fractal and fractional

Article

Scaled Conjugate Gradient for the Numerical Simulations of the Mathematical Model-Based Monkeypox Transmission

Suthep Suantai [1,2], Zulqurnain Sabir [3], Muhammad Umar [4] and Watcharaporn Cholamjiak [5,*]

1 Research Group in Mathematics and Applied Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
2 Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
3 Department of Computer Science and Mathematics, Lebanese American University, Beirut P.O. Box 13-5053/S-14, Lebanon
4 Department of Mathematics and Statistics, Hazara University, Mansehra 21120, Pakistan
5 School of Science, University of Phayao, Phayao 56000, Thailand
* Correspondence: watcharaporn.ch@up.ac.th

Abstract: The current study presents the numerical solutions of a fractional order monkeypox virus model. The fractional order derivatives in the sense of Caputo are applied to achieve more realistic results for the nonlinear model. The dynamics of the monkeypox virus model are categorized into eight classes, namely susceptible human, exposed human, infectious human, clinically ill human, recovered human, susceptible rodent, exposed rodent and infected rodent. Three different fractional order cases have been presented for the numerical solutions of the mathematical monkeypox virus model by applying the stochastic computing performances through the artificial intelligence-based scaled conjugate gradient neural networks. The statics for the system were selected as 83%, 10% and 7% for training, testing and validation, respectively. The exactness of the stochastic procedure is presented through the performances of the obtained results and the reference Adams results. The rationality and constancy are presented through the stochastic solutions together with simulations based on the state transition measures, regression, error histogram performances and correlation.

Keywords: monkeypox virus system; fractional; artificial intelligence; scaled conjugate gradient; numerical solutions

Citation: Suantai, S.; Sabir, Z.; Umar, M.; Cholamjiak, W. Scaled Conjugate Gradient for the Numerical Simulations of the Mathematical Model-Based Monkeypox Transmission. *Fractal Fract.* **2023**, *7*, 63. https://doi.org/10.3390/fractalfract7010063

Academic Editor: Rodica Luca

Received: 23 October 2022
Revised: 16 December 2022
Accepted: 17 December 2022
Published: 5 January 2023

1. Introduction

Monkeypox infection is a type of zoonosis. Africa is a region where this disease has more frequently appeared as compared to other parts of the world. Monkeypox was first identified in 1958 with the condition of mimicking the pox in populations of monkeys. Many nations have informed the World Health Organization (WHO) about monkeypox instances since the start of 2022. As of mid-2022, 2103 positive cases and one death have been reported, according to WHO [1,2]. The spreading process of this disease from one person to another is by direct contact with scabs, infected rash, and bodily fluids. The infection can also spread via respiratory droplets during face-to-face interaction, sex acts, or other close human touch [3,4]. The disease mainly spreads to humans through animals, such as African monkeys and rats. People can contract animal diseases through scratches or bites, the preparation of bush meat, recurrent relations with bodily fluids or eating food tainted by rodents. There are numerous mathematical concepts that identify the natural events using predator–prey studies and interactions between distinct species. The disease can be transferred by immediate communication with wounds and body substances through infected people [5].

The manifestations of monkeypox differ among individuals. The main indications of monkeypox include headache, fever, muscle aches, swollen lymph nodes, backaches, tiredness and anxieties. Patients generally experience a mild disease that subsides on its own in a couple of weeks. People with a weakened immune system can experience serious effects [6]. Currently, there is not any special treatment for monkeypox viral disease. However, smallpox vaccination and antiviral measures have been used to protect against this disease. Due to the worldwide eradication of smallpox, immunization is presently not accessible [7,8].

Monkeypox disease has received little consideration in the past; therefore, the spreading system of this disease is not recognized. Many scientists mathematically performed monkeypox virus formulations and presented the solutions by using deterministic schemes [9]. According to their research, keeping diseased people apart from the general population can prevent the spread of disease. Additionally, a set of nonlinear systems was developed to represent the dynamics of monkeypox infection [10]. The simulation model indicates that a person's immunological health affects whether they recover from the orthopoxviral infection. A variety of computer simulations have been applied on communicable diseases to better comprehend the propagation dynamics and discover new ways to treat the wide-spreading sickness [11].

A design of a mathematical game theory model based on vaccination strategies in a moderate environment was presented to better understand the monkeypox dynamics [12]. Vaccination can effectively eliminate monkeypox in a completely epidemic state. Additionally, the model's theoretical calculations showed that the planned actions will cause the eradication of infectious classes in communities of non-human primates [13]. Nowadays, scientists and engineers from various fields have paid attention to modeling fractional systems, especially mathematical modeling in healthcare. The fractional differential models have a variety of applications in various disciplines [14,15]. Recently, many scientists have used fractional systems to model a variety of infectious as well as non-infectious disorders. The coronavirus infection is an interesting research subject which has been broadly scrutinized, and valuable solutions have been produced based on the fractional Caputo derivative (CD) [16–18]. Studying tuberculosis with spontaneous reactions and external reinfections has already been realized by using a fractional kind of model [19]. A fractional AIDS/HIV epidemic system using the Mittag–Leffler kernel has also been presented [20]. A few more fractional order dynamical systems that have been solved with different techniques are presented in Refs. [21–23].

The current work designates the numerical performances based on the mathematical monkeypox virus system using fractional order equations. The fractional order derivatives in the sense of CD are applied to achieve more realistic results for the nonlinear model. The numerical solutions of the mathematical monkeypox virus model are presented by applying stochastic computing performances through the artificial intelligence-based scaled conjugate gradient. Some well-known applications of the stochastic solvers are nonlinear smoke models [24], singular functional models [25] and delay functional differential models [26]. The minute specifics in the fractional order derivatives provide the super-fast transition as well as super-slow development along with the details of the system dynamics, which are difficult to understand in the integer order model. Moreover, fractional calculus is used to compute the system behavior for the indices. When the circumstance is available, the fractional kind of derivatives performs significantly better than the integer form of the derivative [27,28]. Additionally, a wide range of projects based on engineering, control networks, physical networks and mathematical structures have been researched using fractional differential equations. Fractional calculus has been implemented in the past 3 decades by applying considerable operators, such as CD [29], Weyl-Riesz [30], Riemann-Liouville [31], Grünwald–Letnikov [32] and Erdélyi–Kober [33]. All these operators have their individual strengths and importance; however, CD is simple to measure and has an ability to solve both homogeneous/non-homogeneous input variables. The researchers are motivated to design mathematical models and offer numerical achievements based on these fractional order implementations.

The other sections of this paper are as follows. The fractional order system is shown in Section 2. The proposed methodology using the stochastic performances is given in Section 3. The imitations of the solutions are shown in Section 4. The concluding comments are presented in Section 5.

2. Mathematical System

This section presents the mathematical system based on the monkeypox virus by using the transmission of monkeypox, which involves human and rodent populations. Eight compartmental systems based on five human and three rodent classes have been provided. The dynamics of the nonlinear model are categorized into eight classes: susceptible human $S_h(t)$, exposed human $E_h(t)$, infectious human $I_h(t)$, clinically ill human $C_h(t)$, recovered human $R_h(t)$, susceptible rodent $S_r(t)$, exposed rodent $E_r(t)$ and infected rodent $I_r(t)$. The mathematical illustrations based on fractional monkeypox model are given as follows [34]:

$$\begin{cases} S'_h(t) = \varphi_h - (\mu_h + \lambda_h)S_h(t), & S_h(0) = i_1, \\ E'_h(t) = \lambda_h S_h(t) - (\beta + \mu_h)E_h(t), & E_h(0) = i_2, \\ I'_h(t) = \beta E_h(t) - (\gamma + \psi + \delta_1 + \mu_h)I_h(t), & I_h(0) = i_3, \\ C'_h(t) = \gamma I_h(t) - (\delta_2 + \rho + \mu_h)C_h(t), & C_h(0) = i_4, \\ R'_h(t) = \rho C_h(t) - \mu_h R_h(t) + \psi I_h(t), & R_h(0) = i_5, \\ S'_r(t) = \varphi_r - \lambda_r S_r(t) - \mu_r S_r(t), & S_r(0) = i_6, \\ E'_r(t) = \lambda_r S_r(t) - (\mu_r + \varepsilon)E_r(t), & E_r(0) = i_7, \\ I'_r(t) = \varepsilon E_r(t) - \mu_r I_r(t), & I_r(0) = i_8, \end{cases} \tag{1}$$

where φ_h and φ_r indicate the susceptible population based on human and rodent recruitments, μ_h and μ_r are the natural death rates per capita in humans and rodents, β is the rate of disease progression through exposed infectious individuals, δ_1 and δ_2 are the infectious humans and clinically ill humans, γ presents the clinically ill rate, ρ is the recovery rate based on the clinically ill humans, ψ presents the rate of natural recovery based on the immunity, λ_h is the force of infection, λ_r shows the exposed rodents enhanced with the infection force and ε is the increment rate of infectious rodents.

$$\begin{cases} D^\alpha S_h(t) = \varphi_h - (\mu_h + \lambda_h)S_h(t), & S_h(0) = i_1, \\ D^\alpha E_h(t) = \lambda_h S_h(t) - (\beta + \mu_h)E_h(t), & E_h(0) = i_2, \\ D^\alpha I_h(t) = \beta E_h(t) - (\gamma + \psi + \delta_1 + \mu_h)I_h(t), & I_h(0) = i_3, \\ D^\alpha C_h(t) = \gamma I_h(t) - (\delta_2 + \rho + \mu_h)C_h(t), & C_h(0) = i_4, \\ D^\alpha R_h(t) = \rho C_h(t) - \mu_h R_h(t) + \psi I_h(t), & R_h(0) = i_5, \\ D^\alpha S_r(t) = \varphi_r - \lambda_r S_r(t) - \mu_r S_r(t), & S_r(0) = i_6, \\ D^\alpha E_r(t) = \lambda_r S_r(t) - (\mu_r + \varepsilon)E_r(t), & E_r(0) = i_7, \\ D^\alpha I_r(t) = \varepsilon E_r(t) - \mu_r I_r(t), & I_r(0) = i_8, \end{cases} \tag{2}$$

where α represents the fractional order derivatives in the sense of CD for solving the mathematical monkeypox virus model. The fractional derivatives are obtained in the interval [0, 1]. Recently, fractional calculus has been applied in diverse studies, e.g., pine wilt virus system with the rate of convex [35], irregular rate of heat transfer [36], spatiotemporal patterns using the reactions of Belousov–Zhabotinskii [37], predator/prey system using the performances of herd [38], numerical approximation using the soil animal material content based on visible/near infrared spectroscopy [39], typhoid disease including protection from infection [40] and hepatitis B virus system [41]. Some novel geographies of the proposed approach to solve the monkeypox infection model are described as follows:

- The design of a fractional monkeypox infection model is provided to achieve more accurate performances.

- Stochastic processing is employed to simulate the monkeypox infection model using the fractional derivative between 0 and 1.
- The precision of the proposed method is verified using the comparison of the reference Adams results and the achieved results.
- The negligible absolute error (AE) presents the accuracy and capability of the proposed method.
- The error histograms (EHs), correlation, state transitions (STs) and regression indicate the reliability of the proposed approach to solve the model.

3. Designed Methodology

The stochastic computing approach for the mathematical monkeypox infection model is defined using the network system (1). Figure 1 represents the monkeypox infection model in three steps, namely the mathematical system, methodology and solution presentations. The computing stochastic performances using the noteworthy actions along with the implementation process are presented.

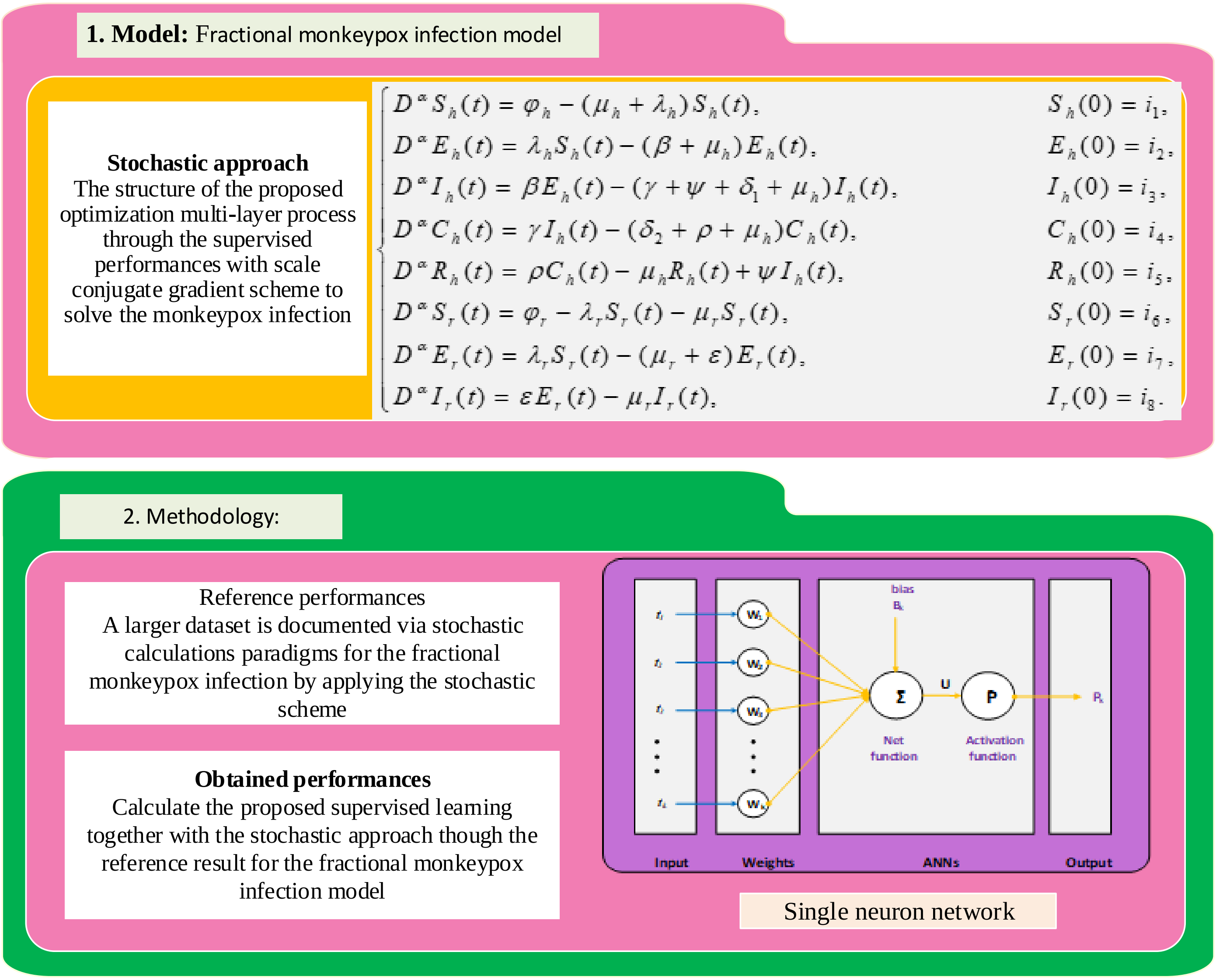


Figure 1. *Cont.*

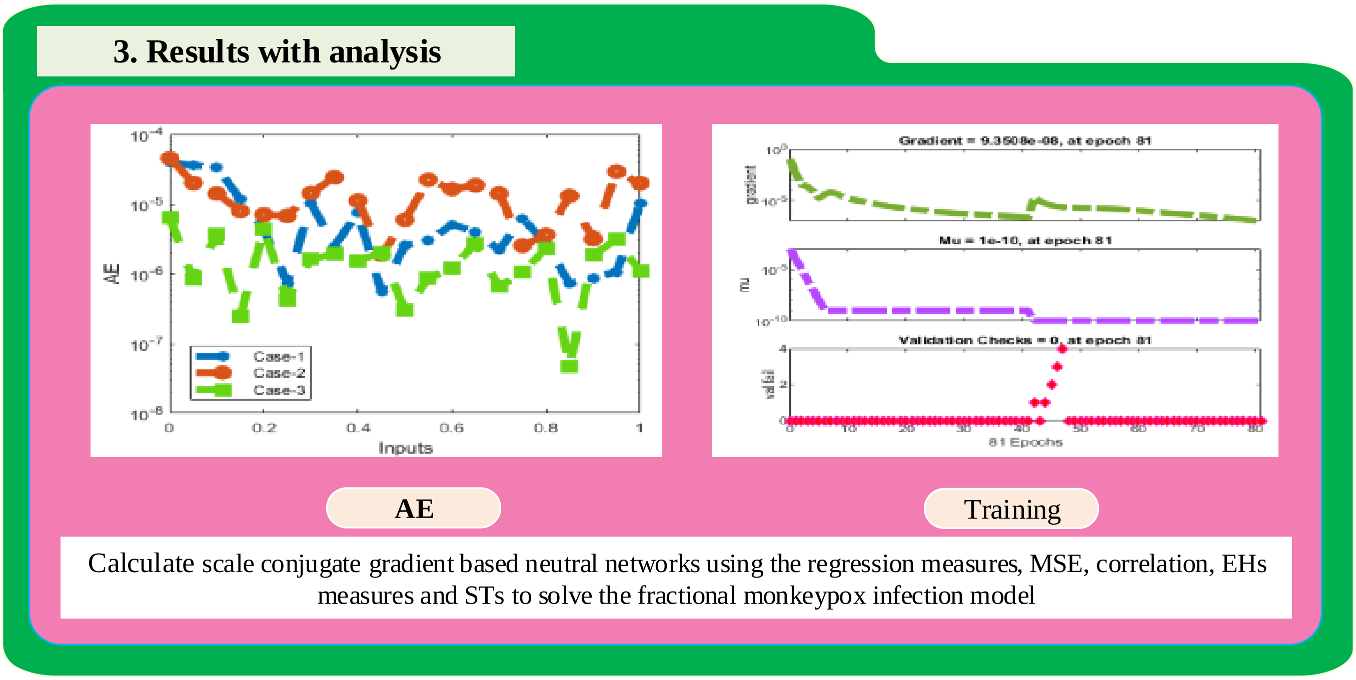


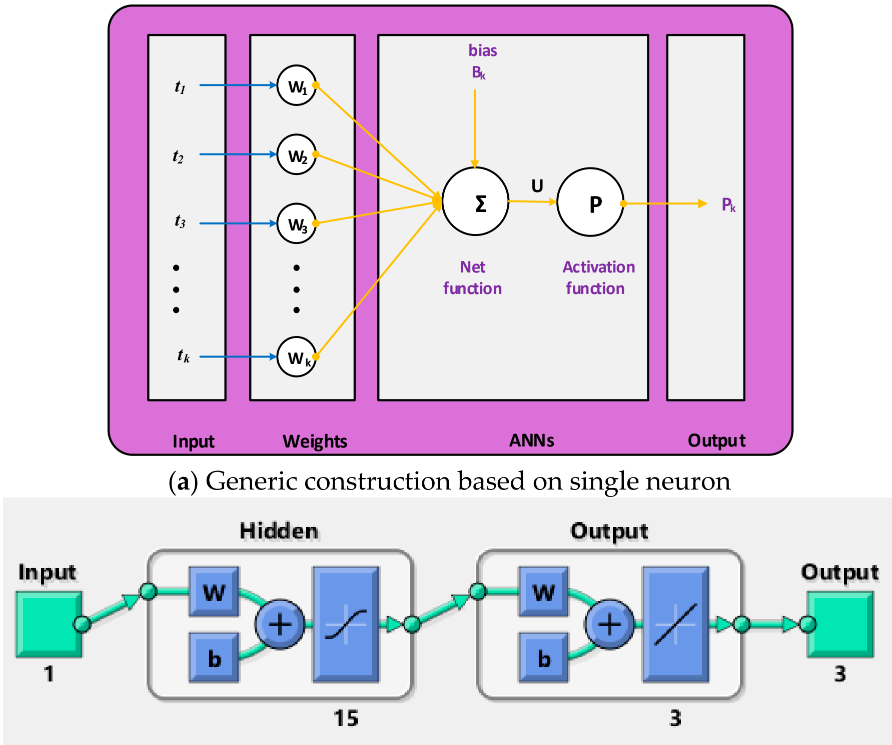
Figure 1. Stochastic illustrations to solve the monkeypox infection system.

The Adams technique has been used to offer major simplification along with default parameter settings to execute simulations numerically [42,43]. The selection of the data was 83% for training, 10% for testing and 7% for validation. Fifteen neurons were used in the hidden layers. The numerical solutions of the mathematical monkeypox virus model were derived using the stochastic computing performances of the artificial intelligence-based scaled conjugate gradient neural networks. The network parameters were used after inclusive simulations and gathering knowledge.

Figure 2a presents the layer structure, whereas the obtained layer design, an input single layer direction with 15 hidden neurons and three outer layer performances are labeled in Figure 2b. The stochastic computing performances are implemented through the MATLAB (nftool) command-based suitable hidden neuron sections, learning approaches, testing and confirmation data. The execution of the stochastic approach for the mathematical monkeypox infection model together with the parameter set is shown in Table 1. The parameter setting values are presented in Table 1, while some of the values have been taken as default.

Table 1. Value setting to perform the stochastic process.

Index	Settings
Fitness goal (MSE)	0
Hidden neurons	15
Maximum mu values	10^9
Decreeing mu	0.1
Increasing mu	09
Adaptive mu	6×10^{-4}
Epochs	880
Minimum values of gradient	10^{-7}
Training values	83%
Testing statics	10%
Validation data	7%
Samples	Random
Hidden and output layers	Single
Adam solver and stopping criteria	Default



(**a**) Generic construction based on single neuron

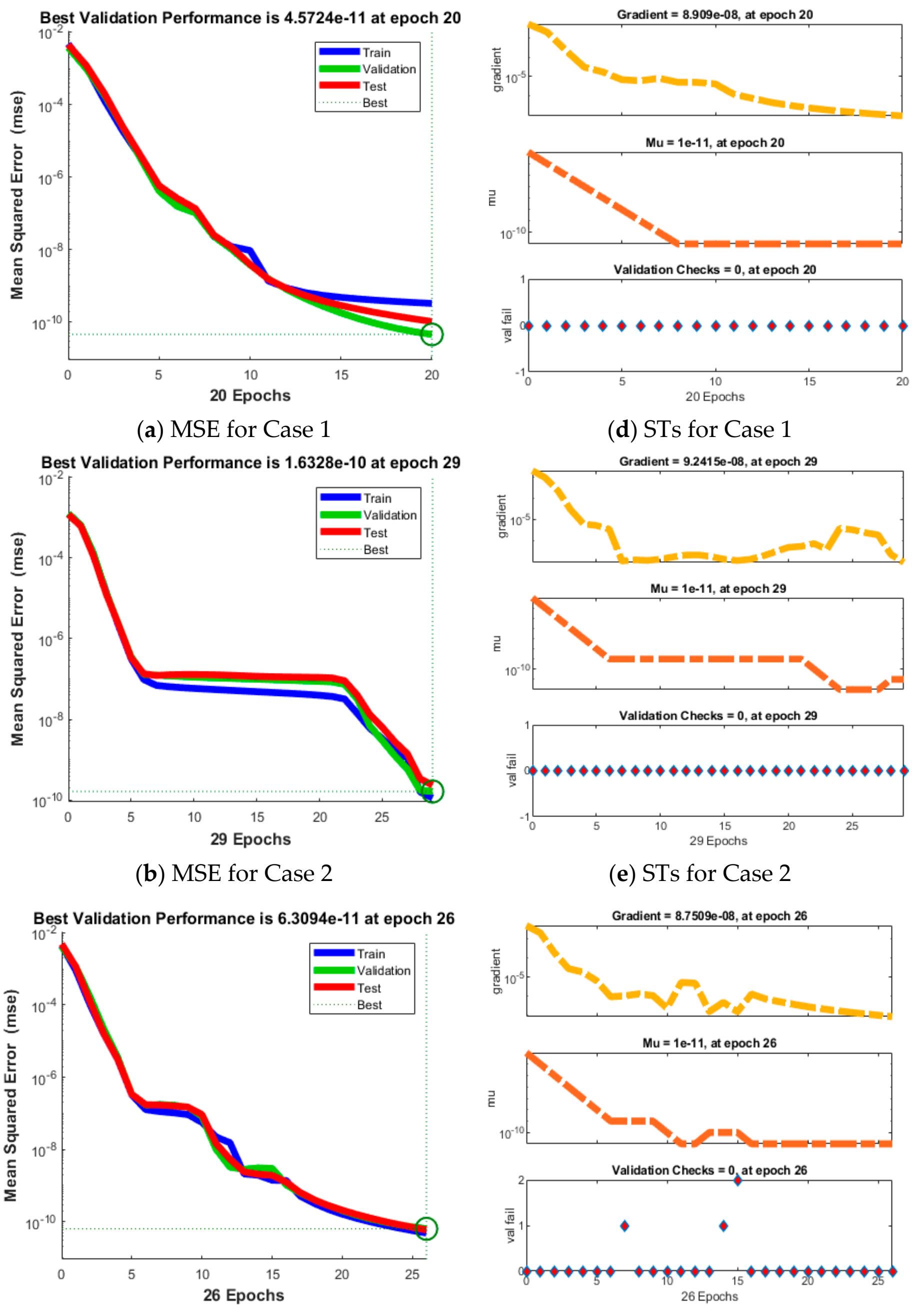
(**b**) A layer construction of the neural network with 15 neurons

Figure 2. A neural network structure to solve the monkeypox infection system.

4. Results of Fractional Monkeypox Infection Model

The solutions of three cases based on the fractional monkeypox infection systems are presented in this section by taking $\alpha = 0.7$ for case 1, $\alpha = 0.8$ for case 2 and $\alpha = 0.9$ for case 3 in system (1). The other values used were $\phi_h = 0.1$, $\phi_r = 0.2$, $\mu_r = 0.002$, $\mu_h = 0.000303$, $\varepsilon = 0.1$, $\rho = 0.036246$, $\beta = 0.016744$, $\delta_1 = 0.003286$, $\gamma = 0.5$, $\delta_2 = 0.055487$, $\psi = 0.088366$, $\lambda_h = 0.000303$, $\lambda_r = 0.00404$, $\sigma = 0.012458$, $i_i = 0.01$, $i_2 = 0.02$, $i_3 = 0.04$, $i_4 = 0.06$, $i_5 = 0.08$, $i_6 = 0.1$, $i_7 = 0.12$ and $i_8 = 0.14$ [44–46]:

The solutions through the stochastic method are shown in Figures 3–5 for the fractional monkeypox infection systems. Figure 3 indicates the STs together with the optimal representations of the fractional monkeypox infection model. ST values and MSE measures through the substantiation and optimal results are presented in Figure 3. The achieved fractional monkeypox infection systems are presented at generations 20, 29 and 26, performed as 4.5724×10^{-11}, 1.6328×10^{-10} and 6.3094×10^{-11}. It is observed that by increasing the number of epochs, the curves of training, testing and validation approach a steady-state position. Figure 3b authenticates the gradient performances through the stochastic approach for solving the monkeypox infection mathematical model. The gradient representations are calculated at around 8.909×10^{-8}, 9.2415×10^{-8} and 8.7509×10^{-8}. An error gradient represents the direction as well as magnitude, obtained during the designed neural network training, which is applied to upgrade the weights of the network in an appropriate direction. Mu presents the training values, showing the momentum constant/parameter. It is included in the process of upgrading the weights, which update the expression to evade the local minimum problem. The assessment of the solutions using the test outputs, training/validations targets and outputs, test targets, fitness curves, validation outputs and errors are presented in Figure 4a. The values of EHs through the training, authentication, zero and test error are provided in Figure 4b. The EH values are plotted as 9.44×10^{-7}, 2.55×10^{-6} and -9.0×10^{-9} to present the numerical solutions of the fractional monkeypox infection mathematical model. Figure 5 shows the correlation illustrations using the train, authentication and test to find the solutions of the fractional monkeypox infection mathematical model. The correlation is reported as 1 for each variation of the fractional monkeypox infection system, which shows the perfect model. The correlation coefficient (R) is applied together with MSE indexes. The values of R exist between -1 and +1. Hence, if the values of R are close to +1, a higher network performance along with a positive linear association can be accomplished. These actions designate the exactness of the stochastic measures based on the monkeypox infection mathematical model. The MSE measures designate the train, complexity, authentication, generation, test and backpropagation measures, which are tabulated in Table 2 to present the fractional monkeypox infection mathematical model.

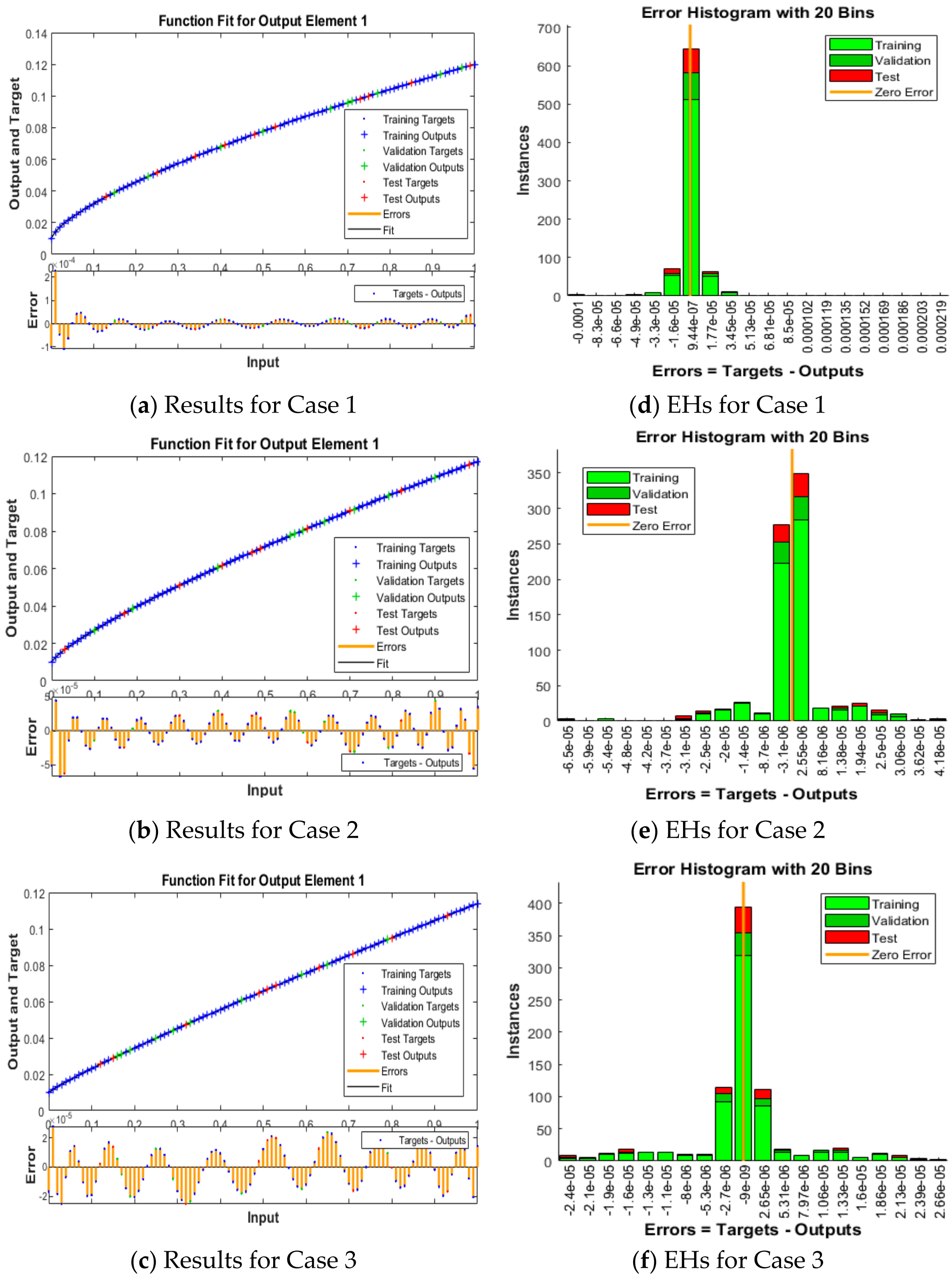


(**a**) MSE for Case 1 (**d**) STs for Case 1

(**b**) MSE for Case 2 (**e**) STs for Case 2

(**c**) MSE for Case 3 (**f**) STs for Case 3

Figure 3. MSE values and ST performances to solve the fractional monkeypox infection mathematical model.



(**a**) Results for Case 1 (**d**) EHs for Case 1

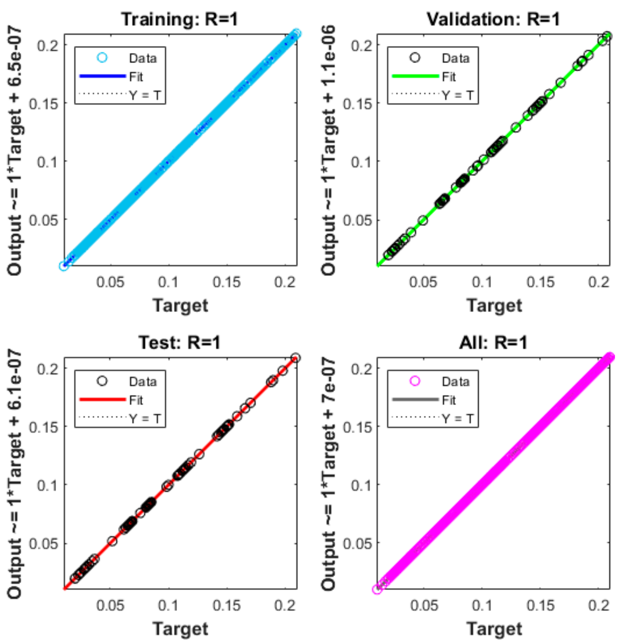
(**b**) Results for Case 2 (**e**) EHs for Case 2

(**c**) Results for Case 3 (**f**) EHs for Case 3

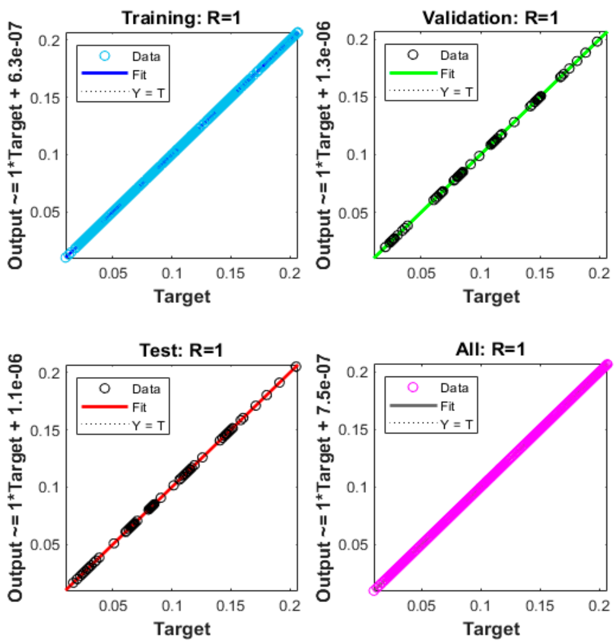
Figure 4. Output assessments and EHs to solve the fractional monkeypox infection mathematical model.

Table 2. Stochastic procedures to solve the fractional monkeypox infection mathematical model.

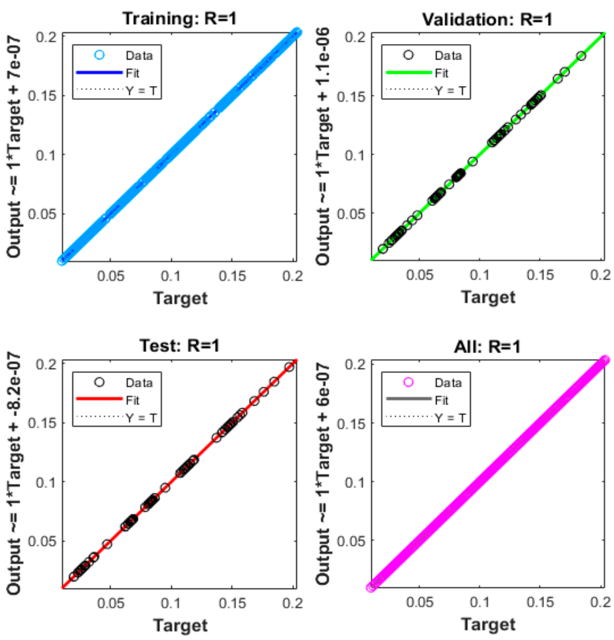
Case	MSE			Iteration	Performance	Gradient	Mu	Complexity
	Validation	Train	Test					
1	4.57×10^{-11}	3.28×10^{-10}	1.03×10^{-10}	20	3.28×10^{-10}	8.91×10^{-8}	1×10^{-11}	2 s
2	1.63×10^{-10}	1.12×10^{-10}	2.35×10^{-10}	29	1.13×10^{-10}	9.24×10^{-8}	1×10^{-11}	2 s
3	6.30×10^{-11}	4.81×10^{-11}	5.90×10^{-11}	26	4.82×10^{-11}	8.75×10^{-8}	1×10^{-11}	2 s



(**a**) Regression (1)



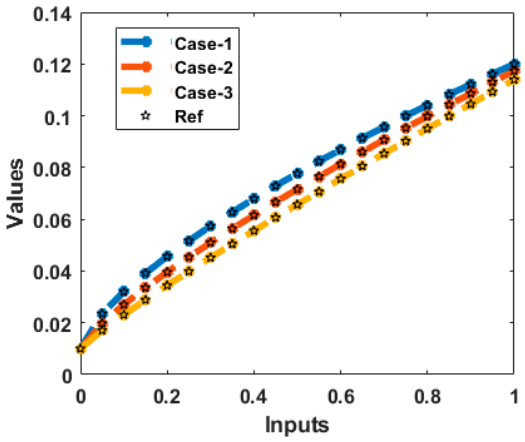
(**b**) Regression (2)



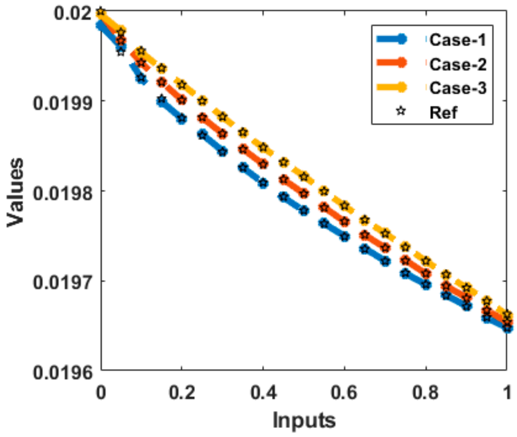
(**c**) Regression (3)

Figure 5. Values of the regression to solve the fractional monkeypox infection mathematical model.

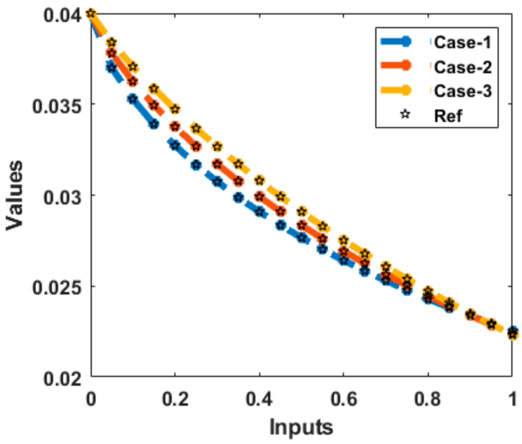
Figures 6 and 7 provide the results comparison and AE performances for the fractional monkeypox infection mathematical model. Figure 6 presents the overlapping of the outcomes for each category of the mathematical model, which authenticates the method's correctness. The AE for the mathematical system is presented in Figure 7. The AE performances for the susceptible human category are found as 10^{-5}–10^{-7}, 10^{-4}–10^{-5} and 10^{-4}–10^{-6} for categories 1–3 of the mathematical monkeypox model. The AE for the second category of exposed humans is 10^{-6} to 10^{-8} for cases 1–3 of the system. The AE values for the infectious humans are calculated as 10^{-5} to 10^{-8}, 10^{-5} to 10^{-6} and 10^{-5} to 10^{-7} for cases 1 to 3. The AE measures for the clinically ill humans are calculated as 10^{-6} to 10^{-7} for cases 1 to 3. The AE values are found as 10^{-6} to 10^{-7} for cases 1 and 2, while these values have been calculated as 10^{-6} to 10^{-8} for the third case based on the recovered humans. The AE for susceptible rodents is found to be 10^{-6} to 10^{-8} for each case of the model. The AE performances for exposed rodents and other classes are found as 10^{-6} to 10^{-8} for cases 1 to 3. These negligible AE values validate the precision of the stochastic method to solve the monkeypox nonlinear model.



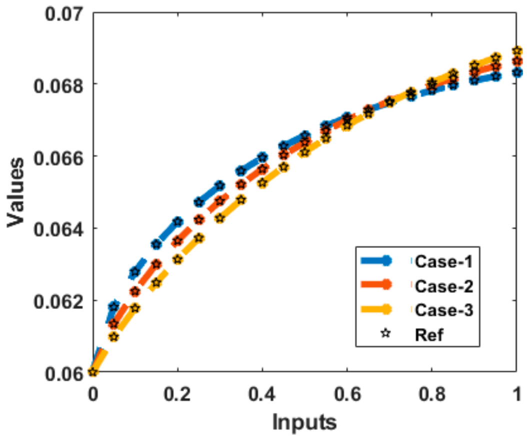
(**a**) Results of $S_h(t)$



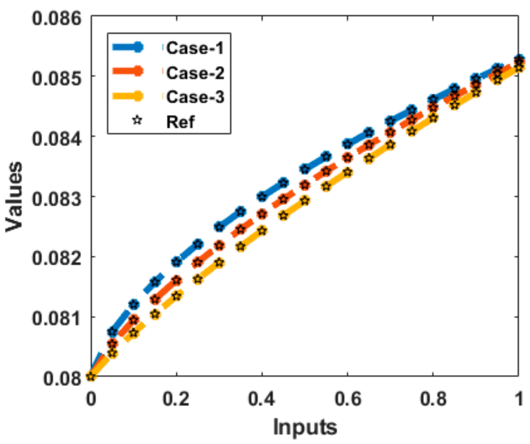
(**b**) Results of $E_h(t)$



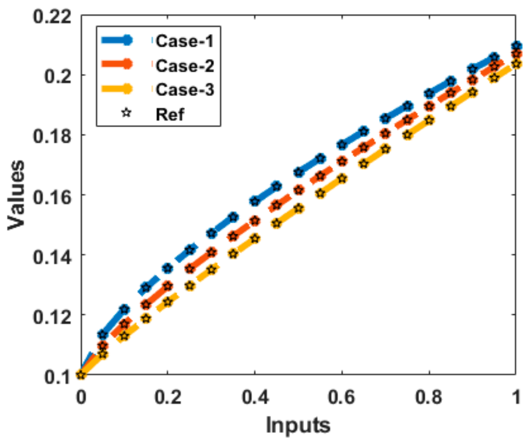
(**c**) Results of $I_h(t)$



(**d**) Results of $C_h(t)$

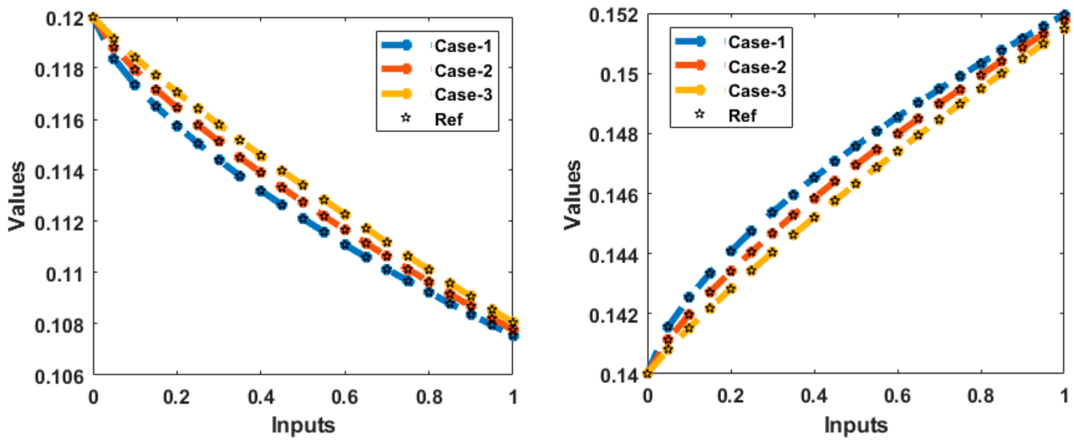


(**e**) Results of $R_h(t)$



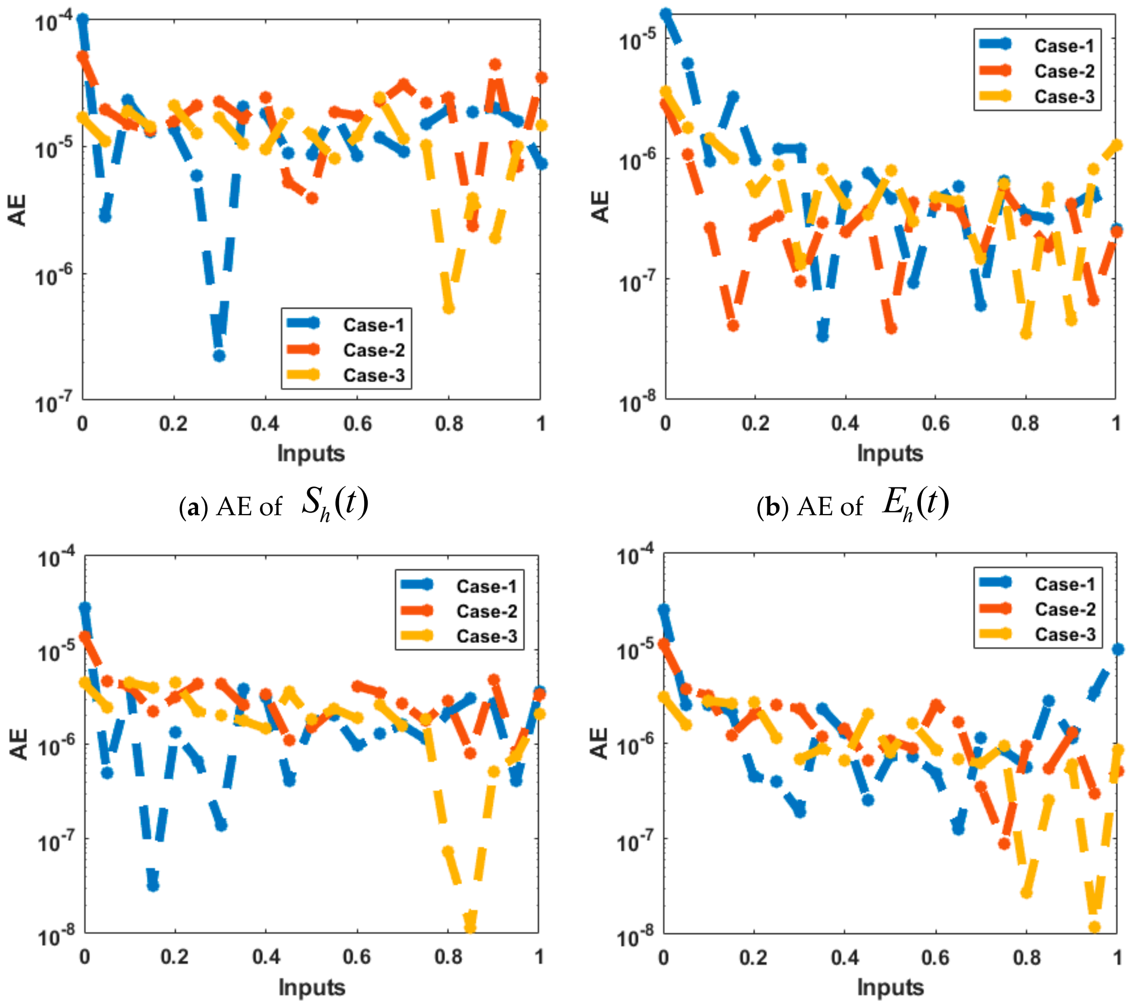
(**f**) Results of $S_r(t)$

Figure 6. *Cont.*



(**g**) Results of $E_r(t)$ (**h**) Results of $I_r(t)$

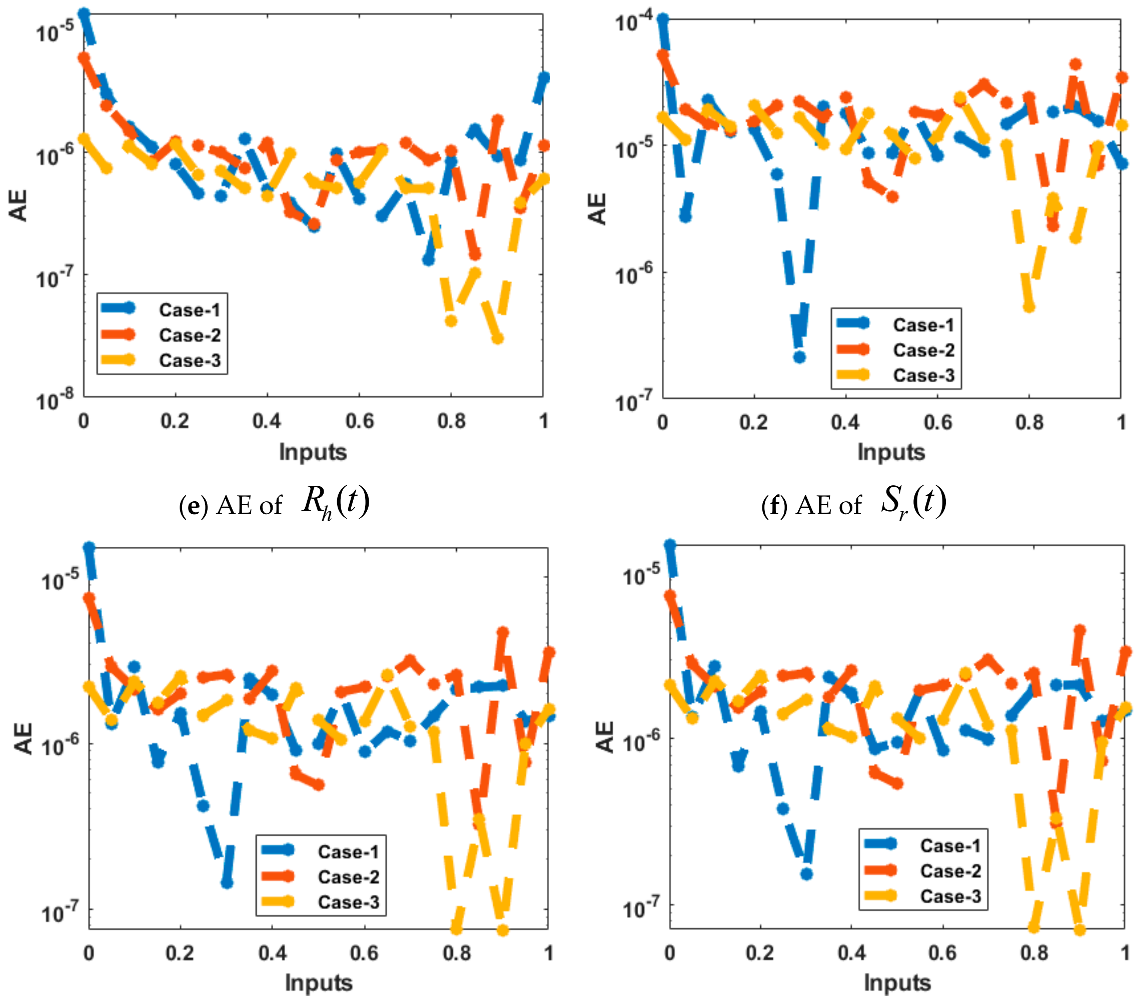
Figure 6. Performances of the results to solve the fractional monkeypox model.



(**a**) AE of $S_h(t)$ (**b**) AE of $E_h(t)$

(**c**) AE of $I_h(t)$ (**d**) AE of $C_h(t)$

Figure 7. *Cont.*



(**e**) AE of $R_h(t)$ (**f**) AE of $S_r(t)$

(**g**) AE of $E_r(t)$ (**h**) AE of $I_r(t)$

Figure 7. AE performances to solve the fractional monkeypox model.

5. Conclusions

The aim of this research was to present the numerical performances of a fractional order mathematical monkeypox virus model. The fractional order derivatives in the sense of Caputo were applied to achieve more realistic results for the nonlinear system. The dynamics of the nonlinear model are sorted into eight classes, i.e., susceptible human, exposed human, infectious human, clinically ill human, recovered human, susceptible rodent, exposed rodent and infected rodent. A few concluding remarks are presented as follows.

- The numerical solutions of the mathematical monkeypox virus model are presented by using the stochastic computing scheme.
- Three different fractional order cases have been used to present the numerical solutions of the mathematical monkeypox virus model.
- The stochastic computing performances through the artificial intelligence-based scaled conjugate gradient neural networks have been chosen as 83%, 10% and 7% for training, testing and validation, respectively.
- The exactness of the stochastic procedure was confirmed through the overlapping of the obtained and reference results.
- The negligible AE performances were presented to verify the accuracy of the proposed method.

- The rationality and constancy were ensured through the stochastic solutions together with simulations based on the state transition measures, regression, error histograms performances, mean square error and correlation.

Author Contributions: S.S., project administration; Z.S., writing—original draft preparation; M.U., software; W.C., writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research received funding support from the NSRF via the program Management Unit for Human Resources & Institutional Development, Research and Innovation [grant number B05F640183] and Chiang Mai University. Watcharaporn Cholamjiak would like to thank National Research Council of Thailand (N42A650334) and Thailand Science Research and Innovation, the University of Phayao (grant no. FF66-UoE). This research project was also supported by Fundamental Fund 2023, Chiang Mai University.

Data Availability Statement: Not applicable.

Conflicts of Interest: All the authors state that there are no potential conflict of interest.

References

1. Weiner, L.M.; Webb, A.K.; Limbago, B.; Dudeck, M.A.; Patel, J.; Kallen, A.J.; Edwards, J.R.; Sievert, D.M. Antimicrobial-Resistant Pathogens Associated With Healthcare-Associated Infections: Summary of Data Reported to the National Healthcare Safety Network at the Centers for Disease Control and Prevention, 2011–2014. *Infect. Control Hospital Epidemiol.* **2016**, *37*, 1288–1301. [CrossRef] [PubMed]
2. Bunge, E.M.; Hoet, B.; Chen, L.; Lienert, F.; Weidenthaler, H.; Baer, L.R.; Steffen, R. The changing epidemiology of human monkeypox—A potential threat? A systematic review. *PLoS Negl. Trop. Dis.* **2022**, *16*, e0010141. [CrossRef] [PubMed]
3. Durski, K.N.; McCollum, A.M.; Nakazawa, Y.; Petersen, B.W.; Reynolds, M.G.; Briand, S.; Djingarey, M.H.; Olson, V.; Damon, I.K.; Khalakdina, A. Emergence of monkeypox—West and central Africa, 1970–2017. *Morb. Mortal. Wkly. Rep.* **2018**, *67*, 306. [CrossRef] [PubMed]
4. Jezek, Z.; Szczeniowski, M.; Paluku, K.M.; Mutombo, M.; Grab, B. Human monkeypox: Confusion with chickenpox. *Acta Trop.* **1988**, *45*, 297–307. [PubMed]
5. Alakunle, E.; Moens, U.; Nchinda, G.; Okeke, M.I. Monkeypox virus in Nigeria: Infection biology, epidemiology, and evolution. *Viruses* **2020**, *12*, 1257. [CrossRef] [PubMed]
6. Botmart, T.; Sabir, Z.; Raja MA, Z.; Sadat, R.; Ali, M.R. Stochastic procedures to solve the nonlinear mass and heat transfer model of Williamson nanofluid past over a stretching sheet. *Ann. Nucl. Energy* **2023**, *181*, 109564. [CrossRef]
7. Latif, S.; Sabir, Z.; Raja, M.A.Z.; Altamirano, G.C.; Núñez, R.A.S.; Gago, D.O.; Sadat, R.; Ali, M.R. IoT technology enabled stochastic computing paradigm for numerical simulation of heterogeneous mosquito model. *Multimed. Tools Appl.* **2022**, 1–16. [CrossRef]
8. Sabir, Z.; Sadat, R.; Ali, M.R.; Said, S.B.; Azhar, M. A numerical performance of the novel fractional water pollution model through the Levenberg-Marquardt backpropagation method. *Arab. J. Chem.* **2022**, *16*, 104493. [CrossRef]
9. Peter, O.J.; Kumar, S.; Kumari, N.; Oguntolu, F.A.; Oshinubi, K.; Musa, R. Transmission dynamics of Monkeypox virus: A mathematical modelling approach. *Model. Earth Syst. Environ.* **2022**, *8*, 3423–3434. [CrossRef]
10. Bhunu, C.P.; Mushayabasa, S. Modelling the Transmission Dynamics of Pox-Like Infections. 2011. Available online: https://www.iaeng.org/IJAM/issues_v41/issue_2/IJAM_41_2_09.pdf (accessed on 1 January 2020).
11. Peter, O.J.; Qureshi, S.; Yusuf, A.; Al-Shomrani, M.; Idowu, A.A. A new mathematical model of COVID-19 using real data from Pakistan. *Results Phys.* **2021**, *24*, 104098. [CrossRef]
12. Bankuru, S.V.; Kossol, S.; Hou, W.; Mahmoudi, P.; Rychtář, J.; Taylor, D. A game-theoretic model of Monkeypox to assess vaccination strategies. *PeerJ* **2020**, *8*, e9272. [CrossRef] [PubMed]
13. Usman, S.; Adamu, I.I. Modeling the transmission dynamics of the monkeypox virus infection with treatment and vaccination interventions. *J. Appl. Math. Phys.* **2017**, *5*, 2335. [CrossRef]
14. Qureshi, S.; Jan, R. Modeling of measles epidemic with optimized fractional order under Caputo differential operator. *Chaos Solitons Fractals* **2021**, *145*, 110766. [CrossRef]
15. Du, M.; Wang, Z.; Hu, H. Measuring memory with the order of fractional derivative. *Sci. Rep.* **2013**, *3*, 3431. [CrossRef]
16. Rahman, G.; Nisar, K.S.; Khan, S.U.; Baleanu, D.; Vijayakumar, V. On the weighted fractional integral inequalities for Chebyshev functionals. *Adv. Differ. Equ.* **2021**, *2021*, 18. [CrossRef]
17. Baba, I.A.; Nasidi, B.A. Fractional order epidemic model for the dynamics of novel COVID-19. *Alex. Eng. J.* **2021**, *60*, 537–548. [CrossRef]
18. Yao, S.W.; Farman, M.; Amin, M.; Inc, M.; Akgül, A.; Ahmad, A. Fractional order COVID 19 model with transmission rout infected through environment. *AIMS Math.* **2022**, *7*, 5156–5174. [CrossRef]

19. Chinnathambi, R.; Rihan, F.A.; Alsakaji, H.J. A fractional-order model with time delay for tuberculosis with endogenous reactivation and exogenous reinfections. *Math. Methods Appl. Sci.* **2021**, *44*, 8011–8025. [CrossRef]
20. Aslam, M.; Murtaza, R.; Abdeljawad, T.; Khan, A.; Khan, H.; Gulzar, H. A fractional order HIV/AIDS epidemic model with Mittag-Leffler kernel. *Adv. Differ. Equ.* **2021**, *2021*, 107. [CrossRef]
21. Qu, H.; ur Rahman, M.; Ahmad, S.; Riaz, M.B.; Ibrahim, M.; Saeed, T. Investigation of fractional order bacteria dependent disease with the effects of different contact rates. *Chaos Solitons Fractals* **2022**, *159*, 112169. [CrossRef]
22. Liu, X.; Rahman, M.u.; Arfan, M.; Tchier, F.; Ahmad, S.; Inc, M.; Akinyemi, L. Fractional Mathematical Modeling to the Spread of Polio with the Role of Vaccination under Non-singular Kernel. *Fractals* **2022**, *30*, 2240144. [CrossRef]
23. Rosa, S.; Torres, D.F. Fractional Modelling and Optimal Control of COVID-19 Transmission in Portugal. *Axioms* **2022**, *11*, 170. [CrossRef]
24. Sabir, Z. Stochastic numerical investigations for nonlinear three-species food chain system. *Int. J. Biomath.* **2022**, *15*, 2250005. [CrossRef]
25. Guirao, J.L.G.; Sabir, Z.; Saeed, T. Design and numerical solutions of a novel third-order nonlinear Emden–Fowler delay differential model. *Math. Probl. Eng.* **2020**, *2020*, 7359242. [CrossRef]
26. Sabir, Z.; Guirao, J.L.G.; Saeed, T. Solving a novel designed second order nonlinear Lane–Emden delay differential model using the heuristic techniques. *Appl. Soft Comput.* **2021**, *102*, 107105. [CrossRef]
27. Yokuş, A.; Gülbahar, S. Numerical solutions with linearization techniques of the fractional Harry Dym equation. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 35–42. [CrossRef]
28. İlhan, E.; Kıymaz, İ.O. A generalization of truncated M-fractional derivative and applications to fractional differential equations. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 171–188. [CrossRef]
29. Momani, S.; Ibrahim, R.W. On a fractional integral equation of periodic functions involving Weyl–Riesz operator in Banach algebras. *J. Math. Anal. Appl.* **2008**, *339*, 1210–1219. [CrossRef]
30. Ibrahim, R.W.; Momani, S. On the existence and uniqueness of solutions of a class of fractional differential equations. *J. Math. Anal. Appl.* **2007**, *334*, 1–10.
31. Yu, F. Integrable coupling system of fractional soliton equation hierarchy. *Phys. Lett. A* **2009**, *373*, 3730–3733. [CrossRef]
32. Bonilla, B.; Rivero, M.; Trujillo, J.J. On systems of linear fractional differential equations with constant coefficients. *Appl. Math. Comput.* **2007**, *187*, 68–78. [CrossRef]
33. Diethelm, K.; Ford, N.J. Analysis of fractional differential equations. *J. Math. Anal. Appl.* **2002**, *265*, 229–248. [CrossRef]
34. Peter, O.J.; Oguntolu, F.A.; Ojo, M.M.; Oyeniyi, A.O.; Jan, R.; Khan, I. Fractional order mathematical model of monkeypox transmission dynamics. *Phys. Scr.* **2022**, *97*, 084005. [CrossRef]
35. Shah, K.; Alqudah, M.A.; Jarad, F.; Abdeljawad, T. Semi-analytical study of Pine Wilt Disease model with convex rate under Caputo–Febrizio fractional order derivative. *Chaos Solitons Fractals* **2020**, *135*, 109754. [CrossRef]
36. Yang, X.J.; Ragulskis, M.; Tana, T. A new general fractional-order derivataive with Rabotnov fractional-exponential kernel applied to model the anomalous heat transfer. *Therm. Sci.* **2019**, *23*, 1677–1681. [CrossRef]
37. Owolabi, K.M.; Hammouch, Z. Spatiotemporal patterns in the Belousov–Zhabotinskii reaction systems with Atangana–Baleanu fractional order derivative. *Phys. A Stat. Mech. Its Appl.* **2019**, *523*, 1072–1090. [CrossRef]
38. Ghanbari, B.; Djilali, S. Mathematical and numerical analysis of a three-species predator-prey model with herd behavior and time fractional-order derivative. *Math. Methods Appl. Sci.* **2020**, *43*, 1736–1752. [CrossRef]
39. Hong, Y.; Liu, Y.; Chen, Y.; Liu, Y.; Yu, L.; Liu, Y.; Cheng, H. Application of fractional-order derivative in the quantitative estimation of soil organic matter content through visible and near-infrared spectroscopy. *Geoderma* **2019**, *337*, 758–769. [CrossRef]
40. Haidong, Q.; Arfan, M.; Salimi, M.; Salahshour, S.; Ahmadian, A. Fractal–fractional dynamical system of Typhoid disease including protection from infection. *Eng. Comput.* **2021**, *34*, 1–10. [CrossRef]
41. Din, A.; Li, Y.; Khan, F.M.; Khan, Z.U.; Liu, P. On Analysis of fractional order mathematical model of Hepatitis B using Atangana–Baleanu Caputo (ABC) derivative. *Fractals* **2022**, *30*, 2240017. [CrossRef]
42. Agarwal, P.; Singh, R.; ul Rehman, A. Numerical solution of hybrid mathematical model of dengue transmission with relapse and memory via Adam–Bashforth–Moulton predictor-corrector scheme. *Chaos Solitons Fractals* **2021**, *143*, 110564. [CrossRef]
43. Abro, K.A. Numerical study and chaotic oscillations for aerodynamic model of wind turbine via fractal and fractional differential operators. *Numer. Methods Partial. Differ. Equ.* **2022**, *38*, 1180–1194. [CrossRef]
44. Jan, R.; Khan, M.A.; Kumam, P.; Thounthong, P. Modeling the transmission of dengue infection through fractional derivatives. *Chaos Solitons Fractals* **2019**, *127*, 189–216. [CrossRef]
45. Jan, R.; Khan, M.A.; Khan, Y.; Ullah, S. A new model of dengue fever in terms of fractional derivative. *Math. Biosci. Eng.* **2020**, *17*, 5267–5288.
46. Bhunu, C.P.; Garira, W.; Magombedze, G. Mathematical analysis of a two strain HIV/AIDS model with antiretroviral treatment. *Acta Biotheor.* **2009**, *57*, 361–381. [CrossRef]