fractal and fractional

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Multi-Peak and Propagation Behavior of *M*-Shape Solitons in (2 + 1)-Dimensional Integrable Schwarz-Korteweg-de Vries Problem

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Abstract: This paper examines the propagation of *M*-shape solitons and their interactions with kink waves to the (2 + 1)-dimensional integrable Schwarz-Korteweg-de Vries (ISKdV) problem by applying the symbolic computation with ansatz functions technique and logarithmic transformation. The governing model usually appears in the nonlinear shallow water waves and fluid mechanics. We discuss various nonlinear waves like multiwave solutions (MSs), homoclinic breather (HB), *M*-shape solitons, single exponential form (one-kink), and double exponential form (two-kink). These waves have lot of applications in fluid dynamics, nonlinear optics, chemical reaction networks, biological systems, climate science, and material science. We also study interaction among *M*-shape solitons with kink wave. At the end, we discuss the stability characteristics of all solutions.

Keywords: nonlinear waves; computational simulations; homoclinic breather; interaction phenomena

Citation: Ahmed, S.; Seadawy, A.R.; Rizvi, S.T.R.; Raza, U. Multi-Peak and Propagation Behavior of *M*-Shape Solitons in (2 + 1)-Dimensional Integrable Schwarz-Korteweg-de Vries Problem. *Fractal Fract.* **2023**, *7*, 709. https://doi.org/10.3390/fractalfract7100709

Academic Editors: Haci Mehmet Baskonus, Lanre Akinyemi and Mehmet Senol

Received: 18 August 2023
Revised: 19 September 2023
Accepted: 20 September 2023
Published: 26 September 2023

1. Introduction

Soliton solutions of nonlinear differential equations (NLPDEs) are a fundamental object in nonlinear sciences. The stability property of a soliton plays a crucial role in unalike mathematical and physical contexts, as it enables the persistence of coherent structures. Solitons are capable of modeling and describing a wide range of complex physical phenomena, including nonlinear optics, plasma physics, and more. Studying solitons in NLPDEs creates opportunities and mathematical challenges for thriving new analytical and numerical techniques. These characteristics advance the field of nonlinear mathematics. Several techniques for finding soliton solutions of NLPDEs have been studied during the past few decades [1–5]. Recently, the interaction among soliton solutions has received a lot of attention from many researchers, due to their uses in biophysics, oceanography, plasma physics, and nonlinear fiber optics [6–11]. An extensive variety of features in soliton dynamics have been studied over the past few decades for nonlinear systems [12–26]. One of the renowned NLPDEs is the Kortweg-de Vries (KdV), given as [24]

$$\Psi_t + \Psi_{xxx} - 6\Psi\Psi_x = 0, \tag{1}$$

where the nonlinear shallow water waves are given in $\Psi = \Psi(x,t)$. The generalized form of Equation (1) is the Schwarz-Kortweg-de Vries (SKdV) equation, given as [25]

$$\Psi_t + \Psi_x\left\{\left(\frac{\Psi_{xx}}{\Psi_x}\right) - \frac{1}{2}\left(\frac{\Psi_{xx}}{\Psi_x}\right)^2\right\} = 0. \tag{2}$$

The component generalization of the SKdV-model in $(2+1)$-dimensions is given as

$$\Psi_t + \frac{1}{4}\Psi_{xxy} - \frac{\Psi_x\Psi_{xy}}{2\Psi} - \frac{\Psi_y\Psi_{xx}}{4\Psi} + \frac{\Psi_x^2\Psi_y}{2\Psi^2} - \frac{\Psi_x}{8}\int\left(\frac{\Psi_x^2}{\Psi^2}\right)_y dx = 0. \tag{3}$$

Equation (3) was originated by Yu and Toda [27]. By the following assumption in Equation (3),

$$\Psi = W_x,\ W = e^g,\ g_x = H,\ g_t = \Theta, \tag{4}$$

where $W = W(x,t)$, $g = g(x,t)$, $H = H(x,t)$, and $\Theta = \Theta(x,t)$ are unknown functions, one can obtain

$$\begin{cases} 4H^2\Theta_x - 4HH_x\Theta + H^2H_{xxy} - HH_{xx}H_y - 3HH_xH_{xy} + 3H_x^2H_y - H^4H_y = 0, \\ H_t - \Theta_x = 0. \end{cases} \tag{5}$$

This equation is due to Pickering and Kudriashov [28–30]. By the Miura transform [30] on Equation (5), one can obtain

$$\Psi_x = \frac{H_{xx}}{4H} - \frac{3H_x^2}{H^2} - \frac{H^2}{8}, \Psi_y = -\frac{\Theta}{H}, \tag{6}$$

and it generates [30]

$$4\Psi_{xt} + \Psi_{xxxy} + 8\Psi_{xy}\Psi_x + 4\Psi_y\Psi_{xx} = 0. \tag{7}$$

Many researchers worked on the ISKDV model. Ramirez et al. investigated multiple solutions for the SKDV equation in (2 + 1)-dimensions [29] via Mobius transformation, Khater worked on new solitary solutions for a $(2+1)$-dimensional ISKDV-model via the Khater technique and Bernoulli sub-equation technique [30], Attia et al. studied numerical and computational solutions for a $(2+1)$-dimensional ISKDV-model with Miura transform [31], Toda et al. investigated the soliton solutions for a governing model in $(2+1)$-dimensions [32] via Lax pairs and well-known higher-dimensional manner, Gandarias et al. founded the classical symmetry reductions for the ISKDV model by using symmetries and arbitrary functions [33], Li et al. studied the soliton solutions of the $(2+1)$-dimensional ISKDV-model by applying Darboux transformation [34], Li et al. evaluated the diversity soliton excitations for the $(2+1)$-dimensional ISKDV-model [35], and Aslan worked on an investigation of analytic solutions for the $(2+1)$-dimensional ISKDV-model via improved mapping approach [36], but the contribution of this document is to evaluate MS, HB, and $M-$shaped solitons by applying the symbolic computation with ansatz functions approach and logarithmic transformation for the $(2+1)$-dimensional ISKDV-model. M-shape rational solitons are described by nonlinear equations that involve both the phase and amplitude of the wave. They often arise in optical fibers, plasma physics, and Bose–Einstein condensates. These solitons are important because of their ability to maintain their stability and shape over long distances. Furthermore, we will study the M-shaped solitons and their interactions among one- and two-kink waves. When M-shape solitons interact, they can undergo complex dynamics. These interactions depend on the amplitudes, velocities, and phases of the solitons. They can show behaviors like fusion, elastic scattering, and bound states. The study of soliton interactions is crucial in signal processing and communication. We will also study the HBs that appear in various systems like Klein–Gordon equations, Hamiltonian systems, and optical systems. Lastly, we will also apply the stability property for attained solutions. All these stated rational solutions via the proposed method are novel and not found in the earlier literature. Our new solutions will be useful for understanding the nonlinear phenomena in the nonlinear shallow water waves and fluid mechanics.

The article template is arranged as follows. In Section 2, we will compute the MS for the $(2+1)$-dimensional ISKdV problem with logarithmic transformation and three waves scheme. In Section 3, we will utilize the HB method and compute some new classes of solitons for the $(2+1)$-dimensional ISKdV problem. In Section 4, we will compute the M-shaped solitons for Equation (1) and we will show some 3D, 2D, contours, and their stability profiles. The precise computation of interaction of M-shaped solitons via one exponential function approach along some 3D, 2D, contours, and their stability graphs is shown in Section 5. We will determine interactional solutions with double exponential form for the stated model in Sections 6 and 7, and this will contain the stability property of solutions and their application to all obtained solutions (see Table 1). In Section 8, we will give results and discussions, and finally, in Section 9, we will state our concluding observations.

Table 1. Stability properties for newly attained solutions $\Psi_i(x,y,t)$ where $(i = 1,2,3,...,14)$.

Solution	Stability	Values of Variables
Ψ_1	Stable	$k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5, x, y, t\ \varepsilon[-7,7]$
Ψ_2	Stable	$k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5, x, y, t\ \varepsilon[-7,7]$
Ψ_3	Unstable	Singular solution
Ψ_4	Stable	$k_1 = 1, k_2 = 1.5, c = 4, a_6 = 5, a_4 = 2, b_0 = 1.5, p_1 = 3, a_2 = 2, p = 7, b_1 = 1, x, y, t\ \varepsilon[-5,5]$
Ψ_5	Stable	$k_1 = 1, k_2 = 1.5, c = 4, a_4 = 2, a_6 = 5, b_0 = 1.5, p_1 = 3, a_2 = 2, p = 2, b_1 = -1, x, y, t\ \varepsilon[-3,3]$
Ψ_6	Stable	$k_1 = 1, k_2 = -1.5, c = 4, a_4 = 2, a_6 = -5, b_0 = -1.5, p_1 = -3, a_2 = 2, p = 2, b_1 = 1, x, y, t\ \varepsilon[-8,8]$
Ψ_7	Unstable	Singular solution
Ψ_8	Stable	$d_3 = 10, k_2 = 5, c = -2.4, k_1 = -3, d_4 = -5, x, y, t\ \varepsilon[-6,6]$
Ψ_9	Stable	$d_1 = -4, k_1 = -1, c = 2.5, x, y, t\ \varepsilon[-10,10]$
Ψ_{10}	Stable	$d_1 = -4, d_3 = 2, d_5 = 4, k_2 = 2, c = 2.5, k_1 = -1, x, y, t\ \varepsilon[-6,6]$
Ψ_{11}	Unstable	Singular solution
Ψ_{12}	Unstable	Singular solution
Ψ_{13}	Stable	$a_2 = 4, a_4 = 1, c = 1.5, a_2 = 2, b_1 = 2, b_2 = 6, x, y, t\ \varepsilon[-5,5]$
Ψ_{14}	Stable	$a_2 = 4, a_4 = 1, c = 1.5, b_2 = 0.2, k_2 = 2, x, y, t\ \varepsilon[-9,9]$

2. MS

For this purpose, we use the ansatz [37]

$$\Psi(x,y,t) = \Upsilon(\eta), \;\; \eta = k_1 x + k_1 y - ct. \tag{8}$$

By applying Equation (8) into Equation (7), we obtain

$$4c\Upsilon'' + 12k_1^2 k_2 \Upsilon' \Upsilon'' + k_1^3 k_2 \Upsilon^{(iv)} = 0. \tag{9}$$

Now with usage of the logarithmic transformation in Equation (9),

$$\Upsilon = 2(\ln f)_\eta, \tag{10}$$

we obtain

$$\begin{aligned} &8ck_1 f^2 f'^3 - 48k_1^2 k_2 f'^5 + 24k_1^3 k_2 f'^5 - 12ck_1 f^3 f' f'' + 120k_1^2 k_2 f f'^3 f'' - 60k_1^3 k_2 f f'^3 f'' - 72k_1^2 k_2 f^2 f' f''^2 \\ &+ 30k_1^3 k_2 f^2 f' f''^2 + 4ck_1 f^4 f''' - 24k_1^2 k_2 f^2 f'^2 f''' + 20k_1^3 k_2 f^2 f'^2 f''' + 24k_1^2 k_2 f^3 f'' f''' - 10k_1^3 k_2 f^3 f'' f''' \\ &- 54k_1^3 k_2 f^3 f' f^{(iv)} + k_1^3 k_2 f^4 f^{(v)} = 0. \end{aligned} \tag{11}$$

For MS, we utilize three waves ansatz on f in Equation (11) as [37]

$$f = b_0 \cosh(a_1\eta + a_2) + b_1 \cos(a_3\eta + a_4) + b_2 \cosh(a_5\eta + a_6), \tag{12}$$

where a_i are any constants. By usage of Equations (11) and (12) and solving the equations from coefficients of cos, sinh, and cosh functions, we have

Set I.

$$a_3 = \sqrt{\frac{15k_1 - 36}{-4 + 5k_1}},\; a_5 = \sqrt{-\frac{15k_1 - 36}{-4 + 5k_1}},\; k_2 = \frac{4(-4+5k_1)c}{3(5k_1 - 12)k_1(5k_1 - 8)},\; a_1 = 0. \tag{13}$$

By using the above values, we have

$$\Psi_1 = \frac{\sqrt{3}\left(-b_1\sqrt{\frac{12-5k_1}{4-5k_1}}\sin\left(2\left(a_4 + \sqrt{\frac{36-15k_1}{4-5k_1}}\left(ct + k_1 x + \frac{4(-4+5k_1)cy}{3(5k_1-12)k_1(5k_1-8)}\right)\right)\right) + b_2\sqrt{\frac{12-5k_1}{4-5k_1}}N_1\right)}{b_1\cos\left(a_4 + \sqrt{\frac{36-15k_1}{4-5k_1}}\left(ct + k_1 x + \frac{4(-4+5k_1)cy}{3(5k_1-12)k_1(5k_1-8)}\right)\right) + b_0\cosh(a_2) + b_2 N_2}, \tag{14}$$

where $N_1 = \sinh\left(2\left(a_6 + \sqrt{\frac{36-15k_1}{-4+5k_1}}\left(ct + k_1 x + \frac{4(-4+5k_1)cy}{3(5k_1-12)k_1(5k_1-8)}\right)\right)\right)$ and $N_2 = \cosh\left(a_6 + \sqrt{\frac{36-15k_1}{-4+5k_1}}\left(ct + k_1 x + \frac{4(-4+5k_1)cy}{3(5k_1-12)k_1(5k_1-8)}\right)\right)$.

Set II. When

$$k_2 = -\frac{4c}{k_1\left(10a_5^2 k_1 - 12a_5^2 + 15k_1 - 36\right)},\; a_1 = a_1,\; a_2 = a_2,\; a_3 = 0. \tag{15}$$

By using the above values,

$$\Psi_2 = \frac{a_1 b_0 \sinh\left(2\left(a_2 + a_1\left(ct + k_1 x - \frac{4c}{k_1\left(10a_5^2 k_1 - 12a_5^2 + 15k_1 - 36\right)}y\right)\right)\right) + N_3}{b_1\cos(a_4) + b_0\cosh\left(a_2 + a_1\left(ct + k_1 x - \frac{4c}{k_1\left(10a_5^2 k_1 - 12a_5^2 + 15k_1 - 36\right)}y\right)\right) + N_4}, \tag{16}$$

where $N_3 = a_5 b_2 sinh\left(2\left(a_6 + a_5\left(ct + k_1 x - \frac{4c}{k_1\left(10a_5^2 k_1 - 12a_5^2 + 15k_1 - 36\right)} y\right)\right)\right)$ and
$N_4 = b_2 cosh\left(2\left(a_6 + a_5\left(k_1 x + c\left(t - \frac{4y}{k_1\left(10a_5^2 k_1 - 12a_5^2 + 15k_1 - 36\right)}\right)\right)\right)\right)$.

Set III.

$$a_1 = \sqrt{-\frac{15k_1 - 36}{-4 + 5k_1}},\ a_3 = \sqrt{-\frac{-15k_1 + 36}{-4 + 5k_1}},\ a_5 = \sqrt{-\frac{15k_1 - 36}{-4 + 5k_1}},\ k_2 = \frac{c(-4 + 5k_1)}{3(5k_1 - 12)k_1(5k_1 - 8)}. \tag{17}$$

By using the above values, we have

$$\Psi_3 = \frac{\left(\sqrt{3}\left(-b_1\sqrt{\frac{12-5k_1}{4-5k_1}}\right)\sin\left(2\left(a_4 + \sqrt{\frac{36-15k_1}{4-5k_1}}\right)\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right)\right) + b_0 N_5 + +b_2 N_6\right)}{b_1 cos\left(a_4 + \sqrt{\frac{36-15k_1}{4-5k_1}}\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right) + b_0 N_7 + +b_2 N_8\right)}, \tag{18}$$

where $N_5 = \sinh\left(2\left(a_2 + \sqrt{\frac{36-15k_1}{4-5k_1}}\right)\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right)\right)$,
$N_6 = \sinh\left(2\left(a_4 + \sqrt{\frac{36-15k_1}{4-5k_1}}\right)\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right)\right)$,
$N_7 = \cosh\left(a_2 + \sqrt{\frac{36-15k_1}{4-5k_1}}\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right)\right)$ and
$N_8 = \cosh\left(a_6 + \sqrt{\frac{36-15k_1}{4-5k_1}}\left(ct + k_1 x + \frac{c(-4+5k_1)y}{3k_1(5k_1-12)(-8+5k_1)}\right)\right)$.

3. HB

For HB, we consider f as follows [37]:

$$f = e^{-p(a_2 + a_1\eta)} + b_1 e^{p(a_4 + a_3\eta)} + b_0\cos(p_1(a_6 + a_5\eta)), \tag{19}$$

where $a_i's, b_i's$ are any constants. Inserting Equation (19) in Equation (11) and taking equations from the coefficients of exp and cos functions and by solving them, we obtain the following:

Set I. When

$$a_1 = \frac{\sqrt{-\frac{5k_1-12}{3k_1-4}}}{p},\ c = -\frac{k_2 k_1^2(5k_1 - 12)}{12k_1 - 16},\ a_5 = 0,\ a_3 = 0,\ b_0 = b_0. \tag{20}$$

Via the above values, we obtain

$$\Psi_4 = -\frac{2e^{-p\left(a_2 + \frac{\sqrt{-\frac{5k_1-12}{3k_1-4}}\left(-\frac{k_2 k_1^2(5k_1-12)t}{12k_1-16} + k_1 x + k_2 y\right)}{p}\right)}\sqrt{-\frac{5k_1-12}{3k_1-4}}}{b_1 e^{a_4 p} + 2e^{-p\left(a_2 + \frac{\sqrt{-\frac{5k_1-12}{3k_1-4}}\left(-\frac{k_2 k_1^2(5k_1-12)t}{12k_1-16} + k_1 x + k_2 y\right)}{p}\right)} + b_0\cos(a_6 p_1)}. \tag{21}$$

Set II. When

$$a_1 = \frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}}{p},\ a_3 = \frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}}{p},\ b_0 = b_0,\ a_5 = 0. \tag{22}$$

Via the above values, we have

$$\Psi_5=\frac{2\left(-e^{-p\left(a_2+\frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}(ct+k_1x+k_2y)}{p}\right)}\sqrt{-\frac{10k_1-24}{15k_1-24}}+b_1e^{-p\left(a_4+\frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}(ct+k_1x+k_2y)}{p}\right)}\sqrt{-\frac{10k_1-24}{15k_1-24}}\right)}{e^{-p\left(a_2+\frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}(ct+k_1x+k_2y)}{p}\right)}+b_1e^{-p\left(a_4+\frac{\sqrt{-\frac{10k_1-24}{15k_1-24}}(ct+k_1x+k_2y)}{p}\right)}+b_0\cos(a_6p_1)}. \tag{23}$$

Set III. When

$$a_1=\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}}{p},\ a_3=\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}}{p},\ a_2=a_2,\ b_0=b_0,\ a_5=0. \tag{24}$$

Via the above values, we have

$$\Psi_6=\frac{2\left(-e^{-p\left(a_2+\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}(ct+k_1x+k_2y)}{p}\right)}\sqrt{-\frac{30k_1+72}{101k_1-216}}+b_1e^{-p\left(a_4+\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}(ct+k_1x+k_2y)}{p}\right)}\sqrt{-\frac{30k_1+72}{101k_1-216}}\right)}{e^{-p\left(a_2+\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}(ct+k_1x+k_2y)}{p}\right)}+b_1e^{-p\left(a_4+\frac{\sqrt{-\frac{30k_1+72}{101k_1-216}}(ct+k_1x+k_2y)}{p}\right)}+b_0\cos(a_6p_1)}. \tag{25}$$

4. *M*-Shape Solitons

For obtaining the M-shape solution, we choose f in the form [37,38]

$$f=(d_1\eta+d_2)^2+(d_3\eta+d_4)^2+d_5, \tag{26}$$

where $d_i(1\le i\le 5)$ are any parameters. Put f into Equation (11) and solve the system of equations, which is achieved from various coefficients of ζ:
Set I. When $d_5=d_2=0$,

$$k_1=\frac{4\left(120d_1^2d_4^2+45d_4^6-16d_1^2\right)}{240d_1^2d_4^4+75d_4^6-32d_1^2},\ k_2=-\frac{\left(57600d_1^4d_4^8+36000d_1^2d_4^{10}+5625d_4^{12}-15360d_1^4d_4^4-4800d_1^2d_4^6+1024d_1^4\right)c}{192\left(120d_1^2d_4^4+45d_4^6-16d_1^2\right)d_1^2}. \tag{27}$$

Using the above values, we obtain

$$\Psi_7=\frac{2\left(2d_1^2\left(ct+\frac{4\left(45d_4^6+8d_1^2\left(-2-15d_4^4\right)\right)x}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}-\frac{c\left(75d_4^6+16d_1^2\left(-2+15d_4^4\right)\right)y}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}\right)+2d_3R_1\right)}{d_1^2\left(ct+\frac{4\left(45d_4^6+8d_1^2\left(-2-15d_4^4\right)\right)x}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}-\frac{c\left(75d_4^6+16d_1^2\left(-2+15d_4^4\right)\right)y}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}\right)^2+R_1^2}, \tag{28}$$

where $R_1=d_4+d_3\left(ct+\frac{4\left(45d_4^6+8d_1^2\left(-2-15d_4^4\right)\right)x}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}-\frac{c\left(75d_4^6+16d_1^2\left(-2+15d_4^4\right)\right)y}{75d_4^6+16d_1^2\left(-2-15d_4^4\right)}\right)$.

Set II. When $d_5=d_2=0$,

$$d_1=\sqrt{-1}d_3,\ k_2=k_2,\ k_1=k_1,\ d_3=d_3,\ d_1=d_1. \tag{29}$$

Through the above values,

$$\Psi_8=\frac{2\left(-2d_3^2(ct+k_1x+k_2y)+2d_3(ct+k_1x+k_2y)\right)}{-d_3^2(ct+k_1x+k_2y)^2+(d_4+d_3(ct+k_1x+k_2y))^2}. \tag{30}$$

Set III. When $d_5 = d_2 = 0$,

$$d_4 = \frac{1}{15} 2^{\frac{1}{4}} 15^{\frac{3}{4}},\ k_2 = -\frac{5}{864} \frac{c\sqrt{2}\sqrt{15}}{d_1^2},\ k_1 = k_1,\ d_3 = 0. \tag{31}$$

Through the above values, we have

$$\Psi_9 = \frac{4d_1^2\left(ct + k_1 x + \frac{5}{44}\sqrt{\frac{5}{6}} c d_1^2 y\right)}{\sqrt{\frac{2}{15}} + d_1^2\left(ct + k_1 x + \frac{5}{44}\sqrt{\frac{5}{6}} c d_1^2 y\right)^2}. \tag{32}$$

Set IV. When $d_4 = d_2 = 0$,

$$c = -\frac{72k_2\left(2d_1^4 + 4d_1^2 d_3^2 + 2d_3^4 + \frac{1}{5}d_1^2\sqrt{15} + \frac{1}{5}d_3^2\sqrt{15}\right)}{48d_1^4 + 96d_1^2 d_3^2 + 48d_3^4 + 8d_1^2\sqrt{15} + 8d_3^2\sqrt{15} + 5},\ k_1 = -\frac{4\left(2d_1^2 + 2d_3^2 + \frac{1}{5}\sqrt{15}\right)}{4d_1^2 + 4d_3^2 + \frac{1}{3}\sqrt{15}},\ d_3 = d_3. \tag{33}$$

Then we obtain

$$\Psi_{10} = \frac{2\left(2d_1^2\left(-\frac{72\left(\sqrt{\frac{3}{5}}d_1^2+2d_1^4+\sqrt{\frac{3}{5}}d_3^2+4d_1^2d_3^2+2d_3^4\right)k_2 t}{5+8\sqrt{15}d_1^2+48d_1^4+8\sqrt{15}d_3^2+96d_1^2d_3^2+48d_3^4} - \frac{4\left(\sqrt{\frac{3}{5}}+2d_1^2+2d_3^2\right)x}{\sqrt{\frac{5}{3}}+4d_1^2+4d_3^2} + k_2 y\right)\right) + 2R_2}{d_5 + d_1^2\left(-\frac{72\left(\sqrt{\frac{3}{5}}d_1^2+2d_1^4+\sqrt{\frac{3}{5}}d_3^2+4d_1^2d_3^2+2d_3^4\right)k_2 t}{5+8\sqrt{15}d_1^2+48d_1^4+8\sqrt{15}d_3^2+96d_1^2d_3^2+48d_3^4} - \frac{4\left(\sqrt{\frac{3}{5}}+2d_1^2+2d_3^2\right)x}{\sqrt{\frac{5}{3}}+4d_1^2+4d_3^2} + k_2 y\right)^2 + R_2^2}, \tag{34}$$

where $R_2 = d_3^2\left(-\frac{72\left(\sqrt{\frac{3}{5}}d_1^2+2d_1^4+\sqrt{\frac{3}{5}}d_3^2+4d_1^2d_3^2+2d_3^4\right)k_2 t}{5+8\sqrt{15}d_1^2+48d_1^4+8\sqrt{15}d_3^2+96d_1^2d_3^2+48d_3^4} - \frac{4\left(\sqrt{\frac{3}{5}}+2d_1^2+2d_3^2\right)x}{\sqrt{\frac{5}{3}}+4d_1^2+4d_3^2} + k_2 y\right)$.

5. *M*-Shape Soliton Interaction with One Kink

For these solutions, we consider one exponential hypothesis on f [37,38],

$$f = \zeta_1^2 + \zeta_2^2 + d_5 + e^{\zeta_3}, \tag{35}$$

where $\zeta_1 = d_1\xi + d_2, \zeta_2 = d_3\xi + d_4, \zeta_3 = d_6\xi + d_7$ $d_i (1 \leq i \leq 7)$ are any parameters. Put f into Equation (11) and evaluate the system of equations, which is attained from the coefficients of ξ and exp functions:

Set I. When $d_4 = d_2 = 0$,

$$c = -\frac{d_6^2\left(5d_5^2 d_6^3 k_1 - 4d_6^3 k_1 + 10d_5^2 k_1 - 24d_5^2\right)k_1 k_2}{12d_6^2 - 16},\ d_1 = d_3\sqrt{-1},\ k_2 = k_2,\ d_7 = d_7. \tag{36}$$

We obtain

$$\Psi_{11} = \frac{2d_6 e^{d_7 + d_6\left(-\frac{d_6^2\left(5d_5^2d_6^3k_1 - 4d_6^3k_1 + 10d_5^2k_1 - 24d_5^2\right)k_1k_2 t}{12d_6^2-16} + k_1 x + k_2 y\right)}}{d_5 + e^{d_7 + d_6\left(-\frac{d_6^2\left(5d_5^2d_6^3k_1 - 4d_6^3k_1 + 10d_5^2k_1 - 24d_5^2\right)k_1k_2 t}{12d_6^2-16} + k_1 x + k_2 y\right)}}. \tag{37}$$

Set II. When $d_4 = d_2 = 0$,

$$d_5 = \frac{2}{3}\sqrt{3},\ k_1 = \frac{12}{d_6^3 + 5},\ d_1 = d_3\sqrt{-1},\ d_7 = d_7,\ c = c. \tag{38}$$

Then we obtain

$$\Psi_{12}(x,y,t) = \frac{2d_6 e^{d_7+d_6\left(ct+\frac{12x}{5+d_6^3}+k_2 y\right)}}{\frac{2}{\sqrt{3}}+e^{d_7+d_6\left(ct+\frac{12x}{5+d_6^3}+k_2 y\right)}}. \tag{39}$$

6. *M*-Shape Soliton Interaction with Two Kinks

We use the following two exponential ansatz [37,38]:

$$f = b_1 e^{-a_1\Delta+a_2} + b_2 e^{a_3\Delta+a_4}, \tag{40}$$

where a_1, a_2, a_3 and a_4 are some constants. By using f in Equation (11) and solving the equations obtained from the coefficients of the exponential functions, we obtain the following:

Set I.

$$k_2 = -\frac{c\left(3a_3^2+5\right)^2}{4a_3^2\left(a_3^2+3\right)^2},\ k_1 = \frac{4\left(a_3^2+3\right)}{3a_3^2+5},\ b_1 = b_1,\ a_5 = a_5,\ b_2 = b_2. \tag{41}$$

Using the above values, we have

$$\Psi_{13} = \frac{2\left(a_1 b_1 e^{a_2+a_1\left(ct+\frac{4\left(a_3^2+3\right)x}{3a_3^2+5}+-\frac{c\left(3a_3^2+5\right)^2 y}{4a_3^2\left(a_3^2+3\right)^2}\right)}+a_3 b_2 e^{a_4+a_3\left(ct+\frac{4\left(a_3^2+3\right)x}{3a_3^2+5}+-\frac{c\left(3a_3^2+5\right)^2 y}{4a_3^2\left(a_3^2+3\right)^2}\right)}\right)}{\left(b_1 e^{a_2+a_1\left(ct+\frac{4\left(a_3^2+3\right)x}{3a_3^2+5}+-\frac{c\left(3a_3^2+5\right)^2 y}{4a_3^2\left(a_3^2+3\right)^2}\right)}+b_2 e^{a_4+a_3\left(ct+\frac{4\left(a_3^2+3\right)x}{3a_3^2+5}+-\frac{c\left(3a_3^2+5\right)^2 y}{4a_3^2\left(a_3^2+3\right)^2}\right)}\right)}. \tag{42}$$

Set II.

$$a_1 = \frac{7}{39}\sqrt{-39},\ k_1 = \frac{26}{19},\ k_1 = \frac{361}{735}c,\ a_3 = -\frac{7}{39}i\sqrt{39},\ a_5 = a_5. \tag{43}$$

Then, we have

$$\Psi_{14} = \frac{2\left(-\frac{7ib_2 e^{a_4-\frac{7i\left(ct+\frac{361cx}{735}+k_2 y\right)}{\sqrt{39}}}}{\sqrt{39}}+\frac{7ib_1 e^{a_2+\frac{7i\left(ct+\frac{361cx}{735}+k_2 y\right)}{\sqrt{39}}}}{\sqrt{39}}\right)}{b_2 e^{a_4-\frac{7i\left(ct+\frac{361cx}{735}+k_2 y\right)}{\sqrt{39}}}+b_1 e^{a_2+\frac{7i\left(ct+\frac{361cx}{735}+k_2 y\right)}{\sqrt{39}}}}. \tag{44}$$

7. Stability Characteristic of Solutions

We now evaluate the stability characteristic via Hamiltonian approach Ψ [30],

$$\Lambda = \frac{1}{2}\int_{-k}^{k}\Psi^2(z)dz. \tag{45}$$

Hence, the stability condition of the solutions can be evaluated as

$$\frac{\partial\Lambda}{\partial c} > 0, \tag{46}$$

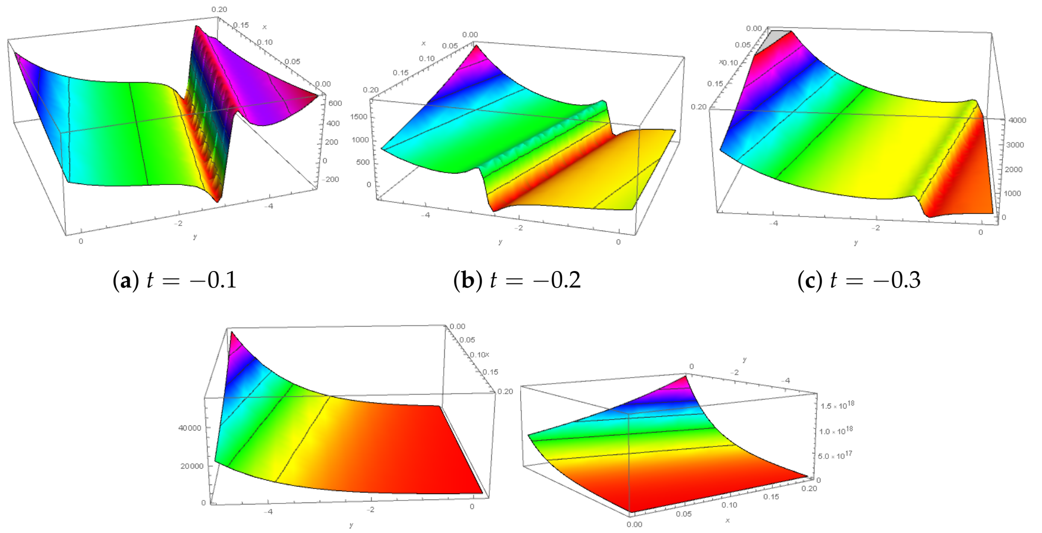
where Λ denotes the momentum in the Hamiltonian system, and c stands for wave velocity. The stability for Equation (28) with appropriate values of constants is given by

$$\left(\frac{\partial \Lambda}{\partial c}\right)_{c=4} = -0.000161749 < 0. \tag{47}$$

In the interval $x, y, t\varepsilon[-9, 9]$ and $c = 4$, we conclude that this solution is unstable. In the same way, we check the stability of all results in the Table 1.

8. Results and Discussions

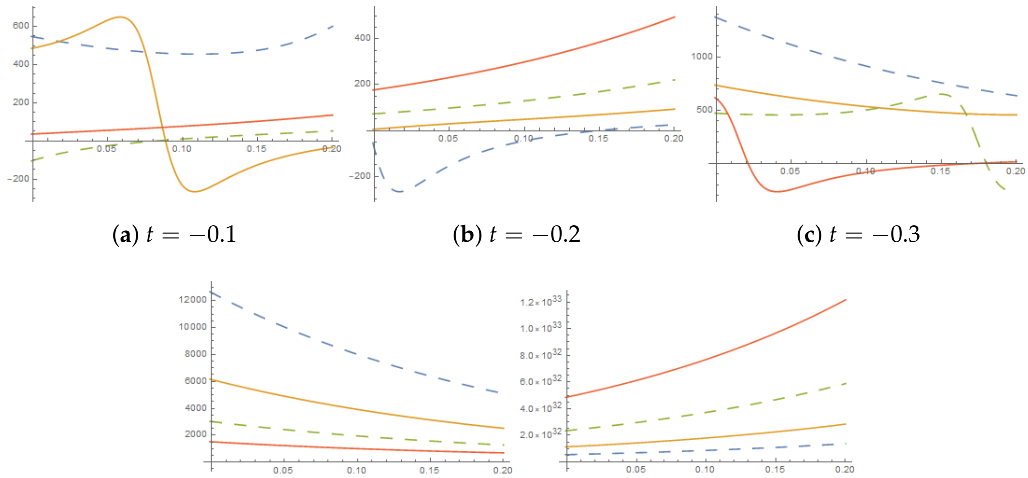
For finding numerical results, we have used the configuration of the Software MATHEMATICA:12.1 and MAPLE:14 to perform the simulation results. We have successfully obtained the stated forms of solutions and they show a discrepancy in the wave when setting the parameters to the appropriate values. First, by applying the three waves approach, we have evaluated three sets of solutions for Equation (1) and their graphs are constructed via appropriate values of involved parameters. Note that Figure 1 shows the 3D multiwave plots for Ψ_1 in Equation (14) presented via numeric values $k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$, respectively. Figure 2 represents the 2D multiwave profiles for Ψ_1 in Equation (14) via numeric values of $k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$. Figures 3 and 4 interpret the density and stream plots for Figure 1. Furthermore, by utilizing the HB technique, we have computed three sets of solutions for Equation (1) and their plots are constructed. We have successfully evaluated the 3D HB profiles for Ψ_4, which are constructed with numeric values of $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$, respectively, in Figure 5. The 2D HB plots for Ψ_4 are displayed via numeric values of $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$, respectively, in Figure 6. Similarly, Figures 7 and 8 interpret the density and stream plots for Figure 5. We have attained four classes of M-shaped solitons via a suitable transformation in Equation (11). We have constructed M-shape evolution plots for Ψ_7, constructed with $d_1 = -5, d_4 = 2, c = 10, d_3 = 3, b_0 = 2$, in Figure 9. We have attained two classes of M-shaped solitons with one exp function via a suitable transformation in Equation (11). The resulting 3D interaction plots for Ψ_{11} are displayed with numerical values of $d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i$ in Figures 10 and 11. Figures 12 and 13 interpret the density and stream plots for Figure 10. We have computed two sets of solutions for M-shaped solitons with double exp function through a suitable transformation in Equation (11). The evolution profiles for Ψ_{13} are constructed with choice of $a_2 = 4, a_4 = 1, c = 1.5, a_2 = 2, b_1 = 2, b_2 = 6$ in Figure 14. Furthermore, their stability characteristics are successfully manipulated. We believe that the results attained in this work will be helpful for recognizing rogue wave-like phenomena and many other novel interactional phenomena in shallow water waves.



(**a**) $t = -0.1$ (**b**) $t = -0.2$ (**c**) $t = -0.3$

(**d**) $t = 0.8$ (**e**) $t = 6$

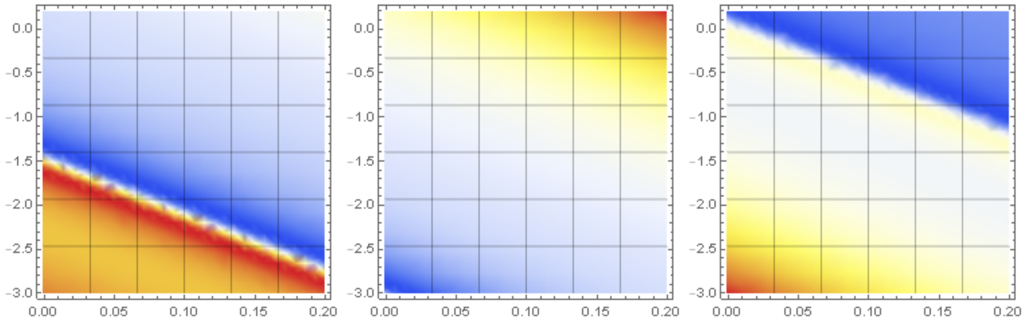
Figure 1. 3D multiwave plots for Ψ_1 in Equation (14) are presented via $k_1 = 1, b_1 = 1.5, c = 2.5$, $a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$.



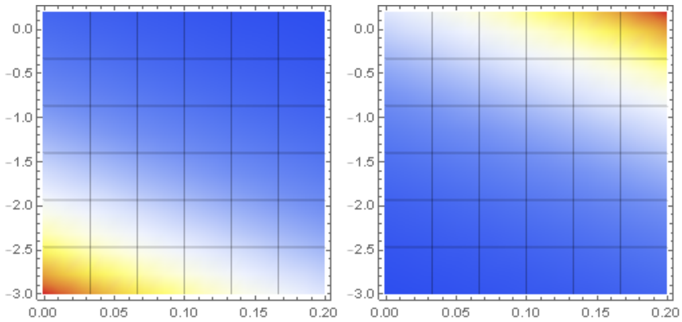
(**a**) $t = -0.1$ (**b**) $t = -0.2$ (**c**) $t = -0.3$

(**d**) $t = 0.8$ (**e**) $t = 6$

Figure 2. 2D multiwave profiles for Ψ_1 in Equation (14) are constructed with $k_1 = 1, b_1 = 1.5, c = 2.5$, $a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$.

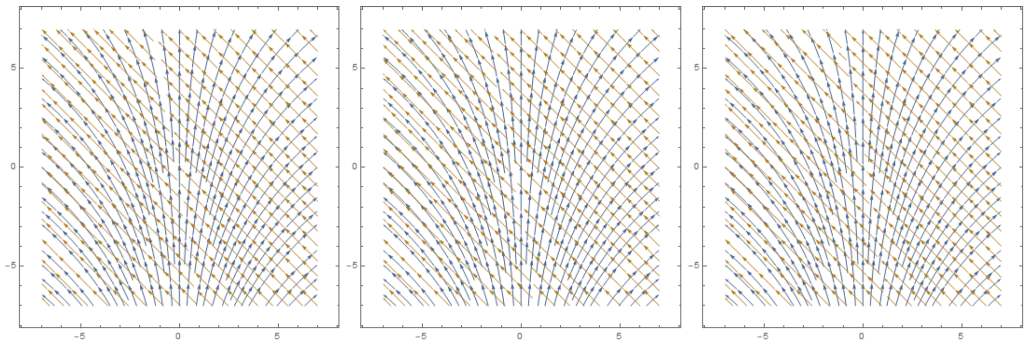


(**a**) $t = -0.1$ (**b**) $t = -0.2$ (**c**) $t = -0.3$

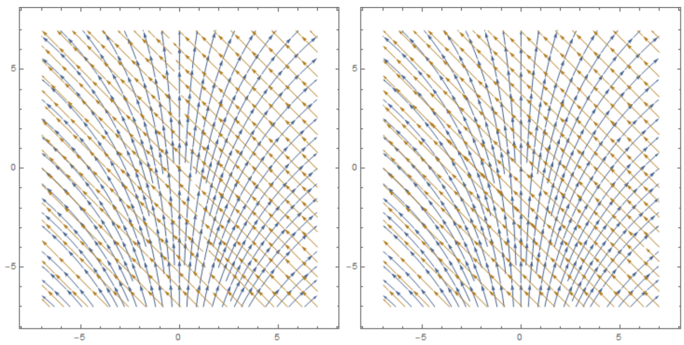


(**d**) $t = 0.8$ (**e**) $t = 6$

Figure 3. Density multiwave plots for Ψ_1 are constructed with $k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$.

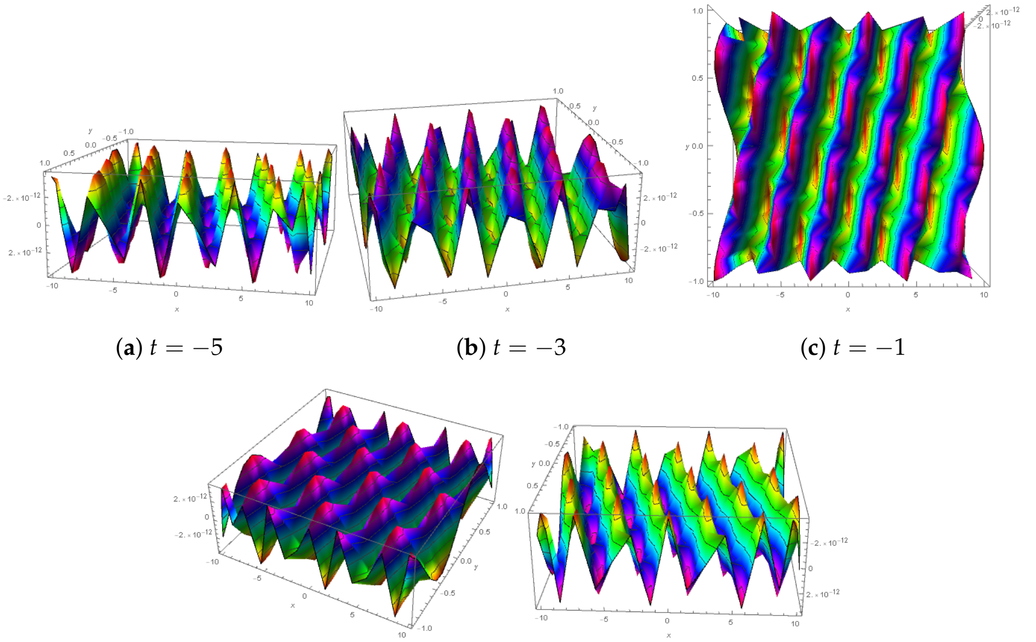


(**a**) $t = -0.1$ (**b**) $t = -0.2$ (**c**) $t = -0.3$



(**d**) $t = 0.8$ (**e**) $t = 6$

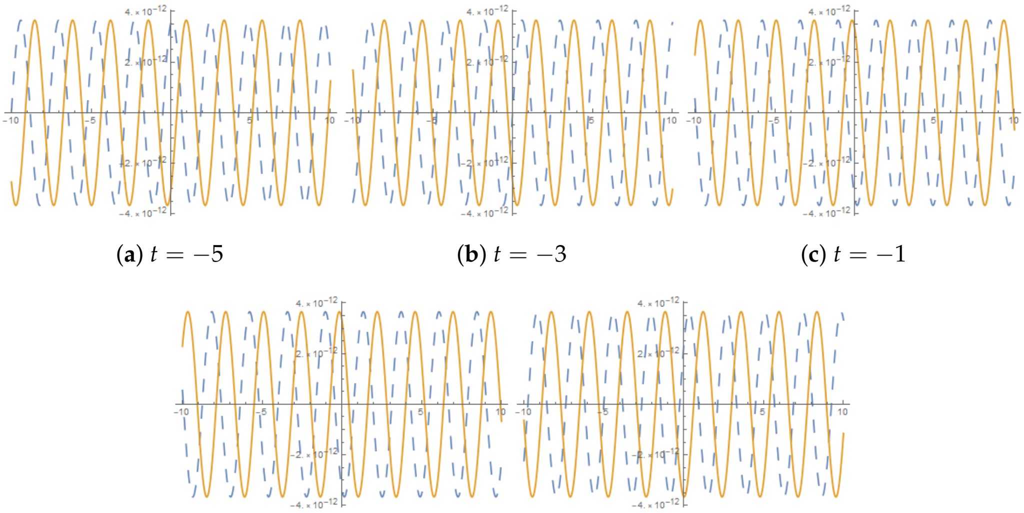
Figure 4. Stream multiwave profiles for Ψ_1 are constructed with $k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 5$ (**e**) $t = 15$

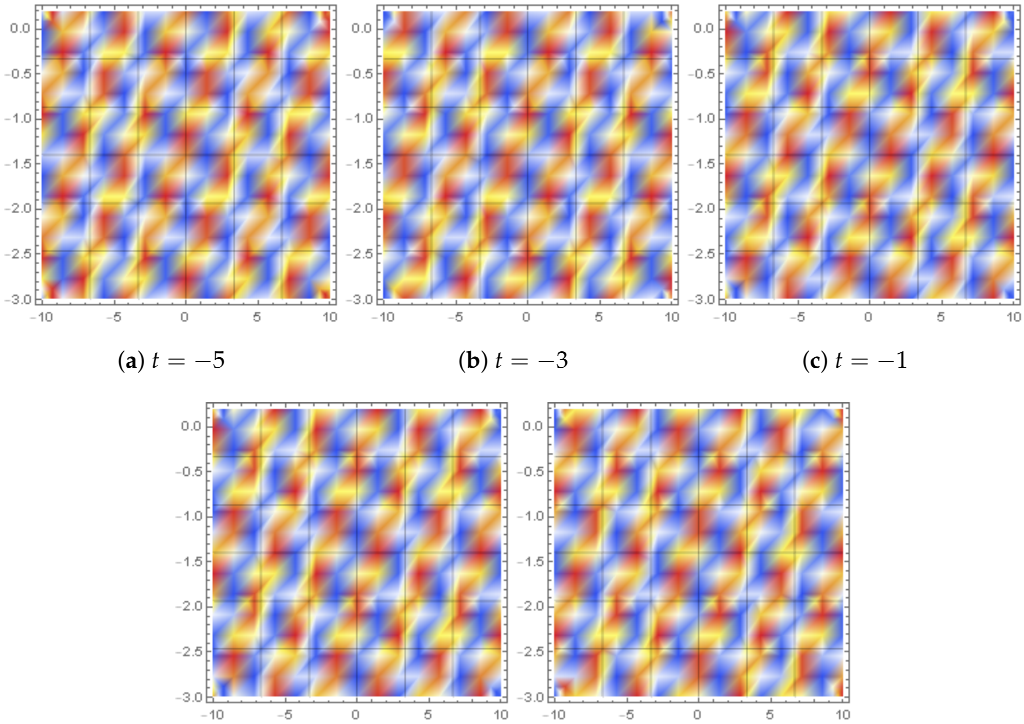
Figure 5. 3D HB profiles for Ψ_4 are constructed with $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 5$ (**e**) $t = 15$

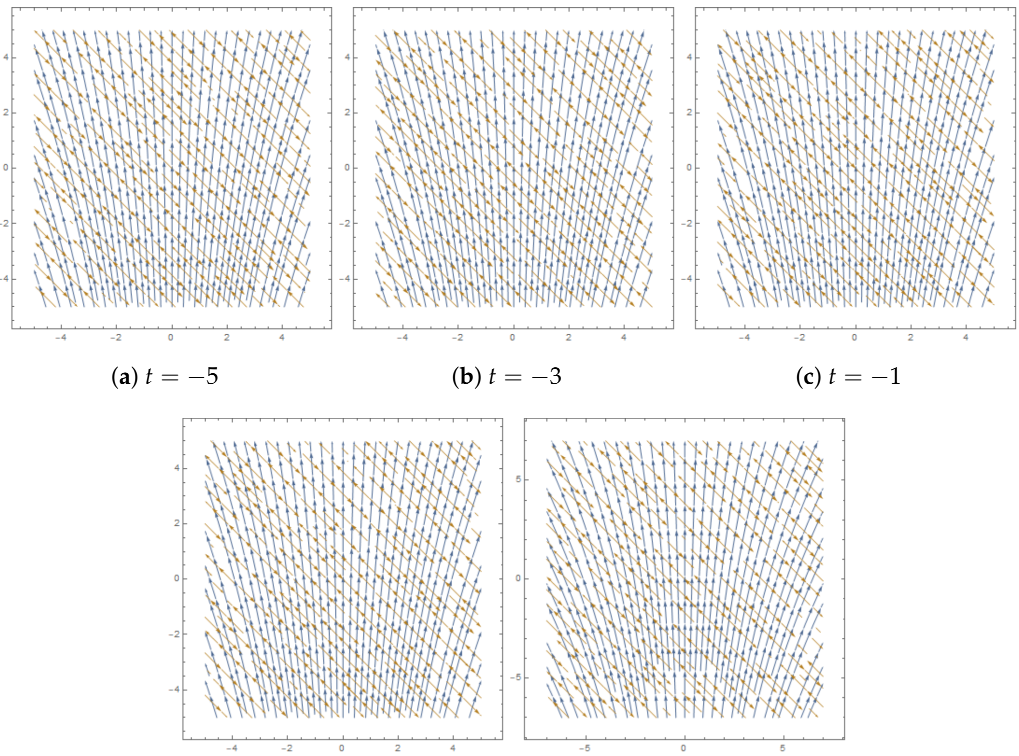
Figure 6. 2D HB plots for Ψ_4 are displayed with $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 5$ (**e**) $t = 15$

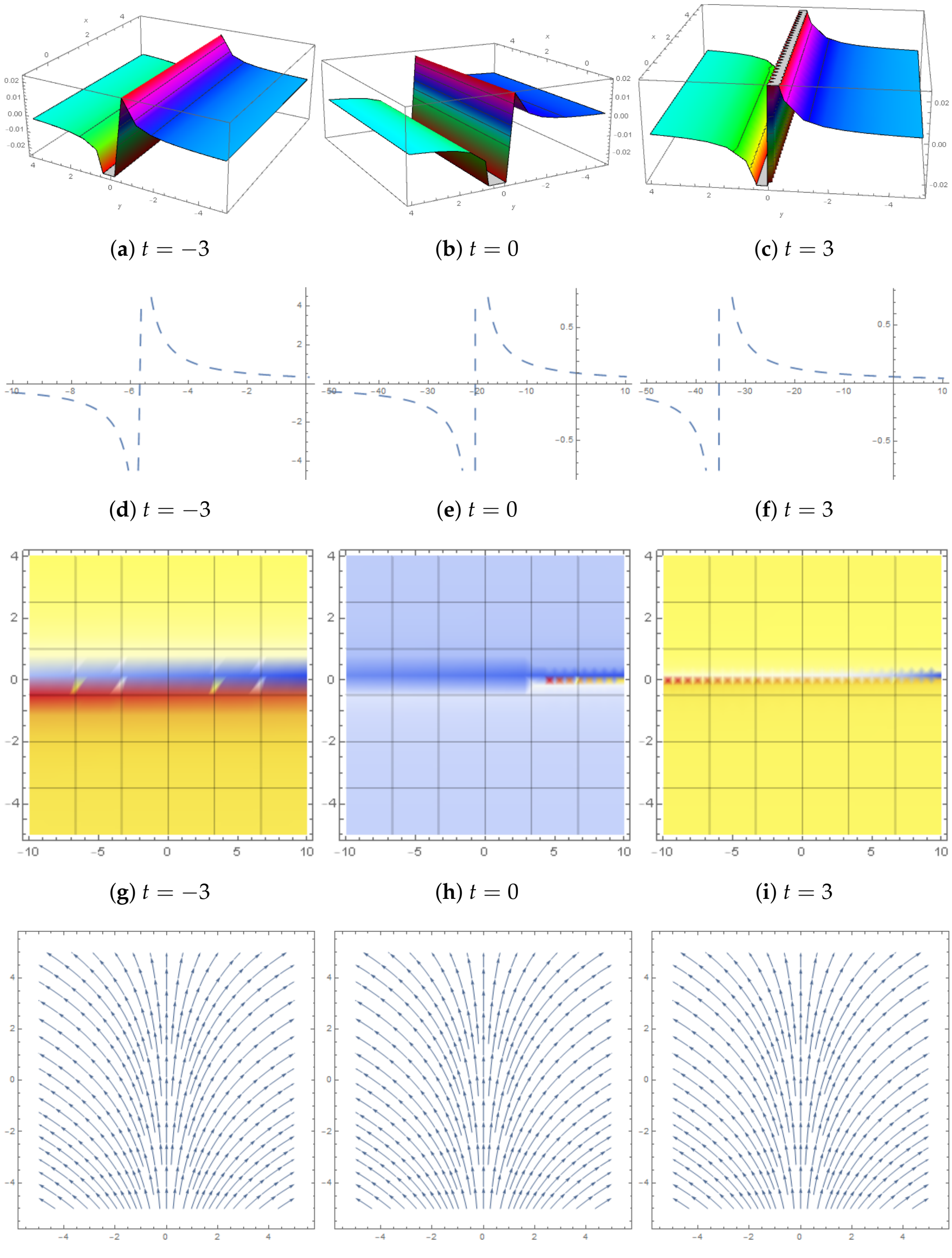
Figure 7. Density profiles for Ψ_4 are constructed with $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 5$ (**e**) $t = 15$

Figure 8. Stream plots for Ψ_4 are constructed with $k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$.



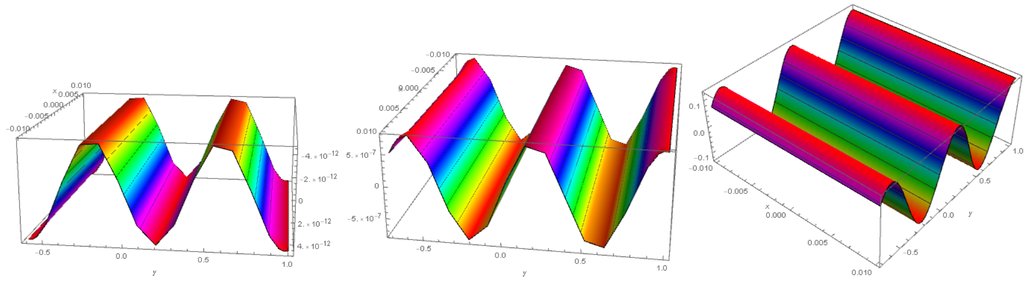
(a) $t = -3$ **(b)** $t = 0$ **(c)** $t = 3$

(d) $t = -3$ **(e)** $t = 0$ **(f)** $t = 3$

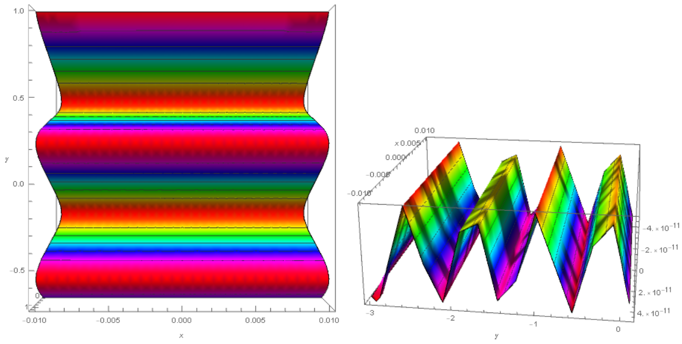
(g) $t = -3$ **(h)** $t = 0$ **(i)** $t = 3$

(j) $t = -3$ **(k)** $t = 0$ **(l)** $t = 3$

Figure 9. M-shape evolution profiles for Ψ_7 are constructed with $d_1 = -5, d_4 = 2, c = 10, d_3 = 3, b_0 = 2$.

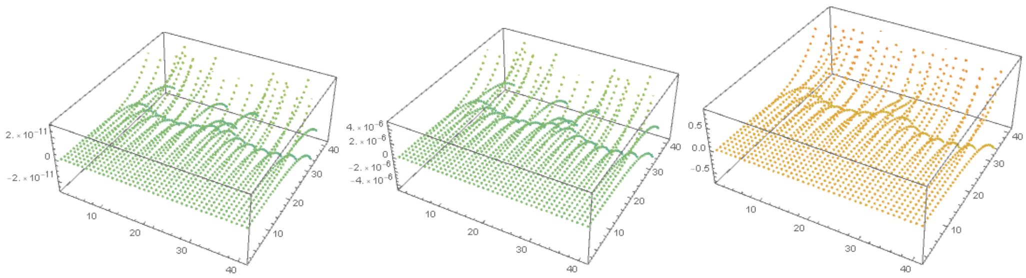


(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

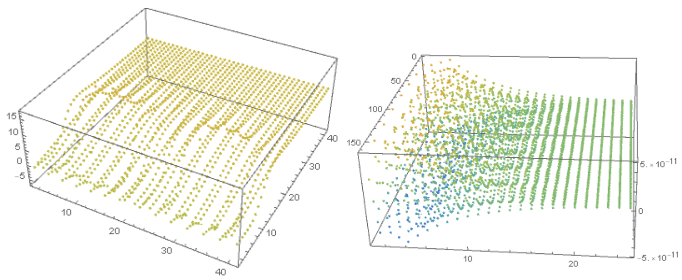


(**d**) $t = 0$ (**e**) $t = 15$

Figure 10. 3D interaction plots for Ψ_{11} are displayed with $d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i$.

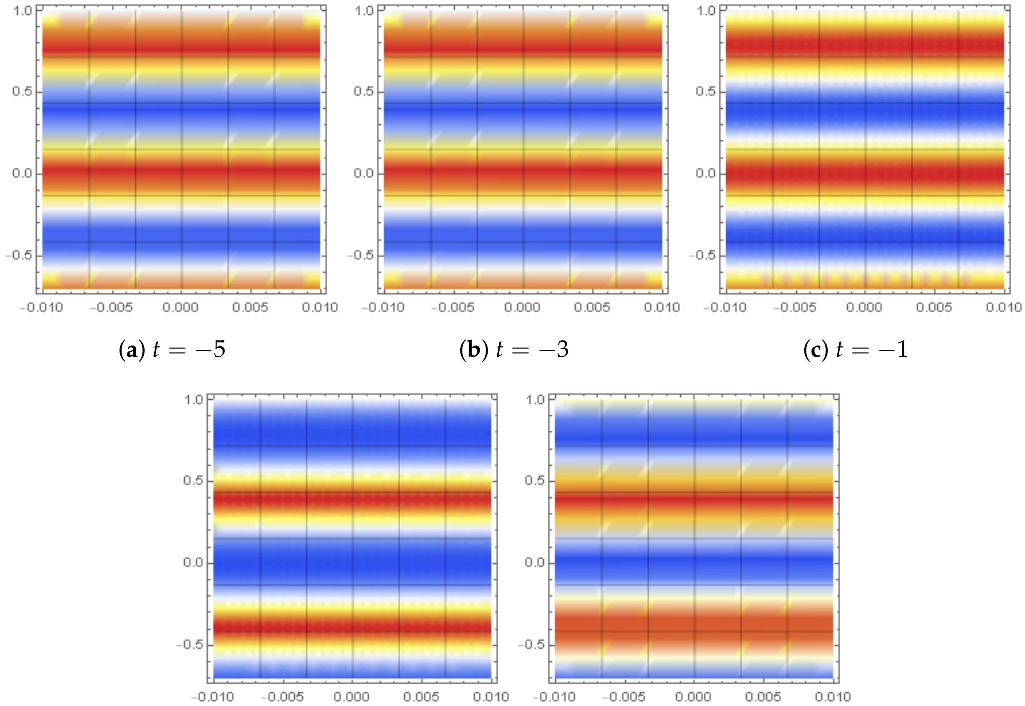


(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$



(**d**) $t = 0$ (**e**) $t = 15$

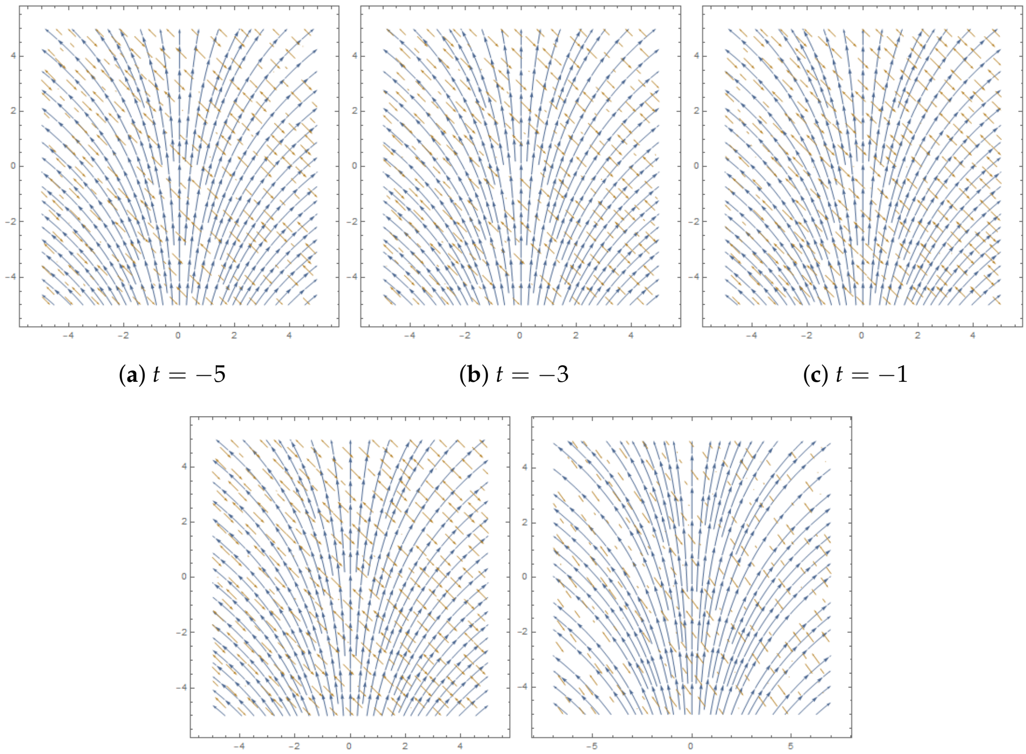
Figure 11. Some interaction slots for Ψ_{11} are shown with $d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 0$ (**e**) $t = 15$

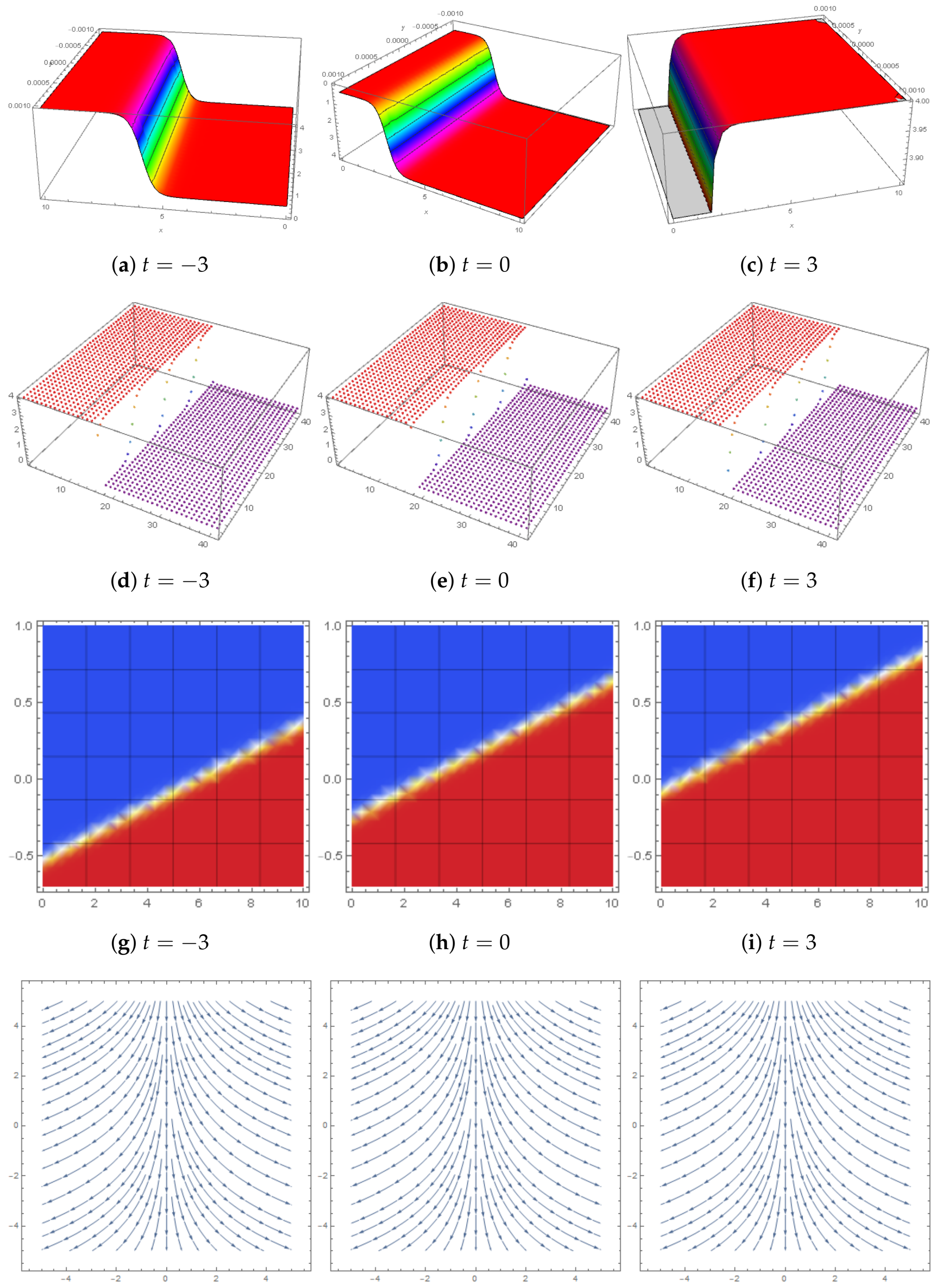
Figure 12. Density interaction graphs for Ψ_{11} are shown with $d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i$.



(**a**) $t = -5$ (**b**) $t = -3$ (**c**) $t = -1$

(**d**) $t = 0$ (**e**) $t = 15$

Figure 13. Stream interaction plots for Ψ_{11} are shown with $d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i$.



(**a**) $t = -3$ (**b**) $t = 0$ (**c**) $t = 3$

(**d**) $t = -3$ (**e**) $t = 0$ (**f**) $t = 3$

(**g**) $t = -3$ (**h**) $t = 0$ (**i**) $t = 3$

(**j**) $t = -3$ (**k**) $t = 0$ (**l**) $t = 3$

Figure 14. Evolution profiles for Ψ_{13} are constructed with $a_2 = 4, a_4 = 1, c = 1.5, a_2 = 2, b_1 = 2, b_2 = 6$.

9. Concluding Remarks

In this article, we studied MS, HB, M-shaped solitons by utilizing the symbolic computation with ansatz functions technique and logarithmic transformation. We also discussed interactions among M-shape solitons with kink waves, like single exponential form (one-kink) and double exponential form (two-kink). These solutions may be used to manage the behavior of pulses. MS and other solutions in shallow water wave theory are valuable tools for understanding the complex dynamics of water waves in shallow environments. They contribute to the fields of environmental science, coastal engineering, and oceanography by providing insights into wave behavior and its impact on coastal areas and ecosystems.

Additionally, we discuss the stability characteristics of the obtained solutions. By using the Hamilton system characteristics, we have come to the conclusion that $\Psi_i(x,y,t)$, where $(i = 1,2,4,5,6,8,10,13,14)$, is a stable solution (see Table 2).

Table 2. Obtained solutions $\Psi_i(x,y,t)$ where $(i = 1,2,3,...,14)$.

Solution	Type of Solution	Values of Variables
Ψ_1	MS	$k_1 = 1, b_1 = 1.5, c = 2.5, a_2 = 3, a_4 = 3, b_0 = 1.5, a_5 = 2, a_6 = 5, b_2 = 2.5$
Ψ_4	MS	$k_1 = 1, a_1 = -1.5, c = -2.5, a_5 = -3, a_4 = -2, a_6 = 3, b_0 = -1.5, b_1 = 3, a_2 = -2, b_2 = -2.5$
Ψ_7	M-shape	$d_1 = -5, d_4 = 2, c = 10, d_3 = 3, b_0 = 2.$
Ψ_{11}	Interactional solution	$d_6 = 6, d_7 = 2, c = 1.5, k_2 = 2i.$
Ψ_{13}	Interactional solution	$a_2 = 4, a_4 = 1, c = 1.5, a_2 = 2, b_1 = 2, b_2 = 6.$

Author Contributions: Methodology, S.A.; Validation, U.R.; Formal analysis, S.T.R.R.; Supervision, A.R.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors extend their appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia for funding this research work through project number: 445-9-358. Furthermore, the authors would like to extend their appreciation to Taibah University for its supervision support.

Conflicts of Interest: The authors declare no conflicts of interest.

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