$+A_1(\sigma_{k+1}-\delta_{2k+1})+\frac{A_2}{\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg)\Bigg\}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\Bigg(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}\\
&+M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_1\\
&+A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)\Bigg]||x-\bar{x}||+\Bigg[N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\\
&\frac{1}{|\Delta|}\Bigg\{N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\\
&+N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)+a_1N_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}\\
&+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Bigg)\\
&+a_2N_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\\
&(\sigma_{k+1}-\delta_{2k+1})+N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}\\
&+\sum_{i=1}^{p}\Bigg(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)\Bigg]||y-\bar{y}||\\
&+\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\Bigg((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)+a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}\\
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Bigg)\\
&+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}\\
&-\delta_{2k+1})+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\}\\
&+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)\Bigg]\epsilon_1.
\end{aligned}
$$

Let

$$\omega_1 =$$

$$\Bigg[M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right)\frac{1}{|\Delta|}\Bigg\{M_{f_1}$$

$$\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_k))^{\beta_1}}{\Gamma(\beta_1+1)}+\sum_{i=1}^{p}\left(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}\right.$$

$$+M_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_1$$

$$\left.+A_2\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\right)+a_1M_{f_1}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}+\frac{\alpha_1(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\left(M_{f_1}\right.$$

$$\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\delta_{2k}-\sigma_k)$$

$$+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_1)}+A_1(\delta_{2k}-\sigma_k)$$

$$\left.+A_2\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\right)+a_2M_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}$$

$$+\frac{\alpha_1((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_1+1})}{\Gamma(\beta_1+2)}+\sum_{i=1}^{p}\left(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})\right.$$

$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}\right.$$

$$\left.-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\Gamma(\beta_1)\psi'(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}+A_1$$

$$\left.(\sigma_{k+1}-\delta_{2k+1})+\frac{A_2}{\psi(\sigma_i)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\right)\Bigg\}+\sum_{i=1}^{p}\left(M_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}\right.$$

$$+\frac{\alpha_1(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1}}{\Gamma(\beta_1+1)}+M_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}$$

$$\left.+\frac{\alpha_1(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_1-1}}{\psi'(\sigma_i)\Gamma(\beta_1)}+A_1+A_2\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\right)\Bigg],$$

$$\omega_2 =$$

$$\Bigg[N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+\left((\psi(\sigma)-\psi(\sigma_k))+\sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right)\frac{1}{|\Delta|}\Bigg\{N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}$$

$$+\sum_{i=1}^{p}\left(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+N_{f_1}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\right)+a_1N_{f_1}$$

$$\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}+\sum_{i=1}^{p}\left(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k)+N_{f_1}\right.$$

$$\left.\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\right)+a_2N_{f_1}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)}$$

$$+\sum_{i=1}^{p}\left(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})+N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}-\right.\right.$$

$$\left.\left.\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\right)\Bigg\}+\sum_{i=1}^{p}\left(N_{f_1}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}+N_{f_1}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\right)\Bigg],$$

$$\omega_3 =$$
$$\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)} + \frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg) + a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1+2)}$$
$$+\sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\delta_{2k}-\sigma_k) + \frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_1+\beta_1)}\Bigg)$$
$$+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_1+\beta_1+1}}{\Gamma(\gamma_1+\beta_1+2)} + \sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}(\sigma_{k+1}-\delta_{2k+1})$$
$$+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\Gamma(\gamma_1+\beta_1)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\} + \sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1}}{\Gamma(\gamma_1+\beta_1+1)}$$
$$+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_1+\beta_1-1}}{\psi'(\sigma_i)\Gamma(\gamma_1+\beta_1)}\Bigg)\Bigg].$$

Then,

$$||\wp-\bar{\wp}|| \leqslant \omega_1||\wp-\bar{\wp}|| + \omega_2||\Im-\bar{\Im}|| + \omega_3\epsilon_1.$$

Similarly,

$$||\Im-\bar{\Im}|| \leqslant \omega_4||\wp-\bar{\wp}|| + \omega_5||\Im-\bar{\Im}|| + \omega_6\epsilon_2,$$

where

$$\omega_4 =$$
$$\Bigg[M_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \frac{\alpha_2(\psi(\sigma)-\psi(\sigma_k))^{\beta_2}}{\Gamma(\beta_2+1)} + \Bigg((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\Bigg)\frac{1}{|\Delta|}\Bigg\{M_{f_2}$$
$$\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \frac{\alpha_2(\psi(\eta)-\psi(\sigma_k))^{\beta_2}}{\Gamma(\beta_2+1)} + \sum_{i=1}^{p}\Bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}$$
$$+M_{f_2}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)} + \frac{\alpha_2(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)} + A_3$$
$$+A_4\frac{(\psi(\eta)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg) + a_1M_{f_2}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)} + \frac{\alpha_2(\psi(\delta_{2k})-\psi(\sigma_k))^{\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\beta_2+2)} + \sum_{i=1}^{p}\Bigg(M_{f_2}$$
$$\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\delta_{2k}-\sigma_k) + \frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\delta_{2k}-\sigma_k)$$
$$+M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)} + \frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\beta_2)}$$
$$+A_3(\delta_{2k}-\sigma_k) + A_4\frac{(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})}\Bigg) + a_2M_{f_2}\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)}$$
$$+\frac{\alpha_2((\psi(\sigma_{k+1})-\psi(\sigma_k))^{\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\beta_2+1})}{\Gamma(\beta_2+2)} + \sum_{i=1}^{p}\Bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1})$$
$$+\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1}) + M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})}$$
$$-\frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\} + \frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\Gamma(\beta_2)\psi'(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\} + A_3(\sigma_{k+1}$$
$$-\delta_{2k+1}) + \frac{A_4}{\psi(\sigma_i)}\Bigg\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\Bigg\}\Bigg)\Bigg\} + \sum_{i=1}^{p}\Bigg(M_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} +$$
$$\frac{\alpha_2(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2}}{\Gamma(\beta_2+1)} + M_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)} + \frac{\alpha_2(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\beta_2-1}}{\psi'(\sigma_i)\Gamma(\beta_2)}$$
$$+A_3 + A_4\frac{(\psi(\sigma)-\psi(\sigma_i))}{\psi'(\sigma_i)}\Bigg)\Bigg],$$

$$
\begin{aligned}
&\omega_5 = \\
&\Bigg[N_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \left((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right)\frac{1}{|\Delta|}\Bigg\{N_{f_2}\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \sum_{i=1}^{p}\Bigg(N_{f_2} \\
&\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + N_{f_2}\frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\Bigg) + a_1N_{f_2}\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)} \\
&+\sum_{i=1}^{p}\left(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\delta_{2k}-\sigma_k) + N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_1-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}\right) + a_2N_{f_2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)} + \sum_{i=1}^{p}\Bigg(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1}) \\
&+N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg\} \\
&+\sum_{i=1}^{p}\left(N_{f_2}\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + N_{f_2}\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\omega_6 = \\
&\Bigg[\frac{(\psi(\sigma)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \left((\psi(\sigma)-\psi(\sigma_k)) + \sum_{i=1}^{p}(\psi(\sigma_i)-\psi(\sigma_{i-1}))\right)\frac{1}{|\Delta|}\Bigg\{\frac{(\psi(\eta)-\psi(\sigma_k))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} \\
&+\sum_{i=1}^{p}\left(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} + \frac{(\psi(\eta)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\right) + a_1\frac{(\psi(\delta_{2k})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2+2)} \\
&+\sum_{i=1}^{p}\left(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\delta_{2k}-\sigma_k) + \frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}(\psi(\delta_{2k})-\psi(\sigma_i))^2}{\psi'(\sigma_i)\psi'(\delta_{2k})\Gamma(\gamma_2+\beta_2)}\right) \\
&+a_2\frac{(\psi(\sigma_{k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}-(\psi(\delta_{2k+1})-\psi(\sigma_k))^{\gamma_2+\beta_2+1}}{\Gamma(\gamma_2+\beta_2+2)} + \sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)}(\sigma_{k+1}-\delta_{2k+1}) \\
&+\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\Gamma(\gamma_2+\beta_2)}\left\{\frac{(\psi(\sigma_{k+1})-\psi(\sigma_i))^2}{\psi'(\sigma_{k+1})} - \frac{(\psi(\delta_{2k+1})-\psi(\sigma_i))^2}{\psi'(\delta_{2k+1})}\right\}\Bigg)\Bigg\} + \sum_{i=1}^{p}\Bigg(\frac{(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2}}{\Gamma(\gamma_2+\beta_2+1)} \\
&+\frac{(\psi(\sigma)-\psi(\sigma_i))(\psi(\sigma_i)-\psi(\sigma_{i-1}))^{\gamma_2+\beta_2-1}}{\psi'(\sigma_i)\Gamma(\gamma_2+\beta_2)}\Bigg)\Bigg].
\end{aligned}
$$

Now, let $\max\{\epsilon_1, \epsilon_2\} = \epsilon$ and $\max\{\omega_1, \omega_2, \omega_4, \omega_5\} = \omega$. Then,

$$
||\wp - \bar{\wp}|| + ||\Im - \bar{\Im}|| \leqslant \omega(||\wp - \bar{\wp}|| + ||\Im - \bar{\Im}||) + (\omega_3 + \omega_6)\epsilon.
$$

$$
||\wp - \bar{\wp}|| + ||\Im - \bar{\Im}|| \leqslant \frac{\omega_3 + \omega_6}{1-\omega}\epsilon.
$$

Let

$$
\vartheta = \frac{\omega_3 + \omega_6}{1-\omega}.
$$

Hence, we have

$$
||(\wp, \Im) - (\bar{\wp}, \bar{\Im})||_{X' \times Y'} \leqslant \vartheta\epsilon.
$$

Thus, system (1) is UH stable. □

6. Example

Example 1.

$$
\begin{aligned}
{}^{c}\mathcal{D}^{\frac{5}{3};\wp}({}^{c}\mathcal{D}^{\frac{1}{3};\wp}+\frac{1}{130})\wp(\sigma) &= \frac{\wp}{220(1+\Im(\sigma))},\ \sigma\in[0,1],\sigma\neq\frac{2}{3},\\
{}^{c}\mathcal{D}^{\frac{7}{5};\Im}({}^{c}\mathcal{D}^{\frac{1}{3};\Im}+\frac{1}{310})\Im(\sigma) &= \frac{\cos(\sigma)\Im}{330(1+\wp)},\ \sigma\in[0,1],\sigma\neq\frac{2}{3},\\
I_1\wp(\frac{2}{3}) &= \frac{\wp(\sigma)}{150+(\,|\,\wp(\sigma)\,|\,)},\ \ J_1\wp(\frac{2}{3})=\frac{\wp(\sigma)}{160+(\,|\,\wp(\sigma)\,|\,)},\\
I_1^{*}\Im(\frac{2}{3}) &= \frac{1}{170+\,|\,\Im(\sigma)\,|\,},\ \ J_1^{*}\Im(\frac{2}{3})=\frac{1}{180+\,|\,\Im(\sigma)\,|\,},\\
\wp(0)=0,\ \ \wp(\frac{9}{10}) &= \frac{1}{2}\int_0^{\frac{2}{5}}\wp(\tau)d\tau+\frac{9}{10}\int_{\frac{3}{5}}^{\frac{7}{10}}\wp(\tau)d\tau,\\
\Im(0)=0,\ \ \Im(\frac{1}{2}) &= \frac{9}{10}\int_0^{\frac{2}{5}}\Im(\tau)d\tau+\frac{9}{10}\int_{\frac{3}{5}}^{\frac{7}{10}}\Im(\tau)d\tau.
\end{aligned}
$$

We see in the proposed problem that $\beta_1=\beta_2=\frac{1}{3}$, $\gamma_1=\frac{5}{3}$, $\gamma_2=\frac{7}{5}$, $\eta=\frac{1}{2}$ *and* $a_1=a_2=\frac{9}{10}$.

$$
\begin{aligned}
|\,F_1(\sigma,\wp(\sigma),\Im(\sigma))-F_1(\sigma,\bar{\wp}(\sigma),\bar{\Im}(\sigma))\,| &\leqslant \frac{1}{220}\,|\,\wp(\sigma)-\bar{\wp}(\sigma)\,|+\frac{1}{220}\,|\,\Im(\sigma)-\bar{\Im}(\sigma)\,|,\ \ \forall\sigma\in[0,e],\\
|\,F_2(\sigma,\wp(\sigma),\Im(\sigma))-F_2(\sigma,\bar{\wp}(\sigma),\bar{\Im}(\sigma))\,| &\leqslant \frac{\pi}{330}\,|\,\wp(\sigma)-\bar{\wp}(\sigma)\,|+\frac{1}{330}\,|\,\Im(\sigma)-\bar{\Im}(\sigma)\,|,\\
|\,I_1\wp(\frac{3}{5})-I_1\bar{\wp}(\frac{3}{5})\,| &\leqslant \frac{1}{150}\,|\,\wp(\sigma)-\bar{\wp}(\sigma)\,|,\\
|\,J_1\wp(\frac{3}{5})-J_1\bar{\wp}(\frac{3}{5})\,| &\leqslant \frac{1}{160}\,|\,\wp(\sigma)-\bar{\wp}(\sigma)\,|,\\
|\,I_1^{*}\Im(\frac{3}{5})-I_1^{*}\bar{\Im}(\frac{3}{5})\,| &\leqslant \frac{1}{170}\,|\,\Im(\sigma)-\bar{\Im}(\sigma)\,|,\\
|\,J_1^{*}\Im(\frac{3}{5})-J_1^{*}\bar{\Im}(\frac{3}{5})\,| &\leqslant \frac{1}{180}\,|\,\Im(\sigma)-\bar{\Im}(\sigma)\,|.
\end{aligned}
$$

From the above inequalities, we obtain that $M_{F_1}=\frac{1}{220}, N_{F_1}=\frac{1}{220}, M_{F_2}=\frac{\pi}{330}$, $N_{F_2}=\frac{\pi}{330}, A_1=\frac{1}{150}, A_2=\frac{1}{160}, A_3=\frac{1}{170}, A_4=\frac{1}{180}$. *On calculating* Ω_1, Ω_2, Ω_3 *and* Ω_4, *we have* $\Omega_1=0.268317<1$, $\Omega_2=0.173521<1$, $\Omega_3=0.321975<1$ *and* $\Omega_4=0.576319<1$. *Then,* $\max\{\Omega_1,\Omega_2,\Omega_3,\Omega_4\}<1$, *and the coupled system* (1) *has a unique solution.*

Furthermore, on calculating $\vartheta=47.29356436833$ *and* $\epsilon=0.01276$, *we get* $\vartheta\epsilon=0.60346588134>0$. *Therefore, the coupled system* (1) *is Ulam–Hyers stable.*

Example 2.

$$
\begin{aligned}
{}^{c}\mathcal{D}^{\frac{4}{3};e^{\wp}}({}^{c}\mathcal{D}^{\frac{1}{2};e^{\wp}}+\frac{1}{110})\wp(\sigma) &= \frac{\sin(\wp+\Im)}{360(\ln(\sigma)+1)},\ \sigma\in[0,1],\sigma\neq\frac{3}{5},\\
{}^{c}\mathcal{D}^{\frac{5}{4};e^{\Im}}({}^{c}\mathcal{D}^{\frac{1}{2};e^{\Im}}+\frac{1}{210})\Im(\sigma) &= \frac{\arctan(\sigma)}{450+|\wp+\Im|},\ \sigma\in[0,1],\sigma\neq\frac{3}{5},\\
I_1\wp(\frac{3}{5}) &= \frac{1}{420+|\wp(\sigma)|},\ \ J_1\wp(\frac{3}{5})=\frac{1}{550+|\wp(\sigma)|},\\
I_1^*\Im(\frac{3}{5}) &= \frac{1}{880+|\Im(\sigma)|},\ \ J_1^*\Im(\frac{3}{5})=\frac{1}{910+|\Im(\sigma)|},\\
\wp(0)=0,\ \ \wp(\frac{1}{3}) &= \int_0^{\frac{1}{4}}\wp(\tau)d\tau+\int_{\frac{1}{2}}^{\frac{2}{3}}\wp(\tau)d\tau,\\
\Im(0)=0,\ \ \Im(\frac{1}{3}) &= \int_0^{\frac{1}{4}}\Im(\tau)d\tau+\int_{\frac{1}{2}}^{\frac{2}{3}}\Im(\tau)d\tau.
\end{aligned}
$$

We see in the proposed problem that $\beta_1=\beta_2=\frac{1}{2},\gamma_1=\frac{4}{3},\gamma_2=\frac{5}{4},\eta=\frac{1}{3}$ *and* $a_1=a_2=1.$

$$
\begin{aligned}
|F_1(\sigma,\wp(\sigma),\Im(\sigma))-F_1(\sigma,\bar{\wp}(\sigma),\bar{\Im}(\sigma))| &\leqslant \frac{1}{360}|\wp(\sigma)-\bar{\wp}(\sigma)|+\frac{1}{360}|\Im(\sigma)-\bar{\Im}(\sigma)|,\ \ \forall\sigma\in[0,e],\\
|F_2(\sigma,\wp(\sigma),\Im(\sigma))-F_2(\sigma,\bar{\wp}(\sigma),\bar{\Im}(\sigma))| &\leqslant \frac{\pi}{450}|\wp(\sigma)-\bar{\wp}(\sigma)|+\frac{\pi}{450}|\Im(\sigma)-\bar{\Im}(\sigma)|,\\
|I_1\wp(\frac{3}{5})-I_1\bar{\wp}(\frac{3}{5})| &\leqslant \frac{1}{420}|\wp(\sigma)-\bar{\wp}(\sigma)|,\\
|J_1\wp(\frac{3}{5})-J_1\bar{\wp}(\frac{3}{5})| &\leqslant \frac{1}{550}|\wp(\sigma)-\bar{\wp}(\sigma)|,\\
|I_1^*\Im(\frac{3}{5})-I_1^*\bar{\Im}(\frac{3}{5})| &\leqslant \frac{1}{880}|\Im(\sigma)-\bar{\Im}(\sigma)|,\\
|J_1^*\Im(\frac{3}{5})-J_1^*\bar{\Im}(\frac{3}{5})| &\leqslant \frac{1}{910}|\Im(\sigma)-\bar{\Im}(\sigma)|.
\end{aligned}
$$

From the above inequalities, we obtain that $M_{F_1}=\frac{1}{360},N_{F_1}=\frac{1}{360},M_{F_2}=\frac{\pi}{450},N_{F_2}=\frac{\pi}{450},A_1=\frac{1}{420},A_2=\frac{1}{550},A_3=\frac{1}{880},A_4=\frac{1}{910}$. *On calculating* Ω_1, Ω_2, Ω_3 *and* Ω_4, *we have* $\Omega_1=0.177405<1$, $\Omega_2=0.04251<1$, $\Omega_3=0.169955<1$ *and* $\Omega_4=0.09639<1$. *Then,* $\max\{\Omega_1,\Omega_2,\Omega_3,\Omega_4\}<1$, *and the coupled system* (1) *has a unique solution.*

Furthermore, on calculating $\vartheta=32.71784770711$ *and* $\epsilon=0.002385$, *we get* $\vartheta\epsilon=0.07803206678>0$. *Therefore, the coupled system* (1) *is Ulam–Hyers stable.*

7. Conclusions

The existence of a unique solution to a coupled system of Langevin fractional problems of ψ-Caputo fractional derivatives with generalized slit-strip-type integral boundary conditions and impulses was examined. We have used Schaefer's fixed point theorem for the existence of at least one solution to our proposed problem. We applied the Banach contraction principle to ensure that the solution of the proposed problem was unique. Additionally, we investigated the Ulam–Hyers stability of the suggested problem. The Ulam–Hyers stability guarantees that we can achieve an exact distinction for any approximation in a given region, allowing us to use the results in approximation theory and numerical analyses of related problems. Lastly, we presented illustrations to support the results.

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