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New Fractional Integral Inequalities via *k*-Atangana–Baleanu Fractional Integral Operators

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Abstract: We propose the definitions of some fractional integral operators called k-Atangana–Baleanu fractional integral operators. These newly proposed operators are generalizations of the well-known Atangana–Baleanu fractional integral operators. As an application, we establish a generalization of the Hermite–Hadamard inequality. Additionally, we establish some new identities involving these new integral operators and obtained new fractional integral inequalities of the midpoint and trapezoidal type for functions whose derivatives are bounded or convex.

Keywords: Hermite–Hadamard inequality; Atangana–Baleanu fractional integral operators; fractional integral inequalities; convex functions; bounded functions

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1. Introduction

Integral inequalities play an indispensable role in mathematical analysis due to their innumerable applications in the analysis of differential equations and approximation theory, amongst others. Several integral inequalities for functions of single variables under various conditions have been established in the literature. Fractional calculus and convex functions have been found to play a fundamental role in the study of integral inequalities in recent years. A great number of researchers have established several integral inequalities under conditions of convexity and its generalizations by utilizing various fractional integral operators. For example, Akdemir et al. established some fractional integral inequalities of the Hermite–Hadamard type for convex and concave functions via the Atagana–Baleanu fractional integral operators in [1], and Ali et al. obtained some parametrized Newton-type inequalities for convex functions using the Riemann–Liouville fractional integrals in [2]. In [3], Butt et al. provided some Hermite–Hadamard-type integral inequalities for convex functions using the ABK-fractional integral operators, and in [4], Chu et al. established some Simpson's type inequalities for n-polynomial convex functions using the Katugampola fractional integrals. Using some generalized fractional integrals by Guzman et al., some Hermite–Hadamard-type inequalities for convex functions were present in [5], and some midpoint-type and trapezoidal-type inequalities for prequasiinvex functions via the Katugampola fractional integrals were provided by Kermausuor and Nwaeze in [6]. Peng et al. established some Simpson's type inequalities for generalized (m, h_1, h_1)-preinvex functions by utilizing the Riemann–Liouville fractional integral operators in [7], and Şanli provided some Simpson's type inequalities for harmonically convex functions via the Katugampola fractional integrals in [8]. In [9], Yu et al. established some Ostrowski-type integral inequalities for p-convex functions by utilizing the Katugampola fractional integral operators, and Zabandan et al. established some integral inequalities of the Hermite–Hadamard type for r-convex functions. For more information about the results mentioned above and other related results, we invite the interested reader to see the papers in [1–22] and the references therein.

The aforementioned results are established via the convexity of the function involved. For this purpose, it is natural to start by first presenting the definition of a convex function.

Definition 1. *A function $\mathcal{J} : I \to \mathbb{R}$ is convex on the interval I if*

$$\mathcal{J}(\lambda\gamma_1 + (1-\lambda)\gamma_2) \leq \lambda\mathcal{J}(\gamma_1) + (1-\lambda)\mathcal{J}(\gamma_2)$$

for all $\gamma_1,\ \gamma_2 \in I$ and $\lambda \in [0,1]$.

The double inequality below holds for convex functions and is known in the literature as the Hermite–Hadamard inequality.

Theorem 1 ([23])**.** *If $\mathcal{J} : I \to \mathbb{R}$ is a convex function, then the double inequality:*

$$\mathcal{J}\left(\frac{\gamma_1+\gamma_2}{2}\right) \leq \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{J}(s)ds \leq \frac{\mathcal{J}(\gamma_1)+\mathcal{J}(\gamma_2)}{2} \tag{1}$$

holds for all $\gamma_1,\ \gamma_2 \in I$ with $\gamma_1 < \gamma_2$.

The Hermite–Hadamard inequality has been one of the most studied integral inequalities. Several generalizations of this inequality have been introduced in the literature in recent years. For instance, in [21], Sarikaya et al. established the following generalization of the Hermite–Hadamard inequality by utilizing the Riemann–Liouville fractional integrals defined in Definition 2.

Theorem 2. *If $\mathcal{J} : [d_1, d_2] \to \mathbb{R}$ is a positive and convex function on $[d_1, d_2]$ with $d_1 < d_2$ and $\mathcal{J} \in L_1([d_1, d_2])$, then the inequalities:*

$$\mathcal{J}\left(\frac{d_1+d_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(d_2-d_1)^{\alpha}}\left[I^{\alpha}_{d_1^+}\mathcal{J}(d_2) + I^{\alpha}_{d_2^-}\mathcal{J}(d_1)\right] \leq \frac{\mathcal{J}(d_1)+\mathcal{J}(d_2)}{2}$$

hold for all $\alpha > 0$, where Γ denotes the gamma function, and $I^{\alpha}_{d_1^+}\mathcal{J}$ and $I^{\alpha}_{d_2^-}\mathcal{J}$, respectively, denotes the left- and right-sided Riemann–Liouville fractional integrals of $\mathcal{J}$ of order α defined in Definition 2.

Similarly, in [24], Wu et al. obtained a generalization of the Hermite–Hadamard inequality by utilizing the k-Riemann–Liouville fractional integrals defined in Definition 3 as follows.

Theorem 3. *If $\mathcal{J} : [d_1, d_2] \to \mathbb{R}$ is a positive and convex function on $[d_1, d_2]$ with $d_1 < d_2$ and $\mathcal{J} \in L_1([d_1, d_2])$, then the inequalities:*

$$\mathcal{J}\left(\frac{d_1+d_2}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(d_2-d_1)^{\frac{\alpha}{k}}}\left[{}_kI^{\alpha}_{d_1^+}\mathcal{J}(d_2) + {}_kI^{\alpha}_{d_2^-}\mathcal{J}(d_1)\right] \leq \frac{\mathcal{J}(d_1)+\mathcal{J}(d_2)}{2}$$

hold for all $\alpha, k > 0$, where Γ_k denotes the k-gamma function, and ${}_kI^{\alpha}_{d_1^+}\mathcal{J}$ and ${}_kI^{\alpha}_{d_2^-}\mathcal{J}$ denotes the left- and right-sided k-Riemann–Liouville fractional integrals of $\mathcal{J}$ of order α defined in Definition 3.

The Riemann–Liouville and k-Riemann–Liouville fractional integrals are defined as follows.

Definition 2 ([25–27])**.** *The left- and right-sided Riemann–Liouville fractional integrals of a real-valued function $\mathcal{J}$ of order $\alpha > 0$ are given by*

$$I^{\alpha}_{d_1^+}\mathcal{J}(\lambda) = \frac{1}{\Gamma(\alpha)}\int_{d_1}^{\lambda}(\lambda-u)^{\alpha-1}\mathcal{J}(u)du$$

and

$$I^{\alpha}_{d_2^-}\mathcal{J}(\lambda) = \frac{1}{\Gamma(\alpha)}\int_{\lambda}^{d_2}(u-\lambda)^{\alpha-1}\mathcal{J}(u)du,$$

where Γ *is the gamma function given by*

$$\Gamma(\nu) := \int_0^\infty u^{\nu-1} e^{-u} du, \quad Re(\nu) > 0.$$

Definition 3 ([28])**.** *The left- and right-sided k-Riemann–Liouville fractional integrals of a real-valued function* $\mathcal{J}$ *of order* $\alpha > 0$ *are given by*

$${}_k I^{\alpha}_{d_1^+} \mathcal{J}(\lambda) = \frac{1}{k\Gamma_k(\alpha)} \int_{d_1}^{\lambda} (\lambda - u)^{\frac{\alpha}{k}-1} \mathcal{J}(u) du$$

and

$${}_k I^{\alpha}_{d_2^-} \mathcal{J}(\lambda) = \frac{1}{k\Gamma_k(\alpha)} \int_{\lambda}^{d_2} (u - \lambda)^{\frac{\alpha}{k}-1} \mathcal{J}(u) du,$$

where Γ_k *is the k-gamma function presented by Diaz et al. [29] as*

$$\Gamma_k(\nu) := \int_0^\infty u^{\nu-1} e^{-\frac{u^k}{k}} du, \quad Re(\nu) > 0.$$

Recently, Atangana and Baleanu introduced a fractional integral which is known in the literature as the left-sided Atangana–Baleanu fractional integral defined below.

Definition 4 ([30])**.** *The left-sided Atangana–Baleanu fractional integral of a real-valued function* $\mathcal{J}$ *of order* $\alpha \in (0,1)$ *is given by*

$${}^{AB} I^{\alpha}_{d_1^+} \mathcal{J}(\lambda) = \frac{1-\alpha}{B(\alpha)} \mathcal{J}(\lambda) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{d_1}^{\lambda} (\lambda - u)^{\alpha-1} \mathcal{J}(u) du,$$

where $B(\alpha) > 0$ *and satisfies the property* $B(0) = B(1) = 1$.

In [31], Abdeljawad and Baleanu introduced the right-sided version of the Atangana–Baleanu fractional integral as follows:

$${}^{AB} I^{\alpha}_{d_2^-} \mathcal{J}(\lambda) = \frac{1-\alpha}{B(\alpha)} \mathcal{J}(\lambda) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\lambda}^{d_2} (u - \lambda)^{\alpha-1} \mathcal{J}(u) du.$$

By utilizing the Atangana–Baleanu fractional integral operators, Fernandez and Mohammed [32] established the following generalization of the Hermite–Hadamard inequality.

Theorem 4. *If* $\mathcal{J} : [d_1, d_2] \to \mathbb{R}$ *is convex on* $[d_1, d_2]$ *and* $\mathcal{J} \in L_1([d_1, d_2])$*, then the inequalities*

$$\mathcal{J}\left(\frac{d_1 + d_2}{2}\right) \le \frac{B(\alpha)\Gamma(\alpha)}{2[(d_2 - d_1)^{\alpha} + (1-\alpha)\Gamma(\alpha)]} \left[{}^{AB} I^{\alpha}_{d_1^+} \mathcal{J}(d_2) + {}^{AB} I^{\alpha}_{d_2^-} \mathcal{J}(d_1) \right] \le \frac{\mathcal{J}(d_1) + \mathcal{J}(d_2)}{2}$$

hold for $\alpha \in (0,1)$.

In [33], Kashuri introduced some fractional integral operators called the ABK-fractional integral operators, which are generalizations of the Atangana–Baleanu fractional operators, and presented the Hermite–Hadamard inequality for these fractional integral operators in the following:

Definition 5. *The left- and right-sided ABK-Atangana–Baleanu fractional integral operators of a real-valued function $\mathcal{J}$ of order $\nu \in (0,1)$ are given by*

$$^{ABK\rho}_{\ \ d_1^+}I^{\nu}_{\lambda}\mathcal{J}(\lambda) = \frac{1-\nu}{B(\nu)}\mathcal{J}(\lambda) + \frac{\rho^{1-\nu}\nu}{B(\nu)\Gamma(\nu)}\int_{d_1}^{\lambda}\frac{u^{\rho-1}}{(\lambda^{\rho}-u^{\rho})^{1-\nu}}\mathcal{J}(u)du, \qquad \lambda > d_1$$

and

$$^{ABK\rho}_{\ \ d_2^-}I^{\nu}_{\lambda}\mathcal{J}(\lambda) = \frac{1-\nu}{B(\nu)}\mathcal{J}(\lambda) + \frac{\rho^{1-\nu}\nu}{B(\nu)\Gamma(\nu)}\int_{\lambda}^{d_2}\frac{u^{\rho-1}}{(u^{\rho}-t^{\rho})^{1-\nu}}\mathcal{J}(u)du, \qquad \lambda < d_2,$$

where $\rho > 0$ and $B(\nu) > 0$ satisfies the property $B(0) = B(1) = 1$.

Theorem 5. *Let $\nu \in (0,1)$ and $\rho > 0$. Let $\mathcal{J} : [d_1^{\rho}, d_2^{\rho}] \to \mathbb{R}$ with $0 \le d_1 < d_2$ be a convex function. Then the following inequalities for the ABK-fractional integrals hold:*

$$\begin{aligned}
&\frac{2(d_2^{\rho}-d_1^{\rho})^{\nu}}{B(\nu)\Gamma(\nu+1)\rho^{2-\rho}}\mathcal{J}\left(\frac{d_1^{\rho}+d_2^{\rho}}{2}\right) + \frac{1-\nu}{B(\nu)}\left[\mathcal{J}(d_1^{\rho}) + \mathcal{J}(d_2^{\rho})\right] \\
&\le {}^{ABK\rho}_{\ \ d_1^+}I^{\nu}_{d_2^{\rho}}\mathcal{J}(d_2^{\rho}) + {}^{ABK\rho}_{\ \ d_2^-}I^{\nu}_{d_1^{\rho}}\mathcal{J}(d_1^{\rho}) \\
&\le \left(\frac{(d_2^{\rho}-d_1^{\rho})^{\nu} + \rho(1-\nu)\Gamma(\nu)}{\rho B(\nu)\Gamma(\nu)}\right)\left[\mathcal{J}(d_1^{\rho}) + \mathcal{J}(d_2^{\rho})\right].
\end{aligned}$$

Driven by the ongoing research work on fractional integrals and the crucial role they play in the study of fractional integral inequalities, our aim in this paper is to propose novel definitions of some generalized fractional integral operators called k-Atangana–Baleanu fractional integral operators. As an application, we will establish the Hermite–Hadamard inequality and some new fractional integral inequalities by using these new operators. Our main results are presented in Section 2, followed by a conclusion in Section 3.

2. Main Results

We begin by providing the definition of the new fractional integrals that presents a generalization of the Atangana–Baleanu fractional integrals.

Definition 6. *The left- and right-sided k-Atangana–Baleanu fractional integral operators of a real-valued function $\mathcal{J}$ of order $\alpha > 0$, are defined as*

$$^{AB}{}_{k}I^{\alpha}_{d_1^+}\mathcal{J}(\lambda) = \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\lambda) + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{d_1}^{\lambda}(\lambda-u)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du, \qquad \lambda > d_1$$

and

$$^{AB}{}_{k}I^{\alpha}_{d_2^-}\mathcal{J}(\lambda) = \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\lambda) + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\lambda}^{d_2}(u-\lambda)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du, \qquad \lambda < d_2,$$

where $k > 0$ and $B(\alpha) > 0$ satisfies the property $B(0) = B(1) = 1$.

Remark 1. *If $k = 1$ in Definition 6, then we have the Atangana–Baleanu fractional integral operators.*

Next, we study the Hermite–Hadamard inequality by utilizing the k-Atangana–Baleanu fractional integral operators.

Theorem 6. *Let $\gamma, \theta \in \mathbb{R}$ with $\gamma < \theta$ and $\mathcal{J} : [\gamma, \theta] \to \mathbb{R}$ be a convex function. Then the inequalities*

$$\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) \leq \frac{B(\alpha)\Gamma_k(\alpha)}{2[(\theta-\gamma)^{\frac{\alpha}{k}} + (1-\alpha)\Gamma_k(\alpha)]}\left[{}^{AB}{}_k I^{\alpha}_{\gamma^+}\mathcal{J}(\theta) + {}^{AB}{}_k I^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right] \leq \frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{2} \tag{2}$$

hold for all $k > 0$ and $\alpha \in (0,1)$.

Proof. Under the condition of convexity of $\mathcal{J}$ on $[\gamma, \theta]$, it follows that for any $u_1, u_2 \in [\gamma, \theta]$

$$\mathcal{J}\left(\frac{u_1+u_2}{2}\right) \leq \frac{\mathcal{J}(u_1)+\mathcal{J}(u_2)}{2}. \tag{3}$$

If we take $u_1 = \lambda\theta + (1-\lambda)\gamma$ and $u_2 = (1-\lambda)\theta + \lambda\gamma$ for $\lambda \in [0,1]$ in (3), then we obtain

$$2\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) \leq \mathcal{J}(\lambda\theta + (1-\lambda)\gamma) + \mathcal{J}((1-\lambda)\theta + \lambda\gamma). \tag{4}$$

Multiply (4) by $\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\lambda^{\frac{\alpha}{k}-1}$ and integrate both sides of the resulting inequality with respect to λ over $[0,1]$ to obtain

$$\begin{aligned}\frac{2}{B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) &\leq \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\theta+(1-\lambda)\gamma)d\lambda \\ &\quad + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\mathcal{J}((1-\lambda)\theta+\lambda\gamma)d\lambda \\ &= \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)(\theta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\theta (\theta-u)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du \\ &\quad + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)(\theta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\theta (v-\gamma)^{\frac{\alpha}{k}-1}\mathcal{J}(v)dv.\end{aligned} \tag{5}$$

Using Definition 6 and (5), we obtain

$$\frac{2(\theta-\gamma)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) + \frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma)+\mathcal{J}(\theta)] \leq {}^{AB}{}_k I^{\alpha}_{\gamma^+}\mathcal{J}(\theta) + {}^{AB}{}_k I^{\alpha}_{\theta^-}\mathcal{J}(\gamma). \tag{6}$$

Now, since $\mathcal{J}$ is convex, we have that $\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) \leq \frac{\mathcal{J}(\theta)+\mathcal{J}(\gamma)}{2}$. Hence, from (6), we have

$$\left[\frac{2(\theta-\gamma)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)} + \frac{2(1-\alpha)}{B(\alpha)}\right]\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) \leq {}^{AB}{}_k I^{\alpha}_{\gamma^+}\mathcal{J}(\theta) + {}^{AB}{}_k I^{\alpha}_{\theta^-}\mathcal{J}(\gamma). \tag{7}$$

By rearranging the terms in (7), we obtain the first inequality in (2). To prove the second inequality, we note that if $\mathcal{J}$ is convex, then for $\lambda \in [0,1]$, we have

$$\mathcal{J}(\lambda\theta + (1-\lambda)\gamma) \leq \lambda\mathcal{J}(\theta) + (1-\lambda)\mathcal{J}(\gamma) \tag{8}$$

and

$$\mathcal{J}((1-\lambda)\theta + \lambda\gamma) \leq (1-\lambda)\mathcal{J}(\theta) + \lambda\mathcal{J}(\gamma). \tag{9}$$

Adding (8) and (9), we obtain

$$\mathcal{J}(\lambda\theta + (1-\lambda)\gamma) + \mathcal{J}((1-\lambda)\theta + \lambda\gamma) \leq \mathcal{J}(\gamma) + \mathcal{J}(\theta). \tag{10}$$

Multiply (10) by $\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\lambda^{\frac{\alpha}{k}-1}$ and integrate both sides of the resulting inequality with respect to λ over $[0,1]$ to obtain

$$\begin{aligned}\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\theta + (1-\lambda)\gamma)d\lambda + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\mathcal{J}((1-\lambda)\theta + \lambda\gamma)d\lambda \\ \leq \frac{1}{B(\alpha)\Gamma_k(\alpha)}[\mathcal{J}(\gamma) + \mathcal{J}(\theta)]. \end{aligned} \tag{11}$$

By using change of variables like in the proof of the first inequality and Definition 6, we deduce from (11) that

$$\begin{aligned} {}^{AB}{}_kI^{\alpha}_{\gamma^+}\mathcal{J}(\theta) + {}^{AB}{}_kI^{\alpha}_{\theta^-}\mathcal{J}(\gamma) &\leq \frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}[\mathcal{J}(\gamma) + \mathcal{J}(\theta)] + \frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma) + \mathcal{J}(\theta)] \\ &= \left[\frac{2(\theta-\gamma)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)} + \frac{2(1-\alpha)}{B(\alpha)}\right]\frac{\mathcal{J}(\gamma) + \mathcal{J}(\theta)}{2}. \end{aligned} \tag{12}$$

Thus, the second inequality in (2) is obtained by rearranging the terms in (12). Hence the proof is complete. □

Remark 2. *If $k = 1$ in Theorem 6, then we have the Hermit–Hadamard inequality via the Atangana–Baleanu fractional integrals as stated in Theorem 4.*

In what follows, we present some novel inequalities of the midpoint type and trapezoidal type for functions with bounded derivatives and functions with convex derivatives in absolute value by utilizing the k-Atangana–Baleanu fractional integrals. To do this, we first establish the following crucial identities involving k-Atangana–Baleanu fractional integrals.

Lemma 1. *Let $\gamma, \theta \in \mathbb{R}$ with $\gamma < \theta$ and $\mathcal{J} : [\gamma, \theta] \to \mathbb{R}$ be a function. If $\mathcal{J}$ is differentiable and $\mathcal{J}' \in L_1([\gamma, \theta])$, then the equality*

$$\begin{aligned}\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}} + (1-\alpha)\Gamma_k(\alpha)}{\Gamma_k(\alpha)}\right)\frac{\mathcal{J}(\gamma) + \mathcal{J}(\theta)}{B(\alpha)} - \left[{}^{AB}{}_kI^{\alpha}_{\gamma^+}\mathcal{J}(\theta) + {}^{AB}{}_kI^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right] \\ = \frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\int_0^1 [(1-\lambda)^{\frac{\alpha}{k}} - \lambda^{\frac{\alpha}{k}}]\mathcal{J}'(\lambda\gamma + (1-\lambda)\theta)d\lambda \end{aligned} \tag{13}$$

holds for all $\alpha, k > 0$.

Proof. Using integration by parts, we obtain

$$\begin{aligned} &\int_0^1 (1-\lambda)^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma + (1-\lambda)\theta)d\lambda \\ &= \frac{(1-\lambda)^{\frac{\alpha}{k}}}{\gamma-\theta}\mathcal{J}(\lambda\gamma + (1-\lambda)\theta)\Big|_0^1 - \frac{\alpha}{k(\theta-\gamma)}\int_0^1 (1-\lambda)^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma + (1-\lambda)\theta)d\lambda \\ &= \frac{1}{\theta-\gamma}\mathcal{J}(\theta) - \frac{\alpha}{k(\theta-\gamma)}\int_0^1 (1-\lambda)^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma + (1-\lambda)\theta)d\lambda \\ &= \frac{1}{\theta-\gamma}\mathcal{J}(\theta) - \frac{\alpha}{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}\int_\gamma^\theta (\theta-u)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du \\ &= \frac{1}{\theta-\gamma}\mathcal{J}(\theta) - \frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_kI^{\alpha}_{\gamma^+}\mathcal{J}(\theta) - \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\theta)\right]. \end{aligned}$$

Thus,

$$\int_0^1 (1-\lambda)^{\frac{\alpha}{k}} \mathcal{J}'(\lambda\gamma + (1-\lambda)\theta) d\lambda$$
$$= \frac{1}{\theta-\gamma}\mathcal{J}(\theta) - \frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_k I^{\alpha}_{\gamma^+}\mathcal{J}(\theta) - \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\theta)\right]. \tag{14}$$

By a similar argument, we deduce that

$$\int_0^1 t^{\frac{\alpha}{k}} \mathcal{J}'(\lambda\gamma + (1-\lambda)\theta) d\lambda$$
$$= -\frac{1}{\theta-\gamma}\mathcal{J}(\gamma) + \frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_k I^{\alpha}_{\theta^-}\mathcal{J}(\gamma) - \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\gamma)\right]. \tag{15}$$

The desired identity in (13) is obtained if we multiply the identities in (14) and (15) by $\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}$, and take the difference between the resulting equations. □

Lemma 2. *Let $\gamma, \theta \in \mathbb{R}$ with $\gamma < \theta$ and $\mathcal{J} : [\gamma, \theta] \to \mathbb{R}$ be a function. If $\mathcal{J}$ is differentiable and $\mathcal{J}' \in L_1([\gamma, \theta])$, then the equality*

$$\left[{}^{AB}{}_k I^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta) + {}^{AB}{}_k I^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]$$
$$- \left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}-1}B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right) + \frac{1-\alpha}{B(\alpha)}\left[\mathcal{J}(\gamma)+\mathcal{J}(\theta)\right]\right)$$
$$= \frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\left[\int_0^{1/2}\lambda^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)d\lambda - \int_{1/2}^1 (1-\lambda)^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)d\lambda\right] \tag{16}$$

holds for all $\alpha, k > 0$.

Proof. Using integration by parts, we obtain

$$\int_0^{1/2}\lambda^{\frac{\alpha}{k}}\mathcal{J}'(t\gamma+(1-\lambda)\theta)d\lambda$$
$$= \frac{\lambda^{\frac{\alpha}{k}}}{\gamma-\theta}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)\Big|_0^{1/2} - \frac{\alpha}{k(\gamma-\theta)}\int_0^{1/2}\lambda^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)d\lambda$$
$$= \frac{1}{2^{\frac{\alpha}{k}}(\gamma-\theta)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right) - \frac{\alpha}{k(\gamma-\theta)}\int_0^{1/2}\lambda^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)d\lambda$$
$$= \frac{1}{2^{\frac{\alpha}{k}}(\gamma-\theta)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right) + \frac{\alpha}{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}\int_{\frac{\gamma+\theta}{2}}^{\theta}(\theta-u)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du$$
$$= -\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right) + \frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_k I^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta) - \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\theta)\right].$$

So,

$$\int_0^{1/2}\lambda^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)d\lambda$$
$$= -\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right) + \frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_k I^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta) - \frac{1-\alpha}{B(\alpha)}\mathcal{J}(\theta)\right]. \tag{17}$$

Similarly,

$$\begin{aligned}
&\int_{1/2}^{1}(1-\lambda)^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)d\lambda \\
&=\frac{(1-\lambda)^{\frac{\alpha}{k}}}{\gamma-\theta}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)\bigg|_{1/2}^{1}-\frac{\alpha}{k(\theta-\gamma)}\int_{1/2}^{1}(1-\lambda)^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)d\lambda \\
&=\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right)-\frac{\alpha}{k(\theta-\gamma)}\int_{1/2}^{1}(1-\lambda)^{\frac{\alpha}{k}-1}\mathcal{J}(\lambda\gamma+(1-\lambda)\theta)d\lambda \\
&=\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right)-\frac{\alpha}{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}\int_{\gamma}^{\frac{\gamma+\theta}{2}}(u-\gamma)^{\frac{\alpha}{k}-1}\mathcal{J}(u)du \\
&=\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right)-\frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_{k}I^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)-\frac{1-\alpha}{B(\alpha)}\mathcal{J}(\gamma)\right].
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{1/2}^{1}(1-\lambda)^{\frac{\alpha}{k}}\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)d\lambda \\
&=\frac{1}{2^{\frac{\alpha}{k}}(\theta-\gamma)}\mathcal{J}\left(\frac{\gamma+\theta}{2}\right)-\frac{B(\alpha)\Gamma_k(\alpha)}{(\theta-\gamma)^{\frac{\alpha}{k}+1}}\left[{}^{AB}{}_{k}I^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)-\frac{1-\alpha}{B(\alpha)}\mathcal{J}(\gamma)\right]. \qquad (18)
\end{aligned}$$

The desired identity in (16) is obtained if we multiply the identities in (17) and (18) by $\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}$, and take the difference between the resulting equations. □

Using Lemma 1, we obtain the following trapezoidal type inequalities.

Theorem 7. *If $\mathcal{J}$ satisfies the conditions of Lemma 1 and $\mathcal{J}'$ is bounded, that is, $\|\mathcal{J}'\|_\infty = \sup_{x\in[\gamma,\theta]}|\mathcal{J}'(x)|<\infty$, then the inequality*

$$\begin{aligned}
&\left|\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}+(1-\alpha)\Gamma_k(\alpha)}{\Gamma_k(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}{}_{k}I^{\alpha}_{\gamma^+}\mathcal{J}(\theta)+{}^{AB}{}_{k}I^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right]\right| \\
&\qquad\leq\frac{2k(\theta-\gamma)^{\frac{\alpha}{k}+1}\|\mathcal{J}'\|_\infty}{(\alpha+k)B(\alpha)\Gamma_k(\alpha)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right) \qquad (19)
\end{aligned}$$

holds for all $\alpha,k>0$.

Proof. By taking the absolute value on both sides of the identity in (13) and using the boundedness of $\mathcal{J}'$, we obtain

$$\begin{aligned}
&\left|\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}+(1-\alpha)\Gamma_k(\alpha)}{\Gamma_k(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}{}_{k}I^{\alpha}_{\gamma^+}\mathcal{J}(\theta)+{}^{AB}{}_{k}I^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right]\right| \\
&\qquad\leq\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}\|\mathcal{J}'\|_\infty}{B(\alpha)\Gamma_k(\alpha)}\int_0^1|(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|d\lambda. \qquad (20)
\end{aligned}$$

Now, we observed that for $\lambda\in[0,1]$,

$$(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}\begin{cases}\geq 0 & \text{for } \lambda\in[0,1/2]\\ <0 & \text{for } \lambda\in(1/2,1]\end{cases}$$

and hence

$$\int_0^1|(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|d\lambda=\int_0^{1/2}(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}d\lambda+\int_{1/2}^{1}\lambda^{\frac{\alpha}{k}}-(1-\lambda)^{\frac{\alpha}{k}}d\lambda$$

$$= \frac{2}{\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right). \tag{21}$$

The desired inequality in (19) follows by substituting (21) in (20). □

Theorem 8. *If $\mathcal{J}$ satisfies the conditions of Lemma 1 and $\mathcal{J}'$ is convex on $[\gamma,\theta]$, then the inequality*

$$\left|\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}+(1-\alpha)\Gamma_k(\alpha)}{\Gamma_k(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}{}_kI^{\alpha}_{\gamma+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{\theta-}\mathcal{J}(\gamma)\right]\right|$$
$$\leq \frac{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}{(\alpha+k)B(\alpha)\Gamma_k(\alpha)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[|\mathcal{J}'(\gamma)|+|\mathcal{J}'(\theta)|\right] \tag{22}$$

holds for all $\alpha,k>0$

Proof. By taking the absolute value on both sides of the identity in (13) and using the convexity of $|\mathcal{J}'|$, we obtain

$$\left|\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}+(1-\alpha)\Gamma_k(\alpha)}{\Gamma_k(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}{}_kI^{\alpha}_{\gamma+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{\theta-}\mathcal{J}(\gamma)\right]\right|$$
$$\leq \frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\int_0^1 |(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|(\lambda|f'(\gamma)|+(1-\lambda)|\mathcal{J}'(\theta)|)d\lambda$$
$$= \frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\left[|\mathcal{J}'(\gamma)|\int_0^1 |(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|\lambda d\lambda\right.$$
$$\left.+|\mathcal{J}'(\theta)|\int_0^1 |(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|(1-\lambda)d\lambda\right]. \tag{23}$$

With an argument similar to the one used in the proof of Theorem 7, we deduce that

$$\int_0^1 |(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|\lambda d\lambda = \frac{1}{\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right) \tag{24}$$

and

$$\int_0^1 |(1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}|(1-\lambda)d\lambda = \frac{1}{\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right). \tag{25}$$

The desired inequality in (22) follows by substituting (24) and (25) in (23).
□

Using Lemma 2, we obtain the following midpoint type inequalities.

Theorem 9. *If $\mathcal{J}$ satisfies the conditions of Lemma 2 and $\mathcal{J}'$ is bounded, that is, $\|\mathcal{J}'\|_\infty = \sup_{x\in[\gamma,\theta]}|\mathcal{J}'(x)|<\infty$, then the inequality*

$$\left|\left[{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})-}\mathcal{J}(\gamma)\right]\right.$$
$$\left.-\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}-1}B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}\left[\mathcal{J}(\gamma)+\mathcal{J}(\theta)\right]\right)\right|$$
$$\leq \frac{k(\theta-\gamma)^{\frac{\alpha}{k}+1}\|\mathcal{J}'\|_\infty}{2^{\frac{\alpha}{k}}(\alpha+k)B(\alpha)\Gamma_k(\alpha)} \tag{26}$$

holds for all $\alpha,k>0$.

Proof. Taking the absolute value on both sides of the identity in (16) and using the boundedness of $\mathcal{J}'$, we obtain

$$\begin{aligned}
&\Bigg|\left[{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]\\
&\qquad-\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}-1}B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma)+\mathcal{J}(\theta)]\right)\Bigg|\\
&\leq\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\left[\int_0^{1/2}\lambda^{\frac{\alpha}{k}}|\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)|d\lambda+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}}|\lambda'(\lambda\gamma+(1-\lambda)\theta)|d\lambda\right]\\
&\leq\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}\|\mathcal{J}'\|_\infty}{B(\alpha)\Gamma_k(\alpha)}\left[\int_0^{1/2}\lambda^{\frac{\alpha}{k}}d\lambda+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}}d\lambda\right]\\
&=\frac{k(\theta-\gamma)^{\frac{\alpha}{k}+1}\|\mathcal{J}'\|_\infty}{2^{\frac{\alpha}{k}}(\alpha+k)B(\alpha)\Gamma_k(\alpha)}.
\end{aligned}$$

Hence the proof is complete. □

Theorem 10. *If $\mathcal{J}$ satisfies the conditions of Lemma 1 and $\mathcal{J}'$ is convex on $[\gamma,\theta]$, then the inequality*

$$\begin{aligned}
&\Bigg|\left[{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]\\
&\qquad-\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}-1}B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}\Big[\mathcal{J}(\gamma)+\mathcal{J}(\theta)\Big]\right)\Bigg|\\
&\leq\frac{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}(\alpha+k)B(\alpha)\Gamma_k(\alpha)}\Big[|\mathcal{J}'(\gamma)|+|\mathcal{J}'(\theta)|\Big]
\end{aligned}\tag{27}$$

holds for all $\alpha,k>0$.

Proof. Taking the absolute value on both sides of the identity in (16) and using the convexity of $|\mathcal{J}'|$, we obtain

$$\begin{aligned}
&\Bigg|\left[{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta)+{}^{AB}{}_kI^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]\\
&\qquad-\left(\frac{(\theta-\gamma)^{\frac{\alpha}{k}}}{2^{\frac{\alpha}{k}-1}B(\alpha)\Gamma_k(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma)+\mathcal{J}(\theta)]\right)\Bigg|\\
&\leq\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\left[\int_0^{1/2}\lambda^{\frac{\alpha}{k}}|\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)|d\lambda+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}}|\mathcal{J}'(\lambda\gamma+(1-\lambda)\theta)|d\lambda\right]\\
&\leq\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\Bigg[\int_0^{1/2}\lambda^{\frac{\alpha}{k}}\Big(\lambda|\mathcal{J}'(\gamma)|+(1-\lambda)|\mathcal{J}'(\theta)|\Big)d\lambda\\
&\qquad+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}}\Big(\lambda|\mathcal{J}'(\gamma)|+(1-\lambda)|\mathcal{J}'(\theta)|\Big)d\lambda\Bigg]\\
&=\frac{(\theta-\gamma)^{\frac{\alpha}{k}+1}}{B(\alpha)\Gamma_k(\alpha)}\Bigg[|\mathcal{J}'(\gamma)|\left(\int_0^{1/2}\lambda^{\frac{\alpha}{k}+1}d\lambda+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}}\lambda d\lambda\right)\\
&\qquad+|\mathcal{J}'(\theta)|\left(\int_0^{1/2}\lambda^{\frac{\alpha}{k}}(1-\lambda)d\lambda+\int_{1/2}^1(1-\lambda)^{\frac{\alpha}{k}+1}d\lambda\right)\Bigg]\\
&=\frac{k(\theta-\gamma)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}(\alpha+k)B(\alpha)\Gamma_k(\alpha)}\Big[|\mathcal{J}'(\gamma)|+|\mathcal{J}'(\theta)|\Big].
\end{aligned}$$

Hence, the proof is complete. □

Remark 3. *It is worth noting that the inequalities in Theorems 7–10 provide estimates for the absolute of the error between the middle term and left or right terms in the inequality in Theorem 6, with some modifications.*

3. Conclusions

We proposed the definitions of some new generalized fractional integral operators called k-Atangana–Baleanu fractional integrals. The well-known Atangana–Baleanu fractional integrals are special cases of the newly proposed fractional integral operators when $k = 1$; see Remark 1. The Hermite–Hadamard inequality and some fractional integral inequalities involving these new generalized fractional integral operators have been established as an application of these new operators. It has been noted that certain results already provided can be obtained as particular cases from some of our results; see Remark 2. In addition, if we set $k = 1$ in the inequalities of Theorems 7–10, we obtain, respectively, the following inequalities involving the Atangana–Baleanu fractional integrals:

1.
$$\left|\left(\frac{(\theta-\gamma)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\Gamma(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}I^{\alpha}_{\gamma^+}\mathcal{J}(\theta)+{}^{AB}I^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right]\right|$$
$$\leq \frac{2(\theta-\gamma)^{\alpha+1}\|\mathcal{J}'\|_{\infty}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(1-\frac{1}{2^{\alpha}}\right).$$

2.
$$\left|\left(\frac{(\theta-\gamma)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\Gamma(\alpha)}\right)\frac{\mathcal{J}(\gamma)+\mathcal{J}(\theta)}{B(\alpha)}-\left[{}^{AB}I^{\alpha}_{\gamma^+}\mathcal{J}(\theta)+{}^{AB}I^{\alpha}_{\theta^-}\mathcal{J}(\gamma)\right]\right|$$
$$\leq \frac{(\theta-\gamma)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|\mathcal{J}'\gamma|+|\mathcal{J}'(\theta)|\right].$$

3.
$$\Bigg|\left[{}^{AB}I^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta)+{}^{AB}I^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]$$
$$-\left(\frac{(\theta-\gamma)^{\alpha}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma)+\mathcal{J}(\theta)]\right)\Bigg|$$
$$\leq \frac{(\theta-\gamma)^{\alpha+1}\|\mathcal{J}'\|_{\infty}}{2^{\alpha}(\alpha+1)B(\alpha)\Gamma(\alpha)}.$$

4.
$$\Bigg|\left[{}^{AB}I^{\alpha}_{(\frac{\gamma+\theta}{2})^+}\mathcal{J}(\theta)+{}^{AB}I^{\alpha}_{(\frac{\gamma+\theta}{2})^-}\mathcal{J}(\gamma)\right]$$
$$-\left(\frac{(\theta-\gamma)^{\alpha}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}\mathcal{J}\left(\frac{\theta+\gamma}{2}\right)+\frac{1-\alpha}{B(\alpha)}[\mathcal{J}(\gamma)+\mathcal{J}(\theta)]\right)\Bigg|$$
$$\leq \frac{(\theta-\gamma)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[|\mathcal{J}'(\gamma)|+|\mathcal{J}'(\theta)|\right].$$

These special cases are new, to the best of our knowledge. We are of the opinion that the results in this paper will stimulate further research on fractional integral operators and their applications.

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