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Fixed Point for Almost Contractions in v-Generalized b-Metric Spaces

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Abstract: We present a significant example to show that the class of v-generalized b-metric spaces properly contains the class of v-generalized metric spaces as well as b-metric spaces. This is accomplished because the example provided by Došenović et al. (2020) is insufficient to expose the generality of v-generalized b-metric spaces over the existing related spaces. Therefore, we establish fixed point theorems by defining generalized almost contractions of rational type and Reich type in v-generalized b-metric spaces. Moreover, we compare the proven results with the already existing fixed point theorems in this space by presenting suitable examples. As a consequence of these fixed point theorems, we further develop some common fixed-point results that ensure the existence and uniqueness of coincidence points and common fixed points for a pair of self maps. Finally, we use the outcome to check that the given Fredholm integral equation has a solution and that it is also unique.

Keywords: b-metric spaces; generalized metric spaces; v-generalized b-metric spaces

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1. Introduction

There are many generalizations of metric spaces in the literature. Among these, one significant generalization is b-metric spaces, which was introduced by Bakhtin [1] in 1989 (see also Czerwik [2]), who developed the Banach contraction principle in this newly defined extension of metric spaces. In the notion of b-metric spaces, the authors replaced the triangle inequality with s-type inequality as stated below.

Definition 1. *"Let Δ be a nonempty set and $\rho : \Delta \times \Delta \to [0,+\infty)$ be a mapping. Then the map ρ is said to be a b-metric on Δ, if for all $\zeta, \xi, \omega \in \Delta$, the following axioms are satisfied*

1. $\rho(\zeta,\xi) = 0$ *if and only if* $\zeta = \xi$,
2. $\rho(\zeta,\xi) = \rho(\xi,\zeta)$,
3. $\rho(\zeta,\xi) \leq s[\rho(\zeta,\omega) + \rho(\omega,\xi)]$ *for some real number* $s \geq 1$.

The space Δ endowed with metric ρ is called a b-metric space and it is denoted as (Δ,ρ)".

In 2000, Branciari [3] announced another useful generalization of metric spaces, namely, v-generalized metric spaces, defined as follows:

Definition 2. *"Let Δ be a nonempty set and $\rho_v : \Delta \times \Delta \to [0,+\infty)$ be a mapping such that for all $\zeta, \xi \in \Delta$, the following conditions are satisfied:*

1. $\rho_v(\zeta,\xi) = 0$ *if and only if* $\zeta = \xi$,
2. $\rho_v(\zeta,\xi) = \rho_v(\xi,\zeta)$,
3. $\rho_v(\zeta,\xi) \leq \rho_v(\zeta,\omega_1) + \rho_v(\omega_1,\omega_2) + ... + \rho_v(\omega_{v-1},\omega_v) + \rho(\omega_v,\xi)$, *for all distinct points* $\zeta, \xi, \omega_1, \omega_2, ..., \omega_{v-1}, \omega_v$ *in* Δ, *where* $v \in \mathbb{N}$ *(the set of natural numbers).*

Then the map ρ_v is called a v-generalized metric and the space Δ equipped with such metric is called a v-generalized metric space which is denoted by the pair (Δ, ρ_v)".

The analogue of the Banach fixed point theorem was proved by Branciari [3] in the context of v-generalized metric spaces. It is interesting to note that both spaces, viz., b-metric spaces and v-generalized metric spaces, played a key role in the further development of fixed point theory, as they enlarged the domain of the mapping for which the fixed point is being investigated. The domain is enlarged in the sense that every metric space is a b-metric space and a v-generalized metric space, but the converse is not necessarily true (see [4] and Example 39 in [5]). Further, the metric function defined in the case of b-metric spaces and v-generalized metric spaces is not continuous in general. For more details about these spaces, one can refer to [6–18] and the references therein. In 2017, Mitrović and Radenović [19] introduced a new type of space, namely, the v-generalized b-metric space, by unifying both aforementioned spaces and establishing fixed point theorems for Banach and Reich contractions in this newly defined space. Recently, many authors have studied different properties of this space and proved several fixed point theorems for this (refer to [19–22]). In the sequel, the aim of this paper is to define the notions of generalized almost contraction of rational type and generalized almost contraction of Reich type in the context of v-generalized b-metric spaces. Then, we ensure the existence and uniqueness of fixed points for these contractions and expose the generality of the proved results over the existing ones.

2. v-Generalized b-Metric Spaces

The following basic details about the v-generalized b-metric spaces are required in the sequel.

Definition 3. *[19]: "Let Δ be a nonempty set and let $v \in \mathbb{N}$. Assume $\rho_v : \Delta \times \Delta \to [0,+\infty)$, then the pair (Δ, ρ_v) is called v-generalized b-metric space if for all $\zeta, \xi \in \Delta$, the following hold:*

1. $\rho_v(\zeta, \xi) = 0$ *if and only if* $\zeta = \xi$,
2. $\rho_v(\zeta, \xi) = \rho_v(\xi, \zeta)$,
3. $\rho_v(\zeta, \xi) \leq s(\rho_v(\zeta, \omega_1) + \rho_v(\omega_1, \omega_2) + ... + \rho_v(\omega_{v-1}, \omega_v) + \rho_v(\omega_v, \xi))$ *for some real number* $s \geq 1$ *and* $\forall$ $\omega_1, \omega_2, ..., \omega_{v-1}, \omega_v \in \Delta$ *such that* $\zeta, \omega_1, \omega_2, ..., \omega_{v-1}, \omega_v, \xi$ *are all distinct."*

It is clear from the definition that for $v = 1$, the v-generalized b-metric spaces reduce to b-metric spaces, and for $s = 1$, they reduce to v-generalized metric spaces. Therefore, the class of v-generalized b-metric spaces contains the class of b-metric spaces and v-generalized metric spaces; in addition, it is pertinent to mention that there are some v-generalized b-metric spaces that are neither b-metric spaces nor v-generalized metric spaces. This is shown by the following example.

Example 1. *Let Δ be a subset of $\mathbb{R}^m (m \in \mathbb{N})$ such that*

$$\Delta = \{(0,0,0,\cdots,0)\} \cup \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m,$$

$$\begin{aligned} \text{where } \Omega_1 &= \{(\zeta_1,0,0,\cdots,0);\ 0<\zeta_1\leq 1\},\\ \Omega_2 &= \{(\zeta_1,\zeta_2,0,\cdots,0);\ 0<\zeta_1,\zeta_2\leq 1\},\\ \Omega_3 &= \{(\zeta_1,\zeta_2,\zeta_3,\cdots,0);\ 0<\zeta_1,\zeta_2,\zeta_3\leq 1\},\\ &\ \vdots\\ \Omega_m &= \{(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m);\ 0<\zeta_1,\zeta_2,\cdots,\zeta_m\leq 1\}. \end{aligned}$$

In addition, map $\rho_v : \Delta \times \Delta \to [0,+\infty)$ is defined as $\rho_v(\zeta,\zeta) = 0$, $\rho_v(\zeta,\xi) = \rho_v(\xi,\zeta)$ for all $\zeta, \xi \in \Delta$ and

$$\begin{aligned}
\rho_v((0,0,0,\cdots,0),(\zeta_1,0,0,\cdots,0)) &= \zeta_1;\ 0<\zeta_1<1,\ \zeta_1 \neq \frac{1}{n},\ n \in \mathbb{N},\\
\rho_v((0,0,0,\cdots,0),(\zeta_1,0,0,\cdots,0)) &= \frac{1}{2n};\ \zeta_1 = \frac{1}{n},\ n \in \mathbb{N},\\
\rho_v\left(\left(\frac{2}{3},0,0,\cdots,0\right),(\zeta_1,0,0,\cdots,0)\right) &= \frac{1}{2n};\ \zeta_1 = \frac{1}{n},\ n \in \mathbb{N},\\
\rho_v((\zeta_1,0,0,\cdots,0),(\zeta_1,\zeta_2,0,\cdots,0)) &= \zeta_2;\ 0<\zeta_1,\zeta_2 \leq 1,\\
\rho_v((\zeta_1,\zeta_2,0,\cdots,0),(\zeta_1,\zeta_2,\zeta_3,\cdots,0)) &= \zeta_3;\ 0<\zeta_1,\zeta_2,\zeta_3 \leq 1,\\
&\ \vdots\\
\rho_v((\zeta_1,\zeta_2,\zeta_3,\cdots,0),(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m)) &= \zeta_m;\ 0<\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m \leq 1,\\
\rho_v\left(\left(\frac{2}{3},0,0,\cdots,0\right),\left(\frac{3}{4},\zeta_2,\zeta_3,\cdots,\zeta_m\right)\right) &= 1;\ 0<\zeta_2,\zeta_3,\cdots,\zeta_m \leq 1,\\
\rho_v((1,1,1,\cdots,1),(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m)) &= 2;\ 0<\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m < 1,\\
\rho_v(\zeta,\xi) &= 5;\ otherwise.
\end{aligned}$$

Now, we observe that (Δ, ρ_v) is not a v-generalized b-metric space for any $v \in \{1,2,\cdots,m,m+1,\cdots,2m,2m+1\}$ and $s \geq 1$. In point of fact, consider the following $2m+3$ points of Δ:

$$\begin{aligned}
\Lambda_1 &= (\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_m), & \Lambda_2 &= (\zeta_1,\zeta_2,\zeta_3,\cdots,0),\cdots,\\
\Lambda_m &= (\zeta_1,0,0,\cdots,0), & \Lambda_{m+1} &= (0,0,0,\cdots,0),\\
\Lambda_{m+2} &= \left(\frac{1}{n_1},0,0,\cdots,0\right), & \Lambda_{m+3} &= \left(\frac{2}{3},0,0,\cdots,0\right),\\
\Lambda_{m+4} &= \left(\frac{1}{n_2},0,0,\cdots,0\right), & \Lambda_{m+5} &= \left(\frac{1}{n_2},\xi_2,0,\cdots,0\right),\\
\Lambda_{m+6} &= \left(\frac{1}{n_2},\xi_2,\xi_3,\cdots,0\right),\cdots, & \Lambda_{2m+3} &= \left(\frac{1}{n_2},\xi_2,\xi_3,\cdots,\xi_m\right),
\end{aligned}$$

where $0 < \zeta_1,\zeta_2,\cdots,\zeta_m,\xi_2,\cdots,\xi_m < 1$, $\zeta_1 \neq \frac{2}{3}$, $\zeta_1 \neq \frac{1}{n}$; $n \in \mathbb{N}$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \neq n_2$. Then, we have $\rho_v(\Lambda_1,\Lambda_{2m+3}) = 5$ and $\rho_v(\Lambda_1,\Lambda_2) + \rho_v(\Lambda_2,\Lambda_3) + \cdots + \rho_v(\Lambda_m,\Lambda_{m+1})$ $+\rho_v(\Lambda_{m+1},\Lambda_{m+2}) + \rho_v(\Lambda_{m+2},\Lambda_{m+3}) + \rho_v(\Lambda_{m+3},\Lambda_{m+4}) + \rho_v(\Lambda_{m+4},\Lambda_{m+5}) +$ $\rho_v(\Lambda_{m+5},\Lambda_{m+6}) + \cdots + \rho_v(\Lambda_{2m+2},\Lambda_{2m+3}) = \zeta_m + \zeta_{m-1} + \cdots + \zeta_1 + \frac{1}{2n_1} + \frac{1}{2n_1} + \frac{1}{2n_2} +$ $\xi_2 + \xi_3 + \cdots + \xi_m$.

If (Δ, ρ_v) is a $(2m+1)$-generalized b-metric space with some coefficient $s \geq 1$, then we must have

$$5 \leq s\left(\zeta_m + \zeta_{m-1} + \cdots + \zeta_1 + \frac{1}{2n_1} + \frac{1}{2n_1} + \frac{1}{2n_2} + \xi_2 + \xi_3 + \cdots + \xi_m\right),$$

which is not possible for sufficiently small values of $\zeta_m, \zeta_{m-1}, \cdots, \zeta_1, \frac{1}{2n_1}, \frac{1}{2n_1}, \frac{1}{2n_2}, \xi_2, \xi_3, \cdots, \xi_m$; hence, (Δ, ρ_v) is not a $(2m+1)$-generalized b-metric space.

However, (Δ, ρ_v) is a $(2m+2)$-generalized b-metric space for any $s \geq 5$. For this , consider $2m+4$ distinct points $P_1, P_2, \cdots, P_m, P_{m+1}, \cdots, P_{2m+3}, P_{2m+4}$. Clearly, by the definition of ρ_v, one of the distances among $\rho_v(P_1,P_2), \rho_v(P_2,P_3), \cdots, \rho_v(P_m,P_{m+1}), \cdots, \rho_v(P_{2m+3},P_{2m+4})$ must be greater than or equal to 1; thus, for any $s \geq 5$,

$$\begin{aligned}
\rho_v(P_1,P_{2m+4}) \leq 5 &\leq s(\rho_v(P_1,P_2) + \rho_v(P_2,P_3) + \cdots + \rho_v(P_m,P_{m+1})\\
&\quad + \cdots + \rho_v(P_{2m+3},P_{2m+4})).
\end{aligned}$$

Remark 1. *In Example 1 in Došenović et al. [20], the authors claim that (Δ, d) is a v-generalized b-metric for $v = 5$ is not correct. If we take the following seven points in Δ:*

$$\begin{aligned} \Lambda_1 &= (\zeta_1, \zeta_2, \zeta_3), & \Lambda_2 &= (\zeta_1, \zeta_2, 0), \\ \Lambda_3 &= (\zeta_1, 0, 0), & \Lambda_4 &= (0, 0, 0), \\ \Lambda_5 &= (\xi_1, 0, 0), & \Lambda_6 &= (\xi_1, \xi_2, 0), \\ \Lambda_7 &= (\xi_1, \xi_2, \xi_3), \end{aligned}$$

where $0 < \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3 \leq 1$, then there is no $s \geq 1$ such that the following inequality is true for sufficiently small values of $\zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3$:

$$\begin{aligned} d(\Lambda_1, \Lambda_7) = 4 \quad &\leq \quad s(d(\Lambda_1, \Lambda_2) + d(\Lambda_2, \Lambda_3) + d(\Lambda_3, \Lambda_4) + d(\Lambda_4, \Lambda_5) + \\ &\qquad d(\Lambda_5, \Lambda_6) + d(\Lambda_6, \Lambda_7)) \\ &= \quad s(\zeta_3, + \zeta_2 + \zeta_1 + \xi_1 + \xi_2 + \xi_3). \end{aligned}$$

In fact, this is a v-generalized metric space for $v = 6$; hence, this is not a suitable example to show that the class of v-generalized b-metric spaces properly contains the class of v-generalized metric spaces as well as the class of b-metric spaces.

Definition 4. *A sequence $\{x_n\}$ in a v-generalized b-metric space (Δ, ρ_v) is said to be the following:*

1. *A convergent sequence that converges to a point x in Δ if for given $\epsilon > 0$, there exists a positive integer N such that $\rho_v(x_n, x) < \epsilon$ for all $n > N$. It can be written as $x_n \to x$ whenever $n \to +\infty$.*
2. *A Cauchy sequence if for given $\epsilon > 0$, there exists a positive integer N such that $\rho_v(x_n, x_p) < \epsilon$ for all $n, p > N$; it is denoted as $\rho_v(x_n, x_p) \to 0$ whenever $n, p \to +\infty$.*

Space (Δ, ρ_v) is said to be complete if every Cauchy sequence in Δ converges to a point in Δ. It is very interesting to note that in v-generalized b-metric spaces, the convergent sequence is not necessarily Cauchy, and the sequence may converge to two or more distinct points. This is seen in the following example.

Example 2. *If pair (Δ, ρ_v) is as defined in Example 1, then sequence $\left\langle \left(\frac{1}{n}, 0, 0, \cdots, 0\right) \right\rangle_{n \in \mathbb{N}}$ in Δ converges to points $(0, 0, 0, \cdots, 0)$ and $\left(\frac{2}{3}, 0, 0, \cdots, 0\right)$ of Δ; thus, the limit of the convergent sequence is not unique.*
Moreover, for $n, p \in \mathbb{N}$, $\rho_v\left(\left(\frac{1}{n}, 0, 0, \cdots, 0\right), \left(\frac{1}{p}, 0, 0, \cdots, 0\right)\right)$ converges to 5 as $n \to +\infty$. Hence, $\left\langle \left(\frac{1}{n}, 0, 0, \cdots, 0\right) \right\rangle_{n \in \mathbb{N}}$ is not a Cauchy sequence in a $(2m+2)$-generalized b-metric space.

Now, to overcome the situation of the non-uniqueness of the limit of a convergent sequence, we prove the following lemma with some special assumptions that help to obtain our main results.

Lemma 1. *Let (Δ, ρ_v) be a v-generalized b-metric space and $\{\zeta_n\}$ be a Cauchy sequence in Δ such that $\zeta_m \neq \zeta_n$ for all distinct $m, n \in \mathbb{N}$. Then, $\{\zeta_n\}$ can converge to at most one point.*

Proof. On the contrary, suppose that $\lim_{n \to +\infty} \zeta_n = \zeta$ and $\lim_{n \to +\infty} \zeta_n = \xi$ such that $\zeta \neq \xi$. As $\zeta_m \neq \zeta_n$ and $\zeta \neq \xi$, then there exists $l \in \mathbb{N}$ such that all the terms of sequence ζ_n for $n \geq l$ are different from ζ and ξ. If such l does not exist, then $\zeta = \xi$.

In addition, as $\{\zeta_n\}$ is a Cauchy sequence, for given $\epsilon > 0$, there exists $p_1 \in \mathbb{N}$ such that

$$\rho_v(\zeta_n, \zeta_p) < \frac{\epsilon}{s(v+1)} \ \ \forall \, n, p \geq p_1.$$

Since $\zeta_n \to \zeta$ and $\zeta_n \to \xi$, there exist $p_2, p_3 \in \mathbb{N}$ such that

$$\rho_v(\zeta_n, \zeta) < \frac{\epsilon}{s(v+1)} \quad \forall\, n \geq p_2,$$

and

$$\rho_v(\zeta_n, \xi) < \frac{\epsilon}{s(v+1)} \quad \forall\, n \geq p_3.$$

Choosing $r = \max\{l, p_1, p_2, p_3\}$, then for all $n \geq r$ and $v \geq 2$,

$$\begin{aligned} \rho_v(\zeta, \xi) &\leq s[\rho_v(\zeta, \zeta_{n+v-2}) + \rho_v(\zeta_{n+v-2}, \zeta_{n+v-3}) + \cdots \\ &\quad + \rho_v(\zeta_{n+1}, \zeta_n) + \rho_v(\zeta_n, \zeta_r) + \rho_v(\zeta_r, \xi)] \\ &< s\left[\frac{\epsilon}{s(v+1)} + \frac{\epsilon}{s(v+1)} + \cdots + \frac{\epsilon}{s(v+1)}\right] = \epsilon, \end{aligned}$$

and for $v = 1$,

$$\begin{aligned} \rho_v(\zeta, \xi) &\leq s[\rho_v(\zeta, \zeta_m) + \rho_v(\zeta_m, \xi)] \\ &< s\left[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\right] = \epsilon. \end{aligned}$$

For all the values of v, as ϵ is arbitrary, we obtain

$$\rho_v(\zeta, \xi) = 0 \implies \zeta = \xi,$$

which is a contradiction. $\square$

3. Almost Contractions in v-Generalized b-Metric Spaces

In 2004, the notion of weak contraction was initiated by Berinde [23] in the framework of metric spaces. Thereafter, to ensure the uniqueness of fixed point, a slightly stronger contractive condition than a weaker contraction was introduced by Babu et al. [24] in 2008. Berinde [25] retitled the term weak contraction as almost contraction. Now, we start our work by defining a version of almost contraction in a v-generalized b-metric space, which is as follows.

Definition 5. *Let (Δ, ρ_v) be a v-generalized b-metric space. A map $\Gamma : \Delta \to \Delta$ is said to be a generalized almost contraction of rational type if there exists a constant $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) &\leq \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\ &\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}, \end{aligned} \tag{1}$$

for all $\zeta, \xi \in \Delta$.

It is clear from the definition that the Banach contraction in a v-generalized b-metric space is a generalized almost contraction of rational type, but the converse assertion is not true in general. This is shown by the following example.

Example 3. *In Example 1, if we take $m = 2$, then the set $\Delta \subseteq \mathbb{R}^2$ is given as $\Delta = \{(0,0)\} \cup \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{(\zeta_1, 0);\ 0 < \zeta_1 \leq 1\}$ and $\Omega_2 = \{(\zeta_1, \zeta_2);\ 0 < \zeta_1, \zeta_2 \leq 1\}$. In addition, we define $\rho_v : \Delta \times \Delta \to [0, +\infty)$ in such a way that $\rho_v(\zeta, \zeta) = 0$, $\rho_v(\zeta, \xi) = \rho_v(\xi, \zeta)$ for all $\zeta, \xi \in \Delta$ and*

$$\rho_v((0,0), (\zeta_1, 0)) = \zeta_1;\ 0 < \zeta_1 < 1,\ \zeta_1 \neq \frac{1}{n},\ n \in \mathbb{N},$$

$$\begin{aligned}
\rho_v((0,0),(\zeta_1,0)) &= \frac{1}{2n};\ \zeta_1=\frac{1}{n},\ n\in\mathbb{N},\\
\rho_v\left(\left(\frac{2}{3},0\right),(\zeta_1,0)\right) &= \frac{1}{2n};\ \zeta_1=\frac{1}{n},\ n\in\mathbb{N},\\
\rho_v((\zeta_1,0),(\zeta_1,\zeta_2)) &= \zeta_2;\ 0<\zeta_1,\zeta_2\le 1,\\
\rho_v\left(\left(\frac{2}{3},0\right),\left(\frac{3}{4},\zeta_2\right)\right) &= 1;\ 0<\zeta_2\le 1,\\
\rho_v((1,1),(\zeta_1,\zeta_2)) &= 2;\ 0<\zeta_1,\zeta_2<1,\\
\rho_v(\zeta,\xi) &= 5,\ otherwise.
\end{aligned}$$

Then, it is clear from Example 1 that (Δ,ρ_v) is a 6-generalized b-metric space for $s\ge 5$. Now, we define a map $\Gamma:\Delta\to\Delta$ as

$$\Gamma\zeta=\begin{cases}(1,1), & \zeta=(0,0)\\ \left(\frac{3}{4},\frac{1}{2}\right), & \zeta\in\Omega_1\cup\Omega_2.\end{cases}$$

Then, we discuss the following possibilities:

Case 1. *If $\zeta=(0,0)$ and $\xi\in\Omega_1$, then*

$$\rho_v(\Gamma\zeta,\Gamma\xi)=2$$

and

$$\begin{aligned}
\rho_v(\zeta,\xi) &\in (0,1)\\
\rho_v(\zeta,\Gamma\zeta) &= 5\\
\rho_v(\xi,\Gamma\xi) &\in \left\{\frac{1}{2},1,5\right\}\\
\rho_v(\zeta,\Gamma\xi) &= 5\\
\rho_v(\xi,\Gamma\zeta) &\in \{1,5\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_v(\Gamma\zeta,\Gamma\xi) \le\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\
&+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}
\end{aligned}$$

for $\delta\ge\frac{4}{5}\in[0,1)$ and $L\ge 1$.

Case 2. *If $\zeta\in\Omega_1$ and $\xi=(0,0)$, then as in Case 1, by symmetry, we have*

$$\begin{aligned}
\rho_v(\Gamma\zeta,\Gamma\xi) \le\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\
&+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}
\end{aligned}$$

for $\delta\ge\frac{4}{5}\in[0,1)$ and $L\ge 1$.

Case 3. *If $\zeta=(0,0)$ and $\xi\in\Omega_2$, then*

$$\rho_v(\Gamma\zeta,\Gamma\xi)=2$$

and

$$\rho_v(\zeta,\xi) = 5$$

$$\begin{aligned}\rho_v(\zeta,\Gamma\zeta) &= 5\\ \rho_v(\xi,\Gamma\xi) &\in \{0,5\}\\ \rho_v(\zeta,\Gamma\xi) &= 5\\ \rho_v(\xi,\Gamma\zeta) &\in \{0,2\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) \leq\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\ &+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}\end{aligned}$$

for $\delta \geq \frac{2}{5}$ *and* $L \geq 0$.

Case 4. *If* $\zeta \in \Omega_2$ *and* $\xi = (0,0)$, *then as in Case 3, by symmetry, we have*

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) \leq\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\ &+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}\end{aligned}$$

for $\delta \geq \frac{2}{5}$ *and* $L \geq 0$.

Case 5. *If* $\zeta,\xi \in \Omega_1 \cup \Omega_2$, *then we have*

$$\rho_v(\Gamma\zeta,\Gamma\xi) = 0.$$

Therefore,

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) \leq\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\ &+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}\end{aligned}$$

for $\delta \in [0,1)$ *and* $L \geq 0$.

From Cases 1–5, we obtain

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) \leq\ & \delta\max\left\{\rho_v(\zeta,\xi),\frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\\ &+L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_(\xi,\Gamma\zeta)\}\end{aligned}$$

for $\delta \geq \frac{4}{5} \in [0,1)$ *and* $L \geq 1$.

However, Γ *does not satisfy the ordinary Banach contraction—for instance, take* $\zeta = (0,0)$, $\xi = \left(\frac{1}{2},0\right)$, *and* $\rho_v(\Gamma\zeta,\Gamma\xi) = 2 > \frac{1}{4} = \rho_v(\zeta,\xi)$.

Theorem 1. *Let* (Δ,ρ_v) *be a complete v-generalized b-metric space and* $\Gamma : \Delta \to \Delta$ *be a generalized almost contraction of rational type. Then,* Γ *has a unique fixed point.*

Proof. Let $\zeta_0 \in \Delta$ be arbitrary. Define sequence $\{\zeta_n\}$ with $\zeta_{n+1} = \Gamma\zeta_n$ for all $n \geq 0$. Then, by (1),

$$\begin{aligned}\rho_v(\zeta_{n+1},\zeta_n) \leq\ & \delta\max\left\{\rho_v(\zeta_n,\zeta_{n-1}),\frac{\rho_v(\zeta_n,\zeta_{n+1})\cdot\rho_v(\zeta_{n-1},\zeta_n)}{1+\rho_v(\zeta_n,\zeta_{n-1})}\right\}\\ &+L\min\{\rho_v(\zeta_n,\zeta_{n+1})+\rho_v(\zeta_{n-1},\zeta_n),\rho_v(\zeta_n,\zeta_n),\rho_v(\zeta_{n-1},\zeta_{n+1})\}\\ \leq\ & \delta\max\{\rho_v(\zeta_n,\zeta_{n-1}),\rho_v(\zeta_n,\zeta_{n+1})\}. \end{aligned} \tag{2}$$

If $\max\{\rho_v(\zeta_n,\zeta_{n-1}),\rho_v(\zeta_{n+1},\zeta_n)\}=\rho_v(\zeta_{n+1},\zeta_n)$, then (2) becomes

$$\rho_v(\zeta_{n+1},\zeta_n)\le\delta\rho_v(\zeta_{n+1},\zeta_n),$$

which is a contradiction, as $\delta\in[0,1)$; thus,

$$\rho_v(\zeta_{n+1},\zeta_n)\le\delta\rho_v(\zeta_n,\zeta_{n-1}).$$

Therefore, we have

$$\rho_v(\zeta_{n+1},\zeta_n)\le\delta^n\rho_v(\zeta_1,\zeta_0). \tag{3}$$

Suppose $u_n=u_m$ for some $n\neq m$. Without loss of generality, we assume that $n>m$. Then, there exists $k\geq 1$ such that $n=m+k$ and so $\zeta_m=\zeta_{m+k}$. Thus, we have $\Gamma\zeta_m=\Gamma\zeta_{m+k}$, which implies $\zeta_{m+1}=\zeta_{m+k+1}$. It follows that

$$\rho_v(\zeta_{m+1},\zeta_m)=\rho_v(\zeta_{m+k+1},\zeta_{m+k})\le\delta^k\rho_v(\zeta_{m+1},\zeta_m)<\rho_v(\zeta_{m+1},\zeta_m),$$

which is a contradiction. Thus, we obtain $\zeta_n\neq\zeta_m$ for all distinct $n,m\in\mathbb{N}$.

Now, using conditions (1) and (3), we have

$$\begin{aligned}
\rho_v(\zeta_m,\zeta_n) &\le \delta\max\left\{\rho_v(\zeta_{m-1},\zeta_{n-1}),\frac{\rho_v(\zeta_{m-1},\zeta_m)\cdot\rho_v(\zeta_{n-1},\zeta_n)}{1+\rho_v(\zeta_{m-1},\zeta_{n-1})}\right\}\\
&\quad +L\min\{\rho_v(\zeta_{m-1},\zeta_m)+\rho_v(\zeta_{n-1},\zeta_n),\rho_v(\zeta_m,\zeta_{n+1}),\rho_v(\zeta_n,\zeta_{m+1})\}\\
&\le \delta\rho_v(\zeta_{m-1},\zeta_{n-1})+\delta\frac{\rho_v(\zeta_{m-1},\zeta_m)\cdot\rho_v(\zeta_{n-1},\zeta_n)}{1+\rho_v(\zeta_{m-1},\zeta_{n-1})}\\
&\quad +L\{\rho_v(\zeta_{m-1},\zeta_m)+\rho_v(\zeta_{n-1},\zeta_n)\}\\
&\le \delta\rho_v(\zeta_{m-1},\zeta_{n-1})+\delta\rho_v(\zeta_{m-1},\zeta_m)\cdot\rho_v(\zeta_{n-1},\zeta_n)\\
&\quad +L\{\delta^{m-1}\rho_v(\zeta_1,\zeta_0)+\delta^{n-1}\rho_v(\zeta_1,\zeta_0)\}\\
&\le \delta\rho_v(\zeta_{m-1},\zeta_{n-1})+\delta\cdot\delta^{m-1}\rho_v(\zeta_1,\zeta_0)\cdot\delta^{n-1}\rho_v(\zeta_1,\zeta_0)\\
&\quad +L\{\delta^{m-1}\rho_v(\zeta_1,\zeta_0)+\delta^{n-1}\rho_v(\zeta_1,\zeta_0)\}\\
&= \delta\rho_v(\zeta_{m-1},\zeta_{n-1})+\delta^{m+n-1}(\rho_v(\zeta_1,\zeta_0))^2+L(\delta^{m-1}+\delta^{n-1})\rho_v(\zeta_1,\zeta_0)
\end{aligned}$$

and

$$\begin{aligned}
\rho_v(\zeta_{m+1},\zeta_{n+1}) &\le \delta\rho_v(\zeta_m,\zeta_n)+1\cdot\delta^{m+n+1}(\rho_v(\zeta_1,\zeta_0))^2+1\cdot\delta^{1-1}L(\delta^m+\delta^n)\rho_v(\zeta_1,\zeta_0)\\
\rho_v(\zeta_{m+2},\zeta_{n+2}) &\le \delta\rho_v(\zeta_{m+1},\zeta_{n+1})+\delta^{m+n+3}(\rho_v(\zeta_1,\zeta_0))^2+L(\delta^{m+1}+\delta^{n+1})\rho_v(\zeta_1,\zeta_0)\\
&\le \delta[\delta\rho_v(\zeta_m,\zeta_n)+\delta^{m+n+1}(\rho_v(\zeta_1,\zeta_0))^2+L(\delta^m+\delta^n)\rho_v(\zeta_1,\zeta_0)]\\
&\quad +\delta^{m+n+3}(\rho_v(\zeta_1,\zeta_0))^2+L(\delta^{m+1}+\delta^{n+1})\rho_v(\zeta_1,\zeta_0)\\
&\le \delta^2\rho_v(\zeta_m,\zeta_n)+2\cdot\delta^{m+n+2}(\rho_v(\zeta_1,\zeta_0))^2+2\cdot\delta^{2-1}L(\delta^m+\delta^n)\rho_v(\zeta_1,\zeta_0)\\
&\;\;\vdots\\
\rho_v(\zeta_{m+k},\zeta_{n+k}) &\le \delta^k\rho_v(\zeta_m,\zeta_n)+k\delta^{m+n+k}(\rho_v(\zeta_1,\zeta_0))^2+k\delta^{k-1}L(\delta^m+\delta^n)\rho_v(\zeta_1,\zeta_0).
\end{aligned}\tag{4}$$

Since $\rho_v(\zeta_0,\Gamma\zeta_0)=\rho_v(\zeta_0,\zeta_1)<+\infty$, then, by applying $n\to+\infty$ in (3), we obtain

$$\lim_{n\to+\infty}\rho_v(\zeta_n,\zeta_{n+1})=0.$$

Now, we prove that $\{\zeta_n\}$ is a Cauchy sequence in Δ. Meanwhile, if $\{\zeta_n\}$ is a non-Cauchy sequence on the contrary, a positive ϵ and subsequences $\{\zeta_{m_\lambda}\}$ and $\{\zeta_{n_\lambda}\}$ of $\{\zeta_n\}$ can be obtained, such that m_λ is the smallest cardinal, with $m_\lambda>n_\lambda>\lambda$,

$$\rho_v(\zeta_{m_\lambda},\zeta_{n_\lambda})\ge\epsilon$$

and

$$\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda}) < \epsilon.$$

Then, it follows that

$$\begin{aligned}\epsilon \leq \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda}) &\leq s[\rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) + \rho_v(\zeta_{n_\lambda+v}, \zeta_{n_\lambda+v-1}) + \cdots \\ &\quad +\rho_v(\zeta_{n_\lambda+2}, \zeta_{n_\lambda+1}) + \rho_v(\zeta_{n_\lambda+1}, \zeta_{n_\lambda})],\end{aligned}$$

which gives

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}).$$

If $\delta = 0$, the proof is trivial. So, let $\delta \in (0,1)$. Since $\lim_{n\to+\infty} \delta^n = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$0 < \delta^{n_0} \cdot \mu < 1,\ \mu = s^3. \tag{5}$$

Now, consider the following two cases:

Case 6. *When $v \geq 2$, by condition (3), we obtain*

$$\begin{aligned}\rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) &\leq \delta\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda+v-1}) + \delta^{m_\lambda+n_\lambda+v-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +L\{\delta^{m_\lambda-1} + \delta^{n_\lambda+v-1}\}\rho_v(\zeta_1, \zeta_0) \\ &\leq \delta s[\rho_v(\zeta_{m_\lambda-1}, \zeta_{m_\lambda+n_0-1}) + \rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \\ &\quad +\rho_v(\zeta_{n_\lambda+n_0}, \zeta_{n_\lambda+2v-3}) + \rho_v(\zeta_{n_\lambda+2v-3}, \zeta_{n_\lambda+2v-4}) + \cdots \\ &\quad +\rho_v(\zeta_{n_\lambda+v+1}, \zeta_{n_\lambda+v}) + \rho_v(\zeta_{n_\lambda+v}, \zeta_{n_\lambda+v-1})] + \delta^{m_\lambda+n_\lambda+v-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +L\{\delta^{m_\lambda-1} + \delta^{n_\lambda+v-1}\}\rho_v(\zeta_1, \zeta_0) \\ &\leq \delta s[\{\delta^{m_\lambda-1}\rho_v(\zeta_{n_0}, \zeta_0) + (m_\lambda-1)\delta^{m_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +(m_\lambda-1)\delta^{m_\lambda-2}L\{\delta^{n_0}+1\}\rho_v(\zeta_1, \zeta_0)\} + \{\delta^{n_0}\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda}) \\ &\quad +n_0\delta^{m_\lambda+n_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 + n_0\delta^{n_0-1}L\{\delta^{m_\lambda-1} + \delta^{n_\lambda}\}\rho_v(\zeta_1, \zeta_0)\} \\ &\quad +\{\delta^{n_\lambda}\rho_v(\zeta_{n_0}, \zeta_{2v-3}) + n_\lambda\delta^{n_\lambda+n_0+2v-3}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +n_\lambda\delta^{n_\lambda-1}L\{\delta^{n_0} + \delta^{2v-3}\}\rho_v(\zeta_1, \zeta_0)\} + \delta^{n_\lambda+2v-4}\rho_v(\zeta_1, \zeta_0) + \cdots \\ &\quad +\delta^{n_\lambda+v}\rho_v(\zeta_1, \zeta_0) + \delta^{n_\lambda+v-1}\rho_v(\zeta_1, \zeta_0)] + \delta^{m_\lambda+n_\lambda+v-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +L\{\delta^{m_\lambda-1} + \delta^{n_\lambda+v-1}\}\rho_v(\zeta_1, \zeta_0) \\ &< \delta s[\{\delta^{m_\lambda-1}\rho_v(\zeta_{n_0}, \zeta_0) + (m_\lambda-1)\delta^{m_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +(m_\lambda-1)\delta^{m_\lambda-2}L\{\delta^{n_0}+1\}\rho_v(\zeta_1, \zeta_0)\} + \{\delta^{n_0}\epsilon + n_0\delta^{m_\lambda+n_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +n_0\delta^{n_0-1}L\{\delta^{m_\lambda-1} + \delta^{n_\lambda}\}\rho_v(\zeta_1, \zeta_0)\} \\ &\quad +\{\delta^{n_\lambda}\rho_v(\zeta_{n_0}, \zeta_{2v-3}) + n_\lambda\delta^{n_\lambda+n_0+2v-3}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +n_\lambda\delta^{n_\lambda-1}L\{\delta^{n_0} + \delta^{2v-3}\}\rho_v(\zeta_1, \zeta_0)\} + \delta^{n_\lambda+2v-4}\rho_v(\zeta_1, \zeta_0) + \cdots \\ &\quad +\delta^{n_\lambda+v}\rho_v(\zeta_1, \zeta_0) + \delta^{n_\lambda+v-1}\rho_v(\zeta_1, \zeta_0)] + \delta^{m_\lambda+n_\lambda+v-1}(\rho_v(\zeta_1, \zeta_0))^2 \\ &\quad +L\{\delta^{m_\lambda-1} + \delta^{n_\lambda+v-1}\}\rho_v(\zeta_1, \zeta_0).\end{aligned}$$

By taking the upper limit as $\lambda \to +\infty$ on both sides of the above inequality, we obtain

$$\limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) \leq s\delta^{n_0+1}\epsilon < \frac{\epsilon}{s}.$$

Case 7. *When $v = 1$,*

$$\begin{aligned}
\rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+1}) &\leq s[\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + \rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+1})] \\
&\leq s[\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + s\{\rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \\
&\quad +\rho_v(\zeta_{n_\lambda+n_0}, \zeta_{n_\lambda+1})\}] \\
&= s\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + s^2\rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \\
&\quad +s^2\rho_v(\zeta_{n_\lambda+n_0}, \zeta_{n_\lambda+1}) \\
&\leq s\{\delta^{m_\lambda}\rho_v(\zeta_{n_0-1}, \zeta_0) + m_\lambda\delta^{m_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\
&\quad +m_\lambda\delta^{m_\lambda-1}L\{\delta^{n_0-1} + 1\}\rho_v(\zeta_1, \zeta_0)\} + s^2\{\delta^{n_0}\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda}) \\
&\quad +n_0\delta^{m_\lambda+n_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 + n_0\delta^{n_0-1}L\{\delta^{m_\lambda-1} + \delta^n\}\rho_v(\zeta_1, \zeta_0)\} \\
&\quad +s^2\{\delta^{n_\lambda}\rho_v(\zeta_{n_0}, \zeta_1) + n_\lambda\delta^{n_\lambda+n_0+1}(\rho_v(\zeta_1, \zeta_0))^2 \\
&\quad +n_\lambda\delta^{n_\lambda-1}L\{\delta^{n_0} + \delta\}\rho_v(\zeta_1, \zeta_0)\} \\
&< s\{\delta^{m_\lambda}\rho_v(\zeta_{n_0-1}, \zeta_0) + m_\lambda\delta^{m_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\
&\quad +m_\lambda\delta^{m_\lambda-1}L\{\delta^{n_0-1} + 1\}\rho_v(\zeta_1, \zeta_0)\} + s^2\{\delta^{n_0}\epsilon + n_0\delta^{m_\lambda+n_\lambda+n_0-1}(\rho_v(\zeta_1, \zeta_0))^2 \\
&\quad +n_0\delta^{n_0-1}L\{\delta^{m_\lambda-1} + \delta^n\}\rho_v(\zeta_1, \zeta_0)\} \\
&\quad +s^2\{\delta^{n_\lambda}\rho_v(\zeta_{n_0}, \zeta_1) + n_\lambda\delta^{n_\lambda+n_0+1}(\rho_v(\zeta_1, \zeta_0))^2 \\
&\quad +n_\lambda\delta^{n_\lambda-1}L\{\delta^{n_0} + \delta\}\rho_v(\zeta_1, \zeta_0)\}.
\end{aligned}$$

Again, by taking the upper limit as $\lambda \to +\infty$ on both sides of the above inequality, we have

$$\begin{aligned}
\limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) &\leq s^2\delta^{n_0}\epsilon \\
&< \frac{\epsilon}{s}.
\end{aligned}$$

Thus, in both cases, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) < \frac{\epsilon}{s},$$

and thus the presumption of $\{\zeta_n\}$ being a non-Cauchy sequence is imprecise; so, $\{\zeta_n\}$ is Cauchy in Δ. Then, the completeness of Δ provides us with an element $\zeta^* \in \Delta$ such that $\lim_{n\to+\infty} \rho_v(\zeta_n, \zeta^*) = 0$.

Now, we verify that $\Gamma\zeta^* = \zeta^*$. On the contrary, we assume that $\Gamma\zeta^* \neq \zeta^*$. Then, owing to Lemma 1, we can say that the terms of sequence $\{\zeta_n\}$ are different from ζ^* and $\Gamma\zeta^*$ for sufficiently large values of n. Thus, it is obtained that

$$\begin{aligned}
\rho_v(\zeta^*, \Gamma\zeta^*) &\leq s[\rho_v(\zeta^*, \zeta_{n+1}) + \rho_v(\zeta_{n+1}, \zeta_{n+2}) + \rho_v(\zeta_{n+2}, \zeta_{n+3}) + \cdots \\
&\quad +\rho_v(\zeta_{n+v-2}, \zeta_{n+v-1}) + \rho_v(\zeta_{n+v-1}, \zeta_{n+v}) + \rho_v(\zeta_{n+v}, \Gamma\zeta^*)] \\
&= s[\rho_v(\zeta^*, \zeta_{n+1}) + \rho_v(\zeta_{n+1}, \zeta_{n+2}) + \rho_v(\zeta_{n+2}, \zeta_{n+3}) + \cdots \\
&\quad +\rho_v(\zeta_{n+v-2}, \zeta_{n+v-1}) + \rho_v(\zeta_{n+v-1}, \zeta_{n+v}) + \rho_v(\Gamma\zeta_{n+v-1}, \Gamma\zeta^*)] \\
&\leq s[\rho_v(\zeta^*, \zeta_{n+1}) + \rho_v(\zeta_{n+1}, \zeta_{n+2}) + \rho_v(\zeta_{n+2}, \zeta_{n+3}) + \cdots \\
&\quad +\rho_v(\zeta_{n+v-2}, \zeta_{n+v-1}) + \rho_v(\zeta_{n+v-1}, \zeta_{n+v}) \\
&\quad +\delta \max\left\{\rho_v(\zeta_{n+v-1}, \zeta^*), \frac{\rho_v(\zeta_{n+v-1}, \zeta_{n+v}) \cdot \rho_v(\zeta^*, \Gamma\zeta^*)}{1 + \rho_v(\zeta_{n+v-1}, \zeta^*)}\right\} \\
&\quad +L \min\{\rho_v(\zeta_{n+v-1}, \zeta_{n+v}) + \rho_v(\zeta^*, \Gamma\zeta^*), \rho_v(\zeta_{n+v-1}, \Gamma\zeta^*), \rho_v(\zeta^*, \zeta_{n+v})\}].
\end{aligned}$$

Since $\lim_{n\to+\infty} \rho_v(\zeta^*, \zeta_n) = 0$ and $\lim_{n\to+\infty} \rho_v(\zeta_n, \zeta_{n+1}) = 0$, we have

$$\rho_v(\zeta^*, \Gamma\zeta^*) = 0 \implies \Gamma\zeta^* = \zeta^*.$$

Aliter: Consider
$$\zeta_n = \Gamma\zeta_{n-1} = \Gamma^2\zeta_{n-2} = \cdots = \Gamma^n\zeta_0,$$
then
$$\lim_{n\to+\infty} \rho_v(\zeta_n, \zeta^*) = 0 \implies \lim_{n\to+\infty} \rho_v(\Gamma^n\zeta_0, \zeta^*) = 0.$$

Now, take
$$\begin{aligned}\rho_v(\Gamma^{n+1}\zeta_0, \Gamma\zeta^*) \leq\ & \delta \max\left\{\rho_v(\Gamma^n\zeta_0, \zeta^*), \frac{\rho_v(\Gamma^n\zeta_0, \Gamma^{n+1}\zeta_0)\cdot\rho_v(\zeta^*, \Gamma\zeta^*)}{1+\rho_v(\Gamma^n\zeta_0, \zeta^*)}\right\} \\ & + L\min\{\rho_v(\Gamma^n\zeta_0, \Gamma^{n+1}\zeta_0) + \rho_v(\zeta^*, \Gamma\zeta^*), \rho_v(\Gamma^n\zeta_0, \Gamma\zeta^*), \rho_v(\zeta^*, \Gamma^{n+1}\zeta_0)\}.\end{aligned}$$

By taking the limit as $n \to +\infty$, we obtain
$$\lim_{n\to+\infty} \rho_v(\Gamma^{n+1}\zeta_0, \Gamma\zeta^*) = 0.$$

Thus, sequence $u_n \to \Gamma\zeta^*$, but due to Lemma 1, sequence $\{u_n\}$ has a unique limit; hence,
$$\rho_v(\zeta^*, \Gamma\zeta^*) = 0 \implies \Gamma\zeta^* = \zeta^*.$$

Finally, we prove that the fixed point thus obtained is unique. For this, let q be another fixed point of Γ. Then,
$$\begin{aligned}\rho_v(\zeta^*, q) &= \rho_v(\Gamma\zeta^*, \Gamma q) \\ &\leq \delta\max\left\{\rho_v(\zeta^*, q), \frac{\rho_v(\zeta^*, \Gamma\zeta^*)\cdot\rho_v(q, \Gamma q)}{1+\rho_v(\zeta^*, q)}\right\} \\ &\quad + L\min\{\rho_v(\zeta^*, \Gamma\zeta^*) + \rho_v(q, \Gamma q), \rho_v(\zeta^*, \Gamma q), \rho_v(q, \Gamma\zeta^*)\} \\ &= \delta\rho_v(\zeta^*, q) < \rho_v(\zeta^*, q),\end{aligned}$$
which is a contradiction. Therefore, $\rho_v(\zeta^*, q) = 0$, that is, $\zeta^* = q$. □

Now, we present the following example in support of the proved result.

Example 4. *Let* $\Delta = \Omega \cup \Upsilon \cup \Theta$, *where*
$$\begin{aligned}\Omega &= \left\{\frac{1}{n}; n \in \{7, 8, 9, 10\}\right\}, \\ \Upsilon &= \left[\frac{1}{2}, 1\right] \\ \text{and } \Theta &= \left\{\frac{4}{3}\right\}.\end{aligned}$$

Define $\rho_v : \Delta \times \Delta \to [0, +\infty)$ *such that* $\rho_v(\zeta, \xi) = \rho_v(\xi, \zeta)$ *for all* $\zeta, \xi \in \Delta$ *and*
$$\begin{aligned}\rho_v\left(\frac{1}{7}, \frac{1}{9}\right) &= \rho_v\left(\frac{1}{10}, \frac{1}{8}\right) = 0.38, \\ \rho_v\left(\frac{1}{7}, \frac{1}{10}\right) &= \rho_v\left(\frac{1}{8}, \frac{1}{9}\right) = 0.01, \\ \rho_v\left(\frac{1}{7}, \frac{1}{8}\right) &= \rho_v\left(\frac{1}{9}, \frac{1}{10}\right) = 0.015, \\ \rho_v\left(\frac{1}{10}, \frac{4}{3}\right) &= 0.42 \\ \text{with } \rho_v(\zeta, \xi) &= (\zeta-\xi)^2 + |\zeta-\xi|, \text{ otherwise.}\end{aligned}$$

Then (Δ, ρ_v) is a 2-generalized b-metric space with coefficient $s = 2 > 1$. However, (Δ, ρ_v) is not a 2-generalized metric space, as

$$\begin{aligned}
\rho_v\left(\frac{1}{9}, \frac{1}{2}\right) &= \left(\frac{1}{9} - \frac{1}{2}\right)^2 + \left|\frac{1}{9} - \frac{1}{2}\right| = \frac{175}{324} = 0.5401, \\
\rho_v\left(\frac{1}{7}, \frac{1}{2}\right) &= \left(\frac{1}{7} - \frac{1}{2}\right)^2 + \left|\frac{1}{7} - \frac{1}{2}\right| = \frac{95}{196} = 0.4847. \\
\Rightarrow \rho_v\left(\frac{1}{9}, \frac{1}{2}\right) &= 0.5401 > 0.01 + 0.015 + 0.4847 \\
&= \rho_v\left(\frac{1}{9}, \frac{1}{8}\right) + \rho_v\left(\frac{1}{8}, \frac{1}{7}\right) + \rho_v\left(\frac{1}{7}, \frac{1}{2}\right).
\end{aligned}$$

If map $\Gamma : \Delta \to \Delta$ is defined as:

$$\Gamma\zeta = \begin{cases} \frac{1}{8}, & \zeta \in \Omega \\ \frac{4}{3}, & \zeta \in \Upsilon \\ \frac{1}{10}, & \zeta \in \Theta. \end{cases}$$

Then, we have the following cases:

Case 8. *If both ζ, ξ belong to any one of Ω, Θ, or Υ, then*

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) &= 0 \\
\text{and so, } \rho_v(\Gamma\zeta, \Gamma\xi) &\leq \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\
&\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}
\end{aligned}$$

holds for any $\delta \in [0, 1)$ and $L \geq 0$.

Case 9. *If $\zeta \in \Omega$ and $\xi \in \Upsilon$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left(\frac{1}{8} - \frac{4}{3}\right)^2 + \left|\frac{1}{8} - \frac{4}{3}\right| = 2.6684$$

and

$$\begin{aligned}
\rho_v(\zeta, \xi) &= (\zeta - \xi)^2 + |\zeta - \xi| \in [0.4847, 1.71], \\
\rho_v(\zeta, \Gamma\zeta) &= \left(\zeta, \frac{1}{8}\right) \in \{0, 0.01, 0.015, 0.38\}, \\
\rho_v(\xi, \Gamma\xi) &= \left(\xi - \frac{4}{3}\right)^2 + \left|\xi - \frac{4}{3}\right| \in [0.444, 1.5278], \\
\rho_v(\zeta, \Gamma\xi) &= \left(\zeta - \frac{4}{3}\right)^2 + \left|\zeta - \frac{4}{3}\right| \in \{0.42, 2.6077, 2.668, 2.716\}, \\
\rho_v(\xi, \Gamma\zeta) &= \left(\xi - \frac{1}{8}\right)^2 + \left|\xi - \frac{1}{8}\right| \in [0.5156, 1.6406].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) &\leq \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\
&\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}
\end{aligned}$$

for any $\delta \in [0, 1)$ and $L \geq 7$.

Case 10. *If $\zeta \in \Omega$ and $\xi \in \Theta$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left(\frac{1}{8}, \frac{1}{10}\right) = 0.38$$

and

$$\begin{aligned}
\rho_v(\zeta, \xi) &= \left(\zeta - \frac{4}{3}\right)^2 + \left|\zeta - \frac{4}{3}\right| \in \{0.42, 2.6077, 2.668, 2.716\}, \\
\rho_v(\zeta, \Gamma\zeta) &= \left(\zeta, \frac{1}{8}\right) \in \{0, 0.01, 0.015, 0.38\}, \\
\rho_v(\xi, \Gamma\xi) &= \left(\frac{4}{3}, \frac{1}{10}\right) = 0.42, \\
\rho_v(\zeta, \Gamma\xi) &= \left(\zeta, \frac{1}{10}\right) \in \{0, 0.01, 0.015, 0.38\}, \\
\rho_v(\xi, \Gamma\zeta) &= \left(\frac{4}{3} - \frac{1}{8}\right)^2 + \left|\frac{4}{3} - \frac{1}{8}\right| = 2.6684.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) \leq\ & \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\
& + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}
\end{aligned}$$

for $\delta > \frac{19}{21} \in [0, 1)$ and $L \geq 0$.

Case 11. *If $\zeta \in \Upsilon$ and $\xi \in \Theta$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left(\frac{4}{3}, \frac{1}{10}\right) = 0.42$$

and

$$\begin{aligned}
\rho_v(\zeta, \xi) &= \left(\zeta - \frac{4}{3}\right)^2 + \left|\zeta - \frac{4}{3}\right| \in [0.444, 1.528], \\
\rho_v(\zeta, \Gamma\zeta) &= \left(\zeta - \frac{4}{3}\right)^2 + \left|\zeta - \frac{4}{3}\right| \in [0.444, 1.528], \\
\rho_v(\xi, \Gamma\xi) &= \left(\frac{4}{3}, \frac{1}{10}\right) = 0.42, \\
\rho_v(\zeta, \Gamma\xi) &= \left(\zeta - \frac{1}{10}\right)^2 + \left|\zeta - \frac{1}{10}\right| \in [0.56, 1.71], \\
\rho_v(\xi, \Gamma\zeta) &= \left(\frac{4}{3}, \frac{4}{3}\right) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) \leq\ & \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\
& + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}
\end{aligned}$$

for $\delta \geq \frac{21}{22} \in [0, 1)$ and $L \geq 0$.

Case 12. *If $\zeta \in \Upsilon$ and $\xi \in \Omega$, then as in Case 9, by symmetry, we have*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\ &\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \end{aligned}$$

for any $\delta \in [0, 1)$ and $L \geq 7$.

Case 13. *If $\zeta \in \Theta$ and $\xi \in \Omega$, then as in Case 10, by symmetry, we have*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\ &\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \end{aligned}$$

for $\delta > \frac{19}{21} \in [0, 1)$ and $L \geq 0$.

Case 14. *If $\zeta \in \Theta$ and $\xi \in \Upsilon$, then as in Case 11, by symmetry, we have*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\ &\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \end{aligned}$$

for $\delta \geq \frac{21}{22} \in [0, 1)$ and $L \geq 0$.

From Cases 8–14, we obtain

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \delta \max\left\{\rho_v(\zeta, \xi), \frac{\rho_v(\zeta, \Gamma\zeta) \cdot \rho_v(\xi, \Gamma\xi)}{1 + \rho_v(\zeta, \xi)}\right\} \\ &\quad + L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \end{aligned}$$

for $\delta \geq \frac{21}{22} \in [0, 1)$, $L \geq 7$, and all $\zeta, \xi \in \Delta$.

Hence, Γ satisfies all the conditions of Theorem 1; therefore, it has a unique fixed point at $\zeta = \frac{1}{8}$.

The following remarks reveal the importance of Theorem 1 in comparison with the existing fixed point theorems in v-generalized b-metric spaces.

Remark 2. *Let set Δ, metric ρ_v, and map Γ be as defined in Example 4:*

1. *If $\zeta = \frac{1}{7}$ and $\xi = \frac{1}{2}$ then, we have*

$$\rho_v(\zeta, \xi) = \left(\frac{1}{7} - \frac{1}{2}\right)^2 + \left|\frac{1}{7} - \frac{1}{2}\right| = 0.4846$$

and

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left(\frac{1}{8} - \frac{4}{3}\right)^2 + \left|\frac{1}{8} - \frac{4}{3}\right| = 2.6684 > \rho_v(\zeta, \xi).$$

This proves that the ordinary Banach contraction is not satisfied; hence, Theorem 2.1 in [19] is not applicable to Example 4.

2. *If $\zeta = \frac{1}{10}$ and $\xi = \frac{1}{2}$, then we have*

$$\begin{aligned} \rho_v(\zeta, \xi) \quad &= \quad \left(\frac{1}{10} - \frac{1}{2}\right)^2 + \left|\frac{1}{10} - \frac{1}{2}\right| = 0.56, \\ \rho_v(\zeta, \Gamma\zeta) \quad &= \quad 0.38, \\ \rho_v(\xi, \Gamma\xi) \quad &= \quad \left(\frac{1}{2} - \frac{4}{3}\right)^2 + \left|\frac{1}{2} - \frac{4}{3}\right| = 1.5278, \end{aligned}$$

and

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) &= \left(\frac{1}{8}-\frac{4}{3}\right)^2+\left|\frac{1}{8}-\frac{4}{3}\right| = 2.6684\\ &> \delta\max\left\{0.56, \frac{0.38\times 1.5278}{1+0.56}\right\}\\ &= \delta\max\left\{\rho_v(\zeta,\xi), \frac{\rho_v(\zeta,\Gamma\zeta)\cdot\rho_v(\xi,\Gamma\xi)}{1+\rho_v(\zeta,\xi)}\right\}\end{aligned}$$

for any $\delta\in[0,1)$. Thus, Theorem 2.7 in [20] does not ensure the existence of a fixed point for Γ.

3. *If $\zeta=\frac{1}{8}$ and $\xi=1$, then we have*

$$\begin{aligned}\rho_v(\zeta,\xi) &= \left(\frac{1}{8}-1\right)^2+\left|\frac{1}{8}-1\right| = 1.6406,\\ \rho_v(\zeta,\Gamma\zeta) &= 0,\\ \rho_v(\xi,\Gamma\xi) &= \left(1-\frac{4}{3}\right)^2+\left|1-\frac{4}{3}\right| = 0.444,\\ \rho_v(\zeta,\Gamma\xi) &= \left(\frac{1}{8}-\frac{4}{3}\right)^2+\left|\frac{1}{8}-\frac{4}{3}\right| = 2.6684,\\ \rho_v(\xi,\Gamma\zeta) &= \left(1-\frac{1}{8}\right)^2+\left|1-\frac{1}{8}\right| = 1.6406\end{aligned}$$

and

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) &= \left(\frac{1}{8}-\frac{4}{3}\right)^2+\left|\frac{1}{8}-\frac{4}{3}\right| = 2.6684\\ &> \delta\max\{1.6406,0,0.444,2.6684,1.6406\}\\ &= \delta\max\{\rho_v(\zeta,\xi),\rho_v(\zeta,\Gamma\zeta),\rho_v(\xi,\Gamma\xi),\rho_v(\zeta,\Gamma\xi),\rho_v(\xi,\Gamma\zeta)\}.\end{aligned}$$

for any $\delta\in[0,1)$. Thus, Theorem 3 in [21] does not ensure the existence of a fixed point of map Γ.

4. *If $\zeta=\frac{1}{7}$ and $\xi=1$, then we have*

$$\begin{aligned}\rho_v(\zeta,\Gamma\zeta) &= 0.015,\\ \rho_v(\xi,\Gamma\xi) &= \left(1-\frac{4}{3}\right)^2+\left|1-\frac{4}{3}\right| = 0.444\end{aligned}$$

and

$$\begin{aligned}\rho_v(\Gamma\zeta,\Gamma\xi) &= \left(\frac{1}{8}-\frac{4}{3}\right)^2+\left|\frac{1}{8}-\frac{4}{3}\right| = 2.6684\\ &> 0.015\beta+0.444\gamma\\ &= \beta\rho_v(\zeta,\Gamma\zeta)+\gamma\rho_v(\xi,\Gamma\xi).\end{aligned}$$

for any $\beta+\gamma<1$; thus, Theorem 2.5 in [19] is not applicable to Example 4.

Remark 3. *In Example 4, map Γ does not satisfy contractive condition 4.1 in [22] at $\zeta=\frac{1}{8}$ and $\xi=1$, as the inequality*

$$\rho_v(\Gamma\zeta,\Gamma\xi)\leq\delta\max\left\{\rho_v(\zeta,\xi),\tfrac{\rho_v(\xi,\Gamma\xi)[1+\rho_v(\zeta,\Gamma\zeta)]}{1+\rho_v(\zeta,\xi)},\tfrac{\rho_v(\zeta,\Gamma\zeta)[1+\rho_v(\xi,\Gamma\xi)]}{1+\rho_v(\Gamma\zeta,\Gamma\xi)}\right\}+L\min\{\rho_v(\zeta,\Gamma\zeta),\rho_v(\xi,\Gamma\zeta)\}$$

gives $2.6684\leq\delta(1.6406)$, which is not possible for any $\delta\in[0,1)$. Thus, Theorems 4.2 and 4.3 in [22] are not applicable to Example 4.

Now, we state another version of almost contraction in v-generalized b-metric spaces.

Definition 6. *Let (Δ, ρ_v) be a v-generalized b-metric space. A map $\Gamma : \Delta \to \Delta$ is said to be a generalized almost contraction of Reich type if there exist constants $\alpha, \beta, \gamma \in [0,1)$ and $L \geq 0$ such that*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) &\leq \alpha\rho_v(\zeta,\xi) + \beta\rho_v(\zeta,\Gamma\zeta) + \gamma\rho_v(\xi,\Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta,\Gamma\zeta) + \rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\} \end{aligned} \tag{6}$$

for all $\zeta, \xi \in \Delta$, where $\alpha + \beta + \gamma < 1$.

By setting $m = 1$ in Example 1, we present an example of generalized almost contraction of Reich type.

Example 5. *Take $\Delta \subseteq \mathbb{R}$ as $\Delta = \{0\} \cup \Omega_1$, where $\Omega_1 = \{\zeta_1;\ 0 < \zeta_1 \leq 1\}$ and the function $\rho_v : \Delta \times \Delta \to [0,+\infty)$ is defined as $\rho_v(\zeta,\zeta) = 0$, $\rho_v(\zeta,\xi) = \rho_v(\xi,\zeta)$ for all $\zeta, \xi \in \Delta$ and*

$$\begin{aligned} \rho_v(0,\zeta_1) &= \zeta_1;\ 0 < \zeta_1 < 1,\ \zeta_1 \neq \frac{1}{n},\ n \in \mathbb{N}, \\ \rho_v(0,\zeta_1) &= \frac{1}{2n};\ \zeta_1 = \frac{1}{n},\ n \in \mathbb{N}, \\ \rho_v\left(\frac{2}{3},\zeta_1\right) &= \frac{1}{2n};\ \zeta_1 = \frac{1}{n},\ n \in \mathbb{N}, \\ \rho_v\left(\frac{2}{3},\frac{3}{4}\right) &= 1, \\ \rho_v(1,\zeta_1) &= 2;\ 0 < \zeta_1 < 1, \zeta_1 \neq \frac{2}{3}, \\ \rho_v(\zeta,\xi) &= 5,\ otherwise. \end{aligned}$$

Then, clearly, (Δ, ρ_v) is a 4-generalized b-metric space for $s \geq 5$ as shown in Example 1. Now, define $\Gamma : \Delta \to \Delta$ as

$$\Gamma\zeta = \begin{cases} \frac{3}{4}, & \zeta = \Delta - \left\{\frac{2}{5}\right\} \\ 1, & \zeta = \frac{2}{5}. \end{cases}$$

Then, we consider the following cases:

Case 15. *If $\zeta \in \left[0, \frac{2}{5}\right)$ and $\xi = \frac{2}{5}$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = 2$$

and

$$\begin{aligned} \rho_v(\zeta,\xi) &\in \left\{\frac{2}{5}, 5\right\} \\ \rho_v(\zeta,\Gamma\zeta) &\in \left\{\frac{3}{4}, 5\right\} \\ \rho_v(\xi,\Gamma\xi) &= 2 \\ \rho_v(\zeta,\Gamma\xi) &\in \left\{\frac{1}{2}, 2\right\} \\ \rho_v(\xi,\Gamma\zeta) &= 5. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) &\leq \alpha\rho_v(\zeta,\xi) + \beta\rho_v(\zeta,\Gamma\zeta) + \gamma\rho_v(\xi,\Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta,\Gamma\zeta) + \rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_(\xi,\Gamma\zeta)\} \end{aligned}$$

for $\alpha = \frac{2}{5}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{5}$, *and* $L \geq 3$.

Case 16. *If* $\zeta = \frac{2}{5}$ *and* $\xi \in \left[0, \frac{2}{5}\right)$, *then as in Case 15, by symmetry, we have*

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_(\xi, \Gamma\zeta)\}\end{aligned}$$

for $\alpha = \frac{2}{5}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{5}$, *and* $L \geq 3$.

Case 17. *If* $\zeta = \frac{2}{5}$ *and* $\xi \in \left(\frac{2}{5}, 1\right]$, *then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = 2$$

and

$$\begin{aligned}\rho_v(\zeta, \xi) \quad &= \quad 5 \\ \rho_v(\zeta, \Gamma\zeta) \quad &= \quad 2 \\ \rho_v(\xi, \Gamma\xi) \quad &\in \quad \{0, 1, 2, 5\} \\ \rho_v(\zeta, \Gamma\xi) \quad &= \quad 5 \\ \rho_v(\xi, \Gamma\zeta) \quad &\in \quad \left\{0, \frac{1}{2}, 2\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_(\xi, \Gamma\zeta)\}\end{aligned}$$

for $\alpha = \frac{2}{5}$, $\beta = 0$, $\gamma = 0$, *and* $L \geq 0$.

Case 18. *If* $\zeta \in \left(\frac{2}{5}, 1\right]$ *and* $\xi = \frac{2}{5}$, *then as in Case 17, by symmetry, we have*

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_(\xi, \Gamma\zeta)\}\end{aligned}$$

for $\alpha = \frac{2}{5}$, $\beta = 0$, $\gamma = 0$, *and* $L \geq 0$.

Case 19. *If* $\zeta, \xi \in \Delta - \frac{2}{5}$, *then we have*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = 0.$$

Therefore,

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_(\xi, \Gamma\zeta)\}\end{aligned}$$

for any $\alpha + \beta + \gamma \in [0, 1)$ *and* $L \geq 0$.

From Cases 15–19, we obtain

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \\ &\quad + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_(\xi, \Gamma\zeta)\}\end{aligned}$$

for $\alpha = \frac{2}{5}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{5}$, *and* $L \geq 3$.

So, map Γ is a generalized almost contraction of rational type. However, Γ does not satisfy the Reich contraction. For this, we take points $\zeta = 0, \xi = \frac{2}{5}$, which implies that $\Gamma\zeta = \frac{3}{4}, \Gamma\xi = 1$; hence, we obtain that $\rho_v(\zeta,\xi) = \frac{2}{5}$, $\rho_v(\Gamma\zeta,\Gamma\xi) = 2$, and $\rho_v(\zeta,\Gamma\zeta) = \frac{3}{4}, \rho_v(\xi,\Gamma\xi) = 2$. Thus,

$$\rho_v(\Gamma\zeta,\Gamma\xi) \quad > \quad \alpha\rho_(\zeta,\xi) + \beta\rho_v(\zeta,\Gamma\zeta) + \gamma\rho_v(\xi,\Gamma\xi)$$

for any $\alpha+\beta+\gamma \in [0,1)$.

Theorem 2. *Let (Δ,ρ_v) be a complete v-generalized b-metric space with coefficient $s \geq 1$ and $\Gamma : \Delta \to \Delta$ be a generalized almost contraction of Reich type with the condition that $\min\{\beta,\gamma\} < \frac{1}{s}$. Then, Γ has a unique fixed point.*

Proof. Let $\zeta_0 \in \Delta$ be arbitrary. Define sequence $\{\zeta_n\}$ with $\zeta_{n+1} = \Gamma\zeta_n$ for all $n \geq 0$. Then, by (6),

$$\begin{aligned}
\rho_v(\zeta_{n+1},\zeta_n) \quad &\leq \quad \alpha\rho_v(\zeta_n,\zeta_{n-1}) + \beta\rho_v(\zeta_n,\zeta_{n+1}) + \gamma\rho_v(\zeta_{n-1},\zeta_n) \\
&\quad\quad +L\min\{\rho_v(\zeta_n,\zeta_{n+1}) + \rho_v(\zeta_{n-1},\zeta_n), \rho_v(\zeta_n,\zeta_n), \rho_v(\zeta_{n-1},\zeta_{n+1})\} \\
&= \quad (\alpha+\gamma)\rho_v(\zeta_n,\zeta_{n-1}) + \beta\rho_v(\zeta_n,\zeta_{n+1})
\end{aligned}$$

and so for each $n \in \mathbb{N}$, we obtain that

$$\rho_v(\zeta_{n+1},\zeta_n) \leq \frac{\alpha+\gamma}{1-\beta}\rho_v(\zeta_n,\zeta_{n+1}).$$

Thus, we obtain

$$\rho_v(\zeta_{n+1},\zeta_n) \leq \sigma^n\rho_v(\zeta_1,\zeta_0), \tag{7}$$

where $\sigma = \frac{\alpha+\gamma}{1-\beta} \in [0,1)$. By following steps similar to those we followed in Theorem 1, it can be obtained that $\zeta_n \neq \zeta_m$ for all distinct $n, m \in \mathbb{N}$.

Now, using conditions (6) and (7), we have

$$\begin{aligned}
\rho_v(\zeta_m,\zeta_n) \quad &\leq \quad \alpha\rho_v(\zeta_{m-1},\zeta_{n-1}) + \beta\rho_v(\zeta_{m-1},\zeta_m) + \gamma\rho_v(\zeta_{n-1},\zeta_n) \\
&\quad\quad +L\min\{\rho_v(\zeta_{m-1},\zeta_m) + \rho_v(\zeta_{n-1},\zeta_n), \rho_v(\zeta_{m-1},\zeta_n), \rho_v(\zeta_{n-1},\zeta_m)\} \\
&\leq \quad \alpha\rho_v(\zeta_{m-1},\zeta_{n-1}) + \beta\sigma^{m-1}\rho_v(\zeta_1,\zeta_0) + \gamma\sigma^{n-1}\rho_v(\zeta_1,\zeta_0) \\
&\quad\quad +L\{\sigma^{m-1}\rho_v(\zeta_1,\zeta_0) + \sigma^{n-1}\rho_v(\zeta_1,\zeta_0)\} \\
&= \quad \alpha\rho_v(\zeta_{m-1},\zeta_{n-1}) + (\sigma^{m-1}(\beta+L) + \sigma^{n-1}(\gamma+L))\rho_v(\zeta_1,\zeta_0).
\end{aligned}$$

This implies that

$$\rho_v(\zeta_m,\zeta_n) \leq \delta\rho_v(\zeta_{m-1},\zeta_{n-1}) + (\delta^{m-1}(\beta+L) + \delta^{n-1}(\gamma+L))\rho_v(\zeta_1,\zeta_0),$$

where $\delta = \max\{\alpha,\sigma\}$. Similarly, we can have

$$\rho_v(\zeta_{m+1},\zeta_{n+1}) \leq \delta\rho_v(\zeta_m,\zeta_n) + 1\cdot\delta^{1-1}(\delta^m(\beta+L) + \delta^n(\gamma+L))\rho_v(\zeta_1,\zeta_0).$$

Then, by repeating the same argument, we obtain that

$$\begin{aligned}
\rho_v(\zeta_{m+2},\zeta_{n+2}) \quad &\leq \quad \delta\rho_v(\zeta_{m+1},\zeta_{n+1}) + (\delta^{m+1}(\beta+L) + \delta^{n+1}(\gamma+L))\rho_v(\zeta_1,\zeta_0) \\
&\leq \quad \delta[\delta\rho_v(\zeta_m,\zeta_n) + (\delta^m(\beta+L) + \delta^n(\gamma+L))\rho_v(\zeta_1,\zeta_0)] \\
&\quad\quad +(\delta^{m+1}(\beta+L) + \delta^{n+1}(\gamma+L))\rho_v(\zeta_1,\zeta_0) \\
&= \quad \delta^2\rho_v(\zeta_m,\zeta_n) + 2\cdot\delta^{2-1}(\delta^m(\beta+L) + \delta^n(\gamma+L))\rho_v(\zeta_1,\zeta_0) \\
&\quad\vdots \\
\rho_v(\zeta_{m+k},\zeta_{n+k}) \quad &\leq \quad \delta^k\rho_v(\zeta_m,\zeta_n) + k\delta^{k-1}(\delta^m(\beta+L) + \delta^n(\gamma+L))\rho_v(\zeta_1,\zeta_0).
\end{aligned} \tag{8}$$

Since $\rho_v(\zeta_0, \Gamma\zeta_0) = \rho_v(\zeta_0, \zeta_1) <+\infty$, then by applying $n \to +\infty$ in (7), we obtain

$$\lim_{n\to+\infty} \rho_v(\zeta_n, \zeta_{n+1}) = 0.$$

Now, we prove that $\{\zeta_n\}$ is a Cauchy sequence in Δ. Meanwhile, if $\{\zeta_n\}$ is a non-Cauchy sequence, on the contrary, a positive ϵ and subsequences $\{\zeta_{m_\lambda}\}$ and $\{\zeta_{n_\lambda}\}$ of $\{\zeta_n\}$ can be obtained, such that m_λ is the smallest cardinal, with $m_\lambda > n_\lambda > \lambda$,

$$\rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda}) \geq \epsilon$$

and

$$\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda}) < \epsilon. \tag{9}$$

Then, it follows that

$$\begin{aligned} \epsilon \leq \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda}) \quad &\leq \quad s[\rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) + \rho_v(\zeta_{n_\lambda+v}, \zeta_{n_\lambda+v-1}) + \cdots \\ &\quad\ +\rho_v(\zeta_{n_\lambda+2}, \zeta_{n_\lambda+1}) + \rho_v(\zeta_{n_\lambda+1}, \zeta_{n_\lambda})], \end{aligned}$$

which gives

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}).$$

If $\delta = 0$, the proof is trivial. So, let $\delta \in (0,1)$. Since $\lim_{n\to+\infty} \delta^n = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$0 < \delta^{n_0} \cdot \mu < 1, \ \mu = s^3. \tag{10}$$

Now, consider the following two cases:

Case 20. *When $v \geq 2$, by condition (3), we obtain*

$$\begin{aligned} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) \quad &\leq \quad \delta\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda+v-1}) + \{\delta^{m_\lambda-1}(\beta+L) + \delta^{n_\lambda+v-1}(\gamma+L)\}\rho_v(\zeta_1, \zeta_0) \\ &\leq \quad \delta s[\rho_v(\zeta_{m_\lambda-1}, \zeta_{m_\lambda+n_0-1}) + \rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \\ &\quad\ +\rho_v(\zeta_{n_\lambda+n_0}, \zeta_{n_\lambda+2v-3}) + \rho_v(\zeta_{n_\lambda+2v-3}, \zeta_{n_\lambda+2v-4}) + \cdots \\ &\quad\ +\rho_v(\zeta_{n_\lambda+v+1}, \zeta_{n_\lambda+v}) + \rho_v(\zeta_{n_\lambda+v}, \zeta_{n_\lambda+v-1})] \\ &\quad\ +\{\delta^{m_\lambda-1}(\beta+L) + \delta^{n_\lambda+v-1}(\gamma+L)\}\rho_v(\zeta_1, \zeta_0) \\ &\leq \quad \delta s[\{\delta^{m_\lambda-1}\rho_v(\zeta_{n_0}, \zeta_0) + (m_\lambda-1)\delta^{m_\lambda-2}\{\delta^{n_0}(\beta+L) + (\gamma+L)\}\rho_v(\zeta_1, \zeta_0)\} \\ &\quad\ +\{\delta^{n_0}\rho_v(\zeta_{m_\lambda-1}, \zeta_{n_\lambda}) + n_0\delta^{n_0-1}\{\delta^{m_\lambda-1}(\beta+L) + \delta^{n_\lambda}(\gamma+L)\}\rho_v(\zeta_1, \zeta_0)\} \\ &\quad\ +\{\delta^{n_\lambda}\rho_v(\zeta_{n_0}, \zeta_{2v-3}) + n_\lambda\delta^{n_\lambda-1}\{\delta^{n_0}(\beta+L) + \delta^{2v-3}(\gamma+L)\}\rho_v(\zeta_1, \zeta_0)\} \\ &\quad\ +\delta^{n_\lambda+2v-4}\rho_v(\zeta_1, \zeta_0) + \cdots + \delta^{n_\lambda+v}\rho_v(\zeta_1, \zeta_0) \\ &\quad\ +\delta^{n_\lambda+v-1}\rho_v(\zeta_1, \zeta_0)] + \{\delta^{m_\lambda-1}(\beta+L) + \delta^{n_\lambda+v-1}(\gamma+L)\}\rho_v(\zeta_1, \zeta_0). \end{aligned}$$

Now, firstly using relations (9) and (10), then taking the upper limit $\lambda \to +\infty$, we obtain

$$\begin{aligned} \limsup_{\lambda\to+\infty} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+v}) \quad &< \quad s\delta^{n_0+1}\epsilon \\ &< \quad \frac{\delta\epsilon}{s^2} < \frac{\delta\epsilon}{s} < \frac{\epsilon}{s}. \end{aligned}$$

Case 21. *When $v = 1$,*

$$\begin{aligned} \rho_v(\zeta_{m_\lambda}, \zeta_{n_\lambda+1}) \quad &\leq \quad s[\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + \rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+1})] \\ &\leq \quad s[\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + s\{\rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \\ &\quad\ +\rho_v(\zeta_{n_\lambda+n_0}, \zeta_{n_\lambda+1})\}] \\ &= \quad s\rho_v(\zeta_{m_\lambda}, \zeta_{m_\lambda+n_0-1}) + s^2\rho_v(\zeta_{m_\lambda+n_0-1}, \zeta_{n_\lambda+n_0}) \end{aligned}$$

$$\begin{aligned}
& +s^2\rho_v(\zeta_{n_\lambda+n_0},\zeta_{n_\lambda+1}) \\
\leq\ & s\{\delta^{m_\lambda}\rho_v(\zeta_{n_0-1},\zeta_0)+m_\lambda\delta^{m_\lambda-1}\{\delta^{n_0-1}(\beta+L)+(\gamma+L)\}\rho_v(\zeta_1,\zeta_0)\} \\
& +s^2\{\delta^{n_0}\rho_v(\zeta_{m_\lambda-1},\zeta_{n_\lambda})+n_0\delta^{n_0-1}\{\delta^{m_\lambda-1}(\beta+L)+\delta^{n}(\gamma+L)\}\rho_v(\zeta_1,\zeta_0)\} \\
& +s^2\{\delta^{n_\lambda}\rho_v(\zeta_{n_0},\zeta_1)+n_\lambda\delta^{n_\lambda-1}\{\delta^{n_0}(\beta+L)+\delta(\gamma+L)\}\rho_v(\zeta_1,\zeta_0)\}.
\end{aligned}$$

Again, using relations (9) and (10) and then taking the upper limit $\lambda\to+\infty$, we have

$$\begin{aligned}
\limsup_{\lambda\to+\infty}\rho_v(\zeta_{m_\lambda},\zeta_{n_\lambda+v}) &< s^2\delta^{n_0}\epsilon \\
&< \frac{\epsilon}{s}.
\end{aligned}$$

Therefore, in both the cases, we obtain that

$$\frac{\epsilon}{s}\leq\limsup_{\lambda\to+\infty}\rho_v(\zeta_{m_\lambda},\zeta_{n_\lambda+v})<\frac{\epsilon}{s},$$

and thus the presumption of $\{\zeta_n\}$ being a non-Cauchy sequence is imprecise; so, $\{\zeta_n\}$ is Cauchy in Δ. Then, the completeness of Δ provides us with an element $\zeta^*\in\Delta$ such that $\lim_{n\to+\infty}\rho_v(\zeta_n,\zeta^*)=0$. Now, we verify that $\Gamma\zeta^*=\zeta^*$. If we take $\Gamma\zeta^*\neq\zeta^*$, then on account of Lemma 1, it follows that $\zeta_n\neq\Gamma\zeta^*$ and $\zeta_n\neq\zeta^*$ for sufficiently large n. Thus, we have

$$\begin{aligned}
\rho_v(\zeta^*,\Gamma\zeta^*) \leq\ & s[\rho_v(\zeta^*,\zeta_{n+1})+\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\rho_v(\zeta_{n+v},\Gamma\zeta^*)] \\
=\ & s[\rho_v(\zeta^*,\zeta_{n+1})+\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\rho_v(\Gamma\zeta_{n+v-1},\Gamma\zeta^*)] \\
\leq\ & s[\rho_v(\zeta^*,\zeta_{n+1})+\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v}) \\
& +\alpha\rho_v(\zeta_{n+v-1},\zeta^*)+\beta\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\gamma\rho_v(\zeta^*,\Gamma\zeta^*) \\
& +L\min\{\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\rho_v(\zeta^*,\Gamma\zeta^*),\rho_v(\zeta_{n+v-1},\Gamma\zeta^*),\rho_v(\zeta^*,\zeta_{n+v})\}].
\end{aligned}$$

Since $\lim_{n\to+\infty}\rho_v(\zeta^*,\zeta_n)=0$ and $\lim_{n\to+\infty}\rho_v(\zeta_n,\zeta_{n+1})=0$, we have

$$\frac{1}{s}\rho_v(\zeta^*,\Gamma\zeta^*)\leq\gamma\rho_v(\zeta^*,\Gamma\zeta^*). \tag{11}$$

Again, if we consider the following:

$$\begin{aligned}
\rho_v(\Gamma\zeta^*,\zeta^*) \leq\ & s[\rho_v(\Gamma\zeta^*,\zeta_{n+1})+\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\rho_v(\zeta_{n+v},\zeta^*)] \\
=\ & s[\rho_v(\Gamma\zeta^*,\Gamma\zeta_n)+\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\rho_v(\zeta_{n+v},\zeta^*)] \\
\leq\ & s[\alpha\rho_v(\zeta^*,\zeta_n)+\beta\rho_v(\zeta^*,\Gamma\zeta^*)+\gamma\rho_v(\zeta_n,\zeta_{n+1}) \\
& L\min\{\rho_v(\zeta^*,\Gamma\zeta^*)+\rho_v(\zeta_n,\zeta_{n+1}),\rho_v(\zeta^*,\zeta_{n+1}),\rho_v(\zeta_n,\Gamma\zeta^*)\} \\
& +\rho_v(\zeta_{n+1},\zeta_{n+2})+\rho_v(\zeta_{n+2},\zeta_{n+3})+\cdots \\
& +\rho_v(\zeta_{n+v-2},\zeta_{n+v-1})+\rho_v(\zeta_{n+v-1},\zeta_{n+v}) \\
& +\alpha\rho_v(\zeta_{n+v-1},\zeta^*)+\beta\rho_v(\zeta_{n+v-1},\zeta_{n+v})+\gamma\rho_v(\zeta^*,\Gamma\zeta^*)].
\end{aligned}$$

Then, by letting $n\to+\infty$ on both sides of the above inequality, it follows that

$$\frac{1}{s}\rho_v(\Gamma\zeta^*,\zeta^*)\leq\beta\rho_v(\zeta^*,\Gamma\zeta^*). \tag{12}$$

Due to inequalities (11) and (12), we have $\frac{1}{s}\rho_v(\Gamma\zeta^*,\zeta^*) \leq \min\{\beta,\gamma\}\rho_v(\zeta^*,\Gamma\zeta^*) < \frac{1}{s}\rho_v(\Gamma\zeta^*,\zeta^*)$, which is a contradiction; hence, $\Gamma\zeta^* = \zeta^*$. Finally, we prove that the fixed point thus obtained is unique. For this, let q be another fixed point of Γ. Then,

$$\begin{aligned}
\rho_v(\zeta^*,q) &= \rho_v(\Gamma\zeta^*,\Gamma q) \\
&\leq \alpha\rho_v(\zeta^*,q) + \beta\rho_v(\zeta^*,\Gamma\zeta^*) + \gamma\rho_v(q,\Gamma q) \\
&\quad + L\min\{\rho_v(\zeta^*,\Gamma\zeta^*) + \rho_v(q,\Gamma q), \rho_v(\zeta^*,\Gamma q), \rho_v(q,\Gamma\zeta^*)\} \\
&= \alpha\rho_v(\zeta^*,q) < \rho_v(\zeta^*,q),
\end{aligned}$$

which is a contradiction. Therefore, $\rho_v(\zeta^*,q) = 0$, that is, $\zeta^* = q$.

Hence, the proof. $\square$

Corollary 1. *Let (Δ,ρ_v) be a complete v-generalized b-metric space with coefficient $s \geq 1$ and $\Gamma : \Delta \to \Delta$ be a mapping such that*

$$\rho_v(\Gamma\zeta,\Gamma\xi) \leq \alpha\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta) + \rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}, \quad (13)$$

for all $\zeta,\xi \in \chi$, where $0 \leq \alpha < 1$ and $L \geq 0$. Then, Γ has a unique fixed point.

We present the following example in favor of Corollary 1; hence, it supports Theorem 2.

Example 6. *Let $\Delta = \Omega \cup \Theta \cup \Upsilon$, where*

$$\begin{aligned}
\Omega &= \left\{\frac{1}{n}; n \in \{2,3,4,5\}\right\}, \\
\Theta &= \left\{\frac{5}{12}\right\} \\
\text{and } \Upsilon &= \left[\frac{2}{3},1\right].
\end{aligned}$$

Define $\rho_v : \Delta \times \Delta \to [0,+\infty)$ such that $\rho_v(\zeta,\xi) = \rho_v(\xi,\zeta)$ for all $\zeta,\xi \in \Delta$ and

$$\begin{aligned}
\rho_v\left(\frac{1}{2},\frac{1}{3}\right) &= \rho_v\left(\frac{1}{4},\frac{1}{5}\right) = 0.02, \\
\rho_v\left(\frac{1}{2},\frac{1}{5}\right) &= \rho_v\left(\frac{1}{3},\frac{1}{4}\right) = 0.03, \\
\rho_v\left(\frac{1}{2},\frac{1}{4}\right) &= \rho_v\left(\frac{1}{5},\frac{1}{3}\right) = 0.56, \\
\rho_v\left(\frac{2}{3},\frac{3}{4}\right) &= 0.005,\ \rho_v\left(\frac{5}{12},\frac{3}{4}\right) = 0.007, \\
\text{with } \rho_v(\zeta,\xi) &= |\zeta-\xi|^3 + (\zeta,\xi)^2, \text{ otherwise.}
\end{aligned}$$

Then, (Δ,ρ_v) is a 2-generalized b-metric space with coefficient $s = 5 > 1$. However, it is not a 2-generalized metric space, as

$$\begin{aligned}
\rho_v\left(\frac{1}{4},\frac{2}{3}\right) &= \left|\frac{1}{4}-\frac{2}{3}\right|^3 + \left(\frac{1}{4}-\frac{2}{3}\right)^2 = \frac{425}{1728} = 0.246, \text{ and} \\
\rho_v\left(\frac{1}{2},\frac{2}{3}\right) &= \left|\frac{1}{2}-\frac{2}{3}\right|^3 + \left(\frac{1}{2}-\frac{2}{3}\right)^2 = \frac{7}{216} = 0.032 \\
\Longrightarrow \rho_v\left(\frac{1}{4},\frac{2}{3}\right) &= 0.246 > 0.03 + 0.02 + 0.032
\end{aligned}$$

$$= \rho_v\left(\frac{1}{4}, \frac{1}{3}\right) + \rho_v\left(\frac{1}{3}, \frac{1}{2}\right) + \rho_v\left(\frac{1}{2}, \frac{2}{3}\right). \tag{14}$$

We define map $\Gamma : \Delta \to \Delta$ as

$$\Gamma\zeta = \begin{cases} \frac{5}{12}, & \zeta \in \Omega \\ \frac{3}{4}, & \zeta \in \Theta \\ \frac{2}{3}, & \zeta \in \Upsilon. \end{cases}$$

Now, we discuss the following possible cases:

Case 22. *If both $\zeta, \xi \in \Omega$ or Θ or Υ, then*

$$\begin{aligned} \rho_v(\Gamma\zeta, \Gamma\xi) &= 0 \\ so,\ \rho_v(\Gamma\zeta, \Gamma\xi) &\leq \delta\rho_v(\zeta, \xi) + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \end{aligned}$$

holds for any $\delta \in [0, 1)$ and $L \geq 0$.

Case 23. *If $\zeta \in \Omega$ and $\xi \in \Theta$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left(\frac{5}{12}, \frac{3}{4}\right) = 0.007$$

and

$$\begin{aligned} \rho_v(\zeta, \xi) &= \left|\zeta - \frac{5}{12}\right|^3 + \left(\zeta - \frac{5}{12}\right)^2 \in \{0.0075, 0.0324, 0.057\}, \\ \rho_v(\zeta, \Gamma\zeta) &= \left|\zeta - \frac{5}{12}\right|^3 + \left(\zeta - \frac{5}{12}\right)^2 \in \{0.0075, 0.0324, 0.057\}, \\ \rho_v(\xi, \Gamma\xi) &= \left(\frac{5}{12}, \frac{3}{4}\right) = 0.007, \\ \rho_v(\zeta, \Gamma\xi) &= \left|\zeta - \frac{3}{4}\right|^3 + \left(\zeta - \frac{3}{4}\right)^2 \in \{0.0781, 0.2459, 0.375, 0.4688\}, \\ \rho_v(\xi, \Gamma\zeta) &= \left(\frac{5}{12}, \frac{5}{12}\right) = 0. \end{aligned}$$

Therefore,

$$\rho_v(\Gamma\zeta, \Gamma\xi) \leq \delta\rho_v(\zeta, \xi) + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}$$

for $\delta > \frac{14}{15} \in [0, 1)$ and $L \geq 0$.

Case 24. *If $\zeta \in \Theta$ and $\xi \in \Omega$, then as in Case 23, by symmetry, we have*

$$\rho_v(\Gamma\zeta, \Gamma\xi) \leq \delta\rho_v(\zeta, \xi) + L\min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\}$$

for $\delta > \frac{14}{15} \in [0, 1)$ and $L \geq 0$.

Case 25. *If $\zeta \in \Omega$ and $\xi \in \Upsilon$, then*

$$\rho_v(\Gamma\zeta, \Gamma\xi) = \left|\frac{5}{12} - \frac{2}{3}\right|^3 + \left(\frac{5}{12} - \frac{2}{3}\right)^2 = 0.078$$

and

$$\rho_v(\zeta, \xi) = |\zeta - \xi|^3 + (\zeta - \xi)^2 \in [0.032, 1.152],$$

$$
\begin{aligned}
\rho_v(\zeta,\Gamma\zeta) &= \left|\zeta-\frac{5}{12}\right|^3+\left(\zeta-\frac{5}{12}\right)^2 \in \{0.0075, 0.0324, 0.057\},\\
\rho_v(\xi,\Gamma\xi) &= \left|\xi-\frac{2}{3}\right|^3+\left(\xi-\frac{2}{3}\right)^2 \in [0, 0.148],\\
\rho_v(\zeta,\Gamma\xi) &= \left|\zeta-\frac{2}{3}\right|^3+\left(\zeta-\frac{2}{3}\right)^2 \in \{0.0324, 0.1481, 0.2459, 0.3194\},\\
\rho_v(\xi,\Gamma\zeta) &= \left|\xi-\frac{5}{12}\right|^3+\left(\xi-\frac{5}{12}\right)^2 \in \{0.007\} \cup [0.078, 0.539].
\end{aligned}
$$

Therefore,

$$\rho_v(\Gamma\zeta,\Gamma\xi) \le \delta\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$$

for any $\delta \in [0,1)$ and $L \ge 12$.

Case 26. *If $\zeta \in \Upsilon$ and $\xi \in \Omega$, then as in Case 25, by symmetry, we have*

$$\rho_v(\Gamma\zeta,\Gamma\xi) \le \delta\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$$

for $\delta \in [0,1)$ and $L \ge 12$.

Case 27. *If $\zeta \in \Theta$ and $\xi \in \Upsilon$, then*

$$\rho_v(\Gamma\zeta,\Gamma\xi) = \left(\frac{3}{4},\frac{2}{3}\right) = 0.005$$

and

$$
\begin{aligned}
\rho_v(\zeta,\xi) &= \left|\frac{5}{12}-\xi\right|^3+\left(\frac{5}{12}-\xi\right)^2 \in \{0.007\} \cup [0.078, 0.539],\\
\rho_v(\zeta,\Gamma\zeta) &= \left(\frac{5}{12},\frac{3}{4}\right) = 0.007,\\
\rho_v(\xi,\Gamma\xi) &= \left|\xi-\frac{2}{3}\right|^3+\left(\xi-\frac{2}{3}\right)^2 \in [0, 0.148],\\
\rho_v(\zeta,\Gamma\xi) &= \left|\frac{5}{12}-\frac{2}{3}\right|^3+\left(\frac{5}{12}-\frac{2}{3}\right)^2 = 0.078,\\
\rho_v(\xi,\Gamma\zeta) &= \left|\xi-\frac{3}{4}\right|^3+\left(\xi-\frac{3}{4}\right)^2 \in [0, 0.078].
\end{aligned}
$$

Therefore,

$$\rho_v(\Gamma\zeta,\Gamma\xi) \le \delta\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$$

for $\delta > \frac{5}{7} \in [0,1)$ and $L \ge 0$.

Case 28. *If $\zeta \in \Upsilon$ and $\xi \in \Theta$, then as in Case 27, by symmetry, we have*

$$\rho_v(\Gamma\zeta,\Gamma\xi) \le \delta\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$$

for $\delta > \frac{5}{7} \in [0,1)$ and $L \ge 0$.

From Cases 22–28, we obtain

$$\rho_v(\Gamma\zeta,\Gamma\xi) \le \delta\rho_v(\zeta,\xi) + L\min\{\rho_v(\zeta,\Gamma\zeta)+\rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$$

for $\delta > \frac{14}{15} \in [0, 1)$ and $L \geq 12$ and all $\zeta, \xi \in \Delta$.

Hence, Γ satisfies all the conditions of Corollary 1; therefore, Γ has a unique fixed point at $\zeta = \frac{2}{3}$.

Remark 4. *In Example 6, we examine that at points $\zeta = \frac{1}{2}$ and $\xi = \frac{2}{3}$,*

$$\begin{aligned}
\rho_v(\zeta, \xi) &= \left|\frac{1}{2} - \frac{2}{3}\right|^3 + \left(\frac{1}{2} - \frac{2}{3}\right)^2 = 0.032, \\
\rho_v(\zeta, \Gamma\zeta) &= \left|\frac{1}{2} - \frac{5}{12}\right|^3 + \left(\frac{1}{2} - \frac{5}{12}\right)^2 = 0.0075, \\
\rho_v(\xi, \Gamma\xi) &= \left|\frac{2}{3} - \frac{2}{3}\right|^3 + \left(\frac{2}{3} - \frac{2}{3}\right)^2 = 0, \\
\rho_v(\zeta, \Gamma\xi) &= \left|\frac{1}{2} - \frac{2}{3}\right|^3 + \left(\frac{1}{2} - \frac{2}{3}\right)^2 = 0.032, \\
\rho_v(\xi, \Gamma\zeta) &= \left|\frac{2}{3} - \frac{5}{12}\right|^3 + \left(\frac{2}{3} - \frac{5}{12}\right)^2 = 0.078.
\end{aligned}$$

Then, we can draw conclusions on the following inequalities:

1. $\rho_v(\Gamma\zeta, \Gamma\xi) \leq \alpha\rho_v(\zeta, \xi) + \beta\rho_v(\zeta, \Gamma\zeta) + \gamma\rho_v(\xi, \Gamma\xi) \implies 0.078 \leq 0.032\alpha + 0.0075\beta$ *is absurd; thus, the Reich contraction is not satisfied. Therefore, Theorem 2.4 in [19] does not guarantee the existence of a fixed point for map Γ.*
2. $\rho_v(\Gamma\zeta, \Gamma\xi) \leq \delta \max\{\rho_v(\zeta, \xi), \rho_v(\zeta, \Gamma\zeta), \rho_v(\xi, \Gamma\xi)\} + L \min\{\rho_v(\zeta, \Gamma\zeta), \rho_v(\xi, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \implies 0.078 \leq \delta \max\{0.032, 0.0075, 0\} + L \min\{00075, 0, 0.078\} = 0.032\delta$, *which is not true for any $\delta \in [0, 1)$ and $L \geq 0$. So, Corollary 2.6.2 in [20] does not work to ensure the existence of a fixed point for the given map Γ.*

4. Consequences

Suppose that Δ is a nonempty set and $\Gamma, I : \Delta \to \Delta$ are self maps. A point $\zeta \in \Delta$ is called a coincidence point (common fixed point) for maps Γ and I if $\Gamma\zeta = I\zeta$ ($\zeta = \Gamma\zeta = I\zeta$). Moreover, maps Γ and I are called weakly compatible if they commute at every coincidence point. In 2011, Haghi et al. [26] proved that some common fixed point theorems are the consequences of existing fixed point theorems. This was ensured by proving the following lemma.

Lemma 2. *Let Γ be a self map defined on a nonempty set Δ. Then, there exists a subset Π of Δ such that $\Gamma(\Pi) = \Gamma(\Delta)$ and the map $\Gamma : \Pi \to \Delta$ is one-to-one.*

As a consequence of the main results proved in the previous section, we obtain some common fixed point theorems.

Theorem 3. *Let (Δ, ρ_v) be a v-generalized b-metric space and Γ, I be self maps defined on Δ such that*

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) \leq{}& \alpha\rho_v(I\zeta, I\xi) + \beta\rho_v(I\zeta, \Gamma\zeta) + \gamma\rho_v(I\xi, \Gamma\xi) \\
&+ L \min\{\rho_v(I\zeta, \Gamma\zeta) + \rho_v(I\xi, \Gamma\xi), \rho_v(I\zeta, \Gamma\xi), \rho_v(I\xi, \Gamma\zeta)\} \qquad (15)
\end{aligned}$$

for all $\zeta, \xi \in \Delta$, where $\alpha + \beta + \gamma < 1, L \geq 0$ with $\min\{\beta, \gamma\} < \frac{1}{s}$. If $\Gamma(\Delta) \subseteq I(\Delta)$ and $I(\Delta)$ is complete, then Γ and I have a unique coincidence point. Moreover, Γ and I have a unique common fixed point if they are weakly compatible maps.

Proof. By Lemma 2, there exists a subset Π of Δ such that $I : \Pi \to \Delta$ is one-to-one and $I(\Pi) = I(\Delta)$. Consider a map $p : I(\Pi) \to I(\Pi)$ defined as $p(I\zeta) = \Gamma\zeta$. Then, clearly, p is well defined, as I is one-to-one. Further, we obtain that

$$\begin{aligned}
\rho_v(p(I\zeta), p(I\xi)) &= \rho_v(\Gamma\zeta, \Gamma\xi) \\
&\leq \alpha\rho_v(I\zeta, I\xi) + \beta\rho_v(I\zeta, \Gamma\zeta) + \gamma\rho_v(I\xi, \Gamma\xi) \\
&\quad + L\min\{\rho_v(I\zeta, \Gamma\zeta) + \rho_v(I\xi, \Gamma\xi), \rho_v(I\zeta, \Gamma\xi), \rho_v(I\xi, \Gamma\zeta)\} \\
&= \alpha\rho_v(I\zeta, I\xi) + \beta\rho_v(I\zeta, p(I\zeta)) + \gamma\rho_v(I\xi, p(I\xi)) \\
&\quad + L\min\{\rho_v(I\zeta, p(I\zeta)) + \rho_v(I\xi, p(I\xi)), \rho_v(I\zeta, p(I\xi)), \rho_v(I\xi, p(I\zeta))\}
\end{aligned}$$

for all $I\zeta, I\xi \in I(\Pi) = I(\Delta)$. As $\alpha + \beta + \gamma < 1$, $L \geq 0$ with $\min\{\beta, \gamma\} < \frac{1}{s}$, then p is a generalized almost contraction of Reich type on $I(\Delta)$. Furthermore, since $I(\Delta)$ is complete, then on account of Theorem 2, there exists a unique point $x^* \in \Pi \subseteq \Delta$ such that $p(Ix^*) = Ix^*$ that gives $Ix^* = \Gamma x^*$. Thus, x^* is a unique coincidence point of Γ and I. Denote $w = Ix^* = \Gamma x^*$, then as Γ and I are weakly compatible, it follows that $\Gamma w = \Gamma I x^* = I\Gamma x^* = Iw$. Therefore,

$$\begin{aligned}
\rho_v(\Gamma w, w) &= \rho_v(\Gamma w, \Gamma x^*) \\
&\leq \alpha\rho_v(Iw, Ix^*) + \beta\rho_v(Iw, \Gamma w) + \gamma\rho_v(Ix^*, \Gamma x^*) \\
&\quad + L\min\{\rho_v(Iw, \Gamma w) + \rho_v(Ix^*, \Gamma x^*), \rho_v(Iw, \Gamma x^*), \rho_v(Ix^*, \Gamma w)\} \\
&\leq \alpha\rho_v(\Gamma w, w),
\end{aligned} \tag{16}$$

which is true for $\alpha \in [0, 1)$ if $w = \Gamma w = Iw$; hence, w is a common fixed point of Γ and I, and it is also unique. □

If we take $L = 0$ in Theorem 3, then we obtain the following corollary which is an extension of Theorem 2.4 in [19].

Corollary 2. *Let (Δ, ρ_v) be a v-generalized b-metric space and Γ, I be self maps defined on Δ such that*

$$\rho_v(\Gamma\zeta, \Gamma\xi) \leq \alpha\rho_v(I\zeta, I\xi) + \beta\rho_v(I\zeta, \Gamma\zeta) + \gamma\rho_v(I\xi, \Gamma\xi) \tag{17}$$

for all $\zeta, \xi \in \Delta$, where $\alpha + \beta + \gamma < 1$ with $\min\{\beta, \gamma\} < \frac{1}{s}$. If $\Gamma(\Delta) \subseteq I(\Delta)$ and $I(\Delta)$ is complete, then Γ and I have a unique coincidence point. Moreover, maps Γ and I have unique common fixed points provided that they are weakly compatible.

Remark 5. *If (Δ, ρ_v) is a v-generalized b-metric space and Γ, I are self maps on X satisfying $\rho_v(\Gamma\zeta, \Gamma\xi) \leq k\max\{\rho_v(I\zeta, I\xi), \rho_v(I\zeta, \Gamma\zeta), \rho_v(I\xi, \Gamma\xi)\}$, for all $\zeta, \xi \in \Delta$, where $k \in [0, 1)$, then it is obvious that $\rho_v(\Gamma\zeta, \Gamma\xi) \leq \alpha\rho_v(I\zeta, I\xi) + \beta\rho_v(I\zeta, \Gamma\zeta) + \gamma\rho_v(I\xi, \Gamma\xi)$ for some non-negative α, β, γ such that $\alpha + \beta + \gamma < 1$. Thus, it is concluded that Corollary 2.6.1 in [20] is a particular case of the above corollary.*

Theorem 4. *Let (Δ, ρ_v) be a v-generalized b-metric space and Γ, I be self maps defined on Δ such that*

$$\begin{aligned}
\rho_v(\Gamma\zeta, \Gamma\xi) &\leq \delta\max\left\{\rho_v(I\zeta, I\xi), \frac{\rho_v(I\zeta, \Gamma\zeta)\cdot\rho_v(I\xi, \Gamma\xi)}{1 + \rho_v(I\zeta, I\xi)}\right\} \\
&\quad + L\min\{\rho_v(I\zeta, \Gamma\zeta) + \rho_v(I\xi, \Gamma\xi), \rho_v(I\zeta, \Gamma\xi), \rho_v(I\xi, \Gamma\zeta)\}
\end{aligned} \tag{18}$$

for all $\zeta, \xi \in \Delta$, where $\delta \in [0, 1)$ and $L \geq 0$. If $\Gamma(\Delta) \subseteq I(\Delta)$ and $I(\Delta)$ is complete. Then, Γ and I have a unique coincidence point. Moreover, Γ and I have a unique common fixed point if they are weakly compatible maps.

Proof. By following the argument as we mention in Theorem 3, it is easy to verify that map $p : I(\Pi) \to I(\Pi)$ defined as $p(I\zeta) = \Gamma\zeta$ is a generalized almost contraction of rational type on $I(\Delta)$. Then, on account of Theorem 1, there exists a unique coincidence point, e.g., x^*, of Γ and I. As maps Γ and I are weakly compatible, it is easy to examine that they have a unique common fixed point. $\square$

5. Application to Fredholm Integral Equation

Now, we shall discuss the existence and uniqueness of the solution of the Fredholm type of integral equation.

Let us consider space $\Delta = C[a,b] = \{\zeta |\ \zeta : [a,b] \to \mathbb{R}$ is continuous on $[a,b]\}$ and integral equation

$$\zeta(t) = \tau(t) + \int_a^b k(t,s,\zeta(s))\, ds \text{ for } t,s \in [a,b] \tag{19}$$

where $\tau \in \Delta$ and $k : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$. Define $\rho_v : \Delta \times \Delta \to [0,+\infty)$ with

$$\rho_v(\zeta,\xi) = \sup_{t\in[a,b]} |\zeta(t) - \xi(t)|^3,\ \forall\, \zeta,\xi \in \Delta.$$

Then, (Δ,ρ_v) is a complete v-generalized b-metric space for $v = 1$ and $s \geq 4$. In addition, consider a self map $\Gamma : \Delta \to \Delta$ defined as

$$\Gamma\zeta(t) = \tau(t) + \int_a^b k(t,s,\zeta(s))\, ds \text{ for } t,s \in [a,b].$$

Then, it is clear that the solution of integral Equation (19) is nothing but the fixed point of map Γ.

Theorem 5. *Suppose that*

$$|k(t,s,\zeta(s)) - k(t,s,\xi(s))| \leq \left(\kappa_1 N(\zeta,\xi) + \kappa_2|\zeta - \xi|^3\right)^{\frac{1}{3}} \tag{20}$$

for some $\kappa_1 \geq \frac{1}{(b-a)^3}$, $\kappa_2 < \frac{1}{(b-a)^3}$ *and for all* $t,s \in [a,b]$; $\zeta,\xi \in \Delta$, *where* $N(\zeta,\xi) = \min\{\rho_v(\zeta,\Gamma\zeta) + \rho_v(\xi,\Gamma\xi), \rho_v(\zeta,\Gamma\xi), \rho_v(\xi,\Gamma\zeta)\}$*.Then, there exists a unique solution for integral Equation (19).*

Proof. We consider

$$\begin{aligned}
|\Gamma\zeta(t) - \Gamma\xi(t)|^3 &\leq \left(\int_a^b |k(t,s,\zeta(s)) - k(t,s,\xi(s))|\, ds\right)^3 \\
&\leq \left(\int_a^b \left(\kappa_1 N(\zeta,\xi) + \kappa_2|\zeta - \xi|^3\right)^{\frac{1}{3}} ds\right)^3 \\
&\leq \left(\kappa_1 N(\zeta,\xi) + \kappa_2|\zeta - \xi|^3\right)\left(\int_a^b ds\right)^3 \\
&\leq (b-a)^3[\kappa_1 N(\zeta,\xi) + \kappa_2\rho_v(\zeta,\xi)].
\end{aligned}$$

Therefore, map Γ satisfies inequality (13) for any $\kappa_1 \geq \frac{1}{(b-a)^3}$, $\kappa_2 < \frac{1}{(b-a)^3}$; hence, all the hypotheses of Corollary 1 are satisfied. Thus, Fredholm Equation (19) has a unique solution. $\square$

6. Conclusions and Open Problem

In the present work, we deal with v-generalized b-metric spaces of [19]. Firstly, by providing a relevant example, it is made clear that v-generalized b-metric spaces are proper extensions of standard metric spaces, b-metric spaces, rectangular b-metric spaces [27], and v-generalized metric spaces. Thereafter, we discuss the existence and uniqueness of fixed points for generalized almost contractions of rational type and Reich type defined in v-generalized b-metric spaces. Consequently, the coincidence point and common fixed point are guaranteed to exist and to be unique for any pair of mappings fulfilling certain hypotheses in these spaces. Additionally, it is made obvious by providing a number of significant examples that the contractions addressed in this paper are crucial in extending the fixed point theorems of the aforementioned spaces. Furthermore, we deduce that the proven results ensure the existence and uniqueness of the solution of the Fredholm integral equation. Moreover, this work leads to the following open problems for the possible scope of research on v-generalized b-metric spaces:

1. Is it feasible to relax the hypothesis $\min\{\beta, \gamma\} < \frac{1}{s}$ in Theorem 2?
2. Is the existence and uniqueness of fixed point for a generalized almost contraction of Cirić type in v-generalized b-metric spaces, i.e., a map $\Gamma : \Delta \to \Delta$ satisfying

$$\begin{aligned}\rho_v(\Gamma\zeta, \Gamma\xi) \quad &\leq \quad \lambda \max\{\rho_v(\zeta, \xi), \rho_v(\zeta, \Gamma\zeta), \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\} \\ &\quad +L \min\{\rho_v(\zeta, \Gamma\zeta) + \rho_v(\xi, \Gamma\xi), \rho_v(\zeta, \Gamma\xi), \rho_v(\xi, \Gamma\zeta)\},\end{aligned}$$

   for some $\lambda \in [0, 1), L \geq 0$ and all $\zeta, \xi \in \Delta$, guaranteed or not?

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