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Stepanov Almost Periodic Type Functions and Applications to Abstract Impulsive Volterra Integro-Differential Inclusions

Marko Kostić [1] and Wei-Shih Du [2,*]

[1] Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia; marco.s@verat.net
[2] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
* Correspondence: wsdu@mail.nknu.edu.tw

Abstract: In this paper, we analyze various classes of Stepanov-p-almost periodic functions and Stepanov-p-almost automorphic functions ($p > 0$). The class of Stepanov-p-almost periodic (automorphic) functions in norm ($p > 0$) is also introduced and analyzed. Some structural results for the introduced classes of functions are clarified. We also provide several important theoretical examples, useful remarks and some new applications of Stepanov-p-almost periodic type functions to the abstract (impulsive) first-order differential inclusions and the abstract (impulsive) fractional differential inclusions.

Keywords: Stepanov-p-almost periodic (automorphic) function; Stepanov-p-almost periodic (automorphic) function in norm; abstract (impulsive) first-order differential inclusion; abstract (impulsive) fractional differential inclusion

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1. Introduction

The concept of almost periodicity was introduced and studied by the Danish mathematician H. Bohr around 1924–1926. Since then, a number of generalizations in various different directions have been investigated by many authors. Suppose that $(U, \|\cdot\|)$ is a complex Banach space and $G : \mathbb{R}^n \to U$ is a continuous function, where $n \in \mathbb{N}$. Recall that $G(\cdot)$ is called *almost periodic* if and only if for each $\epsilon > 0$, there exists $r > 0$, such that for each $\mathbf{t}_0 \in \mathbb{R}^n$, there exists

$$\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{a} \in \mathbb{R}^n : |\mathbf{a} - \mathbf{t}_0| \leq r\}$$

such that

$$\left\|G(\mathbf{z} + \tau) - G(\mathbf{z})\right\| \leq \epsilon, \quad \mathbf{z} \in \mathbb{R}^n,$$

where $|\cdot - \cdot|$ is the Euclidean distance in $\mathbb{R}^n$. In light of the definition, it is also equivalent to say that for any sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$, there exists a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$, such that the sequence of functions $(G(\cdot + \mathbf{a}_k))$ converges in $C_b(\mathbb{R}^n : U)$, the Banach space of all bounded continuous functions on $\mathbb{R}^n$, equipped with the sup-norm. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic and any almost periodic function $G(\cdot)$ is bounded and uniformly continuous. It is well known that a continuous function $G(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$, which converges to $G(\cdot)$ in $C_b(\mathbb{R}^n : U)$.

Suppose now that $G : \mathbb{R}^n \to U$ is continuous. The function $G(\cdot)$ is called *almost automorphic* if and only if for any sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$, there exists a subsequence $(\mathbf{c}_k)$ of $(\mathbf{b}_k)$ and a mapping $H : \mathbb{R}^n \to U$, such that

$$\lim_{k\to\infty} G(\mathbf{x}+\mathbf{c}_k) = H(\mathbf{x}) \text{ and } \lim_{k\to\infty} H(\mathbf{x}-\mathbf{c}_k) = G(\mathbf{x}), \tag{1}$$

Pointwisely, for $\mathbf{x} \in \mathbb{R}^n$. In this case, then the range of $G(\cdot)$ is relatively compact in U and the limit function $H(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. Moreover, if the convergence of limits appearing in (1) is uniform on compact subsets of $\mathbb{R}^n$, then the function $F(\cdot)$ is called *compactly almost automorphic*. It is a well-known fact that an almost automorphic function $G(\cdot)$ is compactly almost automorphic if and only if $G(\cdot)$ is uniformly continuous and any almost periodic function $G(\cdot)$ is compactly almost automorphic. For more details, we refer the readers to the research monographs [1–8].

The concept of metrical almost periodicity has been introduced and analyzed recently in [9]; for more details about various classes of metrically almost periodic functions and their applications, one can refer to our new monograph [10]. In this paper, we continue our previous research analyses of the multi-dimensional metrically Stepanov ρ-almost periodic (automorphic) type functions, where ρ is a general binary relation [11,12]. Let us mention here only that Stepanov-p-almost periodic (automorphic) functions are important for applications because these functions are only p-locally integrable in contrast with the usually considered almost periodic (automorphic) functions, which must be continuous. It would be a very difficult task to provide here a more comprehensive literature review about almost periodic (automorphic) functions, their Stepanov generalizations and various applications made so far; cf. also the research monographs [13–28] and the reference list in our recent monograph [6], which contains more that one thousand one hundred titles quoted. One of the most influential monographs about almost periodic functions and their generalizations was written by M. Levitan in 1953 [7]. In this research article, we provide the proper generalizations of the famous, old result [7] (Theorem 5.3.1, p. 210) concerning the sign of real-valued almost periodic functions, and some proper generalizations of the conclusions established in [7] (pp. 212–213) concerning the Stepanov-p-almost periodicity of functions

$$f(t) = \sin\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R}$$

and

$$g(t) = \cos\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R};$$

cf. Examples 3 and 4 for more details.

The main ideas and organization of our work can be briefly described as follows. In the continuation of this section, we explain the basic notation and terminology used as well as the notion of considered weighted function spaces. One of the main novelties of this work is the analysis of Stepanov-p-almost periodic functions and Stepanov-p-almost automorphic functions, where the exponent p has a value between 0 and 1. Here we would like to note that the class of complex-valued Stepanov-p-almost periodic functions and the class of Stepanov-p-normal functions $f : \mathbb{R} \to \mathbb{C}$, where $p > 0$, was introduced for the first time by H. D. Ursell in [29], a paper which has been cited only six times from 1931 onwards. A Stepanov-p-almost periodic (automorphic) function $g : \mathbb{R} \to [0,\infty)$ need not be locally integrable for $0 < p < 1$. For instance, the function

$$g(x) := \begin{cases} |\sin x|^{-1}, \ x \notin \mathbb{Z}\pi \\ 0, \ x \in \mathbb{Z}\pi, \end{cases}$$

is not locally integrable and therefore not Stepanov-p-almost periodic (automorphic) for any finite exponent $p \geq 1$. On the other hand, it is easy to show that this function is Stepanov-p-almost periodic for any exponent $p \in (0,1)$.

In Section 3, we introduce and study the notion of Stepanov-$(p, \mathcal{B})$-almost periodicity, where $p > 0$ (cf. Definition 5 for the notions of Stepanov-$(p, \rho, \mathcal{B}, \Lambda')$-almost periodicity, (strong) Stepanov-$(p, \mathrm{R}, \mathcal{B})$-almost periodicity and their particular cases, the notions of Stepanov-p-almost periodicity and Bochner–Stepanov-p-almost periodicity, where $p > 0$). If $0 < p < 1$, then the notion of a (Bochner–)Stepanov-p-almost periodic function $F : \Lambda \to Y$ is new, even in the case where $\Lambda = \mathbb{R}$ and $Y = \mathbb{C}$. The main structural results of Section 3 are Proposition 2 and Corollary 1. We also propose here many illustrative examples (without any doubt, the most important are the already-mentioned Examples 3 and 4); an interesting open problem is also proposed in this context.

Section 3.1 investigates the relations between piecewise continuous almost periodic functions and metrically Stepanov-p-almost periodic functions ($p > 0$). In this subsection, we provide proper generalizations of [30] (Theorem 1, Theorem 2) concerning the relations between the class of pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic functions and the class $S^{(\mathbb{F}, T, \mathcal{P}_t, \mathcal{P})}_{\Omega, \Lambda'}(\Lambda : Y)$ with $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\Omega = [0,1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L^p_\nu(t + [0,1] : \mathbb{C})$ for all $t \in \mathbb{R}$ (cf. Theorem 1). The main aim of Theorem 2 is to show that if $\rho = T \in L(Y)$, $p > 0$ and $F : \Lambda \times X \to Y$ is a Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda')$-almost periodic function, where $\Lambda = \mathbb{R}$ or $\Lambda = [0, \infty)$, $\Omega = [0,1]$ and $\Lambda' = \Lambda$, then the validity of a quasi-uniformly convergent type condition (QUC) clarified below implies that the considered function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic. In Corollary 2(ii), we particularly show that any uniformly continuous Stepanov-p-almost periodic function $F : \mathbb{R}^n \to Y$ is almost periodic ($p > 0$), thus extending the well-known Bochner theorem.

Section 3.2 investigates the invariance of Stepanov-p-almost periodicity under the actions of the infinite convolution products ($0 < p < 1$) and provides certain applications of the introduced notion to the abstract Volterra integro-differential equations. The main result of this subsection are Theorem 3 and Proposition 3 (concerning applications, we thought it necessary to emphasize at the very beginning that the situation is very complicated in the case where $0 < p < 1$, since the reverse Hölder inequality is valid in our new framework (for more detailed information, see, e.g., [31] (Proposition 3))).

The main purpose of Section 4 is to analyze various classes of Stepanov-p-almost automorphic type functions ($p > 0$). In Definition 7 and Definition 8, we introduce the notions of Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphy, Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, W_{\mathcal{B},\mathrm{R}})$-almost automorphy, Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, P_{\mathcal{B},\mathrm{R}})$-almost automorphy and some special subnotions of them. After that, we quote some statements established recently for the general classes of metrically Stepanov almost automorphic functions, which can be reformulated in our special framework (see, e.g., Proposition 4). Section 4.1 investigates certain relations between piecewise continuous almost automorphic functions and metrically Stepanov-p-almost automorphic functions ($p > 0$); the main structural results presented here are Proposition 5, Theorems 4 and 5.

In Section 5, we analyze the notion of Stepanov-p-almost periodicity in norm and the notion of Stepanov-p-almost automorphy in norm ($p > 0$). The considered function spaces are introduced in Definitions 11 and 12. The main result established in this section, where we also propose some open problems for our readers, is Theorem 6; cf. also Remark 4.

The main aim of Section 6 is to present applications of the obtained results to the abstract (impulsive) Volterra integro-differential inclusions in Banach spaces. This section contains two separate subsections: Section 6.1 considers certain applications to the abstract impulsive first-order differential inclusions, while Section 6.2 considers certain applications to the abstract fractional differential inclusions. The conclusions and final remarks about the introduced classes of functions are given in the final section of paper.

2. Preliminaries

In this section, we review some basic definitions, notations and known results that will be needed, and further discussion throughout this paper. Assume that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X and Y is any subset $\rho \subseteq X \times Y$. The domain $D(\rho)$ and range $R(\rho)$ of ρ are defined in the usual way as are the set $\rho(x)$ $(x \in X)$ and the set $\rho(X')$, where $X' \subseteq X$.

We will always assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $n \in \mathbb{N}, \emptyset \neq I \subseteq \mathbb{R}^n, \emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X and R is a non-empty collection of sequences in $\mathbb{R}^n$. Furthermore, we assume that for any $z \in X$, there exists $B \in \mathcal{B}$, such that $z \in B$. Denote by I the identity operator of Y. By $L(X, Y)$, we denote the Banach space of all bounded linear operators from X into Y. If $Y = X$, we will use $L(X)$ instead of $L(X, X)$. Define $\mathbb{N}_m := \{1, \ldots, m\}$ and $\mathbb{N}_m^0 := \{0, 1, \ldots, m\}$ $(m \in \mathbb{N})$. Let A and B be non-empty sets. We define $\mathrm{B}^{\mathrm{A}} := \{h | h : \mathrm{A} \to \mathrm{B}\}$, and by $\chi_A(\cdot)$ $[A^c]$ we denote the characteristic function of the set A (the complement of A). Set $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(s \in \mathbb{R})$. For further information concerning the multivalued operator families and solution operator families subgenerated by them, we refer the reader to the recent research monograph [32].

Suppose now that $T > 0$. Then, the space of X-valued piecewise continuous functions on $[0, T]$ is defined by

$$PC([0,T] : X) := \big\{h : [0,T] \to X : h \in C\big((s_i, s_{i+1}] : X\big),\ h(s_i-) = h(s_i) \text{ exist for any } i \in \mathbb{N}_l,\ h(s_i+) \text{ exist for any } i \in \mathbb{N}_l^0 \text{ and } h(0) = h(0+)\big\},$$

where $0 = s_0 < s_1 < s_2 < \ldots < s_l < T = s_{l+1}$ and the symbols $h(s_i-)$ and $h(s_i+)$ denote the left and the right limits of the function $h(x)$ at the point $x = s_i$, $i \in \mathbb{N}_{l-1}^0$, respectively. It is well-known that $PC([0, T] : X)$ is a Banach space endowed with the norm

$$\|h\| := \max\left\{\sup_{t\in[0,T)} \|h(t+)\|, \sup_{t\in(0,T]} \|h(t-)\|\right\}.$$

The space of X-valued piecewise continuous functions on $[0, \infty)$, denoted by $PC([0, \infty) : X)$, if defined as the union of those functions $f : [0, \infty) \to X$, such that the discontinuities of $g(\cdot)$ form a discrete set, and that for each $T > 0$ we have $g_{|[0,T]}(\cdot) \in PC([0, T] : X)$. We similarly define the space $PC(\mathbb{R} : X)$. If $\omega \in \mathbb{R}$, then $C_\omega([0, \infty) : X)$ denotes the space of all continuous functions $h : [0, \infty) \to X$, such that the function $x \mapsto e^{-\omega x}\|h(x)\|$, $x \geq 0$ is bounded; the space $PC_\omega([0, \infty) : X)$ denotes the space of all piecewise continuous functions $h : [0, \infty) \to X$, such that the function $x \mapsto e^{-\omega x}\|h(x)\|$, $x \geq 0$ is bounded.

The following classes of weighted function spaces will be important for us:

1. Suppose that the set $I \subseteq \mathbb{R}^n$ is Lebesgue measurable and $\nu : I \to (0, \infty)$ is a Lebesgue measurable function. We deal with the Banach space

$$L_\nu^{p(\mathbf{t})}(I : Y) := \big\{k : I \to Y\,;\, k(\cdot) \text{ is Lebesgue measurable and } ||k||_{p(\mathbf{t})} < \infty\big\},$$

   where $p \in \mathcal{P}(I)$, the collection of all Lebesgue measurable mappings from I into $[1, +\infty]$, and $\|u\|_{p(\mathbf{t})} := \|\nu(\mathbf{t})u(\mathbf{t})\|_{L^{p(\mathbf{t})}(I:Y)}$.
2. If $\nu : I \to (0, \infty)$ is any function, such that the function $1/\nu(\cdot)$ is locally bounded, then the vector space $C_{b,\nu}(I : Y)$ consists of all continuous functions $u : I \to Y$, satisfying that $\sup_{\mathbf{t}\in I} \|u(\mathbf{t})\|_Y \nu(\mathbf{t}) < +\infty$. When equipped with the norm, $\|\cdot\| := \sup_{\mathbf{t}\in I} \|\nu(\mathbf{t}) \cdot (\mathbf{t})\|_Y$, $C_{b,\nu}(I : Y)$ is a Banach space; if $\nu \equiv 1$, then we set $C_b(I : Y) \equiv C_{b,\nu}(I : Y)$.
3. Suppose that $\nu : I \to [0, \infty)$ is any non-trivial function. Then, we define the vector space $C_{b,\nu}(I : Y)$ as above; equipped with the pseudometric, $d(u, v) := \sup_{\mathbf{t}\in I} \|\nu(\mathbf{t})[u(\mathbf{t}) - v(\mathbf{t})]\|_Y$, $(C_{b,\nu}(I : Y), d)$ becomes a pseudometric space.

2.1. Stepanov Almost Periodic Functions and Stepanov Almost Automorphic Functions in General Metric

The main goal of this subsection is to review the fundamental definitions and results about metrically Stepanov almost periodic type functions.

First of all, we will recall the notion of (strong) Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodicity. Assume that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq Z \subseteq Y^{\Omega}$ and $\Lambda + \Omega \subseteq \Lambda$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a fixed compact set with positive Lebesgue measure. Let $P_Z \subseteq Z^{\Lambda}$, $0 \in P_Z$, let (P_Z, d_{P_Z}) be a pseudometric space, and let $\|f\|_{P_Z} := d_{P_Z}(f, 0)$, $f \in P_Z$. If $F : \Lambda \times X \to Y$, then we introduce the multi-dimensional Bochner transform $\hat{F}_{\Omega} : \Lambda \times X \to Y^{\Omega}$ by

$$\left[\hat{F}_{\Omega}(\mathbf{t}; x)\right](u) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

We need the following notion from [11]:

Definition 1. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function and the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. Then, we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic, resp. strongly Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic in the case that $\Lambda = \mathbb{R}^n$, if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$, there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot\cdot\cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F_{\Omega}^* : \Lambda \times X \to Z$, such that for every $l \in \mathbb{N}$ and $x \in B$, we have $\hat{F}_{\Omega}(\cdot + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - F_{\Omega}^*(\cdot; x) \in P_Z$ and*

$$\lim_{l \to +\infty} \sup_{x \in B} \left\| \hat{F}_{\Omega}(\cdot + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - F_{\Omega}^*(\cdot; x) \right\|_{P_Z} = 0, \tag{2}$$

resp. $\hat{F}(\cdot + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - F^*(\cdot; x) \in P_Z$, $F^*(\cdot - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - \hat{F}(\cdot; x) \in P_Z$, (2) holds and

$$\lim_{l \to +\infty} \sup_{x \in B} \left\| F_{\Omega}^*(\cdot - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - \hat{F}_{\Omega}(\cdot; x) \right\|_{P_Z} = 0.$$

Consider now the following conditions:

(SM-1): Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\Lambda' + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$ and $\mathbb{F} : \Lambda \to (0, \infty)$.

(SM-2): For every $\mathbf{t} \in \Lambda$, $\mathcal{P}_{\mathbf{t}} = (P_{\mathbf{t}}, d_{\mathbf{t}})$ is a metric space of functions from $Y^{\mathbf{t}+\Omega}$, containing the zero function. We set $\|f\|_{P_{\mathbf{t}}} := d_{\mathbf{t}}(f, 0)$ for all $f \in P_{\mathbf{t}}$. We also assume that $\mathcal{P} = (P, d)$ is a metric space of functions from $\mathbb{C}^{\Lambda}$ containing the zero function and set $\|f\|_P := d(f, 0)$ for all $f \in P$. The argument from Λ will be denoted by $\cdot\cdot$ and the argument from $\mathbf{t} + \Omega$ will be denoted by $\cdot$.

The following concept was recently introduced in [33] (Definition 2.2):

Definition 2. *Assume that (SM-1)-(SM-2) hold. By $S_{\Omega,\Lambda',\mathcal{B}}^{(\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}(\Lambda \times X : Y)$, we denote the set consisting of all functions $F : \Lambda \times X \to Y$, such that for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$, such that for each $\mathbf{t}_0 \in \Lambda'$, there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$, such that for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined and*

$$\sup_{x \in B} \left\| \mathbb{F}(\cdot\cdot) \| F(\tau + \cdot; x) - G_x(\cdot) \|_{P_{\cdot\cdot}} \right\|_P < \epsilon.$$

Concerning the multi-dimensional almost automorphic functions, we will recall the following definition from [6]:

Definition 3. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function and R is a certain collection of sequences in $\mathbb{R}^n$. Then, we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$, there exist* *a subsequence* $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot\cdot\cdot, b_{k_l}^n))$ *of* $(\mathbf{b}_k)$ *and a function* $F^* : \mathbb{R}^n \times X \to Y$*, such that* $\lim_{l\to+\infty} F(\mathbf{t} + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) = F^*(\mathbf{t}; x)$ *and* $\lim_{l\to+\infty} F^*(\mathbf{t} - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) = F(\mathbf{t}; x)$*, pointwise for all* $x \in B$ *and* $\mathbf{t} \in \mathbb{R}^n$*. If for each* $x \in B$ *the above limits converge uniformly on compact subsets of* $\mathbb{R}^n$*, then we say that* $F(\cdot;\cdot)$ *is compactly* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic.*

Suppose, finally, that $\emptyset \neq Z \subseteq Y^\Omega$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a fixed compact set with positive Lebesgue measure. Let $Z \subseteq Y^\Omega$, $0 \in Z$ and let (Z, d_Z) be a pseudometric space. Set $\|f\|_Z := d_Z(f, 0)$, $f \in Z$. Further on, we assume that $W_{B;(\mathbf{b}_k)}(x)$ is a certain collection of non-empty subsets of $\mathbb{R}^n$ and that $P_{B;(\mathbf{b}_k)}$ a certain collection of non-empty subsets of $\mathbb{R}^n \times B$ ($B \in \mathcal{B}$, $(\mathbf{b}_k) \in \mathrm{R}$, $x \in B$).

We need the following concept, which was recently introduced in our joint research article with B. Chaouchi and H. C. Koyuncuoğlu [12]:

Definition 4. *Suppose that* $F : \mathbb{R}^n \times X \to Y$ *is a given function and* R *is a collection of sequences in* $\mathbb{R}^n$*. Then, we say that the function* $F(\cdot;\cdot)$ *is Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}})$*-multi-almost automorphic if and only if, for every* $B \in \mathcal{B}$ *and for every sequence* $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n)) \in \mathrm{R}$*, there exist a subsequence* $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot\cdot\cdot, b_{k_l}^n))$ *of* $(\mathbf{b}_k)$ *and a function* $F_B^* : \mathbb{R}^n \times X \to Z$*, such that, for every* $\mathbf{t} \in \mathbb{R}^n$*,* $l \in \mathbb{N}$ *and* $x \in B$*, we have* $F(\mathbf{t} + \cdot + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x) - [F_B^*(\mathbf{t}; x)](\cdot) \in Z$*,* $[F_B^*(\mathbf{t} - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x)](\cdot) - F(\mathbf{t} + \cdot; x) \in Z$*,*

$$\lim_{l\to+\infty} \left\| F\left(\mathbf{t} + \cdot + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x\right) - [F_B^*(\mathbf{t}; x)](\cdot) \right\|_Z = 0, \tag{3}$$

and

$$\lim_{l\to+\infty} \left\| \left[F_B^*\left(\mathbf{t} - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x\right)\right](\cdot) - F(\mathbf{t} + \cdot; x) \right\|_Z = 0. \tag{4}$$

Furthermore, if for each $x \in B$ *the convergence in* (3) *and* (4) *is uniform in* $\mathbf{t}$ *for any element of the collection* $W_{B;(\mathbf{b}_k)}(x)$ *(the convergence in* (3) *and* (4) *is uniform in* $(\mathbf{t}; x)$ *for any set of the collection* $P_{B;(\mathbf{b}_k)}$*), then we say that* $F(\cdot;\cdot)$ *is Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic [Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic].*

2.2. On L^p*-Spaces* $(p > 0)$

Let $0 < p < 1$ and let Ω' be any Lebesgue measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure. Then, the space $L^p(\Omega' : Y)$ consists of all Lebesgue measurable functions $f : \Omega' \to Y$, such that $\int_{\Omega'} \|f(\mathbf{u})\|^p \, d\mathbf{u} < +\infty$. The metric $d(\cdot;\cdot)$ on $L^p(\Omega' : Y)$ is given by $d(f, g) := \int_{\Omega'} \|f(\mathbf{u}) - g(\mathbf{u})\|^p \, d\mathbf{u}$ for all f, $g \in L^p(\Omega' : Y)$. Equipped with this metric, $L^p(\Omega' : Y)$ is a complete quasi-normed metric space. If $\nu : \Omega' \to (0, \infty)$ is a Lebesgue measurable function, then we define the pseudometric space the space $L_\nu^p(\Omega' : Y)$ as in the case where $p \geq 1$, when the basic properties of $L_\nu^p(\Omega' : Y)$ are well known.

Before proceeding any further, we would like to emphasize that the theory of Lebesgue spaces $L^{p(x)}$ with variable exponent $0 < p(x) < 1$ has not still been constituted. Because of this, we will work with the constant coefficients $p > 0$ in the sequel (cf. [6] for many results concerning the generalized almost periodic type functions in the Lebesgue spaces with variable exponent $L^{p(x)}$, where $p(x) \geq 1$, and the important research monograph [34]). We will use the following lemma, which might be known in the existing literature:

Lemma 1. *Suppose that* $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ *is a Lebesgue measurable set,* $p > 0$ *and* $f \in L^p(\Omega)$*. Then, we have*

$$\int_\Omega |f(x)|^p \, dx = p \int_0^\infty y^{p-1} m\left(\{x \in \Omega : |f(x)| > y\}\right) dy. \tag{5}$$

Proof. The Formula (5) is well known in the case that $p \geq 1$: see, e.g., [35] (pp. 7–8). If $0 < p < 1$, then we can use (5) with $p = 1$ and the elementary change of variables $z = y^{1/p}$:

$$\begin{aligned}\int_{\Omega} |f(x)|^p \, dx &= \int_0^{\infty} m\big(\{x \in \Omega : |f(x)|^p > y\}\big) \, dy \\ &= \int_0^{\infty} m\big(\{x \in \Omega : |f(x)| > y^{1/p}\}\big) \, dy \\ &= p \int_0^{\infty} z^{p-1} m(\{x \in \Omega : |f(x)| > z\}) \, dz.\end{aligned}$$

□

Remark 1. *The Formula (5) will play an important role in our further work. We would like to notice that this formula does not have a satisfactory analogue in the theory of the Lebesgue spaces with variable exponent (see, e.g., the introductory part of [34]).*

For more details concerning L^p-spaces for $0 < p < 1$, the interested readers may refer to the lectures of K. Conrad [36] and M. Rosenzweig [31].

3. Stepanov-p-Almost Periodic Type Functions ($p > 0$)

We will consider the following special kinds of Stepanov-p-almost periodic type functions ($p > 0$) in this section.

Definition 5.

(i) *Suppose that $p > 0$, (SM-1) holds true and $\nu_{\mathbf{t}} : \mathbf{t} + \Omega \to (0, \infty)$ is a Lebesgue measurable function ($\mathbf{t} \in \Lambda$). Then, we say that a function $F : \Lambda \times X \to Y$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu_{\cdot})$-almost periodic if and only if $F(\cdot;\cdot)$ belongs to the class $S^{(\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ with $\mathbb{F}(\cdot\cdot) \equiv 1$, $P = C_b(\Lambda : \mathbb{C})$ and $P_{\mathbf{t}} = L^p_{\nu_{\mathbf{t}}}(\mathbf{t} + \Omega : Y)$ for all $\mathbf{t} \in \Lambda$. If there exists a Lebesgue measurable function $\nu : \Lambda \to (0, \infty)$, such that $\nu_{\mathbf{t}}(\cdot) \equiv \nu_{|\mathbf{t}+\Omega}(\cdot)$ for all $\mathbf{t} \in \Lambda$, then we also say that $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$-almost periodic; furthermore, if $\nu_{\mathbf{t}}(\cdot) \equiv 1$ for all $\mathbf{t} \in \Lambda$, then we omit the term "$\nu_{\cdot}$" from the notation.*

(ii) *Suppose that $p > 0$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function and the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. If $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function, then we say that the function $F(\cdot;\cdot)$ is (strongly) Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic if and only if $F(\cdot;\cdot)$ is (strongly) Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic with $Z = L^p_{\nu}(\Omega : Y)$ and $P_Z = C_b(\Lambda : Z)$. If $\nu(\cdot) \equiv 1$, then we omit the term "ν" from the notation.*

(iii) *Suppose that $p > 0$, (SM-1) holds true, $\nu_{\mathbf{t}} : \mathbf{t} + \Omega \to (0, \infty)$ is a Lebesgue measurable function ($\mathbf{t} \in \Lambda$) and $\nu : \Lambda \to (0, \infty)$ is a Lebesgue measurable function. Then, we say that a function $F : \Lambda \to Y$ is Stepanov-$(\Omega, p, \nu_{\cdot})$-almost periodic [Stepanov-(Ω, p, ν)-almost periodic] if and only if $F(\cdot)$ is Stepanov-$(\Omega, p, \rho, \Lambda', \nu_{\cdot})$-almost periodic [Stepanov-$(\Omega, p, \rho, \Lambda', \nu)$-almost periodic] with $\rho \equiv \mathrm{I}$ and $\Lambda' \equiv \{\tau \in \mathbb{R}^n : \tau + \mathbf{t} \in \Lambda,\ \mathbf{t} \in \Lambda\}$. If $\nu(\cdot) \equiv 1$, then we omit the term "ν" from the notation.*

(iv) *Suppose that $p > 0$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \to Y$ is a given function, $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function and R denotes the collection of all sequences in $\mathbb{R}^n$, such that the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. Then, we say that the function $F(\cdot)$ is Bochner–Stepanov-(Ω, p, ν)-almost periodic if and only if $F(\cdot)$ is Stepanov-$(\Omega, p, \mathrm{R}, \nu)$-almost periodic.*

In all above definitions, we omit the term "Ω" from the notation if $\Omega = [0, 1]^n$. If we denote by $A_{X,Y}$ any of the above introduced classes of function spaces, $c_1 \in \mathbb{R} \setminus \{0\}$, $\tau \in \mathbb{R}^n$, $c,\ c_2 \in \mathbb{C} \setminus \{0\}$ and $x_0 \in X$, then it is not difficult to find some sufficient conditions ensuring that the function $cF(\cdot;\cdot)$, $F(c_1\cdot; c_2\cdot)$, $\|F(\cdot;\cdot)\|_Y$ or $F(\cdot + \tau; \cdot + x_0)$ belongs to $A_{X,Y}$ if $F(\cdot;\cdot)$ belongs to $A_{X,Y}$. Using [11] (Proposition 3.2), the statements of [9] (Proposition 2.3(i), Proposition 2.4, Theorem 2.5, Theorem 2.7) and the conclusions (i)–(iii) established in [9] (pp. 234–235) can be simply reformulated for the (strongly) Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic functions. We also have the following consequence of [11] (Proposition 3.7):

Proposition 1. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ and the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. Suppose further that for each integer $j \in \mathbb{N}$, the function $F_j : \Lambda \times X \to Y$ is Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic, $\hat{F_{j;\Omega}} : \Lambda \times X \to Z$, $\hat{F_{j;\Omega}}(\cdot + b_k; x) \in C_b(\Lambda : L^p_\nu(\Omega : Y))$ and $\hat{F_{j;\Omega}}^*(\cdot; x) \in C_b(\Lambda : L^p_\nu(\Omega : Y))$ $(x \in B;\, (b_k) \in \mathrm{R})$ with the meaning clear, and that for every sequence that belongs to R, any of its subsequence also belongs to R. If $F : \Lambda \times X \to Y$ and for every $B \in \mathcal{B}$ and $(b_k) \in \mathrm{R}$ we have*

$$\lim_{(i,l)\to+\infty} \sup_{x\in B} \Big\| F_i\big(\cdot + b_{k_l}; x\big) - F\big(\cdot + b_{k_l}; x\big) \Big\|_{C_b(\Lambda : L^p_\nu(\Omega:Y))} = 0,$$

then the function $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic, $\hat{F_\Omega}(\cdot + b_k; x) \in C_b(\Lambda : L^p_\nu(\Omega : Y))$ and $\hat{F_\Omega}^*(\cdot; x) \in C_b(\Lambda : L^p_\nu(\Omega : Y))$ $(x \in B;\, (b_k) \in \mathrm{R})$.

Let us recall from the introductory part that a Stepanov-p-almost periodic function $F : \mathbb{R}^n \to Y$ need not be locally integrable for $0 < p < 1$. Moreover, if we assume that $F : \mathbb{R}^n \to Y$ is both locally integrable and Stepanov-p-almost periodic for some $p \in (0, 1)$, then it is not clear (cf. the proof of [10] (Proposition 3.5.9)) whether the expression

$$\varphi \mapsto T(\varphi) \equiv \int_{\mathbb{R}^n} \varphi(\mathbf{t}) F(\mathbf{t})\, d\mathbf{t}, \quad \varphi \in \mathcal{D}\big(\mathbb{R}^n\big)$$

determines a regular almost periodic distribution. Here, $\mathcal{D}(\mathbb{R}^n)$ stands for the space of all smooth test functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ with compact support. We will examine generalized almost periodic functions of this type somewhere else.

We continue by providing the following illustrative example:

Example 1. *In [10] (Example 9.2.7), we have constructed a piecewise continuous almost periodic function $f : \mathbb{R} \to Y$, which is not continuous and satisfies that for each $\epsilon > 0$, there exists a relatively dense subset R of $\mathbb{R}$, such that for each $\tau \in R$ and $t \in \mathbb{R}$, we have $\|f(t+\tau) - f(t)\|_Y \leq \epsilon$. This simply implies that for every $p > 0$ and for every Stepanov-p-bounded function $\nu : \mathbb{R} \to (0, \infty)$, the function $f(\cdot)$ is Stepanov-(p, ν)-almost periodic. Here and hereafter, by the Stepanov-p-boundedness of $\nu(\cdot)$, we mean that $\sup_{t\in\mathbb{R}} \int_t^{t+1} |\nu(s)|^p\, ds < +\infty$.*

Now, we will state the following simple result (cf. also [37] (Theorem 1, Corollary, p. 62)):

Proposition 2. *Suppose that $p > 0$, $q > 0$ and* (SM-1) *holds true.*

(i) If $F : \Lambda \times X \to Y$ is Stepanov-$(\Omega, q, \rho, \mathcal{B}, \Lambda')$-almost periodic and $p \leq q$, then $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda')$-almost periodic.

(ii) Suppose that $F : \Lambda \times X \to Y$, $\rho = T \in L(X)$ and for every set $B \in \mathcal{B}$ we have $\|F\|_{B,\infty} := \sup_{x\in B; \mathbf{t}\in\Lambda} \|F(\mathbf{t}; x)\|_Y < +\infty$. Then, $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda')$-almost periodic if and only if $F(\cdot;\cdot)$ is Stepanov-$(\Omega, q, \rho, \mathcal{B}, \Lambda')$-almost periodic.

Proof. We will prove only (ii) because (i) follows almost directly from the definition of Stepanov-$(\Omega, p, \rho, \mathcal{B})$-almost periodicity and an easy application of the Hölder inequality. Due to (i), it suffices to show that the assumption $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \rho, \mathcal{B})$-almost periodic, and implies that $F(\cdot;\cdot)$ is Stepanov-$(\Omega, q, \rho, \mathcal{B})$-almost periodic. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be given. Then, we know there exists $l > 0$, such that for each $\mathbf{t}_0 \in \Lambda'$, there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$, such that for every $\mathbf{t} \in \Lambda$ and $x \in B$, we have:

$$\int_{\mathbf{t}+\Omega} \big\| F(\tau + \mathbf{s}; x) - TF(\mathbf{s}; x) \big\|^p\, ds \leq \epsilon^p.$$

Since $x^p \geq x^q$ for all $x \in [0,1]$, we have

$$\int_{\mathbf{t}+\Omega}\left(\frac{\left\|F(\tau+\mathbf{s};x)-TF(\mathbf{s};x)\right\|}{\|F\|_{B,\infty}(1+\|T\|)}\right)^q ds$$
$$\leq \int_{\mathbf{t}+\Omega}\left(\frac{\left\|F(\tau+\mathbf{s};x)-TF(\mathbf{s};x)\right\|}{\|F\|_{B,\infty}(1+\|T\|)}\right)^p ds \leq \epsilon^p\Big[\|F\|_{B,\infty}(1+\|T\|)\Big]^{-p} m(\Omega).$$

This simply completes the proof. □

Before proceeding any further, we would like to note that the conclusion established in [5] (Example 2.2.3 (i)) directly follows from the conclusion established in [5] (Example 2.2.2) and Proposition 2. Now, we will state the following important corollary of Proposition 2:

Corollary 1.
(i) *Suppose that $0 < p \leq q < \infty$ and $F \in L^p_{loc}(\mathbb{R}^n : Y)$. If $F(\cdot)$ is Stepanov-q-almost periodic, then $F(\cdot)$ is Stepanov-p-almost periodic.*
(ii) *Suppose that $F \in L^\infty(\mathbb{R}^n : Y)$ and $0 < p \leq q < \infty$. Then, $F(\cdot)$ is Stepanov-p-almost periodic if and only if $F(\cdot)$ is Stepanov-q-almost periodic.*
(iii) *Suppose that $F \in BUC(\mathbb{R}^n : Y)$ and $p > 0$. Then, $F(\cdot)$ is almost periodic if and only if $F(\cdot)$ is Stepanov-p-almost periodic.*

We continue with the following illustrative example, which shows that the class of equi-Weyl-almost periodic functions is essentially larger than the union of all classes of Stepanov-p-almost periodic functions with the exponent $p > 0$; cf. [5,6] for the notion and more details about the Weyl almost periodic type functions:

Example 2.
(i) *Suppose that K is any compact subset of $\mathbb{R}^n$ with a positive Lebesgue measure. Then, we know that the function $F(\mathbf{t}) := \chi_K(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ is equi-Weyl-p-almost periodic for any exponent $p \geq 1$; see also [6] (Example 6.3.8) for a slightly stronger result. Arguing similarly as in [38] (Example 4.27), we may conclude that $F(\cdot)$ cannot be Stepanov-p-almost periodic for any exponent $p > 0$; moreover, this function cannot be Stepanov-p-almost periodic in norm for any exponent $p > 0$ (cf. Section 5 for the notion).*
(ii) *Denote by $E_\alpha(\cdot)$ the Mittag-Leffler function. If $\alpha \in (0,2) \setminus \{1\}$ and $r \in \mathbb{R} \setminus \{0\}$, then the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \in \mathbb{R}$ is not Stepanov-p-almost periodic for any exponent $p > 0$, which follows from the argument contained in the proof of [5] (Lemma 2.6.9). On the other hand, if $1 < \alpha < 2$, then the asymptotic expansion formula for the Mittag-Leffler functions (see, e.g., [5] (Theorem 1.4.1 and the Formulas (16)–(18))) shows that the formula stated on the fourth line of the proof of the above-mentioned lemma continues to hold for $t \leq -1$, as well as that the term $|\epsilon_\alpha((ir)^\alpha t^\alpha)|$ is bounded by Const.$\cdot|t|^{-\alpha}$ for $|t| \geq 1$. Throughout the proof, we have mistakenly used the constant β: we actually have $\beta = 1$ here. Keeping in mind the argument given in [10] (pp. 407–408) and the fact that equi-Weyl-p-almost periodic functions form a vector space with the usual operations, it readily follows that the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \in \mathbb{R}$ is equi-Weyl-p-almost periodic for any exponent $p \geq 1$. The situation is a little bit complicated if $0 < \alpha < 1$ because, in this case, the asymptotic expansion formula for the Mittag-Leffler functions shows that there exists a continuous function $q : \mathbb{R} \to \mathbb{C}$, such that $\lim_{|t| \to +\infty} q(t) = 0$, $E_\alpha((ir)^\alpha t^\alpha) = \alpha^{-1}(ir)^{1-\beta}e^{irt} + q(t)$ for $t \geq 0$ and $E_\alpha((ir)^\alpha t^\alpha) = q(t)$ for $t < 0$. Using again the argument given in [10] (pp. 407–408), it readily follows that the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \in \mathbb{R}$ is equi-Weyl-p-almost periodic if and only if the function $t \mapsto F(t)$, $t \in \mathbb{R}$, given by*

$$F(t) := \alpha^{-1}e^{irt}\epsilon_\alpha\big((ir)^\alpha t^\alpha\big), \quad t \geq 0; \;\; F(t) := 0, \; t < 0,$$

is equi-Weyl-p-almost periodic ($p \geq 1$). But, the last statement is not true on account of the formula given in [6] (l. 8, p. 425), so that the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \in \mathbb{R}$ is not equi-Weyl-p-almost periodic ($0 < \alpha < 1$; $p \geq 1$).

The subsequent examples, in which we provide the proper extensions of the conclusions established in [7] (Theorem 5.3.1, p. 210 and p. 212), indicate the importance of the notion introduced in Definition 2 and this paper (without going into further details, we will only mention in passing that the conclusions established here can be also clarified in the multi-dimensional setting; see, e.g., [6] (Example 6.2.9)):

Example 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an almost periodic function and there exist a finite real number $c > 0$ and an analytic function $g : \{z \in \mathbb{C} : |\Im z| < c\} \to \mathbb{C}$, such that $g(t) = f(t)$ for all $t \in \mathbb{R}$. Then, we know that the function $F(t) := sign(f(t))$, $t \in \mathbb{R}$ is Stepanov-p-almost periodic for any $p \geq 1$. Now, we will improve this result by showing that the function $F(\cdot)$ belongs to the class $S_{\Omega,\Lambda'}^{(\mathbb{F},\rho,\mathcal{P}_t,\mathcal{P})}(\Lambda : \mathbb{C})$ with $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\rho = \mathrm{I}$, $\Omega = [0,1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L_\nu^p(t + [0,1] : \mathbb{C})$ for all $t \in \mathbb{R}$, where $p > 0$ is an arbitrary exponent and $\nu : \mathbb{R} \to (0,\infty)$ is a Lebesgue measurable function satisfying the following condition:*

(LT) *For every $\epsilon > 0$, there exists a sufficiently large integer $d > 0$, such that*

$$\sup_{t\in\mathbb{R}} \int_d^{+\infty} y^{p-1} m(\{x \in [t,t+1] : \nu(x) > y\})\, dy < \epsilon,$$

where $m(\cdot)$ denotes the Lebesgue measure.

In order to achieve that, set $E_\alpha := \{x \in \mathbb{R} : |f(x)| > \alpha\}$ ($\alpha > 0$) and fix a number $\epsilon > 0$. Let $d > 0$ be an integer, such that

$$\sup_{t\in\mathbb{R}} \int_d^{+\infty} y^{p-1} m(\{x \in [t,t+1] : \nu(x) > y\})\, dy < 2^{-p} p^{-1} 2^{-1} \epsilon.$$

If $\tau \in \mathbb{R}$ is an α-almost period of the function $f(\cdot)$, then we know that $F(x+\tau) = F(x)$ for all $x \in E_\alpha$ and, because of that, we have (the Equation (6) is a consequence of (5)):

$$\begin{aligned}
&\int_t^{t+1} \big|F(x+\tau) - F(x)\big|^p \nu^p(x)\, dx \leq 2^p \int_{E_\alpha^c \cap [t,t+1]} \nu^p(x)\, dx \\
&= 2^p p \int_0^{+\infty} y^{p-1} m\big(\{x \in E_\alpha^c \cap [t,t+1] : \nu(x) > y\}\big)\, dy \\
&= 2^p p \int_0^{d} y^{p-1} m\big(\{x \in E_\alpha^c \cap [t,t+1] : \nu(x) > y\}\big)\, dy \\
&+ 2^p p \int_d^{+\infty} y^{p-1} m\big(\{x \in E_\alpha^c \cap [t,t+1] : \nu(x) > y\}\big)\, dy \\
&\leq 2^p p \int_0^{d} y^{p-1} m\big(\{x \in E_\alpha^c \cap [t,t+1] : \nu(x) > y\}\big)\, dy \\
&+ 2^p p \int_d^{+\infty} y^{p-1} m\big(\{x \in [t,t+1] : \nu(x) > y\}\big)\, dy \\
&\leq 2^p p \int_0^{d} y^{p-1} m\big(\{x \in E_\alpha^c \cap [t,t+1] : \nu(x) > y\}\big)\, dy + (\epsilon/2) \\
&\leq 2^p p \cdot m\Big(E_\alpha^c \cap [t,t+1]\Big) \int_0^d y^{p-1}\, dy + (\epsilon/2) \\
&= 2^p d^p m\Big(E_\alpha^c \cap [t,t+1]\Big) + (\epsilon/2)
\end{aligned} \tag{6}$$

for any $t \in \mathbb{R}$. By the proof of [7] (Theorem 5.3.1), we have $\lim_{\alpha\to 0+} m(E_\alpha^c \cap [t,t+1]) = 0$, uniformly in $t \in \mathbb{R}$, which simply implies the required conclusion.

Concerning the condition (LT), we would like to emphasize the following facts:

(LT1): *It is clear that condition (LT) holds provided that there exists a finite real constant $M > 0$, such that $\nu(x) \leq M$ for all $x \in \mathbb{R}$. But, condition (LT) also holds for some unbounded functions $\nu(x)$; for example, set $\nu(x) := |k|$ if $x \in [k, k + 2^{-|k|}]$ for some $k \in \mathbb{Z}$, and $\nu(x) := 1$ otherwise. Then, the validity of (LT) simply follows from the fact that $m(\{x \in [t, t+1] : \nu(x) > y\}) \leq 2^{-|k|}$, $t \in \mathbb{R}$, provided that $y \in [k, k+1]$ for some $k \in \mathbb{Z}$.*

(LT2): *Suppose that $\nu(t) = c + \zeta(t)$, $t \in \mathbb{R}$, where $c \geq 0$, $\zeta : \mathbb{R} \to (0, +\infty)$ is a Lebesgue measurable function and $\int_{-\infty}^{+\infty} \zeta^p(x)\, dx < +\infty$. Then, condition (LT) also holds, which can be shown as follows. Let a number $\epsilon > 0$ be fixed; then, it is clear that there exists a sufficiently large number $d > 1$, such that $c + d \in \mathbb{N}$ and $\int_d^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy < \epsilon$. We will prove that (LT) holds with the integer $c + d$ in place of d. In actual fact, we have:*

$$\begin{aligned}&\int_{c+d}^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \nu(x) > y\})\, dy \\ &\qquad = \int_d^{+\infty} (y + c)^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy.\end{aligned}$$

If $0 < p \leq 1$, then we have

$$\begin{aligned}&\int_d^{+\infty} (y + c)^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy \\ &\qquad \leq \int_d^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy \\ &\qquad \leq \int_d^{+\infty} y^{p-1} m(\{x \in \mathbb{R} : \zeta(x) > y\})\, dy < \epsilon.\end{aligned}$$

On the other hand, if $p > 1$, then there exists a real number $c_p > 0$, such that

$$\begin{aligned}&\int_d^{+\infty} (y + c)^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy \\ &\leq c_p \left[\int_d^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy + \int_d^{+\infty} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy \right] \\ &\leq 2c_p \int_d^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \zeta(x) > y\})\, dy < 2c_p \epsilon.\end{aligned}$$

This simply implies the required conclusion. We can similarly prove that condition (LT) holds for the function $c + \nu(\cdot)$, if (LT) holds for the function $\nu(\cdot)$.

Then, it can be simply shown that condition (LT) does not hold if $\nu(t) = e^{\sigma|t|}$, $t \in \mathbb{R}$ or $\nu(t) = (1 + |t|)^{\sigma}$, $t \in \mathbb{R}$ for some real number $\sigma > 0$. The following condition is similar to (LT) but it holds for any p-locally integrable function $\nu(\cdot)$, as easily explained:

(LT-K) *For every $\epsilon > 0$ and for every compact set $K \subseteq \mathbb{R}$, there exists a sufficiently large integer $d > 0$, such that for every $t \in K$, we have $\int_d^{+\infty} y^{p-1} m(\{x \in [t, t+1] : \nu(x) > y\})\, dy < \epsilon$.*

If (LT-K) holds, then we can similarly prove that the function $F(\cdot)$ is Stepanov-Levitan-(N, ν)-almost periodic in the following sense:

(SL-1) *For every $\epsilon > 0$ and $N > 0$, there exists a relatively dense subset $R_{\epsilon,N} \subseteq \mathbb{R}$ of Stepanov–Levitan-(N, ν, ϵ)-almost periods of $F(\cdot)$, which means that if, $\tau \in R_{\epsilon,N}$ and $|t| \leq N$, then $\int_t^{t+1} \left|F(x + \tau) - F(x)\right|^p \nu^p(x)\, dx \leq \epsilon$.*

(SL-2) *For every $\epsilon > 0$ and $N > 0$, there exist a number $\delta > 0$ and a relatively dense subset $R_{\delta,N} \subseteq \mathbb{R}$ of Stepanov–Levitan-(N, ν, δ)-almost periods of $F(\cdot)$ such that $R_{\delta,N} \pm R_{\delta,N} \subseteq R_{\epsilon,N}$.*

We will consider this class of generalized Levitan N-almost periodic functions somewhere else (cf. [10] for further information in this direction).

Example 4. *Suppose that $\alpha,\ \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. Then, we know that the functions*

$$f(t) = \sin\left(\frac{1}{2+\cos\alpha t + \cos\beta t}\right), \quad t \in \mathbb{R}$$

and

$$g(t) = \cos\left(\frac{1}{2+\cos\alpha t + \cos\beta t}\right), \quad t \in \mathbb{R}$$

are Stepanov-p-almost periodic but not almost periodic ($1 \leq p < \infty$). Suppose now that $p > 0$ is an arbitrary exponent and that the function $\nu : \mathbb{R} \to (0,\infty)$ is Stepanov-p-bounded and satisfies (LT); for example, the function $\nu(\cdot)$ constructed in the final part of the previous example enjoys these features. Then, the functions $f(\cdot)$ and $g(\cdot)$ belong to the class $S^{(\mathbb{F},\rho,\mathcal{P}_t,\mathcal{P})}_{\Omega,\Lambda'}(\Lambda : Y)$ with $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\rho = \mathrm{I}$, $\Omega = [0,1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L^p_\nu(t+[0,1] : \mathbb{C})$ for all $t \in \mathbb{R}$.

We will prove this fact only for the function $f(\cdot)$ with $\alpha = 1$ and $\beta = \sqrt{2}$. In order to achieve that, set $E_\alpha := \{x \in \mathbb{R} : |2+\cos x + \cos(\sqrt{2}x)| > \alpha\}$ ($\alpha > 0$) and fix a number $\epsilon > 0$. Let $\alpha > 0$ and $\delta > 0$ be arbitrary real numbers, such that $\alpha > \delta$. Arguing as in [7] (see p. 212), it readily follows that for every δ-almost period $\tau \in \mathbb{R}$ of the function $2+\cos(\cdot)+\cos(\sqrt{2}\cdot)$ and for every $t \in \mathbb{R}$, we have:

$$\int_t^{t+1} \left|f(x+\tau) - f(x)\right|^p \nu^p(x)\,dx \leq \left[\frac{\delta}{\alpha(\alpha-\delta)}\right]^p \|\nu\|_{S^p} + 2^p \int_{E^c_\alpha \cap [t,t+1]} \nu^p(x)\,dx,$$

where $\|\nu\|_{S^p} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \nu^p(x)\,dx$. Then, the final conclusion follows similarly as in [7] and the previous example (if the function $\nu(\cdot)$ is Stepanov-p-bounded, then the functions $f(\cdot)$ and $g(\cdot)$ are Stepanov–Levitan-(N,ν)-almost periodic).

3.1. Relations between Piecewise Continuous Almost Periodic Functions and Metrically Stepanov-p-Almost Periodic Functions ($p > 0$)

In our recent joint research article [30] with W.-S. Du and D. Velinov, we recently analyzed certain relations between piecewise continuous almost periodic functions (piecewise continuous uniformly recurrent functions) and Stepanov almost periodic functions (Stepanov uniformly recurrent functions). We start this subsection by recalling the following notion:

Definition 6. *(cf. [30] (Definition 6 (i))) Suppose that ρ is a binary relation on Y, the function $F : \mathbb{R} \times X \to Y$ [$F : [0,\infty) \times X \to Y$] satisfies that for every $x \in X$, the function $t \mapsto F(t;x)$, $t \in \mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$ [$(t_k)_{k\in\mathbb{N}}$]. Suppose further that $(t_k)_{k\in\mathbb{Z}}$ [$(t_k)_{k\in\mathbb{N}}$] satisfies that $\delta_0 := \inf_{k\in\mathbb{Z}}(t_{k+1}-t_k) > 0$ [$\delta_0 := \inf_{k\in\mathbb{N}}(t_{k+1}-t_k) > 0$]. Then, we say that the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B},\rho,(t_k))$-piecewise continuous almost periodic if and only if for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a relatively dense set S in $\mathbb{R}$ [in $[0,\infty)$], such that if $\tau \in S$, $x \in B$ and $t \in \mathbb{R}$ satisfies $|t-t_k| > \epsilon$ for all $k \in \mathbb{Z}$ [$k \in \mathbb{N}$], then there exists $y_{t,x} \in \rho(F(t;x))$, such that $\|F(t+\tau;x) - y_{t,x}\| < \epsilon$.*

The following result provides, even for the usually considered exponents $p \geq 1$, an extension of [30] (Theorem 1) for pre-$(\mathcal{B},T,(t_k))$-piecewise continuous almost periodic functions (the extension for pre-$(\mathcal{B},T,(t_k))$-piecewise continuous uniformly recurrent functions can be deduced in a similar manner):

Theorem 1. *Suppose that $\rho = T \in L(Y)$, $p > 0$, $F : \Lambda \times X \to Y$ is pre-$(\mathcal{B},T,(t_k))$-piecewise continuous almost periodic, where $\Lambda = \mathbb{R}$ or $\Lambda = [0,\infty)$, and for every $B \in \mathcal{B}$, $\|F\|_{\infty,B} \equiv \sup_{t\in\Lambda, x\in B}\|F(t;x)\| < +\infty$. Suppose further that the function $\nu : \mathbb{R} \to (0,\infty)$ is Stepanov-p-bounded and satisfies the following condition:*

(LQ) *For every $\epsilon > 0$, there exist $d > 0$ and $\epsilon_0 > 0$, such that for every $x \in \mathbb{R}$ and for every Lebesgue measurable set $\Omega \subseteq [x, x+1]$, such that $m(\Omega) < \epsilon_0$, we have $\int_d^{+\infty} y^{p-1} m(\{x \in \Omega : \nu(x) > y\})\, dy < \epsilon$.*

Then, the function $F(\cdot;\cdot)$ belongs to the class $S_{\Omega,\Lambda'}^{(\mathbb{F},T,\mathcal{P}_t,\mathcal{P})}(\Lambda : Y)$ with $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\Omega = [0,1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L_\nu^p(t + [0,1] : \mathbb{C})$ for all $t \in \mathbb{R}$.

Proof. Without loss of generality, we may assume that $\Lambda = \mathbb{R}$ and $T = c\mathrm{I}$ for some $c \in \mathbb{C}$. Let a number $\epsilon > 0$ and a set $B \in \mathcal{B}$ be given. Suppose that a point $x \in \mathbb{R}$ is fixed and the interval $[x, x+1]$ contains the possible first kind discontinuities of functions $F(\cdot; b)$ at the points $\{t_m, \ldots, t_{m+k}\} \subseteq [x, x+1]$ ($b \in X$); then, we clearly have $k \leq \lceil 1/\delta_0 \rceil$. Let the numbers $d > 0$ and $\epsilon_0 > 0$ be determined from condition (LQ), with the number ϵ replaced therein with the number $\epsilon / (2((1+|c|)\|F\|_{\infty,B})^p p)$. Then, let S be a relatively dense set in $\mathbb{R}$, such that if $\tau \in S$ and $b \in B$, then $\|F(t+\tau; x) - cF(t; x)\| < \epsilon_1$ for all $t \in \mathbb{R}$, such that $|t - t_k| > \epsilon_1$, $k \in \mathbb{Z}$, where the number $\epsilon_1 \in (0, \epsilon_0 / 2\lceil 1/\delta_0 \rceil)$ will be precisely clarified a bit later. The function $t \mapsto F(t+\tau; b) - cF(t; b)$, $t \in [x, x+1]$ is not greater than ϵ_1 if $t \in [x, t_m - \epsilon_1] \cup (t_m + \epsilon_1, t_{m+1} - \epsilon_1] \cup \ldots \cup (t_{m+k}, x+1]$; otherwise, $\|F(t+\tau; b) - cF(t; b)\| \leq (1+|c|)\|F\|_{\infty,B}$. Using Lemma 1, the above implies

$$
\begin{aligned}
&\int_x^{x+1} \|F(t+\tau; b) - cF(t; b)\|^p \nu^p(t)\, dt \\
&\leq \epsilon_1^p \int_{A_x} \nu^p(t)\, dt + ((1+|c|)\|F\|_{\infty,B})^p \int_{[x,x+1] \setminus A_x} \nu^p(t)\, dt \\
&\leq \epsilon_1^p \int_x^{x+1} \nu^p(t)\, dt + ((1+|c|)\|F\|_{\infty,B})^p p \int_0^\infty y^{p-1} m(\{s \in [x, x+1] \setminus A_X : \nu(s) > y\})\, dy \\
&\leq \epsilon_1^p \|\nu\|_{S^p} + ((1+|c|)\|F\|_{\infty,B})^p p \int_0^d y^{p-1} m(\{s \in [x, x+1] \setminus A_X : \nu(s) > y\})\, dy \\
&+ ((1+|c|)\|F\|_{\infty,B})^p p \int_d^\infty y^{p-1} m(\{s \in [x, x+1] \setminus A_X : \nu(s) > y\})\, dy \\
&\leq \epsilon_1^p \|\nu\|_{S^p} + 2((1+|c|)\|F\|_{\infty,B})^p d^p \lceil 1/\delta_0 \rceil \epsilon_1 \\
&+ ((1+|c|)\|F\|_{\infty,B})^p p \int_d^\infty y^{p-1} m(\{s \in [x, x+1] \setminus A_X : \nu(s) > y\})\, dy \\
&\leq \epsilon_1^p \|\nu\|_{S^p} + 2((1+|c|)\|F\|_{\infty,B})^p d^p \lceil 1/\delta_0 \rceil \epsilon_1 + \frac{\epsilon}{2}, \quad b \in B,
\end{aligned}
$$

where $A_x := [x, t_m - \epsilon_0] \cup (t_m + \epsilon_0, t_{m+1} - \epsilon_0] \cup \ldots \cup (t_{m+k}, x+1]$. This simply completes the proof of theorem since we can always find a sufficiently small number $\epsilon_1 > 0$, such that

$$
\epsilon_1^p \|\nu\|_{S^p} + 2((1+|c|)\|F\|_{\infty,B})^p d^p \lceil 1/\delta_0 \rceil \epsilon_1 < \frac{\epsilon}{2}.
$$

□

It is clear that condition (LT) implies condition (LQ), so that we can use the weight function $\nu(\cdot)$ constructed in Example 4 here.

The subsequent result follows from [30] (Theorem 3) and the argument contained in the proof of [30] (Theorem 2). The only thing worth noting is that if $0 < p < 1$, then we should replace the number η_k^p with the number η_k throughout the proof of Theorem 2, and assume that $\eta_k \in (0, (\epsilon/4)^p \delta)$ for all $k \in \mathbb{N}$:

Theorem 2. *Suppose that $\rho = T \in L(Y)$, $p > 0$ and $F : \Lambda \times X \to Y$ is a Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda')$-almost periodic function, where $\Lambda = \mathbb{R}$ or $\Lambda = [0, \infty)$, $\Omega = [0,1]$ and $\Lambda' = \Lambda$. Suppose further that $F(\cdot;\cdot)$ satisfies that for every $x \in X$, the function $t \mapsto F(t; x)$, $t \in \mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k \in \mathbb{Z}}$ [$(t_k)_{k \in \mathbb{N}}$] and $\delta_0 := \inf_{k \in \mathbb{Z}} (t_{k+1} - t_k) > 0$ [$\delta_0 := \inf_{k \in \mathbb{N}} (t_{k+1} - t_k) > 0$]. If the condition*

(QUC) *For every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $\delta > 0$, such that if $x \in B$ and the points t' and t'' belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ [$i \in \mathbb{N}_0$; $t_0 \equiv 0$] and $|t' - t''| < \delta$, then $\|F(t'; x) - F(t''; x)\| < \epsilon$.*

holds, then $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, T, (t_k))$-piecewise continuous almost periodic. Furthermore, if $T = \mathrm{I}$, $F(\cdot;\cdot)$ is continuous and any set of collection $\mathcal{B}$ is compact, then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic, i.e., for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $l > 0$, such that for each $t_0 \in \Lambda$, there exists $\tau \in B(t_0, l) \cap \Lambda$, such that for every $t \in \Lambda$ and $x \in B$, we have $\|F(t+\tau; x) - F(t; x)\| \leq \epsilon$.

If $p \geq 1$, then it is well known that any uniformly continuous, Stepanov-p-almost periodic function $F : \mathbb{R}^n \to Y$ is almost periodic and therefore bounded. Furthermore, we know that there exists a Stepanov-1-almost periodic function $F : \mathbb{R} \to \mathbb{R}$, which is not uniformly continuous (bounded), see, e.g., [39]. In connection with this problem, we would like to state the following result: the proof can be deduced using the argument that is very similar to the argument used for proving [30] (Theorems 2 and 3) given in the one-dimensional setting, and therefore omitted:

Corollary 2.

(i) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is continuous, any set of collection $\mathcal{B}$ is compact and $F(\cdot;\cdot)$ is uniformly continuous on the set $\mathbb{R}^n \times B$ ($B \in \mathcal{B}$). If $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \mathrm{I}, \mathcal{B}, \mathbb{R}^n)$-almost periodic function, where $p > 0$ and $\Omega = [0,1]^n$, then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

(ii) *Suppose that $p > 0$. Then, any uniformly continuous, Stepanov-p-almost periodic function $F : \mathbb{R}^n \to Y$ is almost periodic.*

3.2. The Invariance of Stepanov-p-Almost Periodicity under the Actions of the Infinite Convolution Products ($0 < p < 1$)

In series of our recent research articles, we have examined the invariance of Stepanov-p-almost periodicity under the actions of the infinite convolution products and provide various applications of the usually considered classes of Stepanov-p-almost periodic (automorphic) type functions with the exponent $p \geq 1$. In this subsection, we will analyze the invariance of Stepanov-p-almost periodicity under the action of the infinite convolution product

$$t \mapsto F(t) \equiv \int_{-\infty}^{t} R(t-s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{7}$$

where $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying certain extra conditions ($0 < p < 1$). For the sake of brevity, here we will analyze the one-dimensional setting only.

The consideration from [5] (Proposition 2.6.11), where we have analyzed the case $p \geq 1$, is essential but cannot be replicated or modified in our new framework since the reverse Hölder inequality is valid for $0 < p < 1$. In order to overcome this difficulty, we must impose some new unpleasant conditions; for example, we can prove the following:

Theorem 3. *Suppose that $\nu : \mathbb{R} \to (0,\infty)$ is a Lebesgue measurable function and there exists a function $\omega : \mathbb{R} \to [0,\infty)$, such that $\nu(x+y) \leq \nu(x)\omega(y)$ for all $x,\ y \in \mathbb{R}$. Suppose further that $f : \mathbb{R} \to X$ is Stepanov-(p, T, ν)-almost periodic, where $\rho = T \in L(Y)$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying the following conditions:*

(i) *There exists a finite real number $s \geq 1$, such that $\sum_{k=0}^{\infty} \|\omega(\cdot)R(\cdot)\|_{L^{s'}[k,k+1]} < +\infty$, where $1/s + 1/s' = 1$.*

(ii) *There exists a finite real number $M > 0$, such that for every $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$, we have*

$$\sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|f(t-r)\|\, dr$$
$$+ \sum_{k=0}^{\infty} \max\Biggl\{ \Bigl[\|R(r)\| \cdot \|f(t+\tau-r) - Tf(t-r)\| \nu(t-r)\omega(r) \Bigr] :$$
$$r \in [k,k+1],\ \|f(t+\tau-r) - Tf(t-r)\|\nu(t-r) > 1 \Biggr\} \leq M.$$

Then, the function $F(\cdot)$, given by (7), is (T,ν)-almost periodic, provided that

(iii) $\lim_{\delta\to 0}\sup_{t\in\mathbb{R}}\int_t^{t+1}\|f(s+\delta)-Tf(s)\|^p\nu^p(s)\,ds=0;$

here, by (T,ν)-almost periodicity of $F(\cdot)$, we mean that $F(\cdot)$ is continuous and for each $\epsilon>0$ there exists a relatively dense set R in $\mathbb{R}$, such that for each $\tau\in R$ and $t\in\mathbb{R}$, we have $\|F(t+\tau)-TF(t)\|_Y\nu(t)\leq\epsilon$. Furthermore, the condition (iii) *holds provided that the function $\omega(\cdot)$ is bounded and $T=\mathrm{I}$.*

Proof. It can be simply shown that the function $F(\cdot)$ is well defined and the integral that defines $F(\cdot)$ is absolutely convergent, since we have

$$\left\|\int_{-\infty}^{t}R(t-s)f(s)\,ds\right\|_Y=\left\|\int_0^{\infty}R(s)f(t-s)\,ds\right\|_Y\leq\int_0^{\infty}\|R(s)\|\cdot\|f(t-s)\|\,ds\leq M;$$

cf. (ii). Let R be a relatively dense subset of $\mathbb{R}$, such that for each $\tau\in R$ and $t\in\mathbb{R}$, we have $\int_t^{t+1}\|f(s+\tau)-Tf(s)\|^p\nu^p(s)\,ds\leq\epsilon$. Fix a number $t\in\mathbb{R}$. Then, the Lebesgue measure of the set $B_t\equiv\{s\in[t,t+1]:\|f(s+\tau)-Tf(s)\|\cdot\nu(s)>1\}$ is less than or equal to ϵ; hence, we have $(\tau\in R)$:

$$\begin{aligned}
&\|F(t+\tau)-TF(t)\|_Y\nu(t)\leq\sum_{k=0}^{\infty}\int_k^{k+1}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t)\,dr\\
&=\sum_{k=0}^{\infty}\int_{B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&+\sum_{k=0}^{\infty}\int_{[k,k+1]\setminus B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&\leq\sum_{k=0}^{\infty}\int_{B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&+\sum_{k=0}^{\infty}\int_{[k,k+1]\setminus B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|^{p/s}\nu^{p/s}(t-r)\omega(r)\,dr\\
&\leq\sum_{k=0}^{\infty}\int_{B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&+\sum_{k=0}^{\infty}\int_{[k,k+1]}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|^{p/s}\nu^{p/s}(t-r)\omega(r)\,dr\\
&\leq\sum_{k=0}^{\infty}\int_{B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&+\sum_{k=0}^{\infty}\|\omega(\cdot)R(\cdot)\|_{L^{s'}[k,k+1]}\left(\int_k^{k+1}\|f(t-r+\tau)-Tf(t-r)\|^p\nu^p(t-r)\,dr\right)^{1/s}\\
&\leq\sum_{k=0}^{\infty}\int_{B_{t-k-1}}\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\,dr\\
&+\epsilon\sum_{k=0}^{\infty}\|\omega(\cdot)R(\cdot)\|_{L^{s'}[k,k+1]}\\
&\leq\epsilon\sum_{k=0}^{\infty}\max\Bigl\{\Bigl[\|R(r)\|\cdot\|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)\omega(r)\Bigr]:\\
&r\in[k,k+1],\ \|f(t+\tau-r)-Tf(t-r)\|\nu(t-r)>1\Bigr\}\\
&+\epsilon\sum_{k=0}^{\infty}\|\omega(\cdot)R(\cdot)\|_{L^{s'}[k,k+1]}\leq\left(M+\sum_{k=0}^{\infty}\|\omega(\cdot)R(\cdot)\|_{L^{s'}[k,k+1]}\right)\epsilon.
\end{aligned}$$

The continuity of function $F(\cdot)$ can be proved as above, using the condition (iii) and replacing the number τ with the number δ throughout the above computation. It remains to be proven that (ii) holds provided that the function $\omega(\cdot)$ is bounded and $T = \mathrm{I}$, cf. also [29] (p. 403) for the case $\nu(\cdot) \equiv \omega(\cdot) \equiv 1$. Clearly, for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\delta \in (-1, 1)$, we have:

$$\begin{aligned}
&\int_t^{t+1} \|f(s+\delta) - f(s)\|^p \nu^p(s)\, ds \leq \int_t^{t+1} \|f(s+\delta) - f(s+\delta+\tau)\|^p \nu^p(s)\, ds \\
&+ \int_t^{t+1} \|f(s+\delta+\tau) - f(s+\tau)\|^p \nu^p(s)\, ds + \int_t^{t+1} \|f(s+\tau) - f(s)\|^p \nu^p(s)\, ds \\
&\leq \int_t^{t+1} \|f(s+\delta) - f(s+\delta+\tau)\|^p \nu^p(s+\delta) \omega^p(-\delta)\, ds \\
&+ \int_t^{t+1} \|f(s+\delta+\tau) - f(s+\tau)\|^p \nu^p(s+\tau) \omega^p(-\tau)\, ds \\
&+ \int_t^{t+1} \|f(s+\tau) - f(s)\|^p \nu^p(s)\, ds \\
&\leq \Big(1 + \|\omega\|_\infty\Big) \Bigg[\int_t^{t+1} \|f(s+\delta) - f(s+\delta+\tau)\|^p \nu^p(s+\delta)\, ds \\
&+ \int_t^{t+1} \|f(s+\delta+\tau) - f(s+\tau)\|^p \nu^p(s+\tau)\, ds + \int_t^{t+1} \|f(s+\tau) - f(s)\|^p \nu^p(s)\, ds \Bigg].
\end{aligned}$$

For a given $\epsilon > 0$, we can find a real number $l > 3$, such that any interval $I \subseteq \mathbb{R}$ contains a number $\tau \in I$, such that $\int_t^{t+1} \|f(s+\tau) - f(s)\|^p \nu^p(s)\, ds \leq \epsilon/3$ for all $t \in \mathbb{R}$. Fix now a real number t. Then, we can always find a number $\tau \in \mathbb{R}$, such that the last inequality holds and $s + \tau \subseteq [0, l]$ for all $s \in [t, t+1]$. The first addend and the third addend in the above sum can be simply estimated by $\epsilon/3$. This can be also performed for the second addend in the above sum, since we can argue as in the proof of [5] (Proposition 3.5.3), by choosing a sequence of infinitely differentiable functions (φ_k), which converge to $[f(\cdot+\tau) - f(\cdot)]\nu(\cdot)$ in $L^1[0, l]$ and apply the Hölder inequality and the same procedure after that. □

We continue by stating the following result, which is not so easily comparable to Theorem 3 or [33] (Proposition 2.3):

Proposition 3. *Suppose that* $p \in [1, \infty)$, $1/p + 1/q = 1$, *there exists a finite real constant* $c > 0$, *such that* $\nu(t) \geq c > 0$, $t \in \mathbb{R}$, *that* $\nu : \mathbb{R} \to (0, \infty)$ *is a Lebesgue measurable function and there exists a function* $\omega : \mathbb{R} \to [0, \infty)$, *such that* $\nu(x + y) \leq \nu(x)\omega(y)$ *for all* x, $y \in \mathbb{R}$. *Suppose further that* $f : \mathbb{R} \to X$ *is Stepanov-*(p, T, ν)*-almost periodic, where* $\rho = T \in L(Y)$, *and* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family satisfying the following conditions:*

(i) $\sum_{k=0}^{\infty} \|[1 + \omega(\cdot)]R(\cdot)\|_{L^q[k,k+1]} < +\infty$.

(ii) *For every* $t \in \mathbb{R}$, *we have*

$$\sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|f(t-r)\|\, dr < +\infty.$$

Then, the function $F(\cdot)$, *given by* (7), *is* (T, ν)*-almost periodic. Specifically, the function* $F(\cdot)$ *is* T*-almost periodic and almost periodic.*

Proof. We will provide the main details of the proof since it can be given with the help of the argumentation employed for proving [5] (Proposition 2.6.11) and [33] (Proposition 2.3). Due to (ii), the function $F(\cdot)$ is well defined. Observe that the condition $\nu(t) \geq c > 0$, $t \in \mathbb{R}$ implies that the function $f(\cdot)$ is Stepanov-(p, T)-almost periodic and therefore Stepanov-p-almost periodic in the usual sense. Since $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < +\infty$, the continuity and almost periodicity of $F(\cdot)$ follows directly from an application of [5] (Proposition 2.6.11). The remainder of proof follows similarly as in the proof of the above-mentioned result and therefore omitted. $\square$

Remark 2. *It is worth noting that many statements for Stepanov-p-almost periodic functions with the exponent $p \geq 1$, which can be deduced without the help of the Hölder inequality, continue to hold for Stepanov-p-almost periodic functions with the exponent $p \in (0,1)$; for example, this is the case with the statements of [5] (Theorem 2.6.17 (i)) and [6] (Theorem 6.2.15, Proposition 6.2.22, Theorem 6.2.30, Corollary 6.2.31). On the other hand, the statements of [6] (Proposition 6.2.18, Proposition 6.2.19), where the Hölder inequality is essentially employed in the proofs, cannot be clarified for Stepanov-p-almost periodic functions with the exponent $p \in (0,1)$. Especially, we would like to emphasize that the composition principles established in [40] (Theorem 2.2), [33] (Theorem 2.2) and [6] (Theorem 6.2.32, Theorem 6.2.33) cannot be clarified for Stepanov-p-almost periodic functions with the exponent $p \in (0,1)$.*

Finally, we would like to note that it is almost impossible to state any relevant result concerning the invariance of Stepanov p-almost periodicity (automorphy) in norm under the actions of the infinite convolution product (7); cf. the subsequent two sections for the notion.

4. Stepanov-p-Almost Automorphic Type Functions ($p > 0$)

The main aim of this section is to introduce and analyze the following classes of Stepanov-p-almost automorphic functions:

Definition 7. *Suppose that $p > 0$, $\nu : \Omega \to (0,\infty)$ is a Lebesgue measurable function, $F : \mathbb{R}^n \times X \to Y$ is a given function and R is a certain collection of sequences in $\mathbb{R}^n$. Then, we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic (Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, W_{\mathcal{B},\mathrm{R}})$-almost automorphic; Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, P_{\mathcal{B},\mathrm{R}})$-almost automorphic) if and only if $F(\cdot;\cdot)$ is Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}})$-multi-almost automorphic (Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic; Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic) with $Z = L^p_\nu(\Omega : Y)$.*

Any strongly Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic function $F : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic.

Definition 8. *Suppose that $p > 0$ and that $\nu : \Omega \to (0,\infty)$ is a Lebesgue measurable function. Then, we say that a function $F : \mathbb{R}^n \to Y$ is Stepanov-(Ω, p, ν)-almost automorphic (Stepanov-(Ω, p, ν, W)-almost automorphic) if and only if $F(\cdot)$ is Stepanov-$(\Omega, p, \mathrm{R}, \nu)$-almost automorphic (Stepanov-$(\Omega, p, \mathrm{R}, \nu, W_{\mathrm{R}})$-almost automorphic), with R being the collection of all sequences in $\mathbb{R}^n$. $F(\cdot)$ is Stepanov-p-almost automorphic if and only if $F(\cdot)$ is Stepanov-(Ω, p, ν)-almost automorphic with $\Omega = [0,1]^n$ and $\nu(\cdot) \equiv 1$.*

The statements of [12] (Theorem 2.4, Propositions 2.5, 2.7 and 2.8, Theorem 2.9), established recently for general classes of metrically Stepanov almost automorphic functions, can be simply reformulated for the special classes of Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic functions, since the metric space $Z = L^p_\nu(\Omega : Y)$ satisfies all necessary requirements for the application of these results. For example, we have the following (Proposition 2.8):

Proposition 4. *Suppose that $F_m : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic (Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, W_{\mathcal{B},\mathrm{R}})$-almost automorphic; Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, P_{\mathcal{B},\mathrm{R}})$-almost automorphic) for all $m \in \mathbb{N}$, $F : \mathbb{R}^n \times X \to Y$ and $\lim_{m\to+\infty} F_m(\mathbf{t} + \cdot; x) = F(\mathbf{t} + \cdot; x)$ for the topology of $L^p_\nu(\Omega : Y)$, uniformly on the set $\mathbb{R}^n \times B$ for each $B \in \mathcal{B}$. Suppose further that for every sequence $b \in \mathrm{R}$, all of its subsequences also belong to R. Then, $F(\cdot;\cdot)$ is likewise Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic (Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, W_{\mathcal{B},\mathrm{R}})$-almost automorphic; Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, P_{\mathcal{B},\mathrm{R}})$-almost automorphic).*

The following analogue of Corollary 2 can be deduced using the argument contained in the proof of [41] (Proposition 3.1) and a relatively simple argument involving the compactness of sets of the collection $\mathcal{B}$ and the condition (LB) clarified below (cf. also the proof of Theorem 5):

Corollary 3.

(i) *Suppose that $p > 0$, $\Omega = [0,1]^n$, $\nu : \Omega \to (0,\infty)$ is a Lebesgue measurable function, R is any collection of sequences in $\mathbb{R}^n$ that satisfies that for each sequence in R, any of its subsequences also belong to R, $F : \mathbb{R}^n \times X \to Y$ is continuous, any set of collection $\mathcal{B}$ is compact, $F(\cdot; x)$ is uniformly continuous on $\mathbb{R}^n$ for every fixed element $x \in X$ and the following condition holds:*

(LB) For every set $B \in \mathcal{B}$, there exists a finite real number $L_B > 0$, such that $\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\|_Y \leq L_B \|x - y\|$, $\mathbf{t} \in \mathbb{R}^n$, x, $y \in B$.

If $F(\cdot;\cdot)$ is Stepanov $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost automorphic, then $F(\cdot;\cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.

(ii) *Suppose that $p > 0$, $\Omega = [0,1]^n$ and $\nu : \Omega \to (0,\infty)$ is a Lebesgue measurable function. Then, any uniformly continuous Stepanov-(Ω, p, ν)-almost automorphic function $F : \mathbb{R}^n \to Y$ is almost automorphic.*

Before proceeding to the next subsection, we would like to note that Corollary 1 continues to hold for Stepanov-p-almost automorphic functions ($p > 0$). This essentially follows from the Hölder inequality and the fact that for every fixed number $\mathbf{t} \in \mathbb{R}^n$, the assumptions $\lim_{l \to +\infty} \int_{[0,1]^n} \|F(\mathbf{t} + b_{k_l} + \mathbf{s}) - [F^*(\mathbf{t})](\mathbf{s})\|^p \, d\mathbf{s} = 0$ and $\lim_{l \to +\infty} \int_{[0,1]^n} \|[F^*(\mathbf{t} - b_{k_l})](\mathbf{s}) - F(\mathbf{t} + \mathbf{s})\|^p d\mathbf{s} = 0$ imply the existence of a set $N \subseteq [0,1]^n$ with the Lebesgue zero measure, such that $\lim_{l \to +\infty} F(\mathbf{t} + b_{k_l} + \mathbf{s}) = [F^*(\mathbf{t})](\mathbf{s})$ and $\lim_{l \to +\infty} [F^*(\mathbf{t} - b_{k_l})](\mathbf{s}) = F(\mathbf{t} + \mathbf{s})$ for all $\mathbf{s} \in [0,1]^n \setminus N$. After this, we may conclude that the essential boundedness of $F(\cdot)$ implies the essential boundedness of $F^*(\cdot)$ and argue as in the proof of Proposition 2(ii). This argument also provides the affirmative answer to the problem proposed in our earlier joint research study with T. Diagana.

4.1. Relations between Piecewise Continuous Almost Automorphic Functions and Metrically Stepanov-p-Almost Periodic Functions ($p > 0$)

We start this subsection by observing that the notion of a Bochner spatially almost automorphic sequence $(t_k)_{k \in \mathbb{Z}}$ has recently been introduced by L. Qi and R. Yuan in [42] (Definition 3.1), who proved that any Wekler sequence $(t_k)_{k \in \mathbb{Z}}$ is Bochner spatially almost automorphic and that the converse statement is not true in general. The authors have analyzed the classes of Bohr, Bochner and Levitan piecewise continuous almost automorphic functions; in [42] (Theorem 4.8), the authors proved that these classes coincide (cf. also the research article [43] by W. Dimbour and V. Valmorin for the notion of S-almost automorphy). Moreover, in [42] (Theorem 8.2), the authors proved an essential relationship between piecewise continuous almost automorphic functions and Stepanov-p-almost automorphic functions ($p \geq 1$). In order to further study the relations between piecewise continuous almost automorphic functions and metrically Stepanov-p-almost automorphic functions ($p > 0$), we need to introduce the following, rather general, notion (cf. also condition (iii) in [42] (Definition 4.2)):

Definition 9. *Suppose that R is a certain collection of real sequences and the function $F : \mathbb{R} \times X \to Y$ satisfies that for every $x \in X$, the function $t \mapsto F(t; x)$, $t \in \mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k \in \mathbb{Z}}$. Let (t'_k) be any real sequence. Then, we say that the function $F(\cdot;\cdot)$ is pre-$(\mathrm{R}, \mathcal{B}, (t_k), (t'_k))$-piecewise continuous almost automorphic if and only if for every $B \in \mathcal{B}$ and $(b_k) \in \mathrm{R}$, there exist a subsequence (b_{k_l}) of (b_k) and a function $F^*_B : \mathbb{R} \times X \to Y$, such that for every $x \in B$, the function $t \mapsto F^*_B(t; x)$, $t \in \mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of sequence $(t'_k)_{k \in \mathbb{Z}}$, $\lim_{l \to +\infty} F(t + b_{k_l}; x) = F^*_B(t; x)$ for all $t \in \mathbb{R} \setminus \{t'_{k_l} : l \in \mathbb{Z}\}$ and $\lim_{l \to +\infty} F^*_B(t - b_{k_l}; x) = F(t; x)$ for all $t \in \mathbb{R} \setminus \{t_{k_l} : l \in \mathbb{Z}\}$.*

The next simple result follows directly by applying the dominated convergence theorem with $[F_B^*(t;x)](u) \equiv F_B^*(t+u;x)$:

Proposition 5. *Suppose that $p>0$, R is a certain collection of real sequences and the function $F:\mathbb{R}\times X\to Y$ satisfies that for every $x\in X$, the function $t\mapsto F(t;x)$, $t\in\mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$. Let (t_k') be any real sequence and let $\sup_{t\in\mathbb{R};x\in B}\|F(t;x)\|_Y<+\infty$ ($B\in\mathcal{B}$). If the function $F(\cdot;\cdot)$ is pre-$(\mathrm{R},\mathcal{B},(t_k),(t_k'))$-piecewise continuous almost automorphic, then $F(\cdot;\cdot)$ is Stepanov $(\Omega,p,\mathrm{R},\mathcal{B},\nu)$-almost automorphic for any function $\nu\in L^p(\Omega)$.*

Remark 3. *In [43] (Definition 2.3), the authors introduced the notion of $\mathbb{S}$-almost automorphy for a function $F:\mathbb{R}\to Y$, where $\mathbb{S}$ is any subset of $\mathbb{R}$. In this slightly different approach, the authors have not used the assumption that the limit function $F^*(\cdot)$ is piecewise continuous. Without going into further details, we will only note here that an analogue of Proposition 5 can be simply formulated for $\mathbb{S}$-almost automorphic functions, provided that the Lebesgue measure of the set $\mathbb{S}$ is equal to zero and R is the collection of all sequences with values in $\mathbb{S}$.*

The notion of a Levitan piecewise continuous almost automorphic function was introduced previously in [42] (Definition 4.6). Now, we would like to extend this notion by introducing the following general class of functions (we use the condition (iii) from this definition only in a slightly modified form):

Definition 10. *Suppose that $F:\mathbb{R}\times X\to Y$ satisfies that for every $x\in X$, the function $t\mapsto F(t;x)$, $t\in\mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$. If ρ is a binary relation on Y, then we say that the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B},\rho,(t_k))$-Levitan piecewise continuous almost automorphic if and only if for every $B\in\mathcal{B}$, $\epsilon>0$ and $N>0$, there exists a relatively dense subset S of $\mathbb{R}$, such that if $\tau\in S$, $x\in B$, $|t|\le N$ and $|t-\tau_j|>\epsilon$ for all $j\in\mathbb{Z}$, then there exists $y_{t;x}\in\rho(F(t;x))$, such that $\|F(t+\tau;x)-y_{t,x}\|\le\epsilon$.*

The introduced notion is important for our purposes because a slight modification of the proof of Theorem 1 shows that the following result holds true:

Theorem 4. *Suppose that $p>0$, $\rho=T\in L(Y)$ and $F:\mathbb{R}\times X\to Y$ satisfies that for every $x\in X$, the function $t\mapsto F(t;x)$, $t\in\mathbb{R}$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$, which is strictly monotonically increasing and satisfies that there exists $\delta_0>0$, such that $\inf_{k\in\mathbb{Z}}(t_{k+1}-t_k)>\delta_0$. If $\nu:\mathbb{R}\to(0,\infty)$ is Stepanov-p-bounded and satisfies condition* (LQ), *$F(\cdot;\cdot)$ is pre-$(\mathcal{B},\rho,(t_k))$-Levitan piecewise continuous almost automorphic and $\sup_{t\in\mathbb{R};x\in B}\|F(t;x)\|_Y<+\infty$ ($B\in\mathcal{B}$), then we have the following: For every $B\in\mathcal{B}$, $\epsilon>0$ and $t\in\mathbb{R}$, there exists a relatively dense subset S of $\mathbb{R}$, such that if $\tau\in S$ and $x\in B$, then*

$$\int_t^{t+1}\left\|F(s+\tau;x)-TF(s;x)\right\|^p\nu^p(s)\,ds\le\epsilon.$$

Now we would like to note that Proposition 5 and Theorem 4 can serve to provide several different extensions of [42] (Theorem 8.2). For example, the statement of this result holds for any exponent $p>0$; moreover, we have the following (for the notion of a Levitan s.a.a. sequence, we refer the reader to [42] (Definition 3.12)):

Theorem 5. *Suppose that $p>0$, $\Omega=[0,1]$, $\nu:\Omega\to(0,\infty)$ is a p-integrable function, $F:\mathbb{R}\to Y$ is piecewise continuous with possible discontinuities at the points of a subset of a Levitan s.a.a. sequence $(t_k)_{k\in\mathbb{Z}}$ and condition (QUC1) holds, where (QUC1) For every $\epsilon>0$, there exists $\delta>0$, such that if the points t' and t'' belong to (t_i,t_{i+1}) for some $i\in\mathbb{Z}$ and $|t'-t''|<\delta$, then $\|F(t')-F(t'')\|<\epsilon$.*

Then, $F(\cdot)$ is Stepanov-(Ω, p, ν)-almost automorphic if and only if $F(\cdot)$ is piecewise continuous almost automorphic.

Proof. It is clear that Proposition 5 implies that if $F : \mathbb{R}^n \to Y$ is piecewise continuous almost automorphic, then $F(\cdot)$ is Stepanov-(Ω, p, ν)-almost automorphic. To prove the reverse statement, it suffices to show that for each number $\sigma \in (0, 1]$ the Steklov function $F_\sigma := \sigma^{-1} \int_0^\sigma F(t+s)\, ds$, $t \in \mathbb{R}$ is compactly almost automorphic (see the proof of the above-mentioned result). The uniform continuity of $F_\sigma(\cdot)$ can be simply proved using the condition (QUC1) and we will only prove here that $F_\sigma(\cdot)$ is almost automorphic. If (b_k) is a real sequence, then we can always find a subsequence (b_{k_l}) of (b_k) and a function $F^* : \mathbb{R} \to Y$, such that for every fixed number $t \in \mathbb{R}$, we have $\lim_{l\to+\infty} \int_0^1 \|F(t + b_{k_l} + s) - [F^*(t)](s)\|^p \nu^p(s)\, ds = 0$ and $\lim_{l\to+\infty} \int_0^1 \|[F^*(t - b_{k_l})](s) - F(t+s)\|^p \nu^p(s)\, ds = 0$. This implies simply the existence of a set $N \subseteq [0, 1]$ with the Lebesgue zero measure, such that $\lim_{l\to+\infty} F(t + b_{k_l} + s) = [F^*(t)](s)$ and $\lim_{l\to+\infty}[F^*(t - b_{k_l})](s) = F(t + s)$ for all $s \in [0, 1] \setminus N$. Applying the dominated convergence theorem, we simply find that $\lim_{l\to+\infty} F_\sigma(t + b_{k_l}) = \sigma^{-1} \int_0^\sigma [F^*(t)](s)\, ds$ and $\lim_{l\to+\infty} \sigma^{-1} \int_0^\sigma [F^*(t - b_{k_l})](s)\, ds = F_\sigma(t)$, which implies the required conclusion. □

We will consider two-dimensional analogues of Theorem 5 somewhere else. The interested reader may also try to prove certain analogues of Theorem 3 and Proposition 3 for Stepanov-p-almost automorphic type functions ($0 < p < 1$).

5. Stepanov-p-Almost Periodicity in Norm and Stepanov-p-Almost Automorphy in Norm ($p > 0$)

In this section, the following definitions will be considered:

Definition 11.

(i) *Assume that $p > 0$, (SM-1) holds true and $\nu_{\mathbf{t}} : \mathbf{t} + \Omega \to (0, \infty)$ is a Lebesgue measurable function ($\mathbf{t} \in \Lambda$). Then, we say that a function $F : \Lambda \times X \to Y$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu_{\cdot})$-almost periodic in norm (Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$-almost periodic in norm, where a Lebesgue measurable function $\nu : \Lambda \to (0, \infty)$ satisfies that $\nu_{\mathbf{t}}(\cdot) \equiv \nu_{|\mathbf{t}+\Omega}(\cdot)$ for all $\mathbf{t} \in \Lambda$) if and only if the function $\|F(\cdot; \cdot)\|_Y^p : \Lambda \times X \to \mathbb{C}$ is Stepanov-$(\Omega, 1, \rho_N, \mathcal{B}, \Lambda', \nu_{\cdot}^p)$-almost periodic (Stepanov-$(\Omega, 1, \rho_N, \mathcal{B}, \Lambda', \nu^p)$-almost periodic), where $\rho_N := \{(\|x\|_Y^p, \|y\|_Y^p); (x, y) \in \rho\}$; furthermore, if $\nu_{\mathbf{t}}(\cdot) \equiv 1$ for all $\mathbf{t} \in \Lambda$, then we omit the term "$\nu_{\cdot}$" from the notation.*

(ii) *Assume that $p > 0$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function and the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. If $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function, then we say that the function $F(\cdot; \cdot)$ is (strongly) Stepanov-$(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$-almost periodic in norm if and only if $\|F(\cdot; \cdot)\|_Y^p$ is (strongly) Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic with $Z = L^1_{\nu^p}(\Omega : \mathbb{C})$ and $P_Z = C_b(\Lambda : Z)$; if $\nu(\cdot) \equiv 1$ ($\Omega = [0, 1]^n$), then we omit the term "ν" ("Ω") from the notation.*

(iii) *Assume that $p > 0$, (SM-1) holds true, $\nu_{\mathbf{t}} : \mathbf{t} + \Omega \to (0, \infty)$ is a Lebesgue measurable function ($\mathbf{t} \in \Lambda$) and $\nu : \Lambda \to (0, \infty)$ is a Lebesgue measurable function. Then, we say that a function $F : \Lambda \to Y$ is Stepanov-$(\Omega, p, \nu_{\cdot})$-almost periodic in norm (Stepanov-(Ω, p, ν)-almost periodic in norm) if and only if $\|F(\cdot)\|_Y^p$ is Stepanov-$(\Omega, 1, \rho, \Lambda', \nu_{\cdot}^p)$-almost periodic (Stepanov-$(\Omega, 1, \rho, \Lambda', \nu^p)$-almost periodic) with $\rho \equiv \mathrm{I}$ and $\Lambda' \equiv \{\tau \in \mathbb{R}^n : \tau + \mathbf{t} \in \Lambda, \mathbf{t} \in \Lambda\}$; if $\nu(\cdot) \equiv 1$, then we omit the term "ν" from the notation.*

(iv) *Suppose that $p > 0$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \to Y$ is a given function, $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function and R denotes the collection of all sequences in $\mathbb{R}^n$, such that the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in \mathrm{R}$ and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. Then, we say that the function $F(\cdot)$ is Bochner–Stepanov-(Ω, p, ν)-almost periodic in norm if and only if $\|F(\cdot)\|_Y^p$ is Stepanov-$(\Omega, 1, \mathrm{R}, \nu^p)$-almost periodic.*

In the almost automorphic setting, we will use the following notion:

Definition 12.

(i) *Assume that* $p > 0$, $\nu : \Omega \to (0, \infty)$ *is a Lebesgue measurable function,* $F : \mathbb{R}^n \times X \to Y$ *is a given function and* R *is a certain collection of sequences in* $\mathbb{R}^n$. *Then, we say that the function* $F(\cdot;\cdot)$ *is Stepanov* $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu)$*-almost automorphic in norm (Stepanov* $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, W_{\mathcal{B},\mathrm{R}})$*-almost automorphic in norm; Stepanov* $(\Omega, p, \mathrm{R}, \mathcal{B}, \nu, P_{\mathcal{B},\mathrm{R}})$*-almost automorphic in norm) if and only if* $\|F(\cdot;\cdot)\|_Y^p$ *is Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}})$*-multi-almost automorphic (Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, \mathrm{R}, \mathcal{B}, Z^{\mathcal{P}}, P_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic) with* $Z = L^1_{\nu^p}(\Omega : Y)$.

(ii) *Suppose that* $p > 0$ *and that* $\nu : \Omega \to (0, \infty)$ *is a Lebesgue measurable function. Then, we say that a function* $F : \mathbb{R}^n \to Y$ *is Stepanov-*(Ω, p, ν)*-almost automorphic in norm (Stepanov-*(Ω, p, ν, W)*-almost automorphic in norm) if and only if* $\|F(\cdot)\|_Y^p$ *is Stepanov-*$(\Omega, 1, \mathrm{R}, \nu^p)$*-almost automorphic (Stepanov-*$(\Omega, 1, \mathrm{R}, \nu^p, W_{\mathrm{R}})$*-almost automorphic) with* R *being the collection of all sequences in* $\mathbb{R}^n$. $F(\cdot)$ *is Stepanov-p-almost automorphic in norm if and only if* $\|F(\cdot)\|_Y^p$ *is Stepanov-*$(\Omega, 1, \nu^p)$*-almost automorphic with* $\Omega = [0,1]^n$ *and* $\nu(\cdot) \equiv 1$.

If $0 < p \leq 1$, then we have the following inequality:

$$\|x - y\|_Y^p \geq \Big| \|x\|_Y^p - \|y\|_Y^p \Big|, \quad x,\ y \in Y. \tag{8}$$

Using (8), it can be simply verified that any Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu.)$-almost periodic function is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu.)$-almost periodic in norm; this holds for all other classes of functions introduced in Definition 11 and Definition 12, provided that $0 < p \leq 1$. In particular, if $F : \mathbb{R}^n \to Y$ is a Stepanov-p-almost periodic (automorphic) function and $0 < p \leq 1$, then the function $\|F(\cdot)\|_Y^p$ is Stepanov-1-almost periodic (automorphic).

If $p \geq 1$, then we cannot expect the existence of a finite real constant $c_p > 0$, such that the inequality

$$\|x - y\|_Y^p \leq c_p \Big| \|x\|_Y^p - \|y\|_Y^p \Big|, \quad x,\ y \in Y \tag{9}$$

holds true (consider the case in which $\|x\|_Y = \|y\|_Y$ but $x \neq y$). But, the inequality

$$|x - y|^p \leq \Big| |x|^p - |y|^p \Big|, \quad x,\ y \geq 0 \tag{10}$$

is true, as is easily shown with the help of the elementary differential calculus, and therefore any Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu.)$-almost periodic function in norm with the non-negative real values is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu.)$-almost periodic; this holds for all other classes of functions introduced in Definition 11 and Definition 12, provided that $p \geq 1$. In particular, if $F : \mathbb{R}^n \to [0, \infty)$, $p \geq 1$ and $F^p(\cdot)$ is Stepanov-1-almost periodic (automorphic), then $F(\cdot)$ is Stepanov-p-almost periodic (automorphic).

We continue by recalling that for every exponent $p \geq 1$, H. Bohr and E. Følner have constructed, in [37] (Main example II c), a Stepanov-p-almost periodic function $g : \mathbb{R} \to [0, \infty)$, which is not Stepanov-$q$-bounded and therefore not Stepanov-q-almost periodic (automorphic) for any exponent $q > p$. We will use this important example to show the following result:

Theorem 6. *Assume that* $0 < p < 1$. *Then, there exists a function* $f : \mathbb{R} \to [0, \infty)$, *which is Stepanov-p-almost periodic in norm, not Stepanov-q-bounded and therefore not Stepanov-q-almost periodic (automorphic) for any exponent* $q > p$.

Proof. By the foregoing, there exists a Stepanov-1-almost periodic function $g : \mathbb{R} \to [0, \infty)$, which is not Stepanov-$q$-bounded and therefore not Stepanov-q-almost periodic (automorphic) for any exponent $q > 1$. Define $f(t) := g^{1/p}(t)$, $t \in \mathbb{R}$. Then, it is clear that $f(\cdot)$ is Stepanov-p-almost periodic in norm. If we assume that $f(\cdot)$ is Stepanov-q-bounded for some exponent $q > p$, then $g(\cdot)$ must be Stepanov-(q/p)-bounded, which is a contradiction. This simply implies that $f(\cdot)$ is not Stepanov-q-almost periodic (automorphic) for any exponent $q > 1$, which can be also directly shown as follows: Assuming the contrary, then the function $g^{q/p}(\cdot)$ must be Stepanov-1-almost periodic (automorphic) due to (8). By the conclusion clarified directly before the formulation of Problem 1, this show that the function $g(\cdot)$ is Stepanov-(q/p)-almost periodic (automorphic), which leads to a contradiction. □

Remark 4. *Since $f(\cdot)$ is not Stepanov-q-bounded, we also have that $f(\cdot)$ is not Stepanov-q-almost periodic (automorphic) in norm for any exponent $q > p$.*

Further, we know that any Stepanov-p-almost periodic function $f : \mathbb{R} \to [0, \infty)$ is Stepanov-p-almost periodic in norm $(0 < p < 1)$; hence, it is logical to ask whether the function $f(\cdot)$, considered in the proof of Theorem 6, is Stepanov-p-almost periodic. If this is the case, then for each exponent $p > 0$, we would have an example of a Stepanov-p-almost periodic function that is not Stepanov-q-bounded and therefore not Stepanov-q-almost periodic (automorphic) for any exponent $q > p$.

Concerning [37] (Main example II c) and Theorem 6, we give the following examples as well:

Example 5. *We first slightly modify the example already considered in the introductory part. Suppose that $p > 0$ and*

$$f(t) := \begin{cases} |\sin t|^{-(1/p)}, & t \notin \mathbb{Z}\pi \\ 0, & t \in \mathbb{Z}\pi. \end{cases}$$

Then, the function $F(\cdot)$ is not p-locally integrable and therefore not Stepanov-q-almost periodic (automorphic) for any exponent $q \geq p$. On the other hand, it can be simply shown that the function $f(\cdot)$ is Stepanov-p_0-almost periodic for any exponent $p_0 \in (0, p)$.

In connection with this example, it is also worth noting that H. D. Ursell has constructed many non-trivial examples of functions $f : \mathbb{R} \to [0, \infty)$ that are Stepanov-p_0-almost periodic for any exponent $p_0 \in (0, 1)$ but not Stepanov p_1-bounded for any exponent $p > 1$ (in this case, the number 1 is said to be the critical index of $f(\cdot)$; cf. [29] (pp. 430–440) for more details on the subject).

Example 6. *Suppose that $0 < p < 1$. Let us observe that the function $g(\cdot)$ from the proof of Theorem 6 has the form $g(t) = \sum_{k=1}^{+\infty} g_k(t)$, $t \in \mathbb{R}$, where $g_k(\cdot)$ is an essentially bounded, periodic function with the non-negative values ($k \in \mathbb{N}$) and the above series is convergent in the Stepanov S^1-norm. Consider now the following function*

$$h(t) := \sum_{k=1}^{\infty} g_k^{1/p}(t), \quad t \in \mathbb{R}.$$

The function $h(\cdot)$ is well defined and Stepanov-p-almost periodic because the sequence of functions $\sum_{k=1}^{\infty} g_k^{1/p}(\cdot)$ is Cauchy and therefore convergent in the Stepanov S^p-norm (cf. also [29] (p. 404, l. 1–5)), which can be shown using the following simple calculation ($k,\ m \in \mathbb{N}$, $k < m$):

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \left[\sum_{i=k}^{m} g_i^{1/p}(s) \right]^p ds \leq \sup_{t \in \mathbb{R}} \int_t^{t+1} \left[\sum_{i=k}^{m} g_i(s) \right] ds.$$

It is not clear whether the function $h(\cdot)$ is locally integrable or Stepanov-q-almost periodic for some exponent $q > p$. Let us also emphasize that there is no simple theoretial explanation, which would imply that the above conclusions hold in a general situation of this example and that the function $h(\cdot)$ will be Stepanov-q-almost periodic for every exponent $q > p$ in the case that $g_k(t) \geq 1$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$. This follows simply from the inequality

$$\left[\sum_{i=k}^{m} g_i^{1/q}(s)\right]^q \leq \left[\sum_{i=k}^{m} g_i^{1/p}(s)\right]^q.$$

Finally, we would like to recall that for every almost periodic function $f : \mathbb{R} \to Y$ and for every positive real number $p > 0$, the function $\|f(\cdot)\|^p : \mathbb{R} \to [0, \infty)$ is almost periodic [44], which implies that $f(\cdot)$ is Stepanov-p-almost periodic in norm. In connection with Theorem 6 and this observation, we would like to raise the following issue:

Problem 1. *Suppose that $p > 0$. Can we find some sufficient conditions ensuring that the assumptions $f : \mathbb{R} \to Y$ is an unbounded Stepanov-p-almost periodic (automorphic) function, $r > 0$ and $q > 0$ imply that the function $\|f(\cdot)\|^r : \mathbb{R} \to [0, \infty)$ is Stepanov-q-almost periodic (automorphic)?*

We close this section with the observation that the class of Stepanov-p-almost periodic (automorphic) functions in norm is extremely non-trivial. For example, it seems very plausible that Stepanov-p-almost periodic (automorphic) functions in norm do not form a vector space with the usual operations; furthermore, it is very difficult to state some satisfactory analogs of Proposition 2 and Corollary 1 for this class of functions.

6. Applications to the Abstract (Impulsive) Volterra Integro-Differential Inclusions

The main goal of this section is to present certain applications of our results to the abstract (impulsive) Volterra integro-differential inclusions in Banach spaces.

We will first provide some applications of Theorem 3 and Proposition 3 concerning the invariance of Stepanov-p-almost periodicity under the actions of the infinite convolution products ($0 < p < 1$). It is clear that Theorem 3 can be applied in the analysis of the existence and uniqueness of (T, ν)-almost periodic solutions for various classes of the abstract fractional integro-differential inclusions without initial conditions; cf. [5] for more details about applications of this type (here, the main problem is to find the inhomogeniety $f(\cdot)$, which is not Stepanov-1-almost periodic, such that the requirements of Theorem 3 hold). For example, we can analyze the existence and uniqueness of (T, ν)-almost periodic solutions of the fractional Poisson heat equation

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^{\infty}(\Omega)$, $\gamma \in (0, 1)$ and $1 < p < \infty$.

Concerning Proposition 3, we would like to note that the possible applications of this result can be always made to the abstract differential first-order inclusions provided that the operator family $(R(t))_{t>0}$ is a (degenerate) strongly continuous semigroup of operators satisfying that $\|R(t)\| \leq Me^{-ct}t^{\beta-1}$, $t > 0$ for some real constants $M > 0$, $c > 0$ and $\beta \in (0, 1]$ as well as the function $\nu : \mathbb{R} \to (0, \infty)$ is an admissible weight function with the property that $\nu(t) \leq M'e^{\omega|s|}\nu(t+s)$, $t,\ s \in \mathbb{R}$ for some real constants $M' > 0$ and $\omega < c$. The applications can be simply given in the analysis of the existence and uniqueness of ν-almost periodic solutions of the abstract Poisson heat equation

$$\begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

in the space $X := L^p(\Omega)$; cf. also the first application made in [30] (Subsection 4.4), where we analyzed the almost periodic type solutions to the abstract higher-order impulsive Cauchy problems. In particular, we can use the inhomogenities from Example 4 with $\nu(t) = a + \zeta(t)$, $t \in \mathbb{R}$, where $a \geq 0$ and an admissible weight function $\zeta : \mathbb{R} \to (0, \infty)$ satisfies $\zeta(t) \leq M'e^{\omega|s|}\zeta(t+s)$, $t,\ s \in \mathbb{R}$ and $\int_{-\infty}^{+\infty} \zeta^p(x)\,dx < +\infty$ (for example, we can take $\zeta(t) = e^{-\omega|t|}$, $t \in \mathbb{R}$ with $0 < \omega < c$). Many other examples of such admissible weight functions can be given following the recent investigations of chaotic translation semigroups on weighted L^p-spaces.

Before dividing the remainder of this section into two separate subsections, we would like to note that the argument similar to that one contained in the proofs of Theorem 3 and Proposition 3 can be useful to deduce certain results concerning the invariance of Stepanov p-almost periodicity under the actions of the infinite convolution product

$$t \mapsto H(t) \equiv \int_{-\infty}^{+\infty} h(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

where $h \in L^1(\mathbb{R})$. This can be applied in the analysis of the existence and uniqueness of the Stepanov-p-almost periodic type solutions to the inhomogenous heat equation. We refer the interested readers to [6] (Subsection 6.2.6 and pp. 558–559)) for more detailed information.

6.1. Applications to the Abstract Impulsive First-Order Differential Inclusions

We will analyze the existence and uniqueness of ν-almost periodic type solutions to the abstract impulsive differential inclusions of first order in this subsection. Of concern is the following abstract impulsive Cauchy inclusion

$$(ACP)_{1;1}: \begin{cases} u'(t) \in \mathcal{A}u(t) + f(t), \quad t \in [0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}, \\ (\Delta u)(t_k) = u(t_k+) - u(t_k-) = Cy_k, \quad k \in \mathbb{N}_l, \\ u(0) = Cu_0. \end{cases}$$

We refer the reader to [10] for the notion of a (pre-)solution of $(ACP)_{1;1}$. We need the following result from this paper:

Lemma 2. *Suppose that $\mathcal{A}$ is a closed subgenerator of a global C-regularized semigroup $(R(t))_{t\geq 0}$. Suppose further that $0 < t_1 < \ldots < t_l < \ldots < +\infty$, the sequence $(t_l)_l$ has no accumulation point, the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ are continuous on the set $[0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$ for all $t \in [0,\infty) \setminus \{t_1, \ldots, t_l, \ldots\}$, as well as the right limits and the left limits of the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ exist at any point of the set $\{t_1, \ldots, t_l, \ldots\}$. Define the functions $u(t)$ and $\omega(t)$ for $t \geq 0$ by*

$$u(t) := R(t)u_0 + \int_0^t R(t-s)\big(C^{-1}f\big)(s)\,ds + \omega(t), \quad t \geq 0$$

and

$$\omega(t) := \begin{cases} 0, \quad t \in [0, t_1], \\ \sum_{p=1}^{k} R(t-t_p)y_0^p, \text{ if } t \in (t_k, t_{k+1}] \text{ for some } k \in \mathbb{N}_{l-1}^0, \end{cases} \tag{11}$$

respectively. Then, the function $u(t)$ is a unique solution of the problem $(ACP)_{1;1}$, provided that $u_0 \in D(\mathcal{A})$ and $y_k \in D(\mathcal{A})$ for all $k \in \mathbb{N}$.

In order to formulate our main result, we need to impose the following conditions:

(AS1) $\mathcal{A}$ is a closed subgenerator of a global C-regularized semigroup $(R(t))_{t\geq 0}$ and $\|R(t)\| \leq Me^{\omega_0 t}$, $t \geq 0$ for some real numbers $M > 0$ and $\omega_0 < 0$;

(AS2) $m \in \mathbb{N}$, $f_i : [0,\infty) \to \mathbb{C}$ is an almost periodic function ($1 \leq i \leq m$), $f_1(0) \cdot \ldots \cdot f_m(0) \neq 0$, and the sequence $\{t_1, \ldots, t_l, \ldots\}$ of all possible zeroes of functions $f_0(\cdot), \ldots, f_m(\cdot)$ has no accumulation point;

(AS3) $(x_i, y_i) \in \mathcal{A}$ for $1 \leq i \leq m$, $t \mapsto f_0(t)$, $t \geq 0$ is a piecewise continuous function uniquely determined by the function $t \mapsto \sum_{i=1}^{m} \operatorname{sign}(f_i(t))Cx_i$, $t \geq 0$ and $t \mapsto f_{\mathcal{A}}(t)$,

$t \geq 0$ is a piecewise continuous function uniquely determined by the function $t \mapsto \sum_{i=1}^{m} \operatorname{sign}(f_i(t))y_i$, $t \geq 0$;

(AS4) $p \geq 1$, $\zeta : \mathbb{R} \to (0,\infty)$ is an admissible weight function, $\zeta(t) \leq Me^{\omega|s|}\zeta(t+s)$, t, $s \in \mathbb{R}$ for some real constant $\omega < \omega_0$, $\int_{-\infty}^{+\infty} \zeta^p(x)\,dx < +\infty$, $c > 0$ and $\nu(t) = c + \zeta(t)$, $t \in \mathbb{R}$;

(AS5) $\sum_{k\geq 1} e^{-\omega_0 t_k}\|y_k\| < +\infty$, $u_0 \in D(\mathcal{A})$, $y_k \in D(\mathcal{A})$ for all $k \in \mathbb{N}$ and $q : [0,\infty) \to X$ is a locally integrable function, such that $\|q(t)\| \leq Me^{\omega_1 t}$, $t \geq 0$ for some real number $\omega_1 < \omega_0$;

(AS6) The functions $q(\cdot)$ and $f_{\mathcal{A},q}(\cdot)$ are continuous on the set $[0,\infty) \setminus \{t_1,\ldots,t_l,\ldots\}$, $f_{\mathcal{A},q}(t) \in \mathcal{A}q(t)$ for all $t \in [0,\infty) \setminus \{t_1,\ldots,t_l,\ldots\}$, as well as the right limits and the left limits of the functions $q(\cdot)$ and $f_{\mathcal{A},q}(\cdot)$ that exist at any point of the set $\{t_1,\ldots,t_l,\ldots\}$.

We now establish the following result:

Theorem 7. *Assume that conditions* (AS1)–(AS6) *hold good. Then, there exists a unique solution of the problem* $(ACP)_{1;1}$ *with* $f(t) = f_0(t) + Cq(t)$, $t \geq 0$, *which can be written as a sum of a* ν*-almost periodic function, a function from the space* $PC_{\omega_0}([0,\infty) : X)$ *and a function from the space* $C_{\omega_0+\omega_1}([0,\infty) : X)$.

Proof. It is easy to prove that $\nu(t) \leq Me^{\omega|s|}\nu(t+s)$, t, $s \in \mathbb{R}$. Keeping in mind the first condition in (AS5) and the consideration from [30] (Application 1, Subsection 4.1), we have that the function $\omega(\cdot)$, given by (11), belongs to the space $PC_{\omega_0}([0,\infty) : X)$. Moreover, due to the consideration from Example 3 (cf. (LT2)), we have that $C^{-1}f_0 \in S_{\Omega,\Lambda'}^{(\mathbb{F},\rho,\mathcal{P}_t,\mathcal{P})}(\Lambda : \mathbb{C})$ with $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\rho = \mathrm{I}$, $\Omega = [0,1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L_\nu^p(t+[0,1] : \mathbb{C})$ for all $t \in \mathbb{R}$. Since $p \geq 1$ and $\nu(t) \geq c > 0$, $t \in \mathbb{R}$, by applying Proposition 3, we show that the function $t \mapsto \int_{-\infty}^{t} R(t-s)(C^{-1}f_0(s))\,ds$, $t \in \mathbb{R}$ is ν-almost periodic. Furthermore, it is clear that the function $t \mapsto R(t)u_0$, $t \geq 0$ belongs to the space $C_{\omega_0}([0,\infty) : X)$. Due to [45] (Proposition 1.3.4), the mapping $t \mapsto \int_0^t R(t-s)(C^{-1}f(s))\,ds$, $t \geq 0$ is continuous so that the mapping $t \mapsto \int_t^{+\infty} R(t-s)(C^{-1}f_0(s))\,ds$, $t \geq 0$ is likewise continuous due to the representation formula given in Lemma 2 and the Formula (12) given below. Then, the final conclusion simply follows from a simple computation and the decomposition

$$u(t) = \int_{-\infty}^{t} R(t-s)(C^{-1}f_0(s))\,ds + \left[R(t)u_0 + \int_t^{+\infty} R(t-s)(C^{-1}f_0(s))\,ds + \omega(t)\right] + \int_0^t R(t-s)q(s)\,ds, \quad t \geq 0. \tag{12}$$

□

Before proceeding further, we would like to note that Theorem 7 can be applied in the analysis of the existence and uniqueness of asymptotically ν-almost periodic type solutions for a class of the abstract impulsive first-order (degenerate) differential equations involving the (non-coercive) differential operators with constant coefficients in L^p-spaces (cf. [5,32] for more details).

6.2. Applications to the Abstract (Impulsive) Fractional Differential Inclusions

We use the following notion in this subsection. If $\alpha > 0$, then the Caputo fractional derivative $\mathbf{D}_t^\alpha u$ is defined for those functions $u \in C^{\lceil\alpha\rceil-1}([0,\infty) : X)$, for which $g_{\lceil\alpha\rceil-\alpha} * (u - \sum_{j=0}^{\lceil\alpha\rceil-1} u^{(j)}(0)g_{j+1}) \in C^{\lceil\alpha\rceil}([0,\infty) : X)$ by

$$\mathbf{D}_t^\alpha u(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left[g_{\lceil\alpha\rceil-\alpha} * \left(u - \sum_{j=0}^{\lceil\alpha\rceil-1} u^{(j)}(0)g_{j+1}\right)\right];$$

here, $g_\beta(t) \equiv t^{\beta-1}/\Gamma(\beta)$ for $\beta > 0$ and $g_0(t) \equiv \delta(t)$, the Dirac δ-distribution. We use the following lemma (cf. [32] (Proposition 3.2.15 (i)); for the notion of solutions, cf. [32] (Definition 3.1.1 (ii))):

Lemma 3. *Suppose $\alpha \in (0,\infty) \setminus \mathbb{N}$, $x \in D(\mathcal{A})$, $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\infty) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \in [0,\infty)$ and $\mathcal{A}$ is a closed subgenerator of a global (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0}$. Set $v(t) := (g_{\lceil \alpha \rceil - \alpha} * f)(t)$, $t \in [0,\infty)$. If $v \in C^{\lceil \alpha \rceil - 1}([0,\infty) : X)$ and $v^{(k)}(0) = 0$ for $1 \leq k \leq \lceil \alpha \rceil - 2$, then the function $u(t) := R(t)x + (R * C^{-1}f)(t)$, $t \geq 0$ is a unique solution of the following abstract time-fractional inclusion:*

$$(ACP)^f_\alpha : \begin{cases} u \in C^{\lceil \alpha \rceil}((0,\infty) : X) \cap C^{\lceil \alpha \rceil - 1}([0,\infty) : X), \\ \mathbf{D}^\alpha_t u(t) \in \mathcal{A}u(t) + \frac{d^{\lceil \alpha \rceil - 1}}{dt^{\lceil \alpha \rceil - 1}}\big(g_{\lceil \alpha \rceil - \alpha} * f\big)(t),\ t \geq 0, \\ u(0) = Cx,\ u^{(k)}(0) = 0,\ 1 \leq k \leq \lceil \alpha \rceil - 1. \end{cases}$$

Now, we will use the following conditions:

(AS1)′ $\alpha \in (0,2) \setminus \{1\}$, $\mathcal{A}$ is a closed subgenerator of an exponentially bounded (g_α, C)-regularized resolvent family $(R(t))_{t\geq 0}$ and $\|R(t)\| \leq Me^{\omega_0 t}$, $t \geq 0$ for some real numbers $M > 0$ and $\omega_0 \geq 0$;

(AS2)′ $\omega > \omega_0$, $m \in \mathbb{N}$, $f_i : [0,\infty) \to \mathbb{C}$ is identically equal to the function $\sin(\cdot)$ or $\cos(\cdot)$, α_i and β_i are non-zero real numbers, such that $\alpha_i\beta_i^{-1} \notin \mathbb{Q}$ $(1 \leq i \leq m)$;

(AS3)′ $q : [0,\infty) \to X$ is a locally integrable function, such that $\|q(t)\| \leq Me^{\omega_1 t}$, $t \geq 0$ for some real number $\omega_1 < \omega - \omega_0$;

(AS4)′ $(x,y) \in \mathcal{A}$, $(x_i, y_i) \in \mathcal{A}$ for $1 \leq i \leq m$, and $e^{-\omega t}(C^{-1}f)(t) = \sum_{i=1}^m f_i(1/(2 + \cos(\alpha_i t) + \cos(\beta_i t)))x_i + q(t)$, $t \geq 0$;

(AS5)′ The functions $q(\cdot)$ and $f_{\mathcal{A},q}(\cdot)$ are continuous on $[0,\infty)$, $f_{\mathcal{A},q}(t) \in \mathcal{A}q(t)$ for all $t \geq 0$, and $g_{2-\alpha} * e^{\omega \cdot} q(\cdot) \in C^1([0,\infty) : X)$ if $1 < \alpha < 2$;

(AS6)′ $p \geq 1$, $\zeta : \mathbb{R} \to (0,\infty)$ is an admissible weight function, $\zeta(t) \leq Me^{\omega|s|}\zeta(t+s)$, t, $s \in \mathbb{R}$ for some real constant $\omega < \omega_0$, $\int_{-\infty}^{+\infty} \zeta^p(x)\,dx < +\infty$, $c > 0$ and $\nu(t) = c + \zeta(t)$, $t \in \mathbb{R}$;

Then, we obtain the following result, which can be shown in a similar way to Theorem 7 (cf. also [5] (Remark 2.6.15)):

Theorem 8. *Suppose that conditions* (AS1)′–(AS6)′ *hold good. Then, there exists a unique solution $u(\cdot)$ of the problem $(ACP)^f_\alpha$ with $f(t) = f_0(t) + Cq(t)$, $t \geq 0$; furthermore, the function $e^{-\omega \cdot}u(\cdot)$ can be written as a sum of a ν-almost periodic function, a function from the space $C_{\omega_0 - \omega}([0,\infty) : X)$ and a function from the space $C_{\omega_0 - \omega + \omega_1}([0,\infty) : X)$.*

It is clear that Theorem 8 can be applied in the analysis of the existence and uniqueness of asymptotically ν-almost periodic type solutions for a class of the abstract fractional (degenerate) differential equations involving the (non-coercive) differential operators with constant coefficients in L^p-spaces. Furthermore, we can apply this result in the study of the existence and uniqueness of asymptotically ν-almost periodic type solutions for a class of the abstract fractional (degenerate) differential equations considered in [32] (Section 3.5) and a class of the abstract fractional integro-differential inclusions with impulsive effects considered in [30] (Section 5) (cf. [30] (Theorem 10)).

If a sufficiently small real number, $\sigma > 0$, is given, then it would be interesting to construct an example of a Lebesgue measurable function $\nu : \mathbb{R} \to (0,\infty)$ satisfying the following three conditions:

(i) There exists a finite real number $M_\sigma \geq 1$, such that $\nu(x+y) \leq M_\sigma(1+|y|)^\sigma \nu(x)$ for all x, $y \in \mathbb{R}$;

(ii) $\nu(\cdot)$ is unbounded and satisfies (LT);

(iii) There exists a finite real number $c > 0$, such that $\nu(t) \geq c$ for all $t \in \mathbb{R}$.

If such a function exists, then we can consider the case $\omega = 0$ in Theorem 8 and apply Proposition 3, Lemma 3 and our conclusion established in Example 4 in the study of the existence and uniqueness of asymptotically ν-almost periodic type solutions of the abstract fractional Cauchy inclusion $(ACP)^{f}_{\alpha}$ (cf. also [5] (Remark 2.6.12, Remark 2.6.14(ii))).

7. Conclusions and Recommendations for Future Work

In this paper, we considered various classes of Stepanov-p-almost periodic functions and Stepanov-p-almost automorphic functions, where the exponent p has an arbitrary positive real value. We studied and analyzed the class of Stepanov-p-almost periodic (automorphic) functions in norm ($p > 0$) as well. The main structural properties for the introduced classes of functions were clarified. We also provided many illustrative examples, useful remarks, open problems and some new applications. In summary, our some concepts and results in this paper are quite original. We believe that our new concepts and new results will be widely applied to theoretical methods and applications related to fractional calculus and fractional differential equations.

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