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Asymptotic Behavior of Delayed Reaction-Diffusion Neural Networks Modeled by Generalized Proportional Caputo Fractional Partial Differential Equations

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Abstract: In this paper, a delayed reaction-diffusion neural network model of fractional order and with several constant delays is considered. Generalized proportional Caputo fractional derivatives with respect to the time variable are applied, and this type of derivative generalizes several known types in the literature for fractional derivatives such as the Caputo fractional derivative. Thus, the obtained results additionally generalize some known models in the literature. The long term behavior of the solution of the model when the time is increasing without a bound is studied and sufficient conditions for approaching zero are obtained. Lyapunov functions defined as a sum of squares with their generalized proportional Caputo fractional derivatives are applied and a comparison result for a scalar linear generalized proportional Caputo fractional differential equation with several constant delays is presented. Lyapunov functions and the comparison principle are then combined to establish our main results.

Keywords: reaction-diffusion neural networks; generalized proportional Caputo fractional derivatives; delays; asymptotic behavior

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1. Introduction

The delay in differential equations is widely used for analysis and predictions in population dynamics, epidemiology, immunology, physiology, and neural networks. The time delays or time lags, in these models, can be related to the duration of certain hidden processes such as the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, and so on (see, for example, [1]). Recently, in [2], there was a discussion on how delays in differential equations can be applied to understand fundamental design principles of various natural biological systems, and numerous examples demonstrated that the inclusion of explicit delays simplifies complex systems in comparison to the more widely used mechanistic descriptions with ordinary differential equations.

In many models, reaction-diffusion terms are applied to describe more adequately the time and spatial evolution. Reaction-diffusion equations are traditionally applied to problems in biology, population dynamics, ecology, and neurosciences (see, for example, the books [3–5]), in epidemiology and virology (see, for example, [6–9]). The effects of time delays on the dynamical properties of reaction-diffusion epidemic and virus dynamic models are also investigated in the literature (see, for example, [10–12]).

In many models, various types of fractional derivatives instead of derivatives of integer order are applied to model the system more adequately. Note, fractional calculus arise in various fields, since it was shown to be exceptionally well suited in modeling and describing the complex nature of real world problems in comparison to local derivatives (for some models with fractional derivatives, see, for example [13,14]). As it is pointed out in [15], the physically consistent continuous time random walk is the basis for the application of the Riemann-Liouville fractional derivative. However, it is shown in [16] that a very simple example can have unphysical negative solutions. In [17], the Wright function is applied to study the reaction-diffusion problem with the Caputo type derivative.

Recently, the generalized proportional Caputo fractional derivative was defined, studied and applied (see, [18,19]). It is a generalization of the Caputo fractional derivative and it provides wider possibilities for modeling the more adequate complexity of real world problems.

In this paper, a delayed reaction-diffusion neural network model of fractional order is studied. Note, various types of neural network with the reaction-diffusion term and Caputo fractional derivative are studied in the literature, such as for example in [20] (synchronization for a coupled network), [21] (stability and comparison principle) [22] (synchronization for model with leakage and discrete delays), and [23] (synchronization). In this paper, we consider the model with the generalized proportional Caputo fractional derivative; the model has several constant delays. The stability of fractional order systems with proportional Caputo fractional derivatives was considered recently (see, for example [24,25]). The behavior of the solutions of the given model with time increasing without a bound is studied in this paper by applying Lyapunov functionals and their generalized Caputo proportional fractional derivatives in time. An example is provided to illustrate our theoretical results and the dependence of the applied type of fractional derivative on the behavior of the solutions.

The main novelty of the paper can be summarized in the following way:

- A new model of the delay reaction-diffusion model is defined by the application of generalized proportional Caputo fractional derivatives in the time variable;
- The asymptotic behavior of the solutions when the time variable is increasing without any bounds is studied;
- A comparison result for a linear generalized proportional Caputo fractional differential equation with a constant delay is presented;
- Lyapunov functions defined as a sum of squares and depending sighnificantly on the particular solution of the model are applied;
- Lyapunov functions are combined with the comparison principle to develop the main results;
- Sufficient conditions for approaching zero in time are obtained where the comparison principle for a linear delay generalized proportional Caputo fractional differential equation is used. The applications are deeply connected with the presence of delay.

2. Some Results for Scalar Generalized Proportional Caputo Fractional Differential Equations

Let $v \in C([0,b],\mathbb{R}),\ 0 < b \leq \infty$ (in the case $b = \infty$, the interval is half open).

Consider the *generalized proportional fractional integral* (as long as all integrals are well defined) [18]

$$({}_0\mathcal{I}^{q,\rho}v)(t) = \frac{1}{\rho^q\Gamma(q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1}v(s)\,ds, \quad t \in (0,b],\ \ q \geq 0,\ \rho \in (0,1],$$

and the *generalized proportional Caputo fractional derivative* (as long as all integrals are well defined)

$$\begin{aligned}({}_0^C\mathcal{D}^{q,\rho}v)(t) &= \frac{1-\rho}{\rho^{1-q}\Gamma(1-q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{v(s)}{(t-s)^q}ds \\ &\quad + \frac{\rho^q}{\Gamma(1-q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}v'(s)\,ds, \\ &\quad \text{for } \ t \in (0,b],\ \ q \in (0,1),\ \rho \in (0,1].\end{aligned}$$

Consider the following scalar generalized proportional fractional linear differential equation with multiple delays

$$\begin{aligned} &({}_0^C\mathcal{D}^{q,\rho}v)(t) = -av(t) + \sum_{i=1}^{N} b_i v(t-\tau_i), \quad t > 0 \\ &v(t) = \phi(t), \ \ t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $a, b_i, (i = 1, 2, \dots, N)$ are reals, $q \in (0,1)$, $\rho \in (0,1]$, $\tau_i \geq 0$, $\tau = \max_{i=1,2,\dots,N} \tau_i$, $\phi \in C([-\tau, 0], \mathbb{R})$.

Take the Laplace transform on both sides of (1) (see [18]) and obtain

$$\begin{aligned} &(\rho s + 1 - \rho)^q V_1(s) - \rho(\rho s + 1 - \rho)^{q-1}\phi(0) \\ &= -aV_1(s) + \sum_{i=1}^{N} b_i e^{-s\tau_i}\Big(V_1(s) + \int_{-\tau_i}^{0} e^{-s\xi}\phi(\xi)d\xi\Big), \end{aligned} \tag{2}$$

where V_1 is the Laplace transform of $v(t)$ with $V_1(s) = \mathcal{L}(v(t))$.

The Equation (2) could be written in the form

$$\Big((\rho s + 1 - \rho)^q + a - \sum_{i=1}^{N} b_i e^{-s\tau_i}\Big)V_1(s) = P(s),$$

where $P(s) = \sum_{i=1}^{N} b_i e^{-s\tau_i} \int_{-\tau_i}^{0} e^{-s\xi}\phi(\xi)d\xi + \rho(\rho s + 1 - \rho)^{q-1}\phi(0)$.

The characteristic equation of (2) is

$$(\rho s + 1 - \rho)^q + a - \sum_{i=1}^{N} b_i e^{-s\tau_i} = 0. \tag{3}$$

The distribution of the eigenvalues totally determines the stability of Equation (1). Let $S = \rho s + 1 - \rho$. Then Equation (3) is reduced to

$$S^q + a - \sum_{i=1}^{N} B_i e^{-S\frac{\tau_i}{\rho}} = 0 \tag{4}$$

where $B_i = b_i e^{\frac{1-\rho}{\rho}}$. If $a > \sqrt{2}\sum_{i=1}^{N} B_i$ and $B_i > 0$, $i = 1, 2, \dots, N$, then Equation (4) has no purely imaginary roots (see the proof of Theorem 1 [21]) and $\lim_{t\to\infty} v(t) = 0$ if $\phi(t) \geq 0$, $t \in [-\tau, 0]$ (see the proof of Corollary 3 and Theorem 1 [26]). Thus, we obtain the following result:

Lemma 1. *Let* $a > \sqrt{2}e^{\frac{1-\rho}{\rho}} \sum_{i=1}^{N} b_i$, $b_i, \tau_i > 0$, $i = 1, 2, \dots, N$, *and* $\phi \in C([-\tau, 0], [0, \infty))$. *Then* $\lim_{t\to\infty} v(t) = 0$ *where* $v(t)$, $t \in [-\tau, \infty)$, *is the solution of* (1).

Later, we will use the following result proved in Example 5.7 [18] with a slight modification:

Lemma 2. *The solution of the scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_0^C\mathcal{D}^{q,\rho}u)(t) = \lambda u(t) + f(t), \ \ u(0) = u_0, \ \ q \in (0,1), \ \rho \in (0,1]$$

where $\lambda \in \mathbb{R}$, $f \in C([0,\infty), \mathbb{R})$, *has a solution*

$$u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} E_q(\lambda(\frac{t}{\rho})^q) + \frac{1}{\rho^q}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds, \tag{5}$$

where $E_q(.)$ *and* $E_{q,q}(.)$ *are the Mittag-Leffler functions with one and two parameters, respectively.*

Consider the following scalar generalized proportional fractional linear differential inequality with multiple delays

$$
\begin{aligned}
&({}_0^C\mathcal{D}^{q,\rho}v)(t) \le -av(t) + \sum_{i=1}^{N} b_i v(t-\tau_i), \ \ t>0,\\
&v(t) = \phi(t), \ \ t \in [-\tau, 0].
\end{aligned}
\tag{6}
$$

Lemma 3. *Let $a, b_i, \tau_i > 0$, $i = 1, 2, \dots, N$, $\phi(t) \in C([-\tau, 0], [0, \infty))$.*

Then $v_1(t) \le v_2(t)$, $t \ge 0$, where v_1, v_2 are solutions of the scalar inequality (6) and the scalar Equation (1), respectively.

Proof. There exists a function $\xi \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that (6) is equivalent to

$$
\begin{aligned}
&({}_0^C\mathcal{D}^{q,\rho}v)(t) = -av(t) + \sum_{i=1}^{n} b_i v(t-\tau_i) - \xi(t), \ \ t>0,\\
&v(t) = \phi(t), \ \ t \in [-\tau, 0].
\end{aligned}
\tag{7}
$$

Let $v_1(t)$, $t \in [-\tau, \infty)$, be a solution of the scalar inequality (6). According to Lemma 2, we have

$$
\begin{aligned}
v_1(t) &= \phi(0) e^{\frac{\rho-1}{\rho}(t-a)} E_q(-a(\frac{t}{\rho})^q)\\
&+ \frac{1}{\rho^q} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{q-1} E_{q,q}(-a(t-s)^q) (\sum_{i=1}^{N} b_i v_1(s-\tau_i) - \xi(s)) ds.
\end{aligned}
\tag{8}
$$

Let $v_2(t)$, $t \in [-\tau, \infty)$, be a solution of the scalar Equation (1). Then,

$$
\begin{aligned}
v_2(t) &= \phi(0) e^{\frac{\rho-1}{\rho}(t-a)} E_q(-a(\frac{t}{\rho})^q)\\
&+ \frac{1}{\rho^q} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{q-1} E_{q,q}(-a(t-s)^q) (\sum_{i=1}^{N} b_i v_2(s-\tau_i))) ds.
\end{aligned}
\tag{9}
$$

Let $\tau = \min\{\tau_i : \ 1 \le i \le N\}$.

For $t \in [0, \tau]$ it is clear that $v_1(t-\tau_i) = v_2(t-\tau_i) = \phi(t-\tau_i)$, $i = 1, 2, \dots, N$, and thus, $v_1(t) \le v_2(t)$.

We will use induction with respect to the intervals $[(k-1)\tau, k\tau]$, $k = 2, 3, \dots$, to prove the inequality. Assume the inequality $v_1(t) \le v_2(t)$holds for $t \in [0, (k-1)\tau], k \ge 2$. Then, for $t \in [(k-1)\tau, k\tau]$ the inequalities $v_1(t-\tau_i) \le v_2(t-\tau_i)$, $i = 1, 2, \dots, N$, hold and from (8) and (9) we have the validity of the inequality in the interval $[0, k\tau]$. □

3. Generalized Proportional Fractional Integral and Derivative of Functions of Two Variables

We will use the following notations: $\mathbb{R}_+ = [0, \infty)$, $||x|| = \sqrt{\sum_{i=1}^{N} x_i^2}$, $x = (x_1, x_2, \dots, x_N)$, $\mathcal{G} \subset \mathbb{R}^N$ is a bounded open set containing the origin with smooth boundary $\partial\mathcal{G}$ and $mes\mathcal{G} > 0$.

We will use the following notation for $u \in C(\mathbb{R}_+ \times \mathcal{G}, \mathbb{R}^N)$, $u = (u_1, u_2, \dots, u_N)$:

$$
||u||_t = \sqrt{\int_{\mathcal{G}} \sum_{i=1}^{N} u_i^2(t,x) dx}, \quad t \ge 0.
$$

We will give the definitions for fractional type integrals and derivatives with respect to the time variable of functions of two variables.

Let $u \in C([0, b] \times \mathcal{G}, \mathbb{R}^N)$, $0 < b \le \infty$ (in the case $b = \infty$, the interval is half open).

Consider the *generalized proportional fractional integral with respect to the time variable* (as long as all integrals are well defined)

$$({}_0\mathcal{I}_t^{q,\rho}u)(t,x) = \frac{1}{\rho^q\Gamma(q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1}u(s,x)\,ds, \quad t\in(0,b],\ q\geq 0,\ \rho\in(0,1],\ x\in\mathcal{G},$$

and the *generalized proportional Caputo fractional derivative with respect to the time variable* (as long as all integrals are well defined)

$$\begin{aligned}({}_0^C\mathcal{D}_t^{q,\rho}u)(t,x) = {}& \frac{1-\rho}{\rho^{1-q}\Gamma(1-q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{u(s,x)}{(t-s)^q}ds \\ & + \frac{\rho^q}{\Gamma(1-q)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}\frac{\partial u(s,x)}{\partial s}\,ds, \\ & \text{for } t\in(0,b],\ q\in(0,1),\ \rho\in(0,1],\ x\in\mathcal{G}.\end{aligned}$$

Remark 1. *If $\rho=1$, then the generalized proportional Caputo fractional derivative w.r.t. the time variable is reduced to the Caputo fractional derivative w.r.t. the time variable of order $q\in(0,1)$: ${}_a^CD_t^q u(t,x) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\frac{\partial}{\partial s}u(s,x)\,ds$.*

Definition 1. *We say $u\in C^{q,\rho}([0,b]\times\mathcal{G},\mathbb{R}^N)$ if $u(t,x)$ is differentiable w.r.t. its time variable t (i.e., $\frac{\partial u(t,x)}{\partial t}$ exists) and the generalized proportional Caputo fractional derivative w.r.t. the time variable $({}_0^C\mathcal{D}_t^{q,\rho}u)(t,x)$ exists for all $t\in(0,b]$.*

We will use the following result, whose proof is similar to the one in Lemma 1 [25], so we omit it:

Lemma 4. *Let $q\in(0,1)$, $\rho\in(0,1]$ and $u\in C^{q,\rho}([0,b]\times\mathcal{G},\mathbb{R})$. Then, the inequality*

$$({}_0^C\mathcal{D}_t^{q,\rho}u^2)(t,x)\leq 2\,u(t,x)\,({}_0^C\mathcal{D}_t^{q,\rho}u)(t,x), \quad t\in(0,b],\ x\in\mathcal{G}$$

holds.

Remark 2. *Note, in the special case of functions of one variable and $\rho=1$ in Lemma 4 (i.e., the application of the Caputo fractional derivative) the result was proved in [27].*

In our further investigations, we will use the following Poincare-type integral inequality:

Lemma 5 ([28])**.** *Let $\mathcal{G}=\prod_{k=1}^N[a_k,b_k]$, $a_k,b_k\in\mathbb{R}, a_k<b_k$, $\Lambda=\max_{k=1,2,\dots,N}|b_k-a_k|$, $w\in C^1(\mathcal{G},\mathbb{R}^N)$ and $w|_{\partial\mathcal{G}}=0$. Then $\int_{\mathcal{G}}w^2(x)dx\leq\frac{\Lambda^2}{4N}\int_{\mathcal{G}}\sum_{j=1}^N\left(\frac{\partial w(x)}{\partial x_j}\right)^2dx$.*

4. Reaction-Diffusion Model with Generalized Proportional Fractional Derivative: Notations, Definitions and Preliminary Notes

In this paper, we will consider a class of fractional-order N-dimensional delayed reaction-diffusion neural network model of Cohen-Grossberg type with a generalized proportional Caputo fractional derivative with respect to the time variable for $0<q<1,\rho\in(0,1]$ and several constant delays

$$\begin{aligned}({}_0^C\mathcal{D}_t^{q,\rho}u_i)(t,x) = {}& -a_i(u_i(t,x))b_i(u_i(t,x)) + \sum_{k=1}^m\frac{\partial}{\partial x_k}\left(D_{ik}(t,x)\frac{\partial u_i(t,x)}{\partial x_k}\right) \\ & + a_i(u_i(t,x))\left[\sum_{j=1}^N c_{ij}(t)f_j(u_j(t,x)) + \sum_{j=1}^N w_{ij}(t)g_j(u_j(t-\tau_i,x))\right], \\ & t>0,\ x\in\mathcal{G},\ i=1,2,\dots,N,\end{aligned} \tag{10}$$

where $x \in \mathbb{R}^m, m \geq 2$, is the space variable, $u_i(t,x)$ represents the population density of the i-th neural units at time t and in space x, N is the number of the neural units, the transmission time-delay $\tau_j > 0,\ j = 1,2,\dots,N,\ \tau = \max_{1\leq k\leq N} \tau_k$, the amplification functions $a_i \in C(\mathbb{R},\mathbb{R}_+),\ i = 1,2,\dots,N,\ b_i \in C(\mathbb{R},\mathbb{R}), i = 1,2,\dots,N$, are appropriate behaved functions, $c_{ij}, w_{ij} \in C(\mathbb{R},\mathbb{R}), i,j = 1,2,\dots,N$, are the connection weight matrices and time-varying delay connection weight matrices of j-th neuron on the i-th neuron, respectively, $f_j, g_j \in C(\mathbb{R},\mathbb{R}),\ j = 1,2,\dots,N$, are the activation functions, $D_{ik}(t,x) \geq 0$ for $t \geq 0, x \in \mathcal{G}$, are the transmission diffusion coefficients along the i-th neuron for $i = 1,2,\dots,N,\ k = 1,2,\dots,m$. The term $\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\Big(D_{ik}(t,x)\frac{\partial u_i(t,x)}{\partial x_k}\Big)$ denotes the diffusion.

Remark 3. *Note the delays in the reaction–diffusion models have a simple physical interpretation: the substance transfer in a local non-equilibrium medium possesses inertial properties, i.e., the reaction of the system is not instantaneous but delayed for τ_i (see, for example, [29]).*

We will investigate the behavior of the solutions of the fractional neural network model (10) under the following boundary and initial value conditions:

$$u_i(t,x) = 0,\ \ t \in [-\tau,\infty),\ \ x \in \partial\mathcal{G}, \tag{11}$$

$$u_i(s,x) = \Psi_i(s,x),\ \ \ s \in [-\tau,0],\ x \in \mathcal{G}. \tag{12}$$

We introduce the following assumptions:

(A1). There exist constants $\Theta_j,\ \Xi_j > 0,\ j = 1,2,\dots,N$, such that the activation functions satisfy $|f_j(v_1) - f_j(v_2)| \leq \Theta_j|v_1 - v_2|,\ |g_j(v_1) - g_j(v_2)| \leq \Xi_j|v_1 - v_2|$ for $v_1, v_2 \in \mathbb{R},\ j = 1,2,\dots,N$, and $f_j(0) = g_j(0) = 0$.

(A2). There exist constants $B_j > 0$ such that the well-behaved functions $b_j \in C(\mathbb{R},\mathbb{R}) : vb_j(v) \geq B_j v^2$ for $v \in \mathbb{R},\ j = 1,2,\dots,N$ and $b_j(0) = 0$.

(A3). There exist positive constants α_i, β_i such that the amplification functions $a_i \in C(\mathbb{R},\mathbb{R}_+)$ and $\alpha_i \leq a_i(v) \leq \beta_i,\ i = 1,2,\dots,\mathbb{N},\ v \in \mathbb{R}$.

(A4). The connection functions $c_{ij}, w_{ij} \in C(\mathbb{R}_+,\mathbb{R}), i,j = 1,2,\dots,N$ and there exist constants $\bar{C}, \bar{W}$ such that $\bar{C} \geq |c_{ij}(t)|,\ \bar{W} \geq |w_{ij}(t)|$ for $t \geq 0,\ i,j = 1,2,\dots,N$.

(A5). There exist constants $\bar{D}_i \geq 0,\ i = 1,2,\dots,N$, such that $D_{ij}(t,x) \geq \bar{D}_i$ for $t \geq 0,\ x \in \mathcal{G},\ j = 1,2,\dots,N$.

Remark 4. *Note the inequality $vb_j(v) \geq B_j v^2$ in assumption (A2) could not be replaced by $b_j(v) \geq B_j v$ because this inequality is not applicable in the second line of* (16) *(see also, Assumption A2 [30]).*

The goal of our paper is to study the behaviour of the solutions of (10) when time is increasing without bounds.

Theorem 1. *Let the following conditions be satisfied:*

1. *The set $\mathcal{G} = \{x \in \mathbb{R}^N :\ x = (x_1, x_2, \dots, x_m),\ \ d_k \leq |x_k| \leq p_k,\ k = 1,2,\dots,m\}$ where $d_k, p_k \in \mathbb{R}_+, \Lambda = \max_{1\leq k\leq m}(p_k - d_k)$.*
2. *The assumptions (A1)–(A5) are satisfied.*
3. *The delays $\tau_k > 0,\ k = 1,2,\dots,N$ and $\tau = \max\{\tau_k,\ k = 1,2,\dots,N\}$.*
4. *The functions $\Psi_j \in C([-\tau,0] \times \mathcal{G},(0,\infty)),\ j = 1,2,\dots,N$.*
5. *The inequality*

$$\begin{aligned} &2\min_{1\leq k\leq N} \alpha_k B_k + \frac{8m \min_{1\leq k\leq N} \bar{D}_k}{\Lambda^2} \\ &> \max_{1\leq k\leq N} \beta_k \sum_{j=1}^{N}(\bar{C}\Theta_j + \bar{W}\Xi_j) + \bar{C}\max_{1\leq j\leq N} \Theta_j \sum_{k=1}^{N} \beta_k + \sqrt{2}e^{\frac{1-\rho}{\rho}} N\bar{W} \max_{1\leq k\leq N} \beta_k \max_{1\leq j\leq N} \Xi_j \end{aligned} \tag{13}$$

holds.

Then, any solution $u \in C^{q,\rho}([0,\infty)\times\mathcal{G},\mathbb{R}^N)$ of the model (10) with the boundary and initial value conditions (11) and (12) satisfies $\lim_{t\to\infty}||u||_t = 0$.

Proof. Let $u \in C^{q,\rho}([0,\infty)\times\mathcal{G},\mathbb{R}^N)$, $u = (u_1, u_2, \dots, u_N)$ be a solution of (10) with the boundary and initial value conditions (11), (12) for $t \in [-\tau,\infty)$, $x \in \mathcal{G}$.

Consider the function $V \in C([-\tau,\infty),\mathbb{R}_+)$ defined by

$$V(t) = \begin{cases} 0.5\sum_{k=1}^N \int_{\mathcal{G}} u_k^2(t,x)dx = 0.5\int_{\mathcal{G}}\sum_{k=1}^N u_k^2(t,x)dx & t \geq 0, \\ 0.5\int_{\mathcal{G}}\sum_{k=1}^N \Psi_k^2(t,x)dx & t \in [-\tau,0]. \end{cases}$$

Note $V(t) = 0.5(||u||_t)^2$, $t \geq 0$ and the function V depends on the particularly chosen solution of (10).

According to Lemma 4 we obtain for $t > 0$,

$$({}_0^C\mathcal{D}_t^{q,\rho}V)(t) = 0.5\int_{\mathcal{G}}\sum_{k=1}^N ({}_0^C\mathcal{D}_t^{q,\rho}u_k^2)(t,x)dx \leq \int_{\mathcal{G}}\sum_{k=1}^N u_k(t,x)\Big({}_0^C\mathcal{D}_t^{q,\rho}u_k\Big)(t,x)dx. \tag{14}$$

From the Green's formula for the corresponding boundary condition, condition (A5) and Lemma 5 we have the following estimate

$$\begin{aligned} &\int_{\mathcal{G}} u_k(t,x)\sum_{j=1}^m \frac{\partial}{\partial x_j}\left(D_{kj}(t,x)\frac{\partial u_k(t,x)}{\partial x_j}\right)dx = -\sum_{j=1}^m\int_{\mathcal{G}} D_{kj}(t,x)\Big(\frac{\partial u_k(t,x)}{\partial x_j}\Big)^2 dx \\ &\leq -\frac{4m}{\Lambda^2}\bar{D}_k\int_{\mathcal{G}} u_k^2(t,x)dx, \quad t \geq 0. \end{aligned} \tag{15}$$

From Equation (10), conditions (A1)–(A4), we obtain for any $k = 1, 2, \dots, N$ and $t > 0$, $x \in \mathcal{G}$,

$$\begin{aligned} &u_k(t,x)\ ({}_0^C\mathcal{D}_t^{q,\rho}u_k)(t,x) \\ &= -u_k(t,x)a_k(u_k(t,x))b_k(u_k(t,x)) + u_k(t,x)\sum_{j=1}^m\frac{\partial}{\partial x_j}\left(D_{kj}(t,x)\frac{\partial u_k(t,x)}{\partial x_j}\right) \\ &\quad + u_k(t,x)a_k(u_k(t,x))\left[\sum_{j=1}^N c_{kj}(t)f_j(u_j(t,x)) + \sum_{j=1}^N w_{ij}(t)g_j(u_j(t-\tau_k,x))\right] \\ &\leq -\alpha_k B_k u_k^2(t,x) + u_k(t,x)\sum_{j=1}^m\frac{\partial}{\partial x_j}\left(D_{kj}(t,x)\frac{\partial u_k(t,x)}{\partial x_j}\right) \\ &+ \beta_k\left[\sum_{j=1}^N |c_{kj}(t)|\Theta_j|u_k(t,x)|\ |u_j(t,x)| + \sum_{j=1}^N |w_{kj}(t)|\Xi_k|u_k(t,x)|\ |u_j(t-\tau_k,x)|\right] \\ &\leq -\alpha_k B_k u_k^2(t,x) + u_k(t,x)\sum_{j=1}^m\frac{\partial}{\partial x_j}\left(D_{kj}(t,x)\frac{\partial u_k(t,x)}{\partial x_j}\right) \\ &\quad + 0.5\beta_k u_k^2(t,x)\sum_{j=1}^N(\bar{C}\Theta_j + \bar{W}\Xi_j) \\ &\quad + 0.5\beta_k\sum_{j=1}^N \bar{C}\Theta_j u_j^2(t,x) + 0.5\beta_k\sum_{j=1}^N \bar{W}\Xi_j u_j^2(t-\tau_j,x). \end{aligned} \tag{16}$$

From (14)–(16), we obtain the inequality

$$
\begin{aligned}
&({}_0^C\mathcal{D}_t^{q,\rho}V)(t) \leq -\sum_{k=1}^{N}\alpha_k B_k\int_{\mathcal{G}}u_k^2(t,x)dx \\
&+\sum_{k=1}^{N}\int_{\mathcal{G}}u_k(t,x)\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(D_{kj}(t,x)\frac{\partial u_k(t,x)}{\partial x_j}\right)dx \\
&+0.5\sum_{k=1}^{N}\beta_k\int_{\mathcal{G}}u_k^2(t,x)dx\sum_{j=1}^{N}(\bar{C}\Theta_j+\bar{W}\Xi_j) \\
&+0.5\sum_{k=1}^{N}\beta_k\sum_{j=1}^{N}\bar{C}\Theta_j\int_{\mathcal{G}}u_j^2(t,x)dx+0.5\sum_{k=1}^{N}\beta_k\sum_{j=1}^{N}\bar{W}\Xi_j\int_{\mathcal{G}}u_j^2(t-\tau_j,x)dx \\
&\leq -\Big[\min_{1\leq k\leq N}\alpha_k B_k+\frac{4mB}{\Lambda^2}-0.5\bar{\beta}\sum_{j=1}^{N}(\bar{C}\Theta_j+\bar{W}\Xi_j)-0.5\bar{C}\max_{1\leq j\leq N}\Theta_j\sum_{k=1}^{N}\beta_k\Big]V(t) \\
&+0.5\bar{W}\bar{\beta}\max_{1\leq j\leq N}\Xi_j\sum_{k=1}^{N}V(t-\tau_k),\ \ t>0,
\end{aligned}
\tag{17}
$$

where $B=\min_{1\leq k\leq N}\bar{D}_k$, $\bar{\beta}=\max_{1\leq k\leq N}\beta_k$.

Thus, the function $V(t)$ satisfies the scalar fractional differential inequality (6) with

$$
\begin{aligned}
a &= \min_{1\leq k\leq N}\alpha_k B_k+\frac{4m\min_{1\leq k\leq N}\bar{D}_k}{\Lambda^2} \\
&\quad -0.5\max_{1\leq k\leq N}\beta_k\sum_{j=1}^{N}(\bar{C}\Theta_j+\bar{W}\Xi_j)-0.5\bar{C}\max_{1\leq j\leq N}\Theta_j\sum_{k=1}^{N}\beta_k, \\
b_i &= 0.5\bar{W}\max_{1\leq k\leq N}\beta_k\max_{1\leq j\leq N}\Xi_j,\ i=1,2,\dots,N \\
\phi(t) &= 0.5\int_{\mathcal{G}}\sum_{k=1}^{N}\Psi_k^2(t,x)dx\geq 0,\ t\in[-\tau,0].
\end{aligned}
\tag{18}
$$

From Lemma 3 the inequality $V(t)\leq v_2(t),\ t\in[-\tau,\infty)$ holds where $v_2(t)$ is the solution of the scalar fractional Equation (1) with $a, b_i,\ i=1,2\dots,N$, and $\phi\in C([-\tau,0],[0,\infty))$ defined by (18). Moreover, note $V(t)\geq 0$ for $t\geq 0$ so $0\leq V(t)\leq v_2(t)$ for $t\geq 0$. From Lemma 1 if (13) holds then $\lim_{t\to\infty}v_2(t)=0$ and so $\lim_{t\to\infty}V(t)$ i.e., $0=\lim_{t\to\infty}V(t)=0.5\lim_{t\to\infty}||u||_t$. This completes the proof. □

According to Remark 1 we obtain the following result for the model (10), in which the Caputo fractional derivative is applied instead of the generalized proportional Caputo fractional derivative.

Corollary 1. *Let the Conditions 1, 2, 3, 4 of Theorem 1 be satisfied and the inequality*

$$
\begin{aligned}
&2\min_{1\leq k\leq N}\alpha_k B_k+\frac{8m\min_{1\leq k\leq N}\bar{D}_k}{\Lambda^2} \\
&>\max_{1\leq k\leq N}\beta_k\sum_{j=1}^{N}(\bar{C}\Theta_j+\bar{W}\Xi_j)+\bar{C}\max_{1\leq j\leq N}\Theta_j\sum_{k=1}^{N}\beta_k+\sqrt{2}N\bar{W}\max_{1\leq k\leq N}\beta_k\max_{1\leq j\leq N}\Xi_j
\end{aligned}
\tag{19}
$$

holds.

Then, any solution $u\in C^{q,1}([0,\infty)\times\mathcal{G},\mathbb{R}^N)$ of the model (10) with the Caputo fractional derivative w.r.t. the time variable ${}_a^C D_t^q u(t,x)$ (instead of the generalized proportional Caputo fractional derivative ${}_a^C D_t^{q,\rho}u(t,x)$) and the boundary and initial value conditions (11), (12) satisfies $\lim_{t\to\infty}||u||_t=0$.

Remark 5. *Fractional neural networks with Caputo fractional derivatives and reaction-diffusion terms are studied in [30,31] but the applied derivative of the Lyapunov function has no memory so it could not be connected and replaced in the fractional model (see Definition 5 [31], line 16 page 4 right column [30]). Differently, in our paper we use the corresponding fractional derivative of Lyapunov functions, which can be easily connected with the studied model.*

Remark 6. *The established comparison result (Lemma 2) for the scalar linear Equation (6) and its combination with Lyapunov functions gives us the opportunity to avoid any assumption concerning the behavior of the solution such as $u(t-h) \leq u(t)$, which is necessary for the proof of the main results in some papers.*

Remark 7. *Note some authors apply the function $V(t)$ defined as a sum of absolute values instead of the quadratic function as in our work. However, the equality $({}^{C}_{t_0}\mathcal{D}^{q,\rho}|u(.)|)(t) = \text{sign}\, u(t)\, ({}^{C}_{t_0}\mathcal{D}^{q,\rho}u)(t)$ could be applied only if the sign of $u(.)$ is not changeable in the considered interval (see Lemma 2 [32]). Unfortunately, if we use the solution of the model, it is restrictive to assume its sign to be unchangeable.*

5. Applications

Example 1. *Consider the following model with $q \in (0,1)$, $\rho = 0.7$*

$$
\begin{aligned}
&({}^{C}_{0}\mathcal{D}^{q,0.7}_{t}u_1)(t,x) = -[6 + 0.9\cos(u_1(t,x))]u_1(t,x) \\
&+ (3+\cos(t))\frac{\partial^2 u_1(t,x)}{\partial x_1^2} + \frac{\partial^2 u_1(t,x)}{\partial x_2^2} \\
&+ [2 + 0.3\cos(u_1(t,x))]\Big[(0.2 - 0.1\cos(t))\Big[0.5(|u_1(t,x)+1| - |u_1(t,x)-1|)\Big] \\
&\quad + (0.3 - 0.1\sin(t))\Big[0.5(|u_2(t,x)+1| - |u_2(t,x)-1|)\Big] \\
&\quad + (0.1 + 0.1\cos(t))\Big[0.5(|u_1(t - \frac{e^t}{e^t+1}, x) + 1| - |u_1(t - \frac{e^t}{e^t+1}, x) - 1|)\Big] \\
&\quad + (0.2 - 0.1\sin(t))\Big[0.5(|u_2(t - \frac{e^t}{e^t+1}, x) + 1| - |u_2(t - \frac{e^t}{e^t+1}, x) - 1|)\Big]\Big] \\
&({}^{C}_{0}\mathcal{D}^{q,0.7}_{t}u_2)(t,x) = -[4 + 0.6\cos(u_2(t,x))]u_2(t,x) \\
&+ \frac{\partial^2 u_2(t,x)}{\partial x_1^2} + (4+\sin(t))\frac{\partial^2 u_2(t,x)}{\partial x_2^2} \\
&+ [2 + 0.3\cos(u_2(t,x))]\Big[(0.3 - 0.2\sin(t))[0.5(|u_1(t,x)+1| - |u_1(t,x)-1|)] \\
&\quad + (0.4 - 0.2\cos(t))[0.5(|u_2(t,x)+1| - |u_2(t,x)-1|)] \\
&\quad + (0.3 - 0.1\cos(t))\Big[0.5(|u_1(t - \frac{e^t}{e^t+1}, x) + 1| - |u_1(t - \frac{e^t}{e^t+1}, x) - 1|)\Big] \\
&\quad + (0.3 - 0.1\cos(t))\Big[0.5(|u_2(t - \frac{e^t}{e^t+1}, x) + 1| - |u_2(t - \frac{e^t}{e^t+1}, x) - 1|)\Big]\Big], \\
&\textit{for } t > 0, \ |x_1| \leq 2, |x_2| \leq 2
\end{aligned}
\tag{20}
$$

under the following boundary and initial value conditions:

$$u_1(t, \pm 2, \pm 2) = u_2(t, \pm 2, \pm 2) = 0, \ \ t \in [-1, \infty), \tag{21}$$

$$u_1(s,x) = \Psi_1(s,x), \ \ u_2(s,x) = \Psi_2(s,x). \ \ s \in [-1, 0], \ |x_1| \leq 2, |x_2| \leq 2. \tag{22}$$

where $x = (x_1, x_2)$.

Thus, in the model (20) we have $N = m = 2$, $f_k(v) = g_k(v) = 0.5(|v+1| - |v-1|)$, $\tau_1(t) = \tau_2(t) = \frac{e^t}{e^t+1}$, $a_k(v) = 2 + 0.3\cos(v)$, $k = 1,2$, $b_1(v) = 3v, b_2(v) = 2v$ for $v \in \mathbb{R}$ and

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 0.2 - 0.1\cos(t) & 0.3 - 0.1\sin(t) \\ 0.3 - 0.2\sin(t) & 0.4 - 0.2\cos(t) \end{bmatrix},$$

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} 0.1 + 0.1\cos(t) & 0.2 - 0.1\sin(t) \\ 0.3 - 0.2\sin(t) & 0.3 - 0.1\cos(t) \end{bmatrix},$$

$$\begin{bmatrix} D_{11}(t,x) & D_{12}(t,x) \\ D_{21}(t,x) & D_{22}(t,x) \end{bmatrix} = \begin{bmatrix} 3 + \cos(t) & 1 \\ 1 & 4 + \sin(t) \end{bmatrix}.$$

Assumptions (A1)–(A6) are satisfied with $\Theta_k = \Xi_k = 1, k = 1,2$, $\alpha_k = 1.7$, $\beta_k = 2.3, k = 1,2$, $B_1 = 3, B_2 = 2$, $\bar{C} = 0.6, \bar{W} = 0.4$, $\bar{D}_1 = \bar{D}_2 = 0$. Then

$$\begin{aligned} &2\min_{1\leq k\leq N} \alpha_k B_k + \frac{8N\min_{1\leq k\leq N}\bar{D}_k}{\Lambda^2} = 2(1.7)2 + \frac{8(2)(1)}{2^2} = 10.8 \\ &> \max_{1\leq k\leq N} \beta_k \sum_{j=1}^{N}(\bar{C}\Theta_j + \bar{W}\Xi_j) + \bar{C}\max_{1\leq j\leq N}\Theta_j \sum_{k=1}^{N}\beta_k + \sqrt{2}e^{\frac{1-\rho}{\rho}} N\bar{W}\max_{1\leq k\leq N}\beta_k \max_{1\leq j\leq N}\Xi_j \\ &= 1.3(0.6*2 + 0.4*2) + 0.6*2.6 + \sqrt{2}*2*0.4*1.3\, e^{\frac{1-0.7}{0.7}} = 6.05. \end{aligned} \tag{23}$$

Therefore, all the conditions of Theorem 1 are satisfied and for any initial functions $\Psi_k(s, x_1, x_2) > 0, s \in [-1, 0]$, $|x_1| \leq 2$, $|x_2| \leq 2$, $k = 1,2$ the corresponding solution $u(t,x) = (u_1(t,x), u_2(t,x))$ of the model (20)–(22) satisfies

$$\lim_{t\to\infty} ||u||_t = \lim_{t\to\infty} \int_{\max\{|x_1|,|x_2|\}\leq 2} \left(u_1^2(t,x) + u_2^2(t,x)\right) dx = 0 \tag{24}$$

with $x = (x_1, x_2)$.

Note inequality (23) holds for $\rho \in [0.4, 1]$. Therefore, according to Theorem 1 the solutions of the model (20), (21), (22) satisfy (24) if $\rho = 0.7$ is replaced by any number on the interval $[0.4, 1]$. Moreover, the behavior does not depend on the order $q \in (0,1)$ of the fractional derivative.

Example 2. *Consider the following fractional generalization of the standard reaction–diffusion model [33] with $q \in (0,1)$, $\rho = 0.7$ and*

$$({}_0^C\mathcal{D}_t^{q,0.7} u)(t,x) = -ru^2(t,x) + ru(t,x) + D\frac{\partial^2 u(t,x)}{\partial x^2}, \tag{25}$$

under the following boundary and initial value conditions:

$$u(t, \pm 2, \pm 2) = 0, \;\; t \geq 0, \qquad u(0,x) = \Psi(x), \; |x| \leq 2. \tag{26}$$

Thus, in the model (20) we have $N = m = 1$, $a(v) = v$, $b(v) = v$ for $v \in \mathbb{R}$, $f_k(v) = 1$, $g(v) = 0$, and $c(t) = 1$. Thus, $\Theta = 1$, $b = 1$, $\tilde{C} = 1$, $\alpha = \beta = 1$, and $\tilde{D} = 1$. Moreover, we could consider $\bar{W} = 0$. Then, the inequality (13) is reduced to

$$2 + \frac{8}{2^2} = 4 > (1 + \bar{W}\Xi) + 1 + \sqrt{2}e^{\frac{1-\rho}{\rho}}\bar{W}\Xi = 2. \tag{27}$$

Therefore, according to Theorem 1 the solutions of the model (25), (26) satisfy $\lim_{t\to\infty}|u(t,x)| = 0$ for $|x| \leq 2$.

6. Conclusions

In this paper, a delayed reaction-diffusion neural network model of fractional order is investigated and the model consists of several constant delays. The generalized proportional Caputo fractional derivative with respect to the time variable is applied; thus, it gives opportunities for more adequate modeling of the behavior of the system. The long-term behavior of the solution of the model, when the time is increasing without a bound is studied. The Lyapunov functional approach combined with the generalized proportional Caputo fractional derivative and the comparison principle are employed in the development of the main results. As a special case of our results, we obtain some results for delay Caputo fractional reaction-diffusion models known and studied in the literature. The main goal of the example we provide is to illustrate the application of our obtained sufficient conditions. In a future study, we hope to apply these theoretical results to some particular models in science, biology, engineering or neurocomputing. Another practical question concerns the existence of strictly positive solutions of the considered model, and we hope to study this in the future.

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