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Fourth-Order Numerical Solutions for a Fuzzy Time-Fractional Convection–Diffusion Equation under Caputo Generalized Hukuhara Derivative

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Abstract: The fuzzy fractional differential equation explains more complex real-world phenomena than the fractional differential equation does. Therefore, numerous techniques have been timely derived to solve various fractional time-dependent models. In this paper, we develop two compact finite difference schemes and employ the resulting schemes to obtain a certain solution for the fuzzy time-fractional convection–diffusion equation. Then, by making use of the Caputo fractional derivative, we provide new fuzzy analysis relying on the concept of fuzzy numbers. Further, we approximate the time-fractional derivative by using a fuzzy Caputo generalized Hukuhara derivative under the double-parametric form of fuzzy numbers. Furthermore, we introduce new computational techniques, based on fuzzy double-parametric form, to shift the given problem from one fuzzy domain to another crisp domain. Moreover, we discuss some stability and error analysis for the proposed techniques by using the Fourier method. Over and above, we derive several numerical experiments to illustrate reliability and feasibility of our proposed approach. It was found that the fuzzy fourth-order compact implicit scheme produces slightly better results than the fourth-order compact FTCS scheme. Furthermore, the proposed methods were found to be feasible, appropriate, and accurate, as demonstrated by a comparison of analytical and numerical solutions at various fuzzy values.

Keywords: fuzzy time-fractional equation; convection–diffusion equation; fuzzy Caputo gH-derivative; finite difference methods; implicit scheme method; brownian motion

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1. Introduction

In recent years, the study of solving fractional partial differential equations has attracted the attention of many researchers. This can be appraised through a large number of research articles dealing with such equations in several scientific databases. The time-fractional convection–diffusion equation (TFCDE) differs from the integer convection–diffusion equation in the sense that time-fractional derivative can be replaced by a fractional derivative to describe both the movement and speed of particles that are inconsistent with the classical Brownian motion type [1–5]. The exact solutions are often unobtainable using analytical methods. Thus, mathematicians have resorted to using numerical methods to provide solutions for the governing equations. Several finite difference methods discussed by many authors [6–8] are considered to be one of the most essential numerical techniques for solving the time-fractional convection–diffusion equations. The high-order compact finite difference methods are usually preferred due to their high computational efficiency and accuracy. Lately, a number of research articles have been published regarding high-order compact finite difference schemes to solve the time-fractional convection–diffusion equations.

Gao and Sun [9] developed two different three-point combined compact alternating-direction implicit schemes CC–ADI to solve time-fractional convection–diffusion in the sense of the Riemann–Liouville fractional derivative. The CC–ADI is a combination of compact and alternating-direction implicit schemes that yield a high-order accuracy numerical solution. Several numerical examples are carried out to demonstrate the efficiency of the proposed schemes. The unconditional stabilities of the proposed schemes were proven with the Fourier method. Later, Fazio et al. [10] developed an implicit scheme of non-uniform grids to solve the time-fractional convection–diffusion equation. A spatial non-uniform net was utilized in order to increase the accuracy of the Caputo fractional derivative. The stability analysis of the proposed method showed that the method is unconditionally stable. Several numerical examples were reported to show that the finite difference method is more accurate for the non-uniform grid than the uniform mesh, and the stability is better in the non-uniform mesh than the uniform mesh.

Very recently, Sweilam et al. [11] used the compact finite difference method to obtain a numerical solution for the stochastic fractional convection–diffusion equation. The fractional derivative was approximated by the Caputo definition, and the stability and consistency of the presented method were discussed. Two experiments were also presented to examine the performance of the proposed method. It was found that all of the results obtained are compatible with the analytical exact solutions. Li et al. [12] used a fourth-order compact scheme for solving a fluid dynamic problem, groundwater pollution modeled by two-dimensional TFCDE. The time-fractional derivatives of the considered equation ere approximated by Caputo fractional derivative and fourth-order accuracy compact finite difference discretization that was applied to the spatial derivatives. The solvability, convergence, and stability of the proposed scheme were studied on the basis of the von Neumann method. It was further established that the introduced method has unique solvability and convergence with order $O\left(\tau^{2-\alpha}+{h_1}^4+{h_2}^4\right)$.

In the mainstream investigation of the processes modelled by fractional partial differential equations, the variables and parameters are defined exactly, but these quantities (variables and parameters) may be uncertain and vague due to measurement errors that occur in the real experiments and that lead to fuzzy fractional partial differential equations.

Senol et al. [13] developed the perturbation–iteration algorithm (PIA) for solving fuzzy time-fractional partial differential equations with a generalized Hukuhara derivative. The fuzzy time-fractional derivative was approximated by the use of the Caputo definition. The convergence analysis of the proposed method was discussed and showed that the proposed approach gives a fast convergence rate and high accuracy when compared with the exact analytical solutions of the crisp problem. Shah et al. [14] presented analytical solutions of fuzzy time-fractional partial differential equations under certain conditions. The Laplace transform was used to compute series-type solutions for the considered equations under the fuzzy concept. Some examples were solved to illustrate the feasibility of the proposed method. Recently, two finite difference methods that are implicit backward time center space (BTCS) and implicit schemes were developed by the authors in [15] to solve the fuzzy time-fractional convection–diffusion equation (FTFCDE).

Based on the literature, it seems, as far as we know, that limited research has been done in the field of fuzzy time-fractional convection–diffusion equations by using classical and compact finite difference methods. Our motivation in this article is to examine the solution of the fuzzy time-fractional convection–diffusion equation. In this research article, we will develop and implement compact finite difference methods, in particular the fourth-order compact Crank–Nicholson and the fourth-order forward time center space schemes to obtain an approximate solution for the time-fractional convection–diffusion equation in the double-parametric form of a fuzzy number.

2. Preliminaries and Fundamental Definitions

In this section, the main definitions and theorems that are utilized later in this article are considered as follows:

Definition 1 [16]. *Let $\mathbb{R}_F$ denote the set of fuzzy subsets of the real axis and $u : R \rightarrow [0,\ 1]$ satisfies the following properties:*

1. *u is upper semi-continuous on $\mathbb{R}$,*
2. *u is fuzzy convex,*
3. *u is normal,*
4. *Closure of $\{x \in \mathbb{R} | u(x) > 0\}$ is compact,*

Then, by $\mathbb{R}_F$ we denote the space of fuzzy numbers for every $0 < r \leq 1$.

Denote by $u(r) = \left\{ x\epsilon\mathbb{R}^n \vdots u(x) \geq r \right\} = \underline{u}(r) - \overline{u}(r)$. Then, from Definition 1, it follows that the $r -$ level set $u(r)$ is a closed interval for all r, $0 \leq r \leq 1$. For arbitrary $u,\ v \in \mathbb{R}_F$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $(u \oplus v)(r) = u(r) + v(r)$, $(k \odot u)(r) = [k\underline{u}(r),\ k\overline{u}(r)]$, respectively.

Definition 2 [16]. *The Hausdorff distance between the fuzzy numbers is defined by $d : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}^+ \cup \{0\}$ as $d(u,v) = sup_{r\epsilon[0,1]}\ Max\ \{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$. The $r -$ level representation of the fuzzy-valued function $f : [a,\ b] \rightarrow \mathbb{R}_F$ is given by $(x;\ r) = [f\ (x;\ r),\ f\ (x;\ r)]$,$x \in [a,\ b]$, $0 \leq r \leq 1$.*

Definition 3 [17]. *Let $u,\ v \in \mathbb{R}_F$ and there exist $w \in \mathbb{R}_F$ such that $u = v + w$. Then, w is called the Hukuhara difference of u and v , and it is denoted by $u \ominus v$, fora $\leq x$ and $0 < \alpha \leq 1$.*

Definition 4 [18]. *The generalized Hukuhara difference (gH-differentiable for short) of two fuzzy numbers $u,\ v \in \mathbb{R}_F$, gH- difference for short) is $w \in \mathbb{R}_F$ defined by*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i)\ u = v + w \\ or\ (ii)\ v = u + (-1)w \end{cases}.$$

Definition 5 [19]. *The gH-differentiable of a fuzzy-valued function $f(a,b) \rightarrow \mathbb{R}_F$ at x_0 is defined by*

$$(f')_{gH}(x_0) = \frac{\lim_{h\rightarrow 0} f(x_0 + h) \ominus gH\ f(x_0)}{h}.$$

If $(f')_{gH}(x_0)\epsilon\mathbb{R}_F$, we say that f is gH-differentiable at x_0. In addition, we say that f is $[(i) - gH]$-differentiable at x_0 if:

5. $(f')_{gH}(x_0; r) = [(\underline{f})'(x_0; r)\ ,\ \left(\overline{f}\right)'\ (x_0; r)\Big]$, $r \in [0,\ 1]$, *and that f is $[(ii) - gH]$-differentiable* at x_0 *if*
6. $(f')_{gH}(x_0; r) = [(\overline{f})'(x_0; r)\ ,\ \left(\underline{f}\right)'(x_0; r)\Big]$, $r \in [0,\ 1]$.

Definition 6 [19]. *We say that a point $x_0 \in (a,\ b)$ is a switching point for the differentiability of f if, in any neighborhoodV of x_0 , there exists points $x_1 < x_0 < x_2$ such that: Type (I): at x_1 (i) holds while (ii) does not hold and, at x_2 , (ii) holds and (i) does not hold, Type (II): at x_1 (ii) holds while (i) does not hold and, at x_2 , (i) holds and (ii) does not hold.*

Definition 7 [20]. *The fractional derivative was defined as follows*

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t} f(\xi)}{(t-\xi)^\alpha} \partial\xi,\ 0 < \alpha < 1.$$

The time-fractional derivative term is approximated as [21]

$$\frac{\partial^{\alpha} u(x,t)}{\partial^{\alpha} t} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum\nolimits_{j=1}^{n} b_j \left(u_i^{n+1-j} - u_i^{n-j}\right),$$

where $b_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$, $j = 0, 1, 2, \ldots$.

Definition 8 [22]. *The fuzzy fractional Caputo gH- derivative of a fuzzy valued function* $f(t;r)$ *based on (i) and (ii) of gH-differentiable function f is defined as follows:*

7. $({}_{gH}^{c}D_t^{\alpha} f(t;r) = \left[{}_0^cD_t^{\alpha}\underline{f}(t;r), {}_0^cD_t^{\alpha}\overline{f}(t;r)\right]$, $0 \le r \le 1$,
8. $({}_{gH}^{c}D_t^{\alpha} f(t;r) = \left[{}_0^cD_t^{\alpha}\overline{f}(t;r), {}_0^cD_t^{\alpha}\underline{f}(t;r)\right]$, $0 \le r \le 1$,

where ${}_0^cD_t^{\alpha}\underline{f}(t;r) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t}\underline{f}(\xi;r)}{(t-\xi)^{\alpha}} \partial\xi$ *and* ${}_0^cD_t^{\alpha}\overline{f}(t;r) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t}\overline{f}(\xi;r)}{(t-\xi)^{\alpha}} \partial\xi$.

Definition 9 [15]. *(The double-parametric form of fuzzy numbers) Using the single-parametric form, we write* $\widetilde{U} = [\underline{u}(r), \overline{u}(r)]$ *, which may be written as a crisp number using the double-parametric form as*

$$\widetilde{U}(r,\beta) = \beta[\overline{u}(r) - \underline{u}(r)] + \underline{u}(r), \text{where } r \text{ and } \beta \in [0,1].$$

3. High-Order Compact Finite Difference Method in Fuzzy Environment

Let u_i^n indicate the approximation value of u at (x_i, t_n). Based on the Taylor series expansions $u_{i+1}^n$and u_{i+1}^n , u can be expanded about (x_i, t_n) to derive the fuzzy high-order compact finite difference scheme for the spatial derivatives.

$$\left.\begin{array}{l} \tilde{u}{}_{i+1}^{n} = \tilde{u}{}_{i}^{n} + h\left(\frac{\partial\widetilde{u}}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n + \cdots \\ \tilde{u}{}_{i-1}^{n} = \tilde{u}{}_{i}^{n} - h\left(\frac{\partial\widetilde{u}}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n + \ldots \end{array}\right\}. \tag{1}$$

The first derivatives of u_{i+1}^n and u_{i-1}^n are

$$\left.\begin{array}{l} \left(\frac{\partial\widetilde{u}}{\partial x}\right)_{i+1}^n = \left(\frac{\partial\widetilde{u}}{\partial x}\right)_i^n + h\left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^4\widetilde{u}}{\partial x^4}\right)_i^n + \cdots \\ \left(\frac{\partial\widetilde{u}}{\partial x}\right)_{i-1}^n = \left(\frac{\partial\widetilde{u}}{\partial x}\right)_i^n - h\left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^4\widetilde{u}}{\partial x^4}\right)_i^n + \ldots \end{array}\right\}. \tag{2}$$

The second derivatives of u_{i+1}^n and u_{i-1}^n are

$$\left.\begin{array}{l} \left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_{i+1}^n = \left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n + h\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4\widetilde{u}}{\partial x^4}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^5\widetilde{u}}{\partial x^5}\right)_i^n + \cdots \\ \left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_{i-1}^n = \left(\frac{\partial^2\widetilde{u}}{\partial x^2}\right)_i^n - h\left(\frac{\partial^3\widetilde{u}}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4\widetilde{u}}{\partial x^4}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^5\widetilde{u}}{\partial x^5}\right)_i^n + \ldots \end{array}\right\}. \tag{3}$$

From Equations (1)–(3), the first and second partial derivatives are approximated to give

$$\left(\frac{\partial u}{\partial x}\right)_i^n = \frac{\delta_x/2h}{\left(1 + \frac{1}{6}\,\delta^2{}_x\right)} u_i^n + \frac{h^4}{180}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + O\left(h^5\right), \tag{4}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n = \frac{\delta^2{}_x/h^2}{\left(1 + \frac{1}{12}\,\delta^2{}_x\right)} u_i^n + \frac{h^4}{240}\left(\frac{\partial^6 u}{\partial x^6}\right)_i^n + O\left(h^6\right), \tag{5}$$

where $\delta_x = \widetilde{u}_{i+1}^n - u_{i-1}^n$ and $\delta^2{}_x = \widetilde{u}_{i+1}^n - 2\widetilde{u}_i^n + \widetilde{u}_{i-1}^n$ for $0 \le i \le M$, $0 \le n \le N$.

By taking into account the average function that mentioned in [23], we assume that

$$\frac{1}{\left(1+\frac{1}{6}\,\delta^2{}_x\right)}\widetilde{u}_i^n = \frac{1}{6}\left(\widetilde{u}_{i+1}^n + 4\widetilde{u}_i^n + \widetilde{u}_{i-1}^n\right),\ 1 \leq\ i \leq M-1, \tag{6}$$

$$\frac{1}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}\widetilde{u}_i^n = \frac{1}{12}\left(\widetilde{u}_{i+1}^n + 10\widetilde{u}_i^n + \widetilde{u}_{i-1}^n\right),\ 1 \leq\ i \leq M-1. \tag{7}$$

4. Time Fractional Convection–Diffusion Equation in Fuzzy Environment

A fuzzy fractional convection–diffusion equation is used to describe both the speed and movement of particles that are incompatible with the classical Brownian motion pattern. The fuzzy fractional convection–diffusion equation can be used for modelling some physical problems such as groundwater hydrology and gas transport through heterogeneous soil. The applications also exist in aerodynamics and other fields [24–27].

Let us now consider the general formula of the fuzzy time-fractional convection–diffusion equation involving the boundary and initial conditions [28]

$$\begin{gathered} \frac{\partial^\alpha \widetilde{u}(x,t,\alpha)}{\partial^\alpha t} = -\widetilde{v}(x)\frac{\partial \widetilde{u}(x,t)}{\partial x} - \widetilde{D}(x)\frac{\partial^2 \widetilde{u}(x,t)}{\partial x^2} + \widetilde{q}(x,t) \quad ,\ 0 < x < l, t < 0, \\ \widetilde{u}(x,0) = \widetilde{f}(x),\ \widetilde{u}(0,t) = \widetilde{g}(0,t),\ \widetilde{u}(l,t) = \widetilde{z}\,(l,t), \end{gathered} \tag{8}$$

where $\widetilde{u}(x,t,\alpha)$ is the density of a quantity such as fuzzy energy; the fuzzy mass of crisp variables x, t, and α is an arbitrary order such that $\frac{\partial^\alpha \widetilde{U}(x,t,\alpha)}{\partial^\alpha t}$ denotes the fuzzy time-fractional generalized Hukuhara derivative (gH-derivative) of order α; $\widetilde{v}(x)$ is the average velocity of a fuzzy quantity; $\widetilde{D}(x)$ is the diffusivity coefficient; $\widetilde{q}(x,t)$ is a function for the uncertainty crisp variable x; t, $\widetilde{u}(0,t)$ and $\widetilde{u}(l,t)$ are the fuzzy boundary conditions involving $\widetilde{g}$, $\widetilde{z}$ is defined as a fuzzy convex number; and $\widetilde{u}(x,0)$ is the fuzzy initial condition.

In Equation (8), the fuzzy functions $\widetilde{f}(x)$, $\widetilde{v}(x)$, $\widetilde{D}(x)$, and $\widetilde{q}(x)$ are defined as follows [29]:

$$\begin{cases} \widetilde{D}(x) = \widetilde{\theta}_1 s_1(x) \\ \widetilde{q}(x) = \widetilde{\theta}_2 s_2(x) \\ \widetilde{f}(x) = \widetilde{\theta}_3 s_3(x) \\ \widetilde{v}(x) = \widetilde{\theta}_4 s_4(x) \end{cases} \tag{9}$$

where $s_1(x)$, $s_2(x)$, $s_3(x)$ and $s_4(x)$ are the crisp (classical) functions of the classical variable x where $\widetilde{\theta}_1$, $\widetilde{\theta}_2$, $\widetilde{\theta}_3$ and $\widetilde{\theta}_4$ are introduced as the fuzzy convex numbers. The fuzzy time-fractional convection–diffusion equation is defuzzified based on the double-parametric approach of fuzzy numbers as [14]:

$$[\widetilde{u}(x,t)]_r = \underline{u}(x,t;r), \overline{u}(x,t;r), \tag{10}$$

$$\left[\frac{\partial^\alpha \widetilde{u}(x,t,\alpha)}{\partial^\alpha t}\right]_r = \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}, \frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t}, \tag{11}$$

$$\left[\frac{\partial \widetilde{u}(x,t)}{\partial x}\right]_r = \frac{\partial \underline{u}(x,t;r)}{\partial x}, \frac{\partial \overline{u}(x,t;r)}{\partial x} \tag{12}$$

$$\left[\frac{\partial^2 \widetilde{u}(x,t)}{\partial x^2}\right]_r = \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}, \frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2}, \tag{13}$$

$$[\widetilde{v}(x)]_r = \underline{v}(x;r), \overline{v}(x;r) \tag{14}$$

$$\left[\widetilde{D}(x)\right]_r = \underline{D}(x;r), \overline{D}(x;r) \tag{15}$$

$$[\widetilde{q}(x)]_r = \underline{q}(x;r), \overline{q}(x;r) \tag{16}$$

$$[\widetilde{u}(x,0)]_r = \underline{u}(x,0;r), \overline{u}(x,0;r) \tag{17}$$

$$[\widetilde{u}(0,t)]_r = \underline{u}(0,t;r), \overline{u}(0,t;r) \tag{18}$$

$$[\widetilde{u}(l,t)]_r = \underline{u}(l,t;r), \overline{u}(l,t;r), \tag{19}$$

$$\left[\widetilde{f}(x)\right]_r = \underline{f}(x;r), \overline{f}(x;r), \tag{20}$$

$$\begin{cases} [\widetilde{g}]_r = \underline{g}(r), \overline{g}(r), \\ [\widetilde{z}]_r = \underline{z}(r), \overline{z}(r), \end{cases} \tag{21}$$

such that

$$\begin{cases} [D(x)]_r = \left[\underline{\theta}(r)_1, \overline{\theta}_1(r)\right] s_1(x) \\ [\widetilde{q}(x)]_r = \left[\underline{\theta}(r)_2, \overline{\theta}_2(r)\right] s_2(x) \\ \left[\widetilde{f}(x)\right]_r = \left[\underline{\theta}(r)_3, \overline{\theta}_3(r)\right] s_3(x) \\ [v(x)]_r = \left[\underline{\theta}(r)_4, \overline{\theta}_4(r)\right] s_4(x) \end{cases}. \tag{22}$$

By employing the fuzzy extension principle, the membership function is defined as [10]

$$\begin{cases} \underline{u}(x,t;r) = min\{\widetilde{u}(\widetilde{\mu}(r),t)) | \widetilde{\mu}(r) \in \widetilde{u}(x,t;r)\}, \\ \overline{u}(x,t;r) = max\{\widetilde{u}(\widetilde{\mu}(r),t) | \widetilde{\mu}(r) \in \widetilde{u}(x,t;r)\}. \end{cases} \tag{23}$$

Now, for $0\langle x \rangle l,\ t > 0$ and $r \in [0,\ 1]$, Equation (8) is rewritten to yield the general equation of the fuzzy time-fractional convection–diffusion equation

$$\begin{cases} \frac{\partial^\alpha \underline{u}(x,t,\alpha)}{\partial^\alpha t} = -[\underline{\theta}(r)_4] s_4(x) \frac{\partial \underline{u}(x,t;r)}{\partial x} + [\underline{\theta}(r)_1] s_1(x) \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2} + [\underline{\theta}(r)_2] s_2(x), \\ \underline{u}(x,0;r) = \underline{\theta}(r)_3 s_3(x), \\ \underline{u}(0,t;r) = \underline{g}(r), \underline{u}(l,t;r) = \underline{z}(r), \end{cases} \tag{24}$$

$$\begin{cases} \frac{\partial^\alpha \overline{u}(x,t,\alpha)}{\partial^\alpha t} = -\left[\overline{\theta}_4(r)\right] s_4(x) \frac{\partial \overline{u}(x,t;r)}{\partial x} + \left[\overline{\theta}_1(r)\right] s_1(x) \frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2} + \left[\overline{\theta}_2(r)\right] s_2(x), \\ \overline{u}(x,0;r) = \overline{\theta}_3(r)_3 s_3(x), \\ \overline{u}(0,t;r) = \overline{g}(r), \overline{u}(l,t;r) = \overline{z}(r). \end{cases} \tag{25}$$

Equations (24) and (25) present the lower and upper bounds of the general formula of the fuzzy time-fractional convection–diffusion equation. Now, for defuzzification, Equation (8), based on the double-parametric form of the fuzzy numbers, as per the singular-parametric form, may be expressed as

$$\begin{aligned} &\left[\frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}, \frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right] = -[\ \underline{v}(x,r), \overline{v}(x,r)] \left[\frac{\partial \underline{u}_{i,n}(x,t;r)}{\partial x}, \frac{\partial \overline{u}_{i,n}(x,t;r)}{\partial x}\right] \\ &+[\ \underline{D}(x,r), \overline{D}(x,r)] \left[\frac{\partial^2 \underline{u}_{i,n}(x,t;r)}{\partial x^2}, \frac{\partial^2 \overline{u}_{i,n}(x,t;r)}{\partial x^2}\right] + \left[\ \underline{q}(x,t;r), \overline{q}(x,t;r)\right], \end{aligned} \tag{26}$$

subject to the fuzzy boundary and initial conditions

$$[\ \underline{u}(x,0;r), \overline{u}(x,0;r)] = \left[\underline{f}(x,t;r), \overline{f}(x,t;r)\right],\ [\ \underline{u}(0,t;r), \overline{u}(0,t;r)] = \left[\underline{g}(0,t;r), \overline{g}(0,t;r)\right]$$

and $[\ \underline{u}(l,t;r), \overline{u}(l,t;r)] = [\underline{z}(l,t;r), \overline{z}(l,t;r)]$.

Now, via the double-parametric form (see, e.g., [14]), we rewrite Equation (26) as:

$$\begin{aligned} &\left\{\beta\left(\frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t} - \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right) + \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right\} \\ &= -\{\beta(\overline{v}(x,r) - \underline{v}(x,r)) + \underline{v}(x,r)\}\left\{\beta\left(\frac{\partial \overline{u}_{i,n}(x,t;r)}{\partial x} - \frac{\partial \underline{u}_{i,n}(x,t;r)}{\partial x}\right) + \frac{\partial \underline{u}_{i,n}(x,t;r)}{\partial x}\right\} \\ &+\{\beta(\overline{D}(x,r) - \underline{D}(x,r)) + \underline{D}(x,r)\}\left\{\beta\left(\frac{\partial^2 \overline{u}_{i,n}(x,t;r)}{\partial x^2} - \frac{\partial^2 \underline{u}_{i,n}(x,t;r)}{\partial x^2}\right) + \frac{\partial^2 \underline{u}_{i,n}(x,t;r)}{\partial x^2}\right\} \\ &+\{\beta(\overline{q}(x,t;r) - \underline{q}(x,t;r)) + \underline{q}(x,t;r)\}, \end{aligned} \tag{27}$$

subjected to fuzzy initial and boundary conditions

$$\{\ \beta\ (\overline{u}(x,0;r)-\underline{u}(x,0;r))+\underline{u}(x,0;r)\}=\left\{\ \beta\ \left(\overline{f}(x;r)-\underline{f}(x;r)\right)+\underline{f}(x;r)\right\},$$
$$\{\ \beta\ (\overline{u}(0,t;r)-\underline{u}(0,t;r))+\underline{u}(0,t;r)\}=\left\{\ \beta\ \left(\overline{g}(x;r)-\underline{g}(x;r)\right)+\underline{g}(x;r)\right\},$$

and

$$\{\ \beta\ (\overline{u}(l,t;r)-\underline{u}(l,t;r))+\underline{u}(l,t;r)\}=\{\ \beta\ (\overline{z}(x;r)-\underline{z}(x;r))+\underline{z}(x;r)\},$$

where $\beta\in[0,1]$. Now we donate

$$\begin{aligned}
&\tfrac{\partial^\alpha\widetilde{u}(x,t;r,\beta)}{\partial^\alpha t}=\left\{\ \beta\ \left(\tfrac{\partial^\alpha\overline{u}(x,t,\alpha;r)}{\partial^\alpha t}-\tfrac{\partial^\alpha\underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right)+\tfrac{\partial^\alpha\underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right\}\\
&\widetilde{v}(x)\tfrac{\partial\widetilde{u}(x,t;r,\beta)}{\partial x}=\{\ \beta\ (\overline{v}(x,r)-\underline{v}(x,r))+\underline{v}(x,r)\}\left\{\ \beta\ \left(\tfrac{\partial\overline{u}_{i,n}(x,t;r)}{\partial x}-\tfrac{\partial\underline{u}_{i,n}(x,t;r)}{\partial x}\right)+\tfrac{\partial\underline{u}_{i,n}(x,t;r)}{\partial x}\right\},\\
&\widetilde{a}(x)\tfrac{\partial^2\widetilde{u}(x,t;r,\beta)}{\partial x^2}=\{\ \beta\ (\overline{D}(x,r)-\underline{D}(x,r))+\underline{D}(x,r)\}\left\{\ \beta\ \left(\tfrac{\partial^2\overline{u}_{i,n}(x,t;r)}{\partial x^2}-\tfrac{\partial^2\underline{u}_{i,n}(x,t;r)}{\partial x^2}\right)+\tfrac{\partial^2\underline{u}_{i,n}(x,t;r)}{\partial x^2}\right\},\\
&\widetilde{q}(x,t;r,\beta)=\left\{\ \beta\ \left(\overline{q}(x,t;r)-\underline{q}(x,t;r)\right)+\underline{q}(x,t;r)\right\},\\
&\widetilde{u}(x,0;r,\beta)=\{\ \beta\ (\overline{u}(x,0;r)-\underline{u}(x,0;r))+\underline{u}(x,0;r)\},\\
&\widetilde{f}(x;r,\beta)=\left\{\ \beta\ \left(\underline{f}(x;r)-\overline{f}(x;r)\right)+\underline{f}(x;r)\right\},\\
&\widetilde{u}(0,t;r,\beta)=\{\ \beta\ (\overline{u}(0,t;r)-\underline{u}(0,t;r))+\underline{u}(0,t;r)\},\\
&\widetilde{g}(x;r,\beta)=\left\{\ \beta\ \left(\overline{g}(x;r)-\underline{g}(x;r)\right)+\underline{g}(x;r)\right\},\\
&\widetilde{u}(l,t;r,\beta)=\{\ \beta\ (\overline{u}(l,t;r)-\underline{u}(l,t;r))+\underline{u}(l,t;r)\},\\
&\widetilde{z}(x;r,\beta)=\{\ \beta\ (\overline{z}(x;r)-\underline{z}(x;r))+\underline{z}(x;r)\}.
\end{aligned}$$

Substituting these equations into Equation (26) reveals

$$\begin{gathered}\tfrac{\partial^\alpha\widetilde{u}(x,t,\alpha,\beta)}{\partial^\alpha t}=-\widetilde{v}(x)\tfrac{\partial\widetilde{u}(x,t,\beta)}{\partial x}+\widetilde{a}(x)\tfrac{\partial^2\widetilde{u}(x,t,\beta)}{\partial x^2}+\widetilde{b}(x,t,\beta)\ ,\ 0<x<l,\ 0<\beta<l,\ t<0,\\ \widetilde{u}(x,0,\beta)=\widetilde{f}(x,r,\beta),\ \widetilde{u}(0,t,\beta)=\widetilde{g}\ ,\ \widetilde{u}(l,t,\beta)=\widetilde{z}.\end{gathered} \tag{28}$$

To obtain the lower and upper solutions of equation (28) in the single parametric form, assume $\beta=0$ and $\beta=1$, respectively, to obtain

$$\widetilde{u}(x,t;r,0)=\underline{u}(x,t;r)\text{ and }\widetilde{u}(x,t;r,1)=\overline{u}(x,t;r).$$

5. The Fuzzy Fourth-Order Compact Implicit Scheme Method for the Solution of FTFCDE

In this section, the fourth-order compact implicit scheme method is developed and applied to a double-parametric form of fuzzy numbers, utilizing fourth-order approximation at time level $n+\frac{1}{2}$. In addition, the first- and second-order space derivatives, along with a fuzzy Caputo gH-derivative formula, are discretized to approximate the time-fractional derivative to solve the fuzzy time-fractional convection–diffusion equation.

To obtain an approximate solution to the fuzzy time-fractional convection–diffusion equation based on the fuzzy fourth-order compact implicit scheme method, the fuzzy Caputo gH- derivative formula has been applied to approximate the fuzzy time-fractional derivative given in Equation (8). The first and second space partial derivatives are, respectively, approximated by using Equations (4) and (5) as

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}[\widetilde{u}_{i,n+1}-\widetilde{u}_{i,n}+\sum_{j=1}^{n}b_j\ (\widetilde{u}_{i,n+1-j}-\widetilde{u}_{i,n-j})]=-\widetilde{v}(x,r)\frac{\frac{\delta_x}{2h}}{\left(1+\frac{1}{6}\ \delta^2{}_x\right)}+\widetilde{D}(x,r)\frac{\frac{\delta^2{}_x}{h^2}}{\left(1+\frac{1}{12}\ \delta^2{}_x\right)}+\widetilde{q}(x,r). \tag{29}$$

Hence, from Equations (6), (7) and (29) can be simplified to give

$$
\begin{aligned}
\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \times \tfrac{3}{12}\Big(& (\tilde{u}_{i+1}^{n+1} + 6\tilde{u}_{i}^{n+1} + \tilde{u}_{i-1}^{n+1}) - (\tilde{u}_{i+1}^{n} + 6\tilde{u}_{i}^{n} + \tilde{u}_{i-1}^{n}) \\
& + \sum_{j=1}^{n} b_j [(\tilde{u}_{i+1}^{n+1-j} + 6\tilde{u}_{i}^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}) - (\tilde{u}_{i+1}^{n-j} + 6\tilde{u}_{i}^{n-j} + \tilde{u}_{i-1}^{n-j})]\Big) \\
& = -\tilde{v}(x,r)\frac{\tilde{u}_{i+1,n+\frac{1}{2}} - \tilde{u}_{i-1,n+\frac{1}{2}}}{2h} + \tilde{D}(x,r)\frac{\tilde{u}_{i+1,n+\frac{1}{2}} - 2\tilde{u}_{i,n+\frac{1}{2}} + \tilde{u}^{2}_{i-1,n+\frac{1}{2}}}{h^2} \\
& + \left(\tilde{q}_{i+1}^{n+\frac{1}{2}} + 6\tilde{q}_{i}^{n+\frac{1}{2}} + \tilde{q}_{i-1}^{n+\frac{1}{2}}\right)
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \times \tfrac{3}{12}\Big(& (\tilde{u}_{i+1}^{n+1} + 6\tilde{u}_{i}^{n+1} + \tilde{u}_{i-1}^{n+1}) - (\tilde{u}_{i+1}^{n} + 6\tilde{u}_{i}^{n} + \tilde{u}_{i-1}^{n}) \\
& + \sum_{j=1}^{n} b_j [(\tilde{u}_{i+1}^{n+1-j} + 6\tilde{u}_{i}^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}) - (\tilde{u}_{i+1}^{n-j} + 6\tilde{u}_{i}^{n-j} + \tilde{u}_{i-1}^{n-j})]\Big) \\
& = -\tilde{v}(x,t,r)\tfrac{1}{2} \times \left[\frac{\tilde{u}_{i+1,n+1} - \tilde{u}_{i-1,n+1}}{2h} + \frac{\tilde{u}_{i+1,n} - \tilde{u}_{i-1,n}}{2h}\right] \\
& + \tilde{a}(x,t,r)\tfrac{1}{2}\left[\frac{\tilde{u}_{i+1,n+1} - 2\tilde{u}_{i,n+1} + \tilde{u}_{i-1,n+1}}{h^2} + \frac{\tilde{u}_{i+1,n} - 2\tilde{u}_{i,n} + \tilde{u}_{i-1,n}}{h^2}\right] + \tilde{b}(x,t,r).
\end{aligned}
\tag{31}
$$

Therefore, we have

$$
\begin{aligned}
& \tilde{u}_{i+1}^{n+1} + 6\tilde{u}_{i}^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 6\tilde{u}_{i}^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j [(\tilde{u}_{i+1}^{n+1-j} + 6\tilde{u}_{i}^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}) - (\tilde{u}_{i+1}^{n-j} + 6\tilde{u}_{i}^{n-j} + \tilde{u}_{i-1}^{n-j})] \\
& = -\frac{\tilde{v}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h}[3\tilde{u}_{i+1,n+1} - 3\tilde{u}_{i-1,n+1} + 3\tilde{u}_{i+1,n} - 3\tilde{u}_{i-1,n}] \\
& + \frac{\tilde{a}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h^2}[6\tilde{u}_{i+1,n+1} - 12\tilde{u}_{i,n+1} + 6\tilde{u}_{i-1,n+1} + 6\tilde{u}_{i+1,n} - 12\tilde{u}_{i,n} + 6\tilde{u}_{i-1,n}] \\
& + 12\Delta t^{\alpha}\Gamma(2-\alpha)[(\tilde{b}_{i+1}^{n+1} + 6\tilde{b}_{i}^{n+1} + \tilde{b}_{i-1}^{n+1})].
\end{aligned}
\tag{32}
$$

Now, assume $\tilde{p}_1(r) = \frac{\tilde{v}(x,t;r)\ \Gamma(2-\alpha)\ \Delta t^{\alpha}}{h}$, $\tilde{p}_2(r) = \frac{\tilde{D}(x,t;r)\ \Gamma(2-\alpha)\ \Delta t^{\alpha}}{h^2}$. Then, in view of Equation (32) we derive

$$
\begin{aligned}
& \tilde{u}_{i+1}^{n+1} + 6\tilde{u}_{i}^{n+1} + \tilde{u}_{i-1}^{n+1} - \tilde{u}_{i+1}^{n} - 6\tilde{u}_{i}^{n} - \tilde{u}_{i-1}^{n} + \sum_{j=1}^{n} b_j [(\tilde{u}_{i+1}^{n+1-j} + 6\tilde{u}_{i}^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}) - (\tilde{u}_{i+1}^{n-j} + 6\tilde{u}_{i}^{n-j} + \tilde{u}_{i-1}^{n-j})] \\
& = [-3\tilde{p}_1\tilde{u}_{i+1,n+1} + 3\tilde{p}_1\tilde{u}_{i-1,n+1} - 3\tilde{p}_1\tilde{u}_{i+1,n} + 3\tilde{p}_1\tilde{u}_{i-1,n}] \\
& + [6\tilde{p}_2\tilde{u}_{i+1,n+1} - 12\tilde{p}_2\tilde{u}_{i,n+1} + 6\tilde{p}_2\tilde{u}_{i-1,n+1} + 6\tilde{p}_2\tilde{u}_{i+1,n} - \tilde{p}_2 12\tilde{u}_{i,n} + 6\tilde{p}_2\tilde{u}_{i-1,n}] \\
& + 12\Delta t^{\alpha}\Gamma(2-\alpha)[(\tilde{b}_{i+1}^{n+1} + 6\tilde{b}_{i}^{n+1} + \tilde{b}_{i-1}^{n+1})]
\end{aligned}
\tag{33}
$$

Thus, we simplify Equation (33) to obtain a general formula for the fourth-order compact implicit scheme method of the FTFCDE as follows

$$
\begin{aligned}
\left(1 + 3\tilde{p}_1 - 6\tilde{p}_2\right) \quad & \tilde{u}_{i+1}^{n+1} + \left(6 + 12\tilde{p}_2\right)\tilde{u}_{i}^{n+1} + \left(1 - 3\tilde{p}_1 - 6\tilde{p}_2\right)\tilde{u}_{i-1}^{n+1} \\
& = \left(1 - 3\tilde{p}_1 + 6\tilde{p}_2\right)\tilde{u}_{i+1}^{n} + \left(6 - 12\tilde{p}_2\right)\tilde{u}_{i}^{n} + \left(1 + 3\tilde{p}_1 + 6\tilde{p}_2\right)\tilde{u}_{i-1}^{n} \\
& - \sum_{j=1}^{n} b_j \left[\left(\tilde{u}_{i+1}^{n+1-j} + 6\tilde{u}_{i}^{n+1-j} + \tilde{u}_{i-1}^{n+1-j}\right) - \left(\tilde{u}_{i+1}^{n-j} + 6\tilde{u}_{i}^{n-j} + \tilde{u}_{i-1}^{n-j}\right)\right] \\
& + 12\Delta t^{\alpha}\Gamma(2-\alpha)\left[\left(\tilde{q}_{i+1}^{n+1} + 6\tilde{q}_{i}^{n+1} + \tilde{q}_{i-1}^{n+1}\right)\right.
\end{aligned}
\tag{34}
$$

6. The Fuzzy Fourth-Order FTCS Method for the Solution of FTFCDE

In this section, the fourth-order FTCS method is developed and applied to the double-parametric form of fuzzy numbers implementing the fourth-order approximation at time level n. The, we discretize the first- and second-order space derivatives and the fuzzy

Caputo gH-derivative formula to approximate the time-fractional derivative that gives rise to the solution of the fuzzy time-fractional convection–diffusion equation.

To obtain an approximate solution to the fuzzy time-fractional convection–diffusion equation based on fuzzy fourth-order compact FTCS method, the fuzzy Caputo gH-derivative formula has been applied to approximate the fuzzy time-fractional derivative Equation (8). The first- and second-space partial derivatives are therefore approximated by Equation (5) to yield

$$\begin{aligned}&\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left[\widetilde{u}_{i,n+1}-\widetilde{u}_{i,n}+\sum_{j=1}^{n} b_j\left(\widetilde{u}_{i,n+1-j}-\widetilde{u}_{i,n-j}\right)\right]\\&\qquad=-\widetilde{v}(x,r)\frac{\frac{\delta_x}{2h}}{\left(1+\frac{1}{6}\,\delta^2{}_x\right)}+\widetilde{D}(x,r)\frac{\frac{\delta^2{}_x}{h^2}}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}+\widetilde{q}(x,r).\end{aligned}\tag{35}$$

Hence, simplifying Equation (35) reveals

$$\begin{aligned}&\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\times\frac{3}{12}\Big(\big(\widetilde{u}_{i+1}^{n+1}+6\widetilde{u}_{i}^{n+1}+\widetilde{u}_{i-1}^{n+1}\big)-\big(\widetilde{u}_{i+1}^{n}+6\widetilde{u}_{i}^{n}+\widetilde{u}_{i-1}^{n}\big)\\&\quad+\sum_{j=1}^{n} b_j\big[\big(\widetilde{u}_{i+1}^{n+1-j}+6\widetilde{u}_{i}^{n+1-j}+\widetilde{u}_{i-1}^{n+1-j}\big)-\big(\widetilde{u}_{i+1}^{n-j}+6\widetilde{u}_{i}^{n-j}+\widetilde{u}_{i-1}^{n-j}\big)\big]\Big)\\&\quad=-\widetilde{v}(x,r)\frac{\widetilde{u}_{i+1,n}-\widetilde{u}_{i-1,n}}{2h}+\widetilde{D}(x,r)\frac{\widetilde{u}_{i+1,n}-2\widetilde{u}_{i,n}+\widetilde{u}_{i-1,n}}{h^2}+\big(\widetilde{q}_{i+1}^{n}+6\widetilde{q}_{i}^{n}+\widetilde{q}_{i-1}^{n}\big)\end{aligned}\tag{36}$$

Therefore, we have obtained

$$\begin{aligned}&\widetilde{u}_{i+1}^{n+1}+6\widetilde{u}_{i}^{n+1}+\widetilde{u}_{i-1}^{n+1}-\widetilde{u}_{i+1}^{n}-6\widetilde{u}_{i}^{n}-\widetilde{u}_{i-1}^{n}+\sum_{j=1} b_j\big[\big(\widetilde{u}_{i+1}^{n+1-j}+6\widetilde{u}_{i}^{n+1-j}+\widetilde{u}_{i-1}^{n+1-j}\big)-\big(\widetilde{u}_{i+1}^{n-j}+6\widetilde{u}_{i}^{n-j}+\widetilde{u}_{i-1}^{n-j}\big)\big]\\&\quad=-\frac{\widetilde{v}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h}\big[6\widetilde{u}_{i+1,n}-6\widetilde{u}_{i-1,n}\big]\\&\quad+\frac{\widetilde{D}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h^2}\big[12\widetilde{u}_{i+1,n}-24\widetilde{u}_{i,n}+12\widetilde{u}_{i-1,n}\big]+12\Delta t^{\alpha}\Gamma(2-\alpha)\big[\big(\widetilde{q}_{i+1}^{n}+6\widetilde{q}_{i}^{n}+\widetilde{q}_{i-1}^{n}\big)\big].\end{aligned}\tag{37}$$

Now, let $\widetilde{p}_1(r)=\frac{\widetilde{v}(x,r)\ \Delta t^{\alpha}\Gamma(2-\alpha)}{h}$, $\widetilde{p}_2(r)=\frac{\widetilde{D}(x,r)\Delta t^{\alpha}\Gamma(2-\alpha)}{h^2}$. Then, from Equation (37) we write

$$\begin{aligned}&\widetilde{u}_{i+1}^{n+1}+6\widetilde{u}_{i}^{n+1}+\widetilde{u}_{i-1}^{n+1}-\widetilde{u}_{i+1}^{n}-6\widetilde{u}_{i}^{n}-\widetilde{u}_{i-1}^{n}+\sum_{j=1}^{n} b_j\big[\big(\widetilde{u}_{i+1}^{n+1-j}+6\widetilde{u}_{i}^{n+1-j}+\widetilde{u}_{i-1}^{n+1-j}\big)-\big(\widetilde{u}_{i+1}^{n-j}+6\widetilde{u}_{i}^{n-j}+\widetilde{u}_{i-1}^{n-j}\big)\big]\\&\quad=\big[-6\widetilde{p}_1\widetilde{u}_{i+1,n}+6\widetilde{p}_1\widetilde{u}_{i-1,n}\big]+\big[12\widetilde{p}_2\widetilde{u}_{i+1,n}-24\widetilde{p}_2\widetilde{u}_{i,n}+12\widetilde{p}_2\widetilde{u}_{i-1,n}\big]\\&\quad+12\Delta t^{\alpha}\Gamma(2-\alpha)\big[\big(\widetilde{q}_{i+1}^{n}+6\widetilde{q}_{i}^{n}+\widetilde{q}_{i-1}^{n}\big)\big]\end{aligned}\tag{38}$$

By simplifying Equation (38), we obtain the general formula for the fourth-order compact FTCS of the FTFCDE in the form

$$\begin{aligned}&\widetilde{u}_{i+1}^{n+1}+6\widetilde{u}_{i}^{n+1}+\widetilde{u}_{i-1}^{n+1}\\&\quad=\big(1-6\widetilde{p}_1+12\widetilde{p}_2\big)\widetilde{u}_{i+1}^{n}+\big(6-24\widetilde{p}_2\big)\widetilde{u}_{i}^{n}+\big(1+6\widetilde{p}_1+12\widetilde{p}_2\big)\widetilde{u}_{i-1}^{n}\\&\quad-\sum_{j=1}^{n} b_j\big[\big(\widetilde{u}_{i+1}^{n+1-j}+6\widetilde{u}_{i}^{n+1-j}+\widetilde{u}_{i-1}^{n+1-j}\big)-\big(\widetilde{u}_{i+1}^{n-j}+6\widetilde{u}_{i}^{n-j}+\widetilde{u}_{i-1}^{n-j}\big)\big]\\&\quad+12\Delta t^{\alpha}\Gamma(2-\alpha)\big[\big(\widetilde{b}_{i+1}^{n}+6\widetilde{b}_{i}^{n}+\widetilde{b}_{i-1}^{n}\big)\big]\end{aligned}\tag{39}$$

7. The Truncation Error Analysis

In this section, the truncation error of Equation (34) is considered by employing the Taylor series expansion to give

$$\begin{aligned}\widetilde{T}\left(x_i, t_{n+\frac{1}{2}}\right) &= \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha}} \sum_{j=0}^{n} b_j \left(\widetilde{u}_i^{n+1-j} - \widetilde{u}_i^{n-j}\right) + \frac{\delta_x/2h}{\left(1+\frac{1}{6}\delta^2{}_x\right)} u_i^{n+\frac{1}{2}} - \frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\delta^2\right)} \widetilde{u}_i^{n+\frac{1}{2}} \\ &= \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha}} \sum_{j=0}^{n} b_j \left(\widetilde{u}_i^{n+1-j} - \widetilde{u}_i^{n-j}\right) - \left.\frac{\partial^{\alpha}\widetilde{u}}{\partial^{\alpha} t}\right|_i^{n+\frac{1}{2}} + \frac{\delta_x/2h}{\left(1+\frac{1}{6}\delta^2{}_x\right)} u_i^{n+\frac{1}{2}} - \left.\frac{\partial\widetilde{u}}{\partial x}\right|_i^{n+\frac{1}{2}} + \left.\frac{\partial^2\widetilde{u}}{\partial^2 x}\right|_i^{n+\frac{1}{2}} - \frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\delta^2{}_x\right)} \widetilde{u}_i^{n+\frac{1}{2}} \\ &= \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha}} \sum_{j=0}^{n} b_j \left(\widetilde{u}_i^{n+1-j} - \widetilde{u}_i^{n-j}\right) - \left.\frac{\partial^{\alpha}\widetilde{u}}{\partial \alpha}\right|_i^{n+\frac{1}{2}} + \frac{h^4}{180}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + \frac{h^4}{240}\left(\frac{\partial^6\widetilde{u}}{\partial x^6}\right)_i^{n+\frac{1}{2}} \mathbb{I}_i \\ &= O(\Delta t)^{2-\alpha} + O\left(h^4\right) + O\left(h^4\right) = (\Delta t)^{2-\alpha} + O\left(h^4\right)\end{aligned} \tag{40}$$

It is not of place to mention here that we have to take into consideration that the truncation error for the second-order implicit scheme method of Equation (14) is $O(\Delta t)^{2-\alpha} + O\left(h^2\right)$.

8. Stability Analysis

The Fourier method is used in this section to examine the stability of the presented method for the fuzzy time-fractional convection–diffusion equation. First, suppose that the discretization of the initial condition tends to the fuzzy error $\widetilde{\varepsilon}_i^0$. Assume $\widetilde{u}_i^0 = \acute{\widetilde{u}}_i^0 - \widetilde{\varepsilon}_i^0$, $\widetilde{u}_i^n$, and $\hat{\widetilde{u}}_i^n$ are the numerical fuzzy solutions of the fourth-order compact formula in Equation (14). Let $[\widetilde{u}_{i+1}^n(x,t;\alpha)]_r = \beta[\overline{u}\ (r) - \underline{u}\ (r)] + \underline{u}(r)]$, where $r, \beta \in [0,\ 1]$. Then, we define the fuzzy error bound as

$$[\widetilde{\varepsilon}_i^n]_r = \left[\hat{\widetilde{u}}_i^n - \widetilde{u}_i^n\right]_r,\ \ n = 1,2,3,\ \ldots\ldots, X \times M;\ i = 1,2,3,\ldots\ldots; X-1. \tag{41}$$

Then, by making use of the presented approach of [30], Equation (14) can be read as

$$\begin{aligned}\left(1+3\widetilde{p}_1-6\widetilde{p}_2\right) \quad & \widetilde{u}_{i+1}^{n+1} + \left(6+12\widetilde{p}_2\right)\widetilde{u}_i^{n+1} + \left(1-3\widetilde{p}_1-6\widetilde{p}_2\right)\widetilde{u}_{i-1}^{n+1} \\ & = \left(1-3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\widetilde{u}_{i+1}^{n} + \left(6-12\widetilde{p}_2-6b_1\right)\widetilde{u}_i^{n} + \left(1+3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\widetilde{u}_{i-1}^{n} \\ & - \sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(\widetilde{u}_{i+1}^{n-j} + 6\widetilde{u}_i^{n-j} + \widetilde{u}_{i-1}^{n-j}\right) + b_n\left(\widetilde{u}_{i+1}^{0} + 6\widetilde{u}_i^{0} + \widetilde{u}_{i-1}^{0}.\right)\end{aligned} \tag{42}$$

The error bound of Equation (42) therefore has the form

$$\begin{aligned}\left(1+3\widetilde{p}_1-6\widetilde{p}_2\right) \quad & \widetilde{\varepsilon}_{i+1}^{n+1} + \left(6+12\widetilde{p}_2\right)\widetilde{\varepsilon}_i^{n+1} + \left(1-3\widetilde{p}_1-6\widetilde{p}_2\right)\widetilde{\varepsilon}_{i-1}^{n+1} \\ & = \left(1-3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\widetilde{\varepsilon}_{i+1}^{n} + \left(6-12\widetilde{p}_2-6b_1\right)\widetilde{\varepsilon}_i^{n} + \left(1+3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\widetilde{\varepsilon}_{i-1}^{n} \\ & - \sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(\widetilde{\varepsilon}_{i+1}^{n-j} + 6\widetilde{\varepsilon}_i^{n-j} + \widetilde{\varepsilon}_{i-1}^{n-j}\right) + b_n\left(\widetilde{\varepsilon}_{i+1}^{0} + 6\widetilde{\varepsilon}_i^{0} + \widetilde{\varepsilon}_{i-1}^{0}\right),\end{aligned} \tag{43}$$

provided that $\widetilde{\varepsilon}_0^n = \widetilde{\varepsilon}_X^n = 0,\ n = 1,2,\ldots, T \times M$.

Let $\widetilde{\varepsilon}_i^n = [\widetilde{\varepsilon}_1^n, \widetilde{\varepsilon}_2^n, \ldots\ldots, \widetilde{\varepsilon}_{X-1}^n]$. Then, the fuzzy norm is introduced as

$$\|\widetilde{\varepsilon}^n\|_2 = \sqrt{\sum_{i=1}^{X-1} h\ |\widetilde{\varepsilon}_i^n|^2}$$

Then, it yields

$$\|\widetilde{\varepsilon}^n\|_2^2 = \sum_{i=1}^{X-1} h\ |\widetilde{\varepsilon}_i^n|^2. \tag{44}$$

Suppose that $\widetilde{\varepsilon}_i^n$ can be expressed in the form

$$\widetilde{\varepsilon}_i^n = \widetilde{\lambda}^n\ e^{\sqrt{-\theta_i}}\ ,\ \text{where}\ \widetilde{\theta}_i = qih. \tag{45}$$

Then, substituting Equation (45) into Equation (43) implies

$$\begin{aligned}\left(1+3\widetilde{p}_1-6\widetilde{p}_2\right) \quad & \overset{\sim}{\lambda}^{n+1} e^{\sqrt{-\theta_i}}+\left(1-3\widetilde{p}_1-6\widetilde{p}_2\right)\overset{\sim}{\lambda}^{n+1} e^{\sqrt{-\theta_{i-1}}} \\ & =\left(1-3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\overset{\sim}{\lambda}^{n} e^{\sqrt{-\theta_{i+1}}}+\left(6-12\widetilde{p}_2-6b_1\right)\overset{\sim}{\lambda}^{n} e^{\sqrt{-\theta_i}} \\ & +\left(1+3\widetilde{p}_1+6\widetilde{p}_2-b_1\right)\overset{\sim}{\lambda}^{n} e^{\sqrt{-\theta_{i-1}}} \\ & -\sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(\overset{\sim}{\lambda}^{n-j} e^{\sqrt{-\theta_{i+1}}}+\overset{\sim}{\lambda}^{n-j} e^{\sqrt{-\theta_i}}+\overset{\sim}{\lambda}^{n-j} e^{\sqrt{-\theta_{i-1}}}\right) \\ & +b_n\left(\overset{\sim}{\lambda}^{0} e^{\sqrt{-\theta_{i+1}}}+6\overset{\sim}{\lambda}^{0} e^{\sqrt{-\theta_i}}+\overset{\sim}{\lambda}^{0} e^{\sqrt{-\theta_{i-1}}}\right)\end{aligned} \tag{46}$$

Divide Equation (46) by $e^{\sqrt{-\theta i}}$ to have

$$\begin{aligned}[\left(6+12\widetilde{p}_2\right)+ \quad & \left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)-6\widetilde{p}_2\left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)]\overset{\sim}{\lambda}^{n+1} \\ & =\left[\left(6-12\widetilde{p}_2-6b_1\right)+\left(1-b_1\right)\left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)+6\widetilde{p}_2\left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}^{n} \\ & -\sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left[6+\left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}^{n-j}+b_n\left[6+\left(e^{\sqrt{-\theta_i}}+e^{-\sqrt{-\theta_i}}\right)\right]\overset{\sim}{\lambda}^{0}\end{aligned} \tag{47}$$

Then, simplify Equation (47) to write

$$\begin{aligned}\widetilde{\lambda}^{n+1}= \quad & \left[\frac{8-8b_1-4\sin^2\left(\frac{\theta}{2}\right)+4b_1\sin^2\left(\frac{\theta}{2}\right)-24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}\right]\widetilde{\lambda}^n \\ & -\frac{\sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(8-4\sin^2\left(\frac{\theta}{2}\right)\right)\widetilde{\lambda}^{n-j}+b_n\left(8-4\sin^2\left(\frac{\theta}{2}\right)\right)\widetilde{\lambda}^0}{8-4\sin^2\left(\frac{\theta}{2}\right)+48\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)+12\sqrt{-1}\,\widetilde{p}_1 sin\ \theta}.\end{aligned} \tag{48}$$

Proposition 1. *Let $\widetilde{\lambda}^n$ be the fuzzy numerical solution for Equation (48). Then, we have* $\left|\widetilde{\lambda}^n\right| \leq \left|\widetilde{\lambda}^0\right|$.

Proof. From Equation (48), we, for $n=0$, write

$$\left|\widetilde{\lambda}^1\right|=\left(\frac{8-4\sin^2\left(\frac{\theta}{2}\right)-24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}\right)\left|\widetilde{\lambda}^0\right|.$$

Hence, it follows

$$\left|\widetilde{\lambda}^1\right| \leq \left|\widetilde{\lambda}^0\right|.$$

Now, assume that

$$\left|\widetilde{\lambda}^m\right| \leq \left|\widetilde{\lambda}^0\right|,\ m=1,\ 2\ ,\ 3\ ,\ \ldots, n-1.$$

Therefore, by [30], we state that the standard coefficient $b_j=(j+1)^{1-\alpha}-(j)^{1-\alpha}$, $j=1,2,3,\ldots,$ satisfies

9. $b_j>0, j=1,2,\ldots$
10. $b_j>b_{j+1},\ j=1,2,\ \ldots.$

Hence, in view of Equation (48) and the above statement, we obtain

$$\widetilde{\lambda}^{n+1} \leq \left[\frac{8-8b_1-4\sin^2\left(\frac{\theta}{2}\right)+4b_1\sin^2\left(\frac{\theta}{2}\right)-24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\widetilde{\lambda}^n\right|- \frac{\sum_{j=1}^{n-1}\left(b_{j+1}-b_j\right)\left(8-4\sin^2\left(\frac{\theta}{2}\right)\right)\left|\widetilde{\lambda}^{n-j}\right|+b_n\left(8-4\sin^2\left(\frac{\theta}{2}\right)\right)\left|\widetilde{\lambda}^0\right|}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right).}$$

Thus, we write

$$\widetilde{\lambda}^{n+1} \leq \left[\frac{8-8b_1-4\sin^2\left(\frac{\theta}{2}\right)+4b_1\sin^2\left(\frac{\theta}{2}\right)-24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)-\left[(b_n-b_1)\left(8-4\sin^2\left(\frac{\theta}{2}\right)\right)\right]+8b_n-4b_n\sin^2\left(\frac{\theta}{2}\right)}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\widetilde{\lambda}^0\right|$$

That is,

$$\widetilde{\lambda}^{n+1} \leq \left[\frac{8-4\sin^2\left(\frac{\theta}{2}\right)-24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}{8-4\sin^2\left(\frac{\theta}{2}\right)+24\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)}\right]\left|\widetilde{\lambda}^0\right| \leq \left|\widetilde{\lambda}^0\right|.$$

Theorem 1. *The fourth-order compact implicit scheme method Equation (34) is unconditionally stable.*

Proof. From Proposition 1 and the formula of Equation (44), it can be easily shown that

$$\|\widetilde{\varepsilon}^n\|_2 \leq \|\widetilde{\varepsilon}^0\|_2\,, \quad n = 1,2,3,\ldots\ldots,N-1$$

This means that the fourth-order compact implicit scheme method Equation (14) is unconditionally stable. On the other hand, using the same approach, it is easy to show that the fourth-order compact FTCS scheme Equation (39) is conditionally stable, i.e., there are stability conditions for the time step.

9. Numerical Experiments

Consider the one-dimensional time-fractional convection–diffusion equation [28]

$$\frac{\partial^\alpha \widetilde{u}(x,t,\alpha)}{\partial t^\alpha} = -\frac{\partial \widetilde{u}(x,t;r,\beta)}{\partial x} + \frac{\partial^2 \widetilde{u}(x,,t;r,\beta)}{\partial x^2}, \qquad 0 < x\ < \ L,\ t\ <\ 0, \tag{49}$$

subject to the fuzzy boundary conditions $\widetilde{u}(0,t) = \widetilde{u}(1,t) = 0$ and the fuzzy initial condition

$$\widetilde{u}(x,0) = \widetilde{k}e^{-x},\ \ 0 < x < 1 \tag{50}$$

According to the r-cut approach, the double-parametric is defined as follows

$$\widetilde{k}(r,\beta) = ((\beta(0.2-0.2r)) + 0.1r - 0.1).$$

It is clear that the time-fractional derivative $\frac{\partial^\alpha \widetilde{u}(x,t)}{\partial t^\alpha}$ and the second-order space derivative $\frac{\partial^2 \widetilde{u}(x,t)}{\partial x^2}$ follow the (i) case of generalized differentiability defined in Definition 5. It can be noted from [28] that the analytical solution of Equation (34), which is illustrated in Figures 1 and 2, can be defined by

$$\widetilde{u}(x,t,\alpha;r,\beta) = \sum\nolimits_{n=0}^{\infty}\frac{2^n\ t^{n\alpha}}{\Gamma(n\alpha+1)}\widetilde{k}(r,\beta)e^{-x}. \tag{51}$$

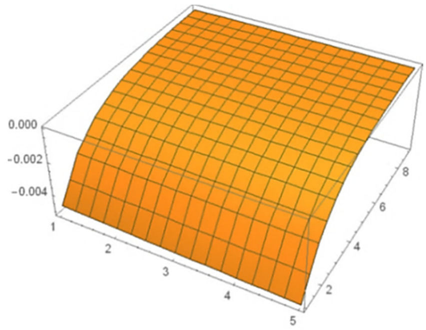


Figure 1. The analytical solution of Equation (49) at $r = 0.6$ and $\beta = 0.4$.

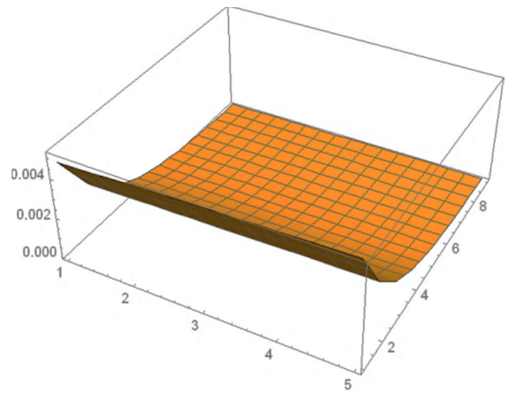


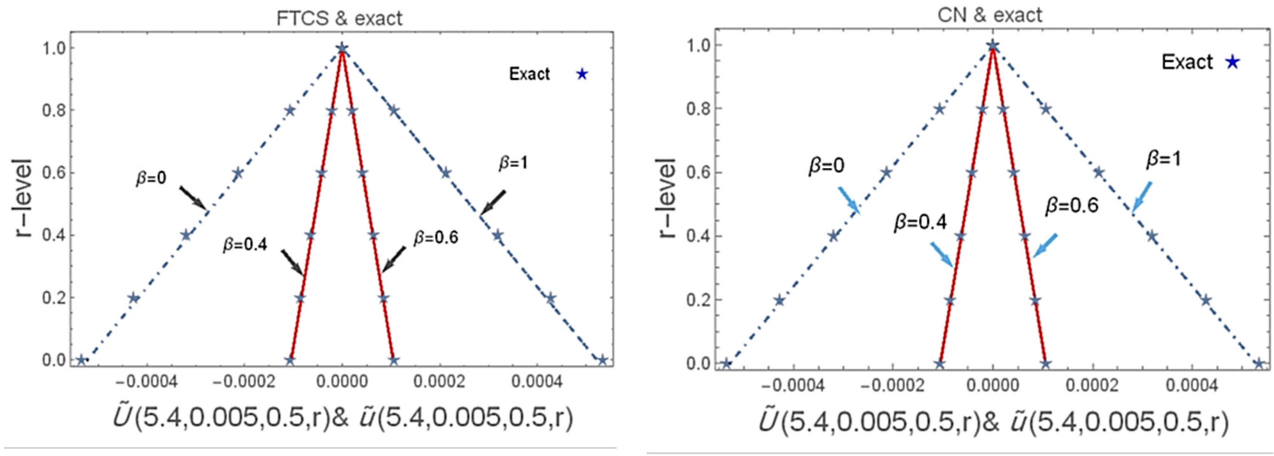
Figure 2. The fuzzy analytical solution of Equation (49) at $r = 0.6$ and $\beta = 0.6$.

The fuzzy absolute error for the numerical solution of Equation (49) can be determined as

$$\left[\widetilde{E}\right]_r = \left|\widetilde{U}(x,t;r,\beta) - \widetilde{u}(x,t;r,\beta)\right| = \begin{cases} [\underline{E}]_r = |\underline{U}(x,t;r,\beta) - \underline{u}(x,t;r,\beta)| \\ [\overline{E}]_r = |\overline{U}(x,t;r,\beta) - \overline{u}(x,t;r,\beta)| \end{cases} \quad (52)$$

At $h = \Delta x = 0.6$ and $\Delta t^{\alpha} = 0.01$, and based on the use of Wolfram Mathematica software, we obtain the following numerical results:

Figures 1–4 and Tables 1 and 2 demonstrated that the fourth-order compact implicit scheme and fourth-order compact explicit FTCS scheme have a good agreement with the exact solution at $x = 5.4$, $t = 0.005$ and for all $r, \beta \in [0,\ 1]$. Additionally, the numerical solutions to the proposed schemes take on the shape of a triangular fuzzy number, which satisfies the fuzzy number properties of the double-parametric form of fuzzy numbers. The fourth-order compact implicit scheme was more accurate than the fourth-order compact explicit FTCS scheme. Furthermore, the double-parametric form was established to be a general and efficient method for converting a fuzzy equation to a crisp equation, as it reduces computational costs and produces more accurate results than the single-parametric form. In Figure 3, we see that the numerical result for the proposed methods is the more accurate solutions at points that are close to the inflection point (β = 0.5). The reason for using a small time step (Δt = 0.001) is that the compact FTCS method is conditionally stable, which means that choosing the value of (Δt and Δx) must be under the stability conditions for the compact FTCS method. However, the compact implicit scheme method handles this problem since it is unconditionally stable (as shown in Section 8), which means that we can use any value of Δt and Δx.



(**a**) (**b**)

Figure 3. Numerical solution of Equation (49) followed by using (**a**) fourth-order compact FTCS and (**b**) fourth-order compact implicit scheme at $t = 0.005$ and $x = 5.4$ for all $r, \beta \in [0,\ 1]$.

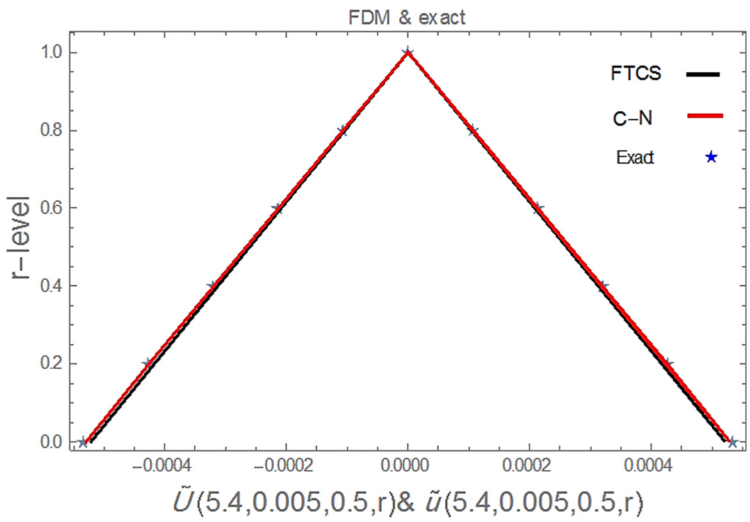


Figure 4. Exact and presented methods of the solution of Equation (49) at $\alpha = 0.5,\ x = 5.4,\ t = 0.005$ and for all $r, \beta \in [0,\ 1]$.

Table 1. Numerical solution of Equation (49) followed by fourth-order compact FTCS and fourth-order compact implicit scheme at $x = 5.4$ and $t = 0.005$ for all $r, \beta \in [0,\ 1]$.

		Fourth Order Compact FTCS		**Fourth Order Compact Implicit Scheme**	
β	r	$\tilde{u}\ (0.9,0.5;r,\beta)$	$\tilde{E}\ (0.9,0.5;r,\beta)$	$\tilde{u}\ (0.9,0.5;r,\beta)$	$\tilde{E}\ (0.9,0.5;r,\beta)$
Lower Solution When $\beta = 0$	0	−0.0005217566	1.20685×10^{-5}	−0.0005274596	6.36552×10^{-6}
	0.2	−0.0004174053	9.65483×10^{-6}	−0.0004219677	5.09242×10^{-6}
	0.4	−0.0003130539	7.24112×10^{-6}	−0.0003164758	3.81931×10^{-6}
	0.6	−0.0002087026	4.82742×10^{-6}	−0.0002109838	2.54621×10^{-6}
	0.8	−0.0001043513	2.41371×10^{-6}	−0.0001054919	1.27310×10^{-6}
	1	0	0	0	0
Upper Solution When $\beta = 1$	0	0.0005217566	1.20685×10^{-5}	0.0005274596	6.36552×10^{-6}
	0.2	0.0004174053	9.65483×10^{-6}	0.0004219677	5.09242×10^{-6}
	0.4	0.0003130539	7.24112×10^{-6}	0.0003164758	3.81931×10^{-6}
	0.6	0.0002087026	4.82742×10^{-6}	0.0002109838	2.54621×10^{-6}
	0.8	0.0001043513	2.41371×10^{-6}	0.0001054919	1.27310×10^{-6}
	1	0	0	0	0

Table 2. Numerical solution of Equation (49) by using fourth-order compact FTCS and fourth-order compact implicit scheme at $x = 5.4$ and $t = 0.005$ for all $r, \beta \in [0,\ 1]$.

		Fourth-Order Compact FTCS		Fourth-Order Compact Implicit Scheme	
β	r	$\tilde{u}\ (0.9,0.5;r,\beta)$	$\widetilde{E}\ (0.9,0.5;r,\beta)$	$\tilde{u}\ (0.9,0.5;r,\beta)$	$\widetilde{E}\ (0.9,0.5;r,\beta)$
$\beta = 0.4$	0	−0.0001043513	2.41371×10^{-6}	−0.0001054919	1.2731×10^{-6}
	0.2	−0.0000834811	1.93097×10^{-6}	−0.0000843935	1.01848×10^{-6}
	0.4	−0.0000626108	1.44822×10^{-6}	−0.0000632952	7.63863×10^{-7}
	0.6	−0.0000417405	9.65483×10^{-7}	−0.0000421968	5.09242×10^{-7}
	0.8	−0.0000208703	4.82742×10^{-7}	−0.0000210984	2.54621×10^{-7}
	1	0	0	0	0
$\beta = 0.6$	0	0.0001043513	2.41371×10^{-6}	0.0001054919	1.2731×10^{-6}
	0.2	0.0000834811	1.93097×10^{-6}	0.0000843935	1.01848×10^{-6}
	0.4	0.0000626108	1.44822×10^{-6}	0.0000632952	7.63863×10^{-7}
	0.6	0.0000417405	9.65483×10^{-7}	0.0000421968	5.09242×10^{-7}
	0.8	0.0000208703	4.82742×10^{-7}	0.0000210984	2.54621×10^{-7}
	1	0	0	0	0

Figure 5 and Tables 3 and 4 demonstrate that the second-order implicit scheme and the fourth-order compact implicit scheme have a good agreement with the analytical solution at $x = 5.4$, $t = 0.005$ and for all $r, \beta \in [0,\ 1]$. The fourth-order compact implicit scheme was more accurate than the second-order classical implicit scheme and thus satisfies and agrees with the theoretical aspects in Section 4.

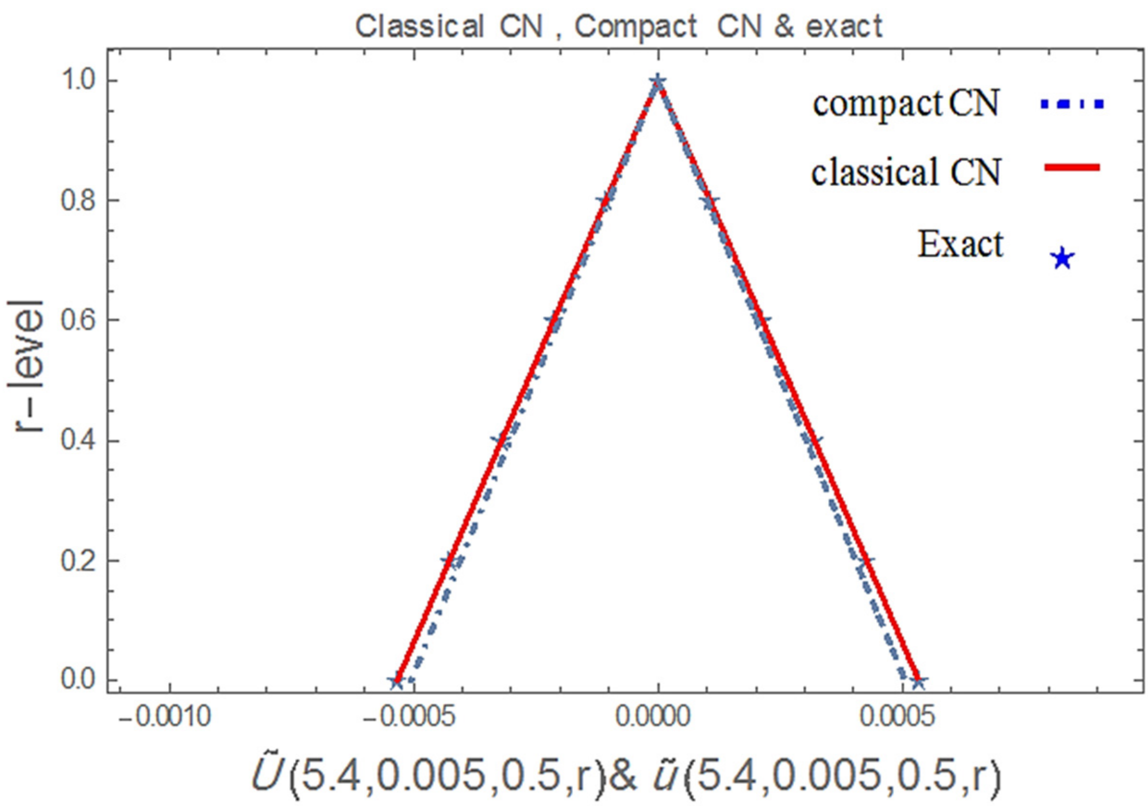


Figure 5. The numerical and exact solution of Equation (14) followed by second-order implicit scheme and fourth-order compact implicit scheme at $= 5.4$, $\beta = 0$ and 1, $t = 0.005$ for all $r \in [0,\ 1]$.

Table 3. Numerical solution of Equation (14) by using second-order implicit scheme and fourth-order compact implicit scheme at $x = 5.4$ and $t = 0.005$ for all $r, \beta \in [0,\ 1]$.

		Second-Order Classical Implicit Scheme		**Fourth-Order Compact Implicit Scheme**	
β	r	$\tilde{u}$ (5.4,0.5;r,β)	$\tilde{E}$ (5.4,0.5;r,β)	$\tilde{u}$ (5.4,0.5;r,β)	$\tilde{E}$ (5.4,0.5;r,β)
Lower Solution When $\beta = 0$	0	−0.0005055836	2.82415×10^{-5}	−0.0005274596	6.36552×10^{-6}
	0.2	−0.0004044669	2.25932×10^{-5}	−0.0004219677	5.09242×10^{-6}
	0.4	−0.0003033501	1.69449×10^{-5}	−0.0003164758	3.81931×10^{-6}
	0.6	−0.0002022334	1.12966×10^{-5}	−0.0002109838	2.54621×10^{-6}
	0.8	−0.0001011167	5.64831×10^{-6}	−0.0001054919	1.27310×10^{-6}
	1	0	0	0	0
Upper Solution When $\beta = 1$	0	0.0005055836	2.82415×10^{-5}	0.0005274596	6.36552×10^{-6}
	0.2	0.0004044669	2.25932×10^{-5}	0.0004219677	5.09242×10^{-6}
	0.4	0.0003033501	1.69449×10^{-5}	0.0003164758	3.81931×10^{-6}
	0.6	0.0002022334	1.12966×10^{-5}	0.0002109838	2.54621×10^{-6}
	0.8	0.0001011167	5.64831×10^{-6}	0.0001054919	1.27310×10^{-6}
	1	0	0	0	0

Table 4. Numerical solution of Equation (14) by using second-order implicit scheme and fourth-order compact implicit scheme at $x = 5.4$ and $t = 0.005$ for all $r, \beta \in [0,\ 1]$.

		Second-Order Classical Implicit Scheme		**Fourth-Order Compact Implicit Scheme**	
β	r	$\tilde{u}$ (5.4,0.5;r,β)	$\tilde{E}$ (5.4,0.5;r,β)	$\tilde{u}$ (5.4,0.5;r,β)	$\tilde{E}$ (5.4,0.5;r,β)
Lower Solution When $\beta = 0.4$	0	−0.0001011167	5.64831×10^{-5}	−0.0001054919	1.2731×10^{-6}
	0.2	−0.0000808933	4.51865×10^{-5}	−0.0000843935	1.01848×10^{-6}
	0.4	−0.0000606700	3.38899×10^{-5}	−0.0000632952	7.63863×10^{-7}
	0.6	−0.00004044669	2.25932×10^{-6}	−0.0000421968	5.09242×10^{-7}
	0.8	−0.0000202233	1.12966×10^{-6}	−0.0000210984	2.54621×10^{-7}
	1	0	0	0	0
Upper Solution When $\beta = 0.6$	0	0.0001011167	5.64831×10^{-5}	0.0001054919	1.2731×10^{-6}
	0.2	0.0000808933	4.51865×10^{-5}	0.0000843935	1.01848×10^{-6}
	0.4	0.0000606700	3.38899×10^{-5}	0.0000632952	7.63863×10^{-7}
	0.6	0.00004044669	2.25932×10^{-6}	0.0000421968	5.09242×10^{-7}
	0.8	0.0000202233	1.12966×10^{-6}	0.0000210984	2.54621×10^{-7}
	1	0	0	0	0

10. Conclusions

Two fourth-order compact finite difference methods for solving a fuzzy time-fractional convection–diffusion equation were developed and implemented in our work. Based on the approach of the double-parametric form of fuzzy number concepts combined with the properties of the fractional derivative of Caputo sense, the considered equation was transferred from the fuzzy domain to the crisp domain with more generalization. The results obtained using the presented methods satisfy the properties of the fuzzy numbers achieving a triangular shape. Furthermore, the stability analysis is illustrated, following from the proof of the stability theorem of the presented schemes under the double-parametric form of fuzzy numbers and has accuracy of order $O\left(\Delta t^{2-\alpha} + \Delta x^4\right)$. A comparison of numerical and exact solutions for the considered examples at various values of the fuzzy level sets reveals that the fourth-order compact implicit scheme produces slightly better results than the fourth-order compact FTCS scheme. The proposed methods for solving the fuzzy time-fractional convection–diffusion equation were found to be feasible, appropriate, and accurate, as demonstrated by a comparison of analytical and numerical solutions at various fuzzy values.

However, in this paper, the authors focused on the solution of fuzzy linear time-fractional convection–diffusion by assuming the solutions are smooth. In future work, the authors plan to discuss the solution of non-linear fuzzy time-fractional convection–diffusion under the reasonable assumptions of the non-smooth solutions as discussed in [31,32]. Furthermore, the authors plan to develop finite difference and finite elements methods to solve the fuzzy linear and nonlinear fuzzy time-fractional convection–diffusion under nonhomogeneous boundary conditions [33,34].

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