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Energy-Preserving AVF Methods for Riesz Space-Fractional Nonlinear KGZ and KGS Equations

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Abstract: The Riesz space-fractional derivative is discretized by the Fourier pseudo-spectral (FPS) method. The Riesz space-fractional nonlinear Klein–Gordon–Zakharov (KGZ) and Klein–Gordon–Schrödinger (KGS) equations are transformed into two infinite-dimensional Hamiltonian systems, which are discretized by the FPS method. Two finite-dimensional Hamiltonian systems are thus obtained and solved by the second-order average vector field (AVF) method. The energy conservation property of these new discrete schemes of the fractional KGZ and KGS equations is proven. These schemes are applied to simulate the evolution of two fractional differential equations. Numerical results show that these schemes can simulate the evolution of these fractional differential equations well and maintain the energy-preserving property.

Keywords: AVF method; Riesz space-fractional KGZ equation; Riesz space-fractional KGS equation; FPS method

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1. Introduction

Fractional differential equations can better describe the behavior of physical phenomena than integral differential equations. Many scholars have taken great interest in studying fractional differential equations and the theory of the fractional derivative. In general, factional differential equations can not have an exact solution. Numerical simulations for fractional nonlinear differential equations have become very important. Many different numerical methods have been to proposed to solve fractional nonlinear partial differential equations (PDEs) [1–4]. In this paper, we numerically investigate the following Riesz space fractional KGZ and KGS equations by the energy preserving method.

The space fractional KGZ equation can be written as [5–8]

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \\ \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^2 m(x,t)}{\partial x^2} - \frac{\partial^2 (|u(x,t)|^2)}{\partial x^2} = 0, \end{cases} \tag{1}$$

which describes the propagation of Langmuir waves in plasma physics. Suppose the finite domain space $\Omega = (a,b) \times (0,T)$ with the initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = u_1 x$, $m(x,0) = m_0(x)$, $m_t(x,0) = m_1(x)$, $x \in [a,b]$ and the boundary conditions $u(a,t) = u(b,t) = 0$, $m(a,t) = m(b,t) = 0$, $t \in [0,T]$, where $\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}$ is the Riesz space fractional derivative with $1 \le \alpha \le 2$.

The KGZ equation can have the following invariant energy conservation

$$E(t) = \int [|u_t|^2 + |\frac{\partial^{\frac{\alpha}{2}} u}{\partial |x|^{\frac{\alpha}{2}}}|^2 + |u|^2 + m|u|^2 + \frac{1}{2}v^2 + \frac{1}{2}|u|^4 + \frac{1}{2}m^2]dx = E(0), \tag{2}$$

where $\frac{\partial v(x,t)}{\partial x} = -\frac{\partial m(x,t)}{\partial t}$. When $\alpha = 2$, the Riesz space fractional KGZ equation becomes the classical KGZ equation [9–11]. Ma et al. proposed a multi-scale integrator method to solve the KGZ equation [12]. Wang et al. solved the KGZ equation based on the lattice Boltzmann model [13]. Mei et al. proposed the Galerkin finite element method to solve the KGZ equation [14]. A lot of work has also been conducted on the Riesz space fractional KGZ equation [8,15]. Macias et al. proposed an energy conserving scheme to solve the fractional KGZ equation [15–19].

The fractional KGS equation can be written in the following form:

$$\begin{cases} if_t - \frac{\beta}{2}(-\Delta)^{\frac{\alpha}{2}} f + \alpha_1 f h = 0, \\ h_{tt} - \gamma h_{xx} + m^2 h - \alpha_1 |f|^2 = 0, \end{cases} \tag{3}$$

where $i = \sqrt{-1}$. The initial conditions are $f(x,0) = f_0(x)$, $h_t(x,0) = h_1(x)$, $x \in R$, and the boundary conditions are $f(x,t) = h(x,t) = 0$, $x \in R \in \Omega$, $\Omega = (x_L, x_R)$, $t \in [0,T]$. The complex function $f(x,t)$ is the complex neutron field and the real function $h(x,t)$ is the meson field. The variables α_1 and β are the coupled constants.

The fractional KGS equation also has invariant energy conservation:

$$\tilde{E}(t) = \int_{-\infty}^{+\infty} h_t^2 + \gamma h_x^2 + m^2 h^2 + \beta |(-\Delta)^{\frac{\alpha}{4}} f|^2 - 2\alpha_1 |f|^2 h dx = \tilde{E}(0). \tag{4}$$

When $\alpha = 2$ and $\beta = \gamma = 1$, Equation (3) is the classical coupled KGS equation. In quantum field theory, the coupled KGS equation is a mathematical model for the interaction of a conservation complex neutron field and a real meson field. Regarding the coupled KGS equation, many important conclusions have been obtained. Ohta et al. analyzed the stability of the coupled KGS equation [20]. Guo et al. investigated the global well-posedness of the KGS equation [21]. Kong et al. proposed a symplectic method to solve the coupled KGS equation [22]. The fractional KGS equation is an extension of the classical coupled KGS equation. Wang et al. proposed to solve the Riesz space fractional KGS equation using the difference method and the spectral method [23,24].

Recently, energy preserving methods have become important numerical methods in simulating energy conservation nonlinear PDEs, and they are structure-preserving numerical methods. Many different energy preserving methods have been derived, such as the discrete gradient method [25,26], the discrete variational derivative method [27] and the Hamiltonian boundary value method [28]. The AVF method, which is a kind of discrete gradient method, has been widely used to solve energy conservation integral PDEs and has achieved great success [29,30]. However, few people have applied the AVF method to solve energy conservation fractional PDEs. In this paper, we apply the AVF method to solve Riesz space fractional KGZ and KGS equations.

The rest of the paper is organized as follows. In Section 2, the definition and properties of the Riesz fractional derivative are given. The Riesz fractional derivative is discretized by the FPS method. In Section 3, the infinite Hamiltonian symplectic structures of the Riesz space fractional KGZ and KGS equations are obtained. These Hamiltonian systems are discretized by the FPS method and the second-order AVF method. Two energy preserving schemes of the Riesz space fractional KGZ and KGS equations are thus obtained. At last, numerical experiments confirm the advantage of the energy preserving schemes of the Riesz space fractional KGZ and KGS equations in simulating solitary wave behavior and preserving the energy conservation property of the equations. The new energy preserving schemes are superior to the existing second-order energy preserving schemes.

2. Discretization of the Riesz Space Fractional Derivative

Definition 1. *When $n-1 \le \alpha \le n$, n is a positive integer, the Riesz space fractional derivative with α order can be defined as [2]*

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \frac{-1}{2cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{+\infty} \frac{u(\xi,t)}{|x-\xi|^{\alpha+1-n}} d\xi, \tag{5}$$

where $\Gamma(.)$ is the Gamma function. $\partial^\alpha_{|x|}$ is denoted as the α order derivative of $u(x,t)$ at (x,t).

Lemma 1. *In the infinite domain interval $(-\infty < x < \infty)$, the function $u(x,t)$ is equivalent to*

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \frac{-1}{2cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{+\infty} \frac{u(\xi,t)}{|x-\xi|^{\alpha+1-n}} d\xi = -(-\triangle)^{\frac{\alpha}{2}} u(x,t), \tag{6}$$

where $n-1 < \alpha < n$.

In the infinite domain interval $(-\infty < x < \infty)$, the fractional Laplace operator is defined as

$$-(-\triangle)^{\frac{\alpha}{2}} u(x,t) = -F^{-1}|x|^\alpha F u(x,t), \tag{7}$$

where F and F^{-1} are represented as the Fourier transformation and the Fourier inverse transform of $u(x,t)$, and we can obtain

$$-(-\triangle)^{\frac{\alpha}{2}} u(x,t) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\xi} |x|^\alpha \int_{-\infty}^{+\infty} e^{i\xi\eta} u(\eta,t) d\eta d\xi. \tag{8}$$

On the other hand, in the finite domain interval $\Omega = (a,b)$, the Fourier series can be defined as

$$-(-\triangle)^{\frac{\alpha}{2}} u(x,t) = -\sum_{l\in Z} |v_l|^\alpha \hat{u}_l e^{iv_l(x-a)}, \tag{9}$$

where $\nu_l = \frac{2l\pi}{b-a}$, and the coefficients of the Fourier series are

$$\hat{u}_l = \frac{1}{b-a} \int_\Omega u(x,t) e^{-iv_l(x-a)} dx. \tag{10}$$

We apply the FPS method to discrete the α order Riesz space fractional derivative. Suppose the space integral interval is $\Omega = [a,b]$, the interval Ω is divided into N equal parts. N is an even number. The space step length is $h_1 = \frac{b-a}{N}$. Take $x_j = a + jh_1$, where $j = 0, 1, \cdots, N-1$ are the space Fourier collocation points. $I_N u(x,t)$ is the approximation of $u(x,t)$, we have

$$(I_N u)(x,t) = u_N(x,t) = \sum_{k=-N/2}^{N/2} \tilde{u}_k e^{ik\mu(x-a)}, \tag{11}$$

where $\tilde{u}_k$, $\mu = \frac{2\pi}{b-a}$ and

$$\tilde{u}_k = \frac{1}{Nc_k} \sum_{j=0}^{N-1} u(x_j,t) e^{-ik\mu(x_j-a)}, \mu = \frac{2\pi}{b-a}, |k| < N/2, c_k = 1, k = \pm N/2 c_k = 2. \tag{12}$$

When $|k| < N/2$, $c_k = 1$, and when $k = \pm N/2$, $c_k = 2$.

We can obtain

$$-(-\triangle)^{\frac{\alpha}{2}} I_N u(x_j, t) = -\sum_{k=-N/2}^{N/2} |k\mu|^{\alpha} \tilde{u}_k e^{ik\mu(x_j - a)}. \tag{13}$$

The α order derivative of the approximation function $I_N u(x,t)$ can be denoted as

$$\begin{aligned}
\frac{\partial^{\alpha} I_N u(x_j, t)}{\partial |x|^{\alpha}} &= -(-\triangle)^{\frac{\alpha}{2}} I_N u(x_j, t) \\
&= -\sum_{k=-N/2}^{N/2} |k\mu|^{\alpha} \Big(\frac{1}{Nc_k} \sum_{l=0}^{N-1} u_l e^{-ik\mu(x_l - a)}\Big) e^{ik\mu(x_j - a)} \\
&= \sum_{l=0}^{N-1} u_l \Big(-\sum_{k=-N/2}^{N/2} \frac{1}{Nc_k} |k\mu|^{\alpha} e^{ik\mu(x_j - x_l)}\Big) \\
&= (\mathbf{D}_2^{\alpha} \mathbf{U})_j,
\end{aligned} \tag{14}$$

where $\mathbf{D}_2^{\alpha}$ is $N \times N$ matrix, $\mathbf{U} = (U_0, U_1, \cdots, U_{N-1})$ and the coefficients of the matrix $\mathbf{D}_2^{\alpha}$ are

$$(\mathbf{D}_2^{\alpha})_{j,l} = -\sum_{k=-N/2}^{N/2} \frac{1}{Nc_k} |k\mu|^{\alpha} e^{ik\mu(x_j - x_l)}. \tag{15}$$

3. Energy Preserving Method of Fractional KGZ and KGS Equations

In this section, we first give the Hamiltonian structures of the Riesz space fractional KGZ and KGS equations. Then, the space fractional derivatives of these two equations are discretized by the FPS method. The AVF method is applied to solve the semi-discrete fractional KGZ and KGS equations.

3.1. Energy Preserving Method of the Fractional KGZ Equation

Let $w(x,t) = u_t(x,t)$, $-2q_{xx}(x,t) = m_t(x,t)$, then the space fractional KGZ Equation (1) is equivalent to

$$\begin{cases}
u_t = w, \\
m_t = -2q_{xx}, \\
w_t = \frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} - u - mu - |u|^2 u, \\
q_t = -\frac{1}{2} m - \frac{1}{2} |u|^2.
\end{cases} \tag{16}$$

Equation (16) can be expressed by the following infinite dimensional Hamiltonian system

$$\frac{d\mathbf{z}}{dt} = \mathbf{J} \frac{\delta H(\mathbf{z})}{\delta \mathbf{z}}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where $\mathbf{z} = (u, m, w, q)^T$, $\mathbf{O}$ is 2×2 zero matrix, $\mathbf{I}$ is 2×2 unite matrix and the corresponding Hamiltonian energy function is

$$H(\mathbf{z}) = \int [\frac{1}{2} w^2 + (q_x)^2 + \frac{1}{2} |\frac{\partial^{\frac{\alpha}{2}} u}{\partial |x|^{\frac{\alpha}{2}}}|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} m |u|^2 + \frac{1}{4} |u|^4 + \frac{1}{4} m^2] dx. \tag{18}$$

The second-order partial derivative q_{xx} of Equation (16) can be approximated by the FPS method [31,32]. Suppose $I_N q(x,t)$ is the FPS approximation of the function $q(x,t)$.

The values of the derivatives $\frac{d}{dx}I_N q(x,t)$ and $\frac{d^2}{dx^2}I_N p(x,t)$ at the collocation points x_j are obtained in terms of the value of q_j, i.e.,

$$\frac{d}{dx}I_N q(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} q_l \frac{dg_l(x_j)}{dx} = (\mathbf{D}_1\mathbf{Q})_j, \tag{19}$$

$$\frac{d^2}{dx^2}I_N q(x,t)|_{x=x_j} = \sum_{l=0}^{N-1} q_l \frac{d^2 g_l(x_j)}{dx^2} = (\mathbf{D}_2\mathbf{Q})_j, \tag{20}$$

The function $g_l(x)$ is the trigonometric polynomial explicitly given by

$$g_l(x) = \frac{1}{N}\sum_{k=-N/2}^{N/2} \frac{1}{c_k} \exp^{ik\mu(x-x_l)}, \tag{21}$$

where $c_k = 1$ $(|k| \neq N/2)$, $c_{-N/2} = c_{N/2} = 2$, $\mu = \frac{2\pi}{L}$ and

$$(\mathbf{D}_2)_{i,j} = \begin{cases} \frac{1}{2}\mu^2(-1)^{i+j+1}\frac{1}{\sin^2(\mu\frac{x_i-x_j}{2})}, & i \neq j, \\ -\mu^2\frac{N^2+2}{12}, & i = j. \end{cases} \tag{22}$$

Based on the above FPS method, we can obtain the semi-discrete system of the space fractional KGZ equation

$$\begin{cases} \frac{d}{dt}u_j = w_j, \\ \frac{d}{dt}m_j = -2(\mathbf{D}_2\mathbf{Q})_j, \\ \frac{d}{dt}w_j = (\mathbf{D}_2^\alpha\mathbf{U})_j - u_j - m_j u_j - |u_j|^2 u_j, \\ \frac{d}{dt}q_j = -\frac{1}{2}m_j - \frac{1}{2}|u_j|^2, \end{cases} \tag{23}$$

where $j = 0, 1, \cdots, N-1$.

Equation (23) can be written as the following semi-discrete Hamiltonian system

$$\frac{d\mathbf{Z}}{dt} = \hat{\mathbf{J}}\nabla_{\mathbf{Z}} H(\mathbf{Z}), \quad \hat{\mathbf{J}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I_N} \\ -\mathbf{I_N} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I_N} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{24}$$

where $\mathbf{Z} = (\mathbf{U}^T, \mathbf{M}^T, \mathbf{W}^T, \mathbf{Q}^T)$, $\mathbf{M} = (m_0, m_1, \cdots, m_{N-1})^T$, $\mathbf{W} = (w_0, w_1, \cdots, w_{N-1})^T$, $\mathbf{Q} = (q_0, q_1, \cdots, q_{N-1})^T$. $\mathbf{0}$ and $\mathbf{I}_N$ are the $N \times N$ zeros matrix and unite matrix. The corresponding Hamiltonian function is

$$H(\mathbf{Z}) = -\mathbf{Q}^T\mathbf{D}_2\mathbf{Q} - \frac{1}{2}\mathbf{U}^T\mathbf{D}_2^\alpha\mathbf{U} + \sum_{j=0}^{N-1}[\frac{1}{2}w_j^2 + \frac{1}{2}|u_j|^2 + \frac{1}{2}m_j|u_j|^2 + \frac{1}{4}|u_j|^4 + \frac{1}{4}m_j^2]. \tag{25}$$

The semi-discrete Hamiltonian system (24) is solved by the following second-order AVF method

$$\frac{\mathbf{Z}^{n+1} - \mathbf{Z}^n}{\tau} = \hat{\mathbf{J}}\int_0^1 \nabla H((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1})d\xi. \tag{26}$$

Equation (26) is equivalent to the following equations

$$\frac{u_j^{n+1}-u_j^n}{\tau} = \int_0^1 ((1-\xi)w_j^n + \xi w_j^{n+1})d\xi, \tag{27}$$

$$\frac{m_j^{n+1}-m_j^n}{\tau} = -2\int_0^1 ((1-\xi)(\mathbf{D}_2\mathbf{Q}^n)_j + \xi(\mathbf{D}_2\mathbf{Q}^{n+1})_j)d\xi, \tag{28}$$

$$\begin{aligned}\frac{w_j^{n+1}-w_j^n}{\tau} =& \int_0^1 [((1-\xi)(\mathbf{D}_2^\alpha\mathbf{U}^n)_j + \xi\left(\mathbf{D}_2^\alpha\mathbf{U}^{n+1}\right)_j) - ((1-\xi)u_j^n + \xi u_j^{n+1})]d\xi \\ &- \int_0^1 \left((1-\xi)m_j^n\right) + \xi m_j^{n+1}\Big)\left((1-\xi)u_j^n + \xi u_j^{n+1}\right)d\xi \\ &- \int_0^1 \left|\left((1-\xi)u_j^n + \xi u_j^{n+1}\right)\right|^2 \left((1-\xi)u_j^n + \xi u_j^{n+1}\right)d\xi,\end{aligned} \tag{29}$$

$$\frac{q_j^{n+1}-q_j^n}{\tau} = -\frac{1}{2}\int_0^1 (((1-\xi)m_j^n + \xi m_j^{n+1}) + \left|((1-\xi)u_j^n + \xi u_j^{n+1})\right|^2)d\xi. \tag{30}$$

From Equations (27)–(30), the auxiliary variables w, q can be deleted. We can obtain

$$\begin{aligned}\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\tau^2} =& (\mathbf{D}_2^\alpha\frac{\mathbf{U}^{n+1}+\mathbf{U}^n}{4})_j - \frac{u_j^{n+1}+u_j^n}{4} - \frac{m_j^{n+1}+m_j^n}{4}u_j^n - \frac{2m_j^{n+1}+m_j^n}{12}(u_j^{n+1}-u_j^n) \\ &- \left|\frac{1}{6}(u_j^{n+1})^2 + \frac{1}{6}(u_j^n)^2 + \frac{1}{6}u_j^{n+1}u_j^n\right|u_j^n - \left|\frac{1}{8}(u_j^{n+1})^2 + \frac{1}{24}(u_j^n)^2 + \frac{1}{12}u_j^{n+1}u_j^n\right| \\ &(u_j^{n+1}-u_j^n) + (\mathbf{D}_2^\alpha\frac{\mathbf{U}^n+\mathbf{U}^{n-1}}{4})_j - \frac{u_j^n+u_j^{n-1}}{4} - \frac{m_j^n+m_j^{n-1}}{4}u_j^{n-1} \\ &- \frac{2m_j^n+m_j^{n-1}}{12}(u_j^n-u_j^{n-1}) - \left|\frac{1}{6}(u_j^n)^2 + \frac{1}{6}(u_j^{n-1})^2 + \frac{1}{6}u_j^n u_j^{n-1}\right|u_j^{n-1} - \\ &\left|\frac{1}{8}(u_j^n)^2 + \frac{1}{24}(u_j^{n-1})^2 + \frac{1}{12}u_j^n u_j^{n-1}\right|(u_j^n-u_j^{n-1}),\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{m_j^{n+1}-2m_j^n+m_j^{n-1}}{\tau^2} =& (\mathbf{D}_2\frac{\mathbf{M}^{n+1}+2\mathbf{M}^n+\mathbf{M}^{n-1}}{4})_j + \sum_{l=1}^N d_{j,l}(\left|\frac{1}{6}(u_j^{n+1})^2 + \frac{1}{6}(u_j^n)^2 + \frac{1}{6}u_j^{n+1}u_j^n\right| \\ &+ \left|\frac{1}{6}(u_j^n)^2 + \frac{1}{6}(u_j^{n-1})^2 + \frac{1}{6}u_j^n u_j^{n-1}\right|).\end{aligned} \tag{32}$$

Theorem 1. *The semi-discrete scheme (26) can preserve the discrete energy conservation of the finite dimensional Hamiltonian system*

$$H(\mathbf{Z}^{n+1}) = H(\mathbf{Z}^n). \tag{33}$$

Proof. From Equation (26), we can obtain that

$$\begin{aligned}&(\int_0^1 \nabla H((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1})d\xi)^T\frac{\mathbf{Z}^{n+1}-\mathbf{Z}^n}{\tau} \\ &= (\int_0^1 \nabla H((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1})d\xi)^T\mathbf{J}\int_0^1 \nabla H((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1})d\xi,\end{aligned} \tag{34}$$

where $\mathbf{J}$ is the skew symmetric matrix. We can find that

$$\frac{1}{\tau}\int_0^1 (\mathbf{Z}^n - \mathbf{Z}^{n+1})^T\nabla H((1-\xi)\mathbf{Z}^n + \xi\mathbf{Z}^{n+1})d\xi = 0. \tag{35}$$

From the basic theorem of the integral theory, we can get

$$\frac{1}{\tau}(H(\mathbf{Z}^{n+1}) - H(\mathbf{Z}^n)) = 0. \tag{36}$$

The proof of the theorem is completed. □

3.2. Energy Preserving Method of the Fractional KGS Equation

Suppose $f(x,t) = p(x,t) + r(x,t)i$, $p(x,t) = p$, $r(x,t) = r$, then Equation (3) can be written as the following differential equations

$$\begin{cases} r_t = -\frac{\beta}{2}(-\triangle)^{\frac{\alpha}{2}} p + \alpha_1 hp, \\ p_t = \frac{\beta}{2}(-\triangle)^{\frac{\alpha}{2}} r - \alpha_1 hr, \\ v_t = \frac{\gamma}{2} h_{xx} - \frac{m^2}{2} h + \frac{\alpha_1}{2}(p^2 + r^2), \\ h_t = 2v. \end{cases} \tag{37}$$

Equation (37) can be transformed into the following infinite dimensional Hamiltonian system

$$\frac{d\tilde{\mathbf{z}}}{dt} = \tilde{\mathbf{J}} \frac{\delta H(\tilde{\mathbf{z}})}{\delta \tilde{\mathbf{z}}}, \tag{38}$$

where $\tilde{\mathbf{z}} = (r, p, v, h)^T$,

$$\tilde{\mathbf{J}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{39}$$

and the corresponding Hamiltonian energy function is

$$H(\tilde{\mathbf{z}}) = \int \frac{\beta}{4}(-(-\triangle)^{\frac{\alpha}{4}} p)^2 - (-\triangle)^{\frac{\alpha}{4}} r)^2) - \frac{\gamma}{4}(h_x)^2 - \frac{1}{4} m^2 h^2 + \frac{\alpha_1}{2} h(p^2 + r^2) - v^2 dx. \tag{40}$$

Equation (37) is discretized by the FPS method, and we can obtain

$$\begin{cases} \frac{d}{dt} r_j = \frac{\beta}{2}(\mathbf{D}_2^{\alpha} \mathbf{p})_j + \alpha_1 h_j p_j, \\ \frac{d}{dt} p_j = -\frac{\beta}{2}(\mathbf{D}_2^{\alpha} \mathbf{r})_j - \alpha_1 h_j r_j, \\ \frac{d}{dt} v_j = \frac{\gamma}{2}(\mathbf{D}_2 \mathbf{h})_j - \frac{m^2}{2} h_j + \frac{\alpha_1}{2}(p_j^2 + r_j^2), \\ \frac{d}{dt} h_j = 2v_j. \end{cases} \tag{41}$$

Equation (41) can be transformed into the finite dimensional Hamiltonian system

$$\frac{d\tilde{\mathbf{Z}}}{dt} = \mathbf{S} \nabla_{\mathbf{z}} H(\tilde{\mathbf{Z}}), \tag{42}$$

where $\tilde{\mathbf{Z}} = (r_0, \cdots, r_{N-1}, p_0, \cdots, p_{N-1}, v_0, \cdots, v_{N-1}, h_0, \cdots, h_{N-1})^T$ and

$$\mathbf{S} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_N & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_N & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_N \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_N & \mathbf{0} \end{pmatrix}. \tag{43}$$

$$H(\tilde{\mathbf{Z}}) = \frac{\beta}{4}(\mathbf{p}^T \mathbf{D}_2^{\alpha} \mathbf{p} + \mathbf{r}^T \mathbf{D}_2^{\alpha} \mathbf{r}) + \frac{\gamma}{4}(\mathbf{h}^T \mathbf{D}_2 \mathbf{h} + \sum_{j=0}^{N-1} [\frac{\alpha_1}{2} h_j (p_j^2 + r_j^2) - \frac{1}{4} m^2 h_j^2 - v_j^2]. \tag{44}$$

The semi-discrete Hamiltonian system (42) is solved by the second order AVF method, and we can get

$$\frac{\tilde{\mathbf{Z}}^{n+1}-\tilde{\mathbf{Z}}^{n}}{\tau}=\mathbf{S}\int_0^1\nabla_{\tilde{\mathbf{Z}}}H((1-\xi)\tilde{\mathbf{Z}}^{n}+\xi\tilde{\mathbf{Z}}^{n+1})d\xi. \tag{45}$$

Equation (45) is equivalent to the following schemes

$$\frac{r_j^{n+1}-r_j^n}{\tau}=\frac{\beta}{2}(\mathbf{D}_2^{\alpha}\frac{\mathbf{p}^{n+1}+\mathbf{p}^n}{2})_j+\alpha_1(\frac{1}{3}h_j^{n+1}p_j^{n+1}+\frac{1}{6}h_j^{n+1}p_j^n+\frac{1}{6}h_j^np_j^{n+1}+\frac{1}{3}h_j^np_j^n), \tag{46}$$

$$\frac{p_j^{n+1}-p_j^n}{\tau}=-\frac{\beta}{2}(\mathbf{D}_2^{\alpha}\frac{\mathbf{r}^{n+1}+\mathbf{r}^n}{2})_j-\alpha_1(\frac{1}{3}h_j^{n+1}r_j^{n+1}+\frac{1}{6}h_j^{n+1}r_j^n+\frac{1}{6}h_j^nr_j^{n+1}+\frac{1}{3}h_j^nr_j^n), \tag{47}$$

$$\begin{aligned}\frac{v_j^{n+1}-v_j^n}{\tau}=&\frac{\gamma}{2}(\mathbf{D}_2\frac{\mathbf{h}^{n+1}+\mathbf{h}^n}{2})_j-\frac{m^2}{2}(\frac{h_j^{n+1}+h_j^n}{2})+\frac{\alpha_1}{6}(p_j^{n+1}p_j^{n+1}+p_j^{n+1}p_j^n+p_j^np_j^n\\&+r_j^{n+1}r_j^{n+1}+r_j^{n+1}r_j^n+r_j^nr_j^n),\end{aligned} \tag{48}$$

$$\frac{h_j^{n+1}-h_j^n}{\tau}=2\frac{v_j^{n+1}+v_j^n}{2}. \tag{49}$$

According to the above Theorem 1, schemes (46)–(49) can also preserve the discrete Hamiltonian energy of the Riesz space fractional KGS equation exactly.

4. Numerical Simulation

4.1. Numerical Simulation of the Fractional KGZ Equation

In this section, we test the following discrete Hamiltonian energy errors

$$Error_H^n=|\frac{H(\mathbf{Z}^n)-H(\mathbf{Z}^0)}{H(\mathbf{Z}^0)}|, \tag{50}$$

by schemes (31) and (32), where

$$\begin{aligned}H(\mathbf{Z}^n)=&-(\mathbf{Q}^n)^T\mathbf{D}_2\mathbf{Q}^n-\frac{1}{2}(\mathbf{U}^n)^T\mathbf{D}_2^{\alpha}\mathbf{U}^n+\sum_{j=0}^{N-1}[\frac{1}{2}(w_j^n)^2+\frac{1}{2}|u_j^n|^2+\frac{1}{2}m_j^n|u_j^2|^2+\\&\frac{1}{4}|u_j^2|^4+\frac{1}{4}(m_j^n)^2],\end{aligned} \tag{51}$$

and $H(\mathbf{Z}^0)$ is the discrete Hamiltonian energy at $t=0$.

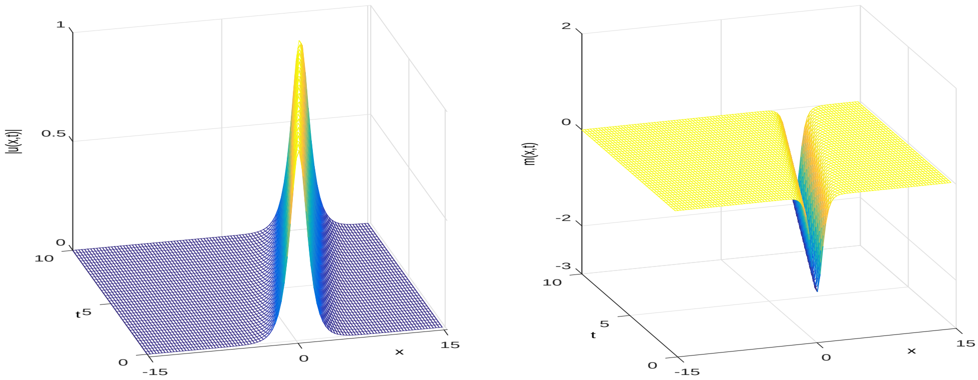
The two level initial conditions can be taken as follows [16]:

$$\begin{cases}u_0=\frac{\sqrt{10}-\sqrt{2}}{2}\mathtt{sech}(\sqrt{\frac{1+\sqrt{5}}{2}}x)\exp(i\sqrt{\frac{2}{1+\sqrt{5}}}x),\\ m_0=-2\mathtt{sech}^2(\sqrt{\frac{1+\sqrt{5}}{2}}x),\end{cases} \tag{52}$$

and

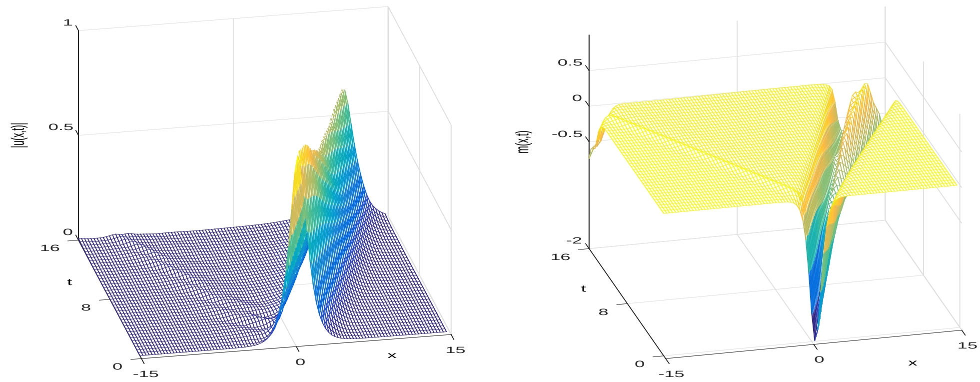
$$\begin{cases}u_1=\frac{\sqrt{10}-\sqrt{2}}{2}(\tanh x-1)\mathtt{sech}(\sqrt{\frac{1+\sqrt{5}}{2}}x)\exp(i\sqrt{\frac{2}{1+\sqrt{5}}}x),\\ m_1=-4\mathtt{sech}^2(\sqrt{\frac{1+\sqrt{5}}{2}}x)\tanh(\sqrt{\frac{1+\sqrt{5}}{2}}x).\end{cases} \tag{53}$$

Figure 1 gives the numerical solution of the KGZ equation with $\alpha=2$, $t\in[0,10]$. Figure 2 gives the numerical solution of the Riesz space fractional KGZ equation with $\alpha=1.7$, $t\in[0,16]$. These numerical results are consistent with the existing numerical results [16]. From Figures 1 and 2, we can see that the new scheme can simulate the evolution of solitary waves of the Riesz space fractional KGZ equation well. The numerical solutions and the numerical schemes are stable. Figure 3 shows the energy error of the Riesz space fractional KGZ equation with (a): $\alpha = 2$, (b): $\alpha = 1.7$ at $t \in [0, 20]$. The energy error is up to 10^{-13}, which can neglected. It is obvious that the new scheme can well preserve the discrete energy of the fractional KGZ equation.



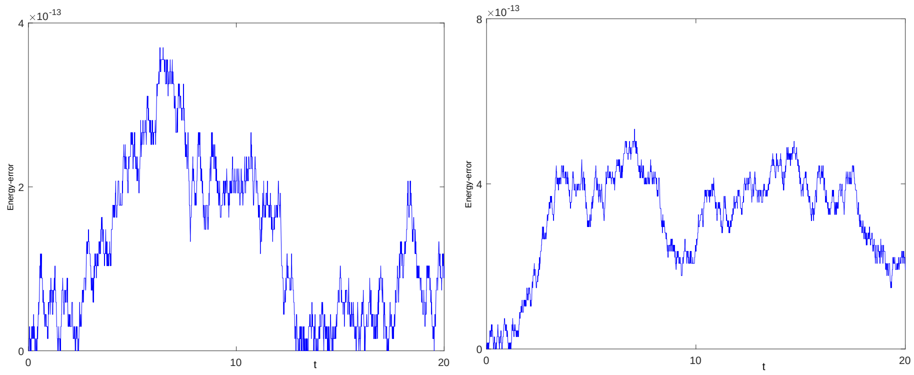
(**a**) (**b**)

Figure 1. Evolution of solitary waves (**a**) $u(x, t)$ and (**b**) $m(x, t)$ at $\alpha = 2$, $t \in [0, 10]$.



(**a**) (**b**)

Figure 2. Evolution of solitary waves (**a**) $u(x, t)$ and (**b**) $m(x, t)$ at $\alpha = 1.7$, $t \in [0, 16]$.



(**a**) (**b**)

Figure 3. Energy error of fractional KGZ equation: (**a**) $\alpha = 2$, (**b**) $\alpha = 1.7$ at $t \in [0, 20]$.

4.2. Numerical Simulation of the Fractional KGS Equation

We apply schemes (46)–(49) to simulate the Riesz space fractional KGS equation. The discrete Hamiltonian energy errors can be defined as

$$Error_H^n = |\frac{H(\tilde{\mathbf{Z}}^n) - H(\tilde{\mathbf{Z}}^0)}{H(\tilde{\mathbf{Z}}^0)}|, \tag{54}$$

where

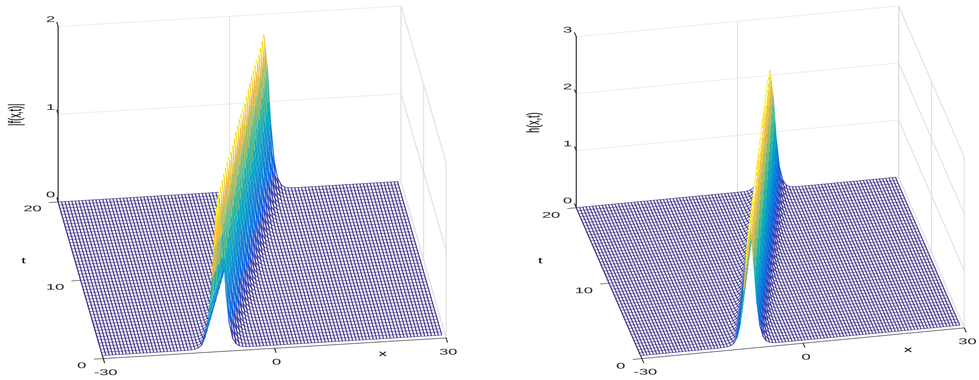
$$H(\tilde{\mathbf{Z}}^n) = \frac{\beta}{4}((\mathbf{p}^n)^T\mathbf{D}_2^{\alpha}\mathbf{p}^n + (\mathbf{r}^n)^T\mathbf{D}_2^{\alpha}\mathbf{r}^n) + \frac{\gamma}{4}(\mathbf{h}^n)^T\mathbf{D}_2\mathbf{h}^n + \sum_{j=0}^{N-1}[\frac{\alpha_1}{2}h_j^n((p_j^n)^2 + (r_j^n)^2) - \frac{1}{4}m^2(h_j^n)^2 - (v_j^n)^2] \tag{55}$$

and $H(\tilde{\mathbf{Z}}^0)$ is the discrete Hamiltonian energy at $t = 0$.

The parameter variables are taken as $v = 0.8$, $x_0 = 10$. First, we consider the evolution of single solitary waves. The initial condition of single solitary wave is taken as follows

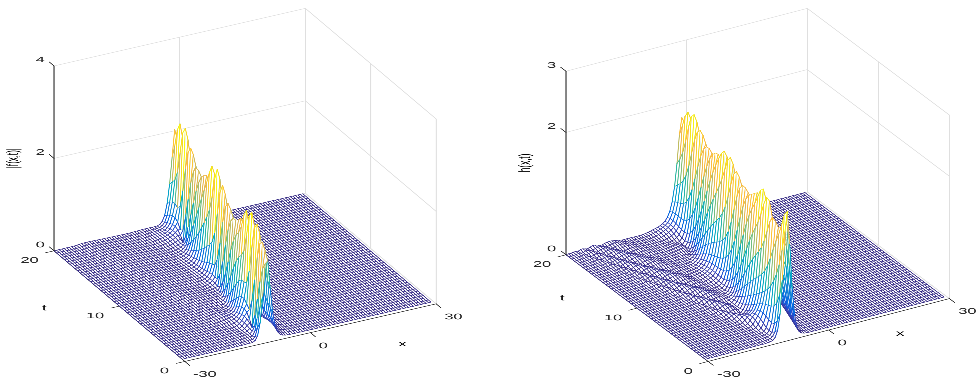
$$\begin{cases} f_0 = f(x - x_0, 0, v) = \frac{3\sqrt{2}}{4(1-v^2)}\mathtt{sech}^2(\frac{1}{2\sqrt{1-v^2}}(x - x_0)\exp(ivx), \\ h_0 = h(x - x_0, 0, v) = \frac{3}{4(1-v^2)}\mathtt{sech}^2(\frac{1}{2\sqrt{1-v^2}}(x - x_0). \end{cases} \tag{56}$$

Figure 4 shows the numerical solution of a single solitary wave of the KGS equation at $\alpha = 2$, $t \in [0, 20]$. Figure 5 shows the numerical solution of solitary waves $f(x, t)$ and $h(x, t)$ of the Riesz space fractional KGS equation at $\alpha = 1.2$, $t \in [0, 20]$. From Figures 4 and 5, we can see that the new scheme of the Riesz space fractional KGS equation can also simulate the evolution of a solitary wave of the equation well. The numerical solutions and the numerical schemes are stable. Figure 6 shows the energy error of the Riesz space fractional KGS equation. The error can be neglected. The new scheme of the equation can also well preserve the discrete energy of the equation.



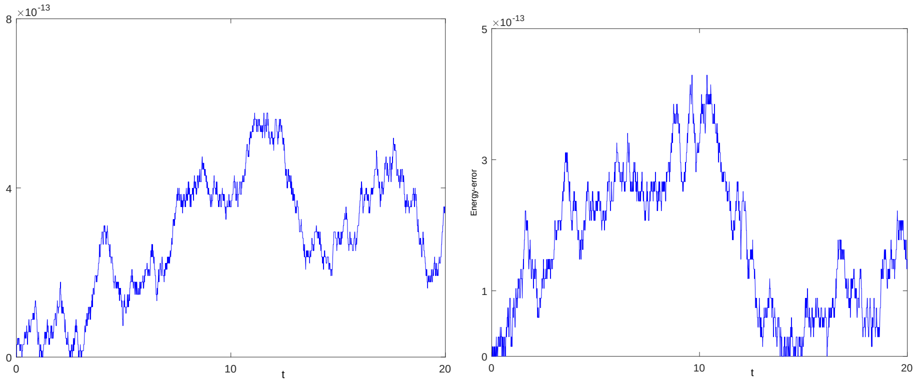
(**a**) (**b**)

Figure 4. Evolution of solitary waves (**a**) $f(x, t)$ and (**b**) $h(x, t)$ at $\alpha = 2$, $t \in [0,\ 20]$.



(**a**) (**b**)

Figure 5. Evolution of solitary waves (**a**) $f(x, t)$ and (**b**) $h(x, t)$ at $\alpha = 1.2$, $t \in [0, 20]$.



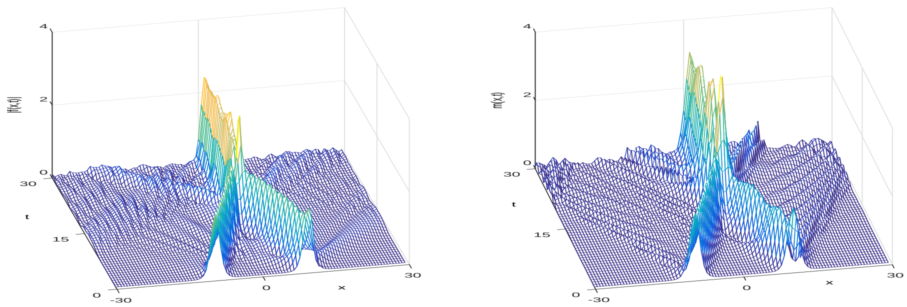
(**a**) (**b**)

Figure 6. Energy error of the fractional KGS equation: (**a**) $\alpha = 2$, (**b**) $\alpha = 1.2$ at $t \in [0, 20]$.

Then, we consider the evolution of two solitary waves with the following initial conditions

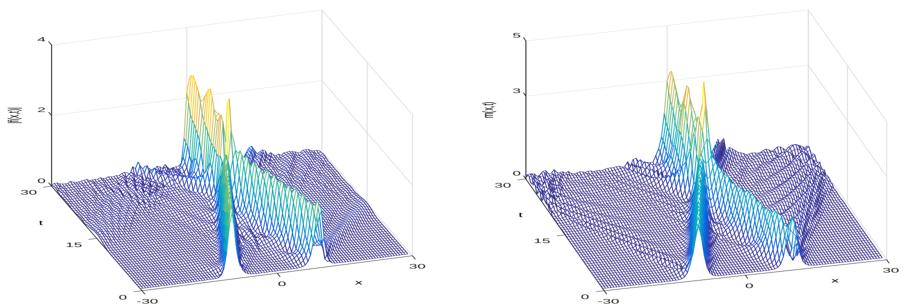
$$\begin{cases} f_0 = f(x - x_0, 0, v) + f(x + x_0, 0, -v), \\ h_0 = h(x - x_0, 0, v) + h(x + x_0, 0, -v). \end{cases} \tag{57}$$

Figure 7 shows the numerical solutions of two solitary waves of the KGS equation at $\alpha = 2$, $t \in [0, 30]$. Figure 8 shows numerical solutions of two solitary waves of the Riesz space fractional KGS equation at $\alpha = 1.6$, $t \in [0, 30]$. The numerical results are consistent with the existing results [23]. From Figures 7 and 8, we can see that the new scheme of the Riesz space fractional KGS equation can also simulate the evolution of multiple solitary waves of the equation well. The numerical solutions and the numerical schemes are stable. Figure 9 shows the energy error of the Riesz space fractional KGS equation. The error can be neglected. The new scheme of the Riesz space fractional KGS equation can also preserve the discrete energy of the equation well.



(**a**) (**b**)

Figure 7. Evolution of solitary waves (**a**) $f(x,t)$ and (**b**) $h(x,t)$ at $\alpha = 2$, $t \in [0, 30]$.



(**a**) (**b**)

Figure 8. Evolution of solitary waves (**a**) $f(x,t)$ and (**b**) $h(x,t)$ at $\alpha = 1.6$, $t \in [0, 30]$.

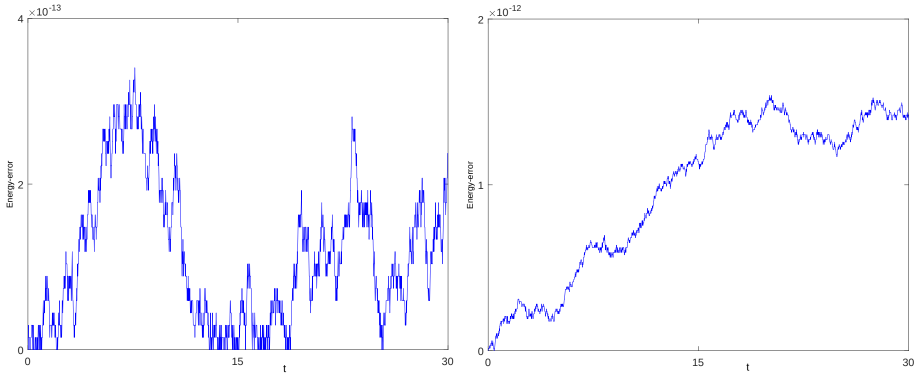


Figure 9. Energy errors of the fractional KGS equations: (**a**) $\alpha = 2$, (**b**) $\alpha = 1.6$ at $t \in [0, 30]$.

5. Conclusions

In this paper, two new energy preserving schemes for the Riesz space fractional KGZ and KGS equations are proposed based on the FPS method and the AVF method. The energy conservation property is proven. Numerical results show that these new schemes can simulate the evolution behavior of the solitary waves of these fractional differential equations well and preserve the discrete energy conservation property. The existing energy conserving schemes of the space fractional KGZ and KGS equations are in general of second-order accuracy. However, these energy conservation schemes based on the second-order AVF method can be easily extended to high-order energy preserving schemes. In future work, we will construct high-order energy preserving schemes for fractional differential equations based on the high-order AVF method and analyze the convergence and stability of these new energy-preserving schemes.

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Abbreviations

The following abbreviations are used in this manuscript:

KGS	Klein–Gordon–Schrödinger
KGZ	Klein–Gordon–Zakharov
AVF	average vector field
PDEs	partial differential equations
FPS	Fourier pseudo-spectral

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