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Article

Some Compound Fractional Poisson Processes

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Abstract: In this paper, we introduce and study fractional versions of the Bell–Touchard process, the Poisson-logarithmic process and the generalized Pólya–Aeppli process. The state probabilities of these compound fractional Poisson processes solve a system of fractional differential equations that involves the Caputo fractional derivative of order $0 < \beta < 1$. It is shown that these processes are limiting cases of a recently introduced process, namely, the generalized counting process. We obtain the mean, variance, covariance, long-range dependence property, etc., for these processes. Further, we obtain several equivalent forms of the one-dimensional distribution of fractional versions of these processes.

Keywords: Bell–Touchard process; Poisson-logarithmic process; Pólya–Aeppli process; compound Poisson process; long-range dependence property

1. Introduction

Di Crescenzo et al. [1] introduced and studied a fractional counting process that performs k kinds of jumps of amplitude $1, 2, \dots, k$ with positive rates $\lambda_1, \lambda_2, \dots, \lambda_k$, respectively. We call it the generalized fractional counting process (GFCP) and denote it by $\{M_\beta(t)\}_{t\geq 0}$, $0 < \beta \leq 1$. It is defined as a counting process whose state probabilities $p_\beta(n,t) = \Pr\{M_\beta(t) = n\}$ satisfy the following system of fractional differential equations:

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} p_\beta(n,t) = -(\lambda_1 + \lambda_2 + \cdots + \lambda_k) p_\beta(n,t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j p_\beta(n-j,t), \ n \geq 0, \tag{1}$$

with initial condition

$$p_\beta(n,0) = \begin{cases} 1, \ n = 0, \\ 0, \ n \geq 1. \end{cases}$$

Here, $\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}$ is the Caputo fractional derivative defined as (see [2])

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t (t-s)^{-\beta} f'(s)\, \mathrm{d}s, \ 0 < \beta < 1, \\ f'(t), \ \beta = 1, \end{cases} \tag{2}$$

and its Laplace transform is given by (see [2], Equation (5.3.3))

$$\mathcal{L}\left(\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t); s\right) = s^\beta \tilde{f}(s) - s^{\beta-1} f(0), \ s > 0. \tag{3}$$

For $\beta = 1$, the GFCP reduces to the generalized counting process (GCP) $\{M(t)\}_{t\geq 0}$ (see [1]). For $k = 1$, the GFCP and the GCP reduce to the time fractional Poisson process (TFPP) (see [3]) and the Poisson process, respectively. For other recently introduced fractional stochastic processes, we refer the reader to Khalaf et al. [4,5].

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Let $\{Y_\beta(t)\}_{t\geq 0}$ be the inverse stable subordinator; that is, the first passage time of a stable subordinator $\{D_\beta(t)\}_{t\geq 0}$. A stable subordinator $\{D_\beta(t)\}_{t\geq 0}$ is a one-dimensional Lévy process with non-decreasing sample paths. It is known that (see [1])

$$M_\beta(t) \stackrel{d}{=} M\big(Y_\beta(t)\big),\ t \geq 0,$$

where $\stackrel{d}{=}$ denotes equality in the distribution and $\{M(t)\}_{t\geq 0}$ is independent of $\{Y_\beta(t)\}_{t\geq 0}$.

Kataria and Khandakar [6] showed that several recently introduced counting processes, such as the Poisson process of order k, Pólya–Aeppli process of order k, Pólya–Aeppli process, negative binomial process, etc., are particular cases of the GCP. In Kataria and Khandakar [7], the authors studied a limiting case of the GFCP, namely, the convoluted fractional Poisson process.

For $j \geq 1$, let $\{N_j(t)\}_{t\geq 0}$ be a Poisson process with intensity λ_j. Jánossy et al. [8] considered a composed process $\left\{\sum_{j=1}^{\infty} jN_j(t)\right\}_{t\geq 0}$ with $\sum_{j=1}^{\infty}\lambda_j < \infty$. The probability mass function (pmf) of $\{\xi(t)\}_{t\geq 0}$ where $\xi(t) = \sum_{j=1}^{\infty} jN_j(t)$ is given by (see [8], Equation (2.18))

$$\Pr\{\xi(t) = n\} = \sum_{\Omega_n}\prod_{j=1}^{n}\frac{(t\lambda_j)^{x_j}}{x_j!}e^{-\sum_{j=1}^{\infty} t\lambda_j}. \tag{4}$$

Here,

$$\Omega_n = \left\{(x_1, x_2, \dots, x_n) : \sum_{j=1}^{n} jx_j = n,\ x_j \in \mathbb{N}\cup\{0\}\right\}. \tag{5}$$

Its equivalent version is given by (see [8], Equation (2.22))

$$\Pr\{\xi(t) = n\} = \sum_{\Omega_k^n}\prod_{j=1}^{k}\lambda_{x_j}\frac{t^k}{k!}e^{-\sum_{j=1}^{\infty} t\lambda_j}, \tag{6}$$

where

$$\Omega_k^n = \left\{(x_1, x_2, \dots, x_k) : \sum_{i=1}^{k} x_i = n,\ k \leq n,\ x_i \geq 1\right\}. \tag{7}$$

Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables such that $\Pr\{X_1 = j\} = c_j$ for all $j \geq 1$. Consider a compound Poisson process

$$Y(t) = \sum_{i=1}^{N(t)} X_i,\ t \geq 0,$$

where $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ that is independent of $\{X_i\}_{i\geq 1}$.

For a suitable choice of λ and c_j's, the compound Poisson process $\{Y(t)\}_{t\geq 0}$ is equal in distribution to a counting process introduced and studied by Freud and Rodriguez [9], namely, the Bell–Touchard process (BTP) $\{\mathcal{M}(t)\}_{t\geq 0}$. For $\lambda = \alpha(e^\theta - 1)$ and $c_j = \theta^j/j!(e^\theta - 1)$, $j \geq 1$ where $\alpha > 0, \theta > 0$ they have shown that

$$\mathcal{M}(t) \stackrel{d}{=} Y(t). \tag{8}$$

For $\lambda_j = \alpha\theta^j/j!$, $j \geq 1$, the process $\{\xi(t)\}_{t\geq 0}$ is equal in distribution to BTP (see [9]); that is, $\xi(t) \stackrel{d}{=} \mathcal{M}(t)$. Therefore, the pmf $q(n,t) = \Pr\{\mathcal{M}(t) = n\}$ of BTP is given by

$$q(n,t) = \sum_{\Omega_n}\prod_{j=1}^{n}\frac{(\alpha t\theta^j/j!)^{x_j}}{x_j!}e^{-\alpha t(e^\theta - 1)}, \tag{9}$$

where Ω_n is given in (5).

For $c_j = -(1-p)^j/j\ln p,\ j \geq 1$ where $p \in (0,1)$, the compound Poisson process $\{Y(t)\}_{t\geq 0}$ reduces to a counting process introduced and studied by Sendova and Minkova [10], namely, the Poisson-logarithmic process (PLP). We denote it by $\{\hat{\mathcal{M}}(t)\}_{t\geq 0}$. It is defined as

$$\hat{\mathcal{M}}(t) = Y(t). \tag{10}$$

As an application, they considered a risk process in which the PLP is used to model the claim numbers.

For $c_j = \binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$, $j \geq 1$ where $r > 0, 0 < \rho < 1$, the compound Poisson process $\{Y(t)\}_{t\geq 0}$ reduces to a counting process introduced and studied by Jacob and Jose [11], namely, the generalized Pólya–Aeppli process (GPAP). We denote it by $\{\bar{\mathcal{M}}(t)\}_{t\geq 0}$. It is defined as

$$\bar{\mathcal{M}}(t) = Y(t). \tag{11}$$

If $r = 1$, then the GPAP reduces to the Pólya–Aeppli process (see [12]).

In this paper, we study, in detail, the fractional versions of BTP, PLP and GPAP. We obtain their Lévy measures. It is shown that these processes are the limiting cases of GCP. We obtain the probability generating function (pgf), mean, variance, covariance, etc., for their fractional variants and establish their long-range dependence (LRD) property. It is known that the process that exhibits an LRD property has applications in several areas, such as finance, econometrics, hydrology, internet data traffic modelling, etc. Several equivalent forms of the pmf of fractional versions of these processes are obtained. We have shown that the fractional variants of these processes are overdispersed and non-renewal. It is observed that the one-dimensional distributions of their fractional variants are not infinitely divisible, and their increments have the short-range dependence (SRD) property. It is shown that these fractional processes are equal in distribution to some particular cases of the compound fractional Poisson process studied by Beghin and Macci [13].

2. Preliminaries

In this section, we present some known results and definitions that will be used later.

2.1. Mittag–Leffler Function

The three-parameter Mittag–Leffler function is defined as (see [2], p. 45)

$$E^{\gamma}_{\alpha,\beta}(x) := \frac{1}{\Gamma(\gamma)}\sum_{j=0}^{\infty}\frac{\Gamma(j+\gamma)x^j}{j!\Gamma(j\alpha+\beta)},\ x \in \mathbb{R},$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.

It reduces to the two-parameter Mittag–Leffler function for $\gamma = 1$, and for $\gamma = \beta = 1$ it reduces to the Mittag–Leffler function.

For any real x, the Laplace transform of the function $t^{\beta-1}E^{\gamma}_{\alpha,\beta}(xt^{\alpha})$ is given by (see [2], Equation (1.9.13)):

$$\mathcal{L}\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(xt^{\alpha});s\right) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-x)^{\gamma}},\ s > |x|^{1/\alpha}. \tag{12}$$

2.2. Inverse Stable Subordinator

The following result holds for the inverse stable subordinator $\{Y_\beta(t)\}_{t\geq 0}$ (see [14]):

$$\mathbb{E}\left(e^{-sY_\beta(t)}\right) = E_{\beta,1}\left(-st^{\beta}\right),\ s > 0. \tag{13}$$

The mean and variance of $\{Y_\beta(t)\}_{t\geq 0}$ are given by (see [14])

$$\mathbb{E}\big(Y_\beta(t)\big) = \frac{t^\beta}{\Gamma(\beta+1)}, \tag{14}$$

$$\mathrm{Var}\big(Y_\beta(t)\big) = \left(\frac{2}{\Gamma(2\beta+1)} - \frac{1}{\Gamma^2(\beta+1)}\right)t^{2\beta}. \tag{15}$$

For fixed s and large t, the following asymptotic result holds (see [15]):

$$\mathrm{Cov}\big(Y_\beta(s), Y_\beta(t)\big) \sim \frac{1}{\Gamma^2(\beta+1)}\left(\beta s^{2\beta}B(\beta,\beta+1) - \frac{\beta^2 s^{\beta+1}}{(\beta+1)t^{1-\beta}}\right), \tag{16}$$

where $B(\beta,\beta+1)$ denotes the beta function.

2.3. Poisson-Logarithmic Process

The state probabilities $\hat{q}(n,t) = \Pr\{\hat{\mathcal{M}}(t) = n\}$ of PLP satisfy the following (see [10]):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}(0,t) &= -\lambda\hat{q}(0,t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}(n,t) &= -\lambda\hat{q}(n,t) - \frac{\lambda}{\ln p}\sum_{j=1}^{n}\frac{(1-p)^j}{j}\hat{q}(n-j,t),\ n \geq 1, \end{aligned} \tag{17}$$

with initial conditions $\hat{q}(0,0) = 1$ and $\hat{q}(n,0) = 0$, $n \geq 1$. Its pgf $\hat{G}(u,t) = \mathbb{E}\Big(u^{\hat{\mathcal{M}}(t)}\Big)$ is given by

$$\hat{G}(u,t) = \exp\left(-\lambda t\left(1 - \frac{\ln(1-(1-p)u)}{\ln p}\right)\right). \tag{18}$$

2.4. Generalized Pólya–Aeppli Process

The state probabilities $\bar{q}(n,t) = \Pr\{\bar{\mathcal{M}}(t) = n\}$ of GPAP satisfy the following (see [11]):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bar{q}(0,t) &= -\lambda\bar{q}(0,t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\bar{q}(n,t) &= -\lambda\bar{q}(n,t) + \frac{\lambda}{(1-\rho)^{-r}-1}\sum_{j=1}^{n}\binom{r+j-1}{j}\rho^j\bar{q}(n-j,t),\ n \geq 1, \end{aligned} \tag{19}$$

with initial conditions $\bar{q}(0,0) = 1$ and $\bar{q}(n,0) = 0$, $n \geq 1$. Its pgf $\bar{G}(u,t) = \mathbb{E}\Big(u^{\bar{\mathcal{M}}(t)}\Big)$ is given by

$$\bar{G}(u,t) = \exp\left(-\lambda t\left(1 - \frac{(1-\rho u)^{-r}-1}{(1-\rho)^{-r}-1}\right)\right). \tag{20}$$

Let $\bar{r}_1 = r\rho\lambda/(1-\rho)(1-(1-\rho)^r)$ and $\bar{r}_2 = r(1+r\rho)\rho\lambda/(1-\rho)^2(1-(1-\rho)^r)$. Its mean and variance are given by

$$\mathbb{E}\big(\bar{\mathcal{M}}(t)\big) = \bar{r}_1 t,\ \mathrm{Var}\big(\bar{\mathcal{M}}(t)\big) = \bar{r}_2 t. \tag{21}$$

2.5. LRD and SRD Properties

The LRD and SRD properties for a non-stationary stochastic process $\{X(t)\}_{t\geq 0}$ are defined as follows (see [15,16]):

Definition 1. *Let $s > 0$ be fixed. The process $\{X(t)\}_{t\geq 0}$ is said to exhibit the LRD property if its correlation function has the following asymptotic behaviour:*

$$\mathrm{Corr}(X(s), X(t)) \sim c(s)t^{-\theta},\ \textit{as } t \to \infty,$$

for some $c(s) > 0$ and $\theta \in (0,1)$. If $\theta \in (1,2)$, then it is said to possess the SRD property.

3. Bell–Touchard Process and Its Fractional Version

Here, we introduce a fractional version of a recently introduced counting process, namely, the Bell–Touchard process (BTP) (see [9]).

First, we study some additional properties of BTP, the BTP is defined as follows:

Definition 2. *A counting process $\{\mathcal{M}(t)\}_{t\geq 0}$ is said to be a BTP with parameters $\alpha > 0$, $\theta > 0$ if*
(a) $\mathcal{M}(0) = 0$;
(b) it has independent and stationary increments;
(c) for $m = 0, 1, \dots$ and $h > 0$ small enough such that $o(h) \to 0$ as $h \to 0$, its state transition probabilities are given by

$$\Pr\{\mathcal{M}(t+h) = n | \mathcal{M}(t) = m\} = \begin{cases} 1 - \alpha(e^{\theta} - 1)h + o(h), \ n = m, \\ \alpha\theta^j h/j! + o(h), \ n = m + j, \ j = 1, 2, \dots. \end{cases}$$

It follows that the state probabilities $q(n,t) = \Pr\{\mathcal{M}(t) = n\}$, $n \geq 0$ of BTP satisfy the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q(0,t) &= -\alpha\left(e^{\theta} - 1\right) q(0,t), \\ \frac{\mathrm{d}}{\mathrm{d}t} q(n,t) &= -\alpha\left(e^{\theta} - 1\right) q(n,t) + \alpha \sum_{j=1}^{n} \frac{\theta^j}{j!} q(n-j,t), \ n \geq 1, \end{aligned} \tag{22}$$

with initial conditions $q(0,0) = 1$ and $q(n,0) = 0$, $n \geq 1$.

Remark 1. *On taking $\beta = 1$, $\lambda_j = \alpha\theta^j/j!$ for all $j \geq 1$ and letting $k \to \infty$, System (1) reduces to System (22). Thus, the BTP is a limiting case of the GCP.*

In view of Remark 1, we note that several results for BTP can be obtained from the corresponding results for GCP. Next, we present a few of them.

The next result gives a recurrence relation for the pmf of BTP. It follows from Proposition 1 of Kataria and Khandakar [6].

Proposition 1. *The state probabilities $q(n,t)$ of BTP satisfy*

$$q(n,t) = \frac{\alpha t}{n} \sum_{j=1}^{n} \frac{\theta^j}{(j-1)!} q(n-j,t), \ n \geq 1.$$

It is known that the GCP is equal in distribution to the following weighted sum of k independent Poisson processes (see [6]):

$$M(t) \overset{d}{=} \sum_{j=1}^{k} j N_j(t),$$

where $\{N_j(t)\}_{t\geq 0}$'s are independent Poisson processes with intensity λ_j's. On taking $\lambda_j = \alpha\theta^j/j!$ for all $j \geq 1$ and letting $k \to \infty$, we get

$$\mathcal{M}(t) \overset{d}{=} \sum_{j=1}^{\infty} j N_j(t),$$

which agrees with the result obtained by Freud and Rodriguez [9]. As $\lim_{t\to\infty} N_j(t)/t = \alpha\theta^j/j!$, $j \geq 1$ almost surely, we have

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\mathcal{M}(t)}{t} &\stackrel{d}{=} \sum_{j=1}^{\infty} j \lim_{t\to\infty}\frac{N_j(t)}{t} \\
&= \alpha\theta e^{\theta}, \text{ in probability.}
\end{aligned} \tag{23}
$$

The next result gives a martingale characterisation for the BTP, whose proof follows from Proposition 2 of Kataria and Khandakar [6].

Proposition 2. *The process $\{\mathcal{M}(t) - \alpha\theta e^{\theta} t\}_{t\geq 0}$ is a martingale with respect to a natural filtration $\mathcal{F}_t = \sigma(\mathcal{M}(s), s \leq t)$.*

From (8), it follows that the BTP is a Lévy process as it is equal in distribution to a compound Poisson process. Therefore, its mean, variance and covariance are given by

$$
\begin{aligned}
E(\mathcal{M}(t)) &= \alpha\theta e^{\theta} t, && (24)\\
\mathrm{Var}(\mathcal{M}(t)) &= \alpha\theta(\theta+1)e^{\theta} t, && (25)\\
\mathrm{Cov}(\mathcal{M}(s), \mathcal{M}(t)) &= \alpha\theta(\theta+1)e^{\theta}\min\{s,t\},
\end{aligned}
$$

respectively. The BTP exhibits the overdispersion property as $\mathrm{Var}(\mathcal{M}(t)) - E(\mathcal{M}(t)) = \alpha\theta^2 e^{\theta} t > 0$ for $t > 0$.

The characteristic function of BTP can be obtained by taking $\lambda_j = \alpha\theta^j/j!$ for all $j \geq 1$ and letting $k \to \infty$ in Equation (12) of Kataria and Khandakar [6]. It is given by

$$
\mathbb{E}\left(e^{\omega\xi\mathcal{M}(t)}\right) = \exp\left(-\alpha t\left(e^{\theta} - e^{\theta e^{\omega\xi}}\right)\right),\ \omega = \sqrt{-1},\ \xi \in \mathbb{R}.
$$

Therefore, its Lévy measure is given by $\mu(\mathrm{d}x) = \alpha\sum_{j=1}^{\infty}\frac{\theta^j}{j!}\delta_j \mathrm{d}x$, where δ_j's are Dirac measures.

Remark 2. *For fixed s and large t, the correlation function of BTP has the following asymptotic behaviour:*

$$
\mathrm{Corr}(\mathcal{M}(s), \mathcal{M}(t)) \sim \sqrt{s}\, t^{-1/2}.
$$

Thus, it exhibits the LRD property.

Fractional Bell–Touchard Process

Here, we introduce a fractional version of the BTP, namely, the fractional Bell–Touchard process (FBTP). We define it as the stochastic process $\{\mathcal{M}_\beta(t)\}_{t\geq 0}$, $0 < \beta \leq 1$ whose state probabilities $q_\beta(n,t) = \Pr\{\mathcal{M}_\beta(t) = n\}$ satisfy the following system of fractional differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\beta(0,t) &= -\alpha\left(e^{\theta} - 1\right) q_\beta(0,t), \\
\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\beta(n,t) &= -\alpha\left(e^{\theta} - 1\right) q_\beta(n,t) + \alpha\sum_{j=1}^{n}\frac{\theta^j}{j!} q_\beta(n-j,t),\ n \geq 1,
\end{aligned} \tag{26}
$$

with initial conditions $q_\beta(0,0) = 1$ and $q_\beta(n,0) = 0$, $n \geq 1$.

Note that the system of Equation (26) is obtained by replacing the integer order derivative in (22) with the Caputo fractional derivative defined in (2).

Remark 3. *On taking $\lambda_j = \alpha\theta^j/j!$ for all $j \geq 1$ and letting $k \to \infty$, System (1) reduces to System (26). Thus, the FBTP is a limiting case of the GFCP.*

Using (26), it can be shown that the pgf $G_\beta(u,t) = \mathbb{E}\left(u^{\mathcal{M}_\beta(t)}\right)$, $|u| \leq 1$ of FBTP satisfies

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}G_\beta(u,t) = -\alpha\left(e^\theta - e^{\theta u}\right)G_\beta(u,t),$$

with $G_\beta(u,0) = 1$. On taking the Laplace transform in the above equation and using (3), we get

$$s^\beta \tilde{G}_\beta(u,s) - s^{\beta-1}G_\beta(u,0) = -\alpha\left(e^\theta - e^{\theta u}\right)\tilde{G}_\beta(u,s),\ s > 0,$$

where $\tilde{G}_\beta(u,s)$ denotes the Laplace transform of $G_\beta(u,t)$. Thus,

$$\tilde{G}_\beta(u,s) = \frac{s^{\beta-1}}{s^\beta + \alpha\left(e^\theta - e^{\theta u}\right)}.$$

On taking the inverse Laplace transform and using (12), we get

$$G_\beta(u,t) = E_{\beta,1}\left(-\alpha\left(e^\theta - e^{\theta u}\right)t^\beta\right). \tag{27}$$

Remark 4. *On taking $\beta = 1$ in (27), we get the pgf $G(u,t) = \exp\left(-\alpha\left(e^\theta - e^{\theta u}\right)t\right)$ of BTP. Further, from (27), we can verify that the pmf $q_\beta(n,t)$ sums up to one; that is, $\sum_{n=0}^{\infty} q_\beta(n,t) = G_\beta(u,t)|_{u=1} = E_{\beta,1}(0) = 1$.*

The next result gives a time-changed relationship between the BTP and its fractional variant, FBTP.

Theorem 1. *Let $\{Y_\beta(t)\}_{t\geq 0}$, $0 < \beta < 1$, be an inverse stable subordinator independent of the BTP $\{\mathcal{M}(t)\}_{t\geq 0}$. Then*

$$\mathcal{M}_\beta(t) \overset{d}{=} \mathcal{M}(Y_\beta(t)),\ t \geq 0. \tag{28}$$

Proof. Let $h_\beta(x,t)$ be the density of $\{Y_\beta(t)\}_{t\geq 0}$. Then,

$$\begin{aligned}
\mathbb{E}\left(u^{\mathcal{M}(Y_\beta(t))}\right) &= \int_0^\infty G(u,x)h_\beta(x,t)\mathrm{d}x \\
&= \int_0^\infty \exp\left(-\alpha\left(e^\theta - e^{\theta u}\right)x\right)h_\beta(x,t)\mathrm{d}x, \quad \text{(using Remark 4)} \\
&= E_{\beta,1}\left(-\alpha\left(e^\theta - e^{\theta u}\right)t^\beta\right) \ \text{(using (13))},
\end{aligned}$$

which agrees with (27). This completes the proof. □

Remark 5. *Let $\{T_{2\beta}(t)\}_{t>0}$ be a random process whose distribution is given by the folded solution of the following Cauchy problem (see [17]):*

$$\frac{\mathrm{d}^{2\beta}}{\mathrm{d}t^{2\beta}}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t),\ x \in \mathbb{R},\ t > 0, \tag{29}$$

with $u(x,0) = \delta(x)$ for $0 < \beta \leq 1$ and $\frac{\partial}{\partial t}u(x,0) = 0$ for $1/2 < \beta \leq 1$. It is known that the density functions of $Y_\beta(t)$ and $T_{2\beta}(t)$ coincide (see [18]). Hence,

$$\mathcal{M}_\beta(t) \overset{d}{=} \mathcal{M}(T_{2\beta}(t)),\ t > 0, \tag{30}$$

where $\{T_{2\beta}(t)\}_{t>0}$ is independent of the BTP.

The random process $\{T_{2\beta}(t)\}_{t>0}$ becomes a reflecting Brownian motion $\{|B(t)|\}_{t>0}$ for $\beta = 1/2$ as Equation (29) reduces to the following heat equation:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t),\ x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x). \end{cases}$$

Therefore, $\mathcal{M}_{1/2}(t)$ is equal in distribution to BTP at a Brownian time; that is, $\mathcal{M}(|B(t)|)$, $t > 0$.

In view of (8) and (28), it follows that the FBTP is equal in distribution to the following compound fractional Poisson process:

$$\mathcal{M}_\beta(t) \stackrel{d}{=} \sum_{i=1}^{N(Y_\beta(t))} X_i \stackrel{d}{=} \sum_{i=1}^{N_\beta(t)} X_i,\ t \geq 0, \tag{31}$$

where $\{N_\beta(t)\}_{t\geq 0}$ is a TFPP with intensity $\alpha\left(e^\theta - 1\right)$ independent of the sequence of independent and identically distributed random variables $\{X_i\}_{i\geq 1}$. Thus, it is neither Markovian nor a Lévy process (see [19]).

Remark 6. *The system of differential equations that governs the state probabilities of the compound fractional Poisson process was obtained by Beghin and Macci [13]. In view of (31), System (26) can alternatively be obtained using Proposition 1 of Beghin and Macci [13].*

Next, we obtain the pmf of FBTP and some of its equivalent versions.
The solution $u_{2\beta}(x,t)$ of (29) is given by (see [3]):

$$u_{2\beta}(x,t) = \frac{1}{2t^\beta} W_{-\beta,1-\beta}\left(-\frac{|x|}{t^\beta}\right),\ t > 0,\ x \in \mathbb{R}, \tag{32}$$

where $W_{\nu,\gamma}(\cdot)$ is the Wright function defined as follows:

$$W_{\nu,\gamma}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(k\nu+\gamma)},\ \nu > -1,\ \gamma > 0,\ x \in \mathbb{R}.$$

Let

$$\bar{u}_{2\beta}(x,t) = \begin{cases} 2u_{2\beta}(x,t),\ x > 0, \\ 0,\ x < 0, \end{cases} \tag{33}$$

be the folded solution to (29).

Theorem 2. *The pmf $q_\beta(n,t) = \Pr\{\mathcal{M}_\beta(t) = n\}$ of FBTP is given by*

$$q_\beta(n,t) = \begin{cases} E_{\beta,1}\left(-\alpha(e^\theta - 1)t^\beta\right),\ n = 0, \\ \displaystyle\sum_{\Omega_n}\prod_{j=1}^{n} \frac{\left(\alpha\theta^j/j!\right)^{x_j}}{x_j!} z_n! t^{z_n\beta} E^{z_n+1}_{\beta,z_n\beta+1}\left(-\alpha(e^\theta - 1)t^\beta\right),\ n \geq 1, \end{cases} \tag{34}$$

where $z_n = x_1 + x_2 + \cdots + x_n$ and Ω_n is given in (5).

Proof. From (30) and (33), we have

$$q_\beta(n,t) = \int_0^\infty q(n,x)\bar{u}_{2\beta}(x,t)\mathrm{d}x.$$

We use (9) and (32) in the above equation to obtain

$$\begin{aligned} q_\beta(n,t) &= \sum_{\Omega_n}\prod_{j=1}^{n} \frac{\left(\alpha\theta^j/j!\right)^{x_j}}{x_j!} t^{-\beta} \int_0^\infty e^{-\alpha x(e^\theta-1)} x^{z_n} W_{-\beta,1-\beta}\left(-\frac{x}{t^\beta}\right)\mathrm{d}x \\ &= \sum_{\Omega_n}\prod_{j=1}^{n} \frac{\left(\alpha\theta^j/j!\right)^{x_j}}{x_j!} \frac{t^{-\beta}}{\left(\alpha(e^\theta-1)\right)^{z_n+1}} \int_0^\infty e^{-y} y^{z_n} W_{-\beta,1-\beta}\left(-\frac{y}{\alpha(e^\theta-1)t^\beta}\right)\mathrm{d}y. \end{aligned}$$

On using the following result (see [20], Equation (2.13)):

$$E_{\beta,z_n\beta+1}^{z_n+1}\left(-\alpha(e^\theta-1)t^\beta\right)=\frac{t^{-\beta(z_n+1)}}{z_n!\left(\alpha(e^\theta-1)\right)^{z_n+1}}\int_0^\infty e^{-y}y^{z_n}W_{-\beta,1-\beta}\left(-\frac{y}{\alpha(e^\theta-1)t^\beta}\right)\mathrm{d}y,$$

the proof follows. □

Remark 7. *An equivalent form of the pmf of BTP can be obtained from (6). It is given by*

$$q(n,t)=\sum_{\Omega_k^n}\prod_{j=1}^{k}\frac{\theta^{x_j}}{x_j!}\frac{(\alpha t)^k}{k!}e^{-\alpha t(e^\theta-1)},\ n\geq 0, \tag{35}$$

where Ω_k^n is given in (7). If we use (35) in the proof of Theorem 2, then we can obtain the following alternate form of the pmf of FBTP:

$$q_\beta(n,t)=\sum_{\Omega_k^n}\prod_{j=1}^{k}\frac{\theta^{x_j}}{x_j!}\left(\alpha t^\beta\right)^k E_{\beta,k\beta+1}^{k+1}\left(-\alpha(e^\theta-1)t^\beta\right). \tag{36}$$

The pmf of TFPP with intensity $\alpha\left(e^\theta-1\right)$ is given by (see [20], Equation (2.5))

$$\Pr\{N_\beta(t)=n\}=\left(\alpha(e^\theta-1)t^\beta\right)^n E_{\beta,n\beta+1}^{n+1}\left(-\alpha(e^\theta-1)t^\beta\right),\ n\geq 0.$$

From (31), we have

$$q_\beta(0,t)=\Pr\{N_\beta(t)=0\}=E_{\beta,1}\left(-\alpha(e^\theta-1)t^\beta\right).$$

We recall that X_i's are independent of $N_\beta(t)$ in (31). Thus, for $n\geq 1$, we have

$$\begin{aligned} q_\beta(n,t)&=\sum_{k=1}^{n}\Pr\{X_1+X_2+\cdots+X_k=n\}\Pr\{N_\beta(t)=k\} && (37)\\ &=\sum_{k=1}^{n}\sum_{\Theta_n^k}k!\prod_{j=1}^{n}\frac{(\theta^j/j!)^{x_j}}{x_j!}\left(\alpha t^\beta\right)^k E_{\beta,k\beta+1}^{k+1}\left(-\alpha(e^\theta-1)t^\beta\right), && (38)\end{aligned}$$

where x_j is the total number of claims of j units and

$$\Theta_n^k=\left\{(x_1,x_2,\ldots,x_n):\sum_{j=1}^{n}x_j=k,\ \sum_{j=1}^{n}jx_j=n,\ x_j\in\mathbb{N}\cup\{0\}\right\}. \tag{39}$$

Again, as X_i's are independent and identically distributed, we have

$$\begin{aligned}\Pr\{X_1+X_2+\cdots+X_k=n\}&=\sum_{\substack{m_1+m_2+\cdots+m_k=n\\ m_j\in\mathbb{N}}}\Pr\{X_1=m_1,X_2=m_2,\ldots,X_k=m_k\}\\ &=\sum_{\substack{m_1+m_2+\cdots+m_k=n\\ m_j\in\mathbb{N}}}\prod_{j=1}^{k}\Pr\{X_j=m_j\}\\ &=\sum_{\substack{m_1+m_2+\cdots+m_k=n\\ m_j\in\mathbb{N}}}\frac{1}{(e^\theta-1)^k}\prod_{j=1}^{k}\frac{\theta^{m_j}}{m_j!}, && (40)\end{aligned}$$

where we have used (8). On substituting (40) into (37), we get an equivalent expression for the pmf of FBTP in the following form:

$$q_\beta(n,t) = \sum_{k=1}^{n} \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j\in\mathbb{N}}} \prod_{j=1}^{k} \frac{\theta^{m_j}}{m_j!} \left(\alpha t^\beta\right)^k E_{\beta,k\beta+1}^{k+1}\left(-\alpha(e^\theta-1)t^\beta\right). \tag{41}$$

The pmf (38) can be written in the following equivalent form by using Lemma 2.4 of Kataria and Vellaisamy [21]:

$$q_\beta(n,t) = \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{j=1}^{n-k+1} \frac{(\theta^j/j!)^{x_j}}{x_j!} \left(\alpha t^\beta\right)^k E_{\beta,k\beta+1}^{k+1}\left(-\alpha(e^\theta-1)t^\beta\right). \tag{42}$$

where

$$\Lambda_n^k = \left\{(x_1,x_2,\ldots,x_{n-k+1}) : \sum_{j=1}^{n-k+1} x_j = k,\ \sum_{j=1}^{n-k+1} jx_j = n,\ x_j \in \mathbb{N}\cup\{0\}\right\}. \tag{43}$$

Thus, we have obtained the five alternate forms of the pmf of FBTP given in (34), (36), (38), (41) and (42).

By using (24), (25) and Theorem 2.1 of Leonenko et al. [14], the mean, variance and covariance of FBTP can be obtained in the following forms:

$$\begin{aligned}
\mathbb{E}\left(\mathcal{M}_\beta(t)\right) &= \alpha\theta e^\theta \mathbb{E}\left(Y_\beta(t)\right), \\
\operatorname{Var}\left(\mathcal{M}_\beta(t)\right) &= \alpha\theta(\theta+1)e^\theta \mathbb{E}\left(Y_\beta(t)\right) + \left(\alpha\theta e^\theta\right)^2 \operatorname{Var}\left(Y_\beta(t)\right), \qquad (44)\\
\operatorname{Cov}\left(\mathcal{M}_\beta(s),\mathcal{M}_\beta(t)\right) &= \alpha\theta(\theta+1)e^\theta \mathbb{E}\left(Y_\beta(\min\{s,t\})\right) + \left(\alpha\theta e^\theta\right)^2 \operatorname{Cov}\left(Y_\beta(s),Y_\beta(t)\right). \qquad (45)
\end{aligned}$$

The FBTP exhibits overdispersion as $\operatorname{Var}\left(\mathcal{M}_\beta(t)\right) - \mathbb{E}\left(\mathcal{M}_\beta(t)\right) > 0$ for all $t > 0$.

Theorem 3. *The FBTP exhibits the LRD property.*

Proof. From (44) and (45), we get

$$\operatorname{Corr}\left(\mathcal{M}_\beta(s),\mathcal{M}_\beta(t)\right) = \frac{\alpha\theta(\theta+1)e^\theta \mathbb{E}\left(Y_\beta(\min\{s,t\})\right) + (\alpha\theta e^\theta)^2 \operatorname{Cov}\left(Y_\beta(s),Y_\beta(t)\right)}{\sqrt{\operatorname{Var}\left(\mathcal{M}_\beta(s)\right)}\sqrt{\alpha\theta(\theta+1)e^\theta \mathbb{E}\left(Y_\beta(t)\right) + (\alpha\theta e^\theta)^2 \operatorname{Var}\left(Y_\beta(t)\right)}}.$$

On using (14)–(16) for fixed s and large t, we get

$$\begin{aligned}
&\operatorname{Corr}\left(\mathcal{M}_\beta(s),\mathcal{M}_\beta(t)\right) \\
&\quad\sim \frac{\alpha\theta(\theta+1)e^\theta\Gamma^2(\beta+1)\mathbb{E}\left(Y_\beta(s)\right) + (\alpha\theta e^\theta)^2\left(\beta s^{2\beta}B(\beta,\beta+1) - \frac{\beta^2 s^{\beta+1}}{(\beta+1)t^{1-\beta}}\right)}{\Gamma^2(\beta+1)\sqrt{\operatorname{Var}\left(\mathcal{M}_\beta(s)\right)}\sqrt{\frac{\alpha\theta(\theta+1)e^\theta t^\beta}{\Gamma(\beta+1)} + \frac{2(\alpha\theta e^\theta)^2 t^{2\beta}}{\Gamma(2\beta+1)} - \frac{(\alpha\theta e^\theta)^2 t^{2\beta}}{\Gamma^2(\beta+1)}}} \\
&\quad\sim c_0(s)t^{-\beta},
\end{aligned}$$

where

$$c_0(s) = \frac{(\theta+1)\Gamma^2(\beta+1)\mathbb{E}\left(Y_\beta(s)\right) + \alpha\theta e^\theta \beta s^{2\beta}B(\beta,\beta+1)}{\Gamma^2(\beta+1)\sqrt{\operatorname{Var}\left(\mathcal{M}_\beta(s)\right)}\sqrt{\frac{2}{\Gamma(2\beta+1)} - \frac{1}{\Gamma^2(\beta+1)}}}.$$

As $0 < \beta < 1$, it follows that the FBTP has the LRD property. □

Remark 8. *For a fixed $h > 0$, the increment process of FBTP is defined as*

$$\mathcal{Z}_\beta^h(t) := \mathcal{M}_\beta(t+h) - \mathcal{M}_\beta(t),\ t \geq 0.$$

It can be shown that the increment process $\{\mathcal{Z}_\beta^h(t)\}_{t\geq 0}$ exhibits the SRD property. The proof follows similar lines to that of Theorem 1 of Maheshwari and Vellaisamy [15].

The factorial moments of the FBTP can be obtained by taking $\lambda_j = \alpha\theta^j/j!$ for all $j \geq 1$ and letting $k \to \infty$ in Proposition 4 of Kataria and Khandakar [6].

Proposition 3. *Let $\psi_\beta(r,t) = \mathbb{E}(\mathcal{M}_\beta(t)(\mathcal{M}_\beta(t)-1)\cdots(\mathcal{M}_\beta(t)-r+1))$, $r \geq 1$ be the rth factorial moment of FBTP. Then,*

$$\psi_\beta(r,t) = r!\sum_{n=1}^{r}\frac{\left(\alpha e^\theta t^\beta\right)^n}{\Gamma(n\beta+1)}\sum_{\substack{\sum_{i=1}^n m_i = r\\ m_i\in\mathbb{N}}}\prod_{\ell=1}^{n}\frac{\theta^{m_\ell}}{m_\ell!}.$$

The proof of the next result follows on using (23), the self-similarity property of $\{Y_\beta(t)\}_{t\geq 0}$ in (28) and the arguments used in Proposition 3 of Kataria and Khandakar [6].

Proposition 4. *The one-dimensional distributions of FBTP are not infinitely divisible.*

Remark 9. *Let the random variable $\mathcal{W}_1$ be the first waiting time of FBTP. Then, the distribution of $\mathcal{W}_1$ is given by*

$$\Pr\{\mathcal{W}_1 > t\} = \Pr\{\mathcal{M}_\beta(t) = 0\} = E_{\beta,1}\left(-\alpha(e^\theta - 1)t^\beta\right),$$

which coincides with the first waiting time of TFPP with intensity $\alpha(e^\theta - 1)$ (see [22], Remark 3.3). However, the one-dimensional distributions of TFPP and FBTP differ. Thus, the fact that the TFPP is a renewal process (see [18]) implies that the FBTP is not a renewal process.

4. Poisson-Logarithmic Process and Its Fractional Version

Here, we introduce a fractional version of the PLP. First, we give some additional properties of it.

On taking $\beta = 1$, $\lambda_j = -\lambda(1-p)^j/j\ln p$ for all $j \geq 1$ and letting $k \to \infty$, the governing system of differential Equation (1) for GCP reduces to the governing system of differential Equation (17) of PLP. Thus, the PLP is a limiting case of the GCP. Thus, we note that several results for PLP can be obtained from the corresponding results for GCP.

The next result gives a recurrence relation for the pmf of PLP that follows from Proposition 1 of Kataria and Khandakar [6].

Proposition 5. *The state probabilities $\hat{q}(n,t) = \Pr\{\hat{\mathcal{M}}(t) = n\}$, $n \geq 1$ of PLP satisfy*

$$\hat{q}(n,t) = -\frac{\lambda t}{n\ln p}\sum_{j=1}^{n}(1-p)^j\hat{q}(n-j,t).$$

The pgf (18) can be rewritten as

$$\hat{G}(u,t) = \prod_{j=1}^{\infty}\exp\left(-\frac{\lambda t}{\ln p}\frac{(1-p)^j}{j}(u^j - 1)\right).$$

It follows that the PLP is equal in distribution to a weighted sum of independent Poisson processes; that is,

$$\hat{\mathcal{M}}(t) \overset{d}{=} \sum_{j=1}^{\infty} jN_j(t), \tag{46}$$

where $\{N_j(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda_j = -\lambda(1-p)^j/j\ln p$. Thus,

$$\begin{aligned}\lim_{t\to\infty}\frac{\hat{\mathcal{M}}(t)}{t} &\overset{d}{=} \sum_{j=1}^{\infty} j \lim_{t\to\infty}\frac{N_j(t)}{t}\\ &= \frac{\lambda(p-1)}{p\ln p}, \text{ in probability,}\end{aligned} \tag{47}$$

where we have used $\lim_{t\to\infty} N_j(t)/t = -\lambda(1-p)^j/j\ln p$, $j\geq 1$ almost surely.

In view of (4) and (46), the pmf of PLP is given by

$$\hat{q}(n,t) = \sum_{\Omega_n}\prod_{j=1}^{n}\frac{\left((1-p)^j/j\right)^{x_j}}{x_j!}\left(\frac{-\lambda t}{\ln p}\right)^{z_n} e^{-\lambda t},$$

where $z_n = x_1 + x_2 + \cdots + x_n$ and Ω_n is given in (5).

Using (6), the pmf of PLP can alternatively be written as

$$\hat{q}(n,t) = \sum_{\Omega_k^n}\prod_{j=1}^{k}\frac{(1-p)^{x_j}}{x_j k!}\left(-\frac{\lambda t}{\ln p}\right)^{k} e^{-\lambda t}, \tag{48}$$

where Ω_k^n is given in (7).

The next result gives a martingale characterisation for the PLP.

Proposition 6. *The process* $\left\{\hat{\mathcal{M}}(t) - \frac{\lambda(p-1)}{p\ln p}t\right\}_{t\geq 0}$ *is a martingale with respect to a natural filtration* $\mathcal{F}_t = \sigma\big(\hat{\mathcal{M}}(s), s\leq t\big)$.

Let $\hat{r}_1 = \lambda(p-1)/p\ln p$ and $\hat{r}_2 = \lambda(p-1)/p^2\ln p$. From (10), it follows that the PLP is a Lévy process. Its Lévy measure can be obtained by taking $\lambda_j = -\lambda(1-p)^j/j\ln p$ for all $j\geq 1$ and letting $k\to\infty$ in Equation (13) of Kataria and Khandakar [6]. It is given by

$$\hat{\mu}(\mathrm{d}x) = -\frac{\lambda}{\ln p}\sum_{j=1}^{\infty}\frac{(1-p)^j}{j}\delta_j \mathrm{d}x,$$

where δ_j's are Dirac measures. Its mean, variance and covariance are given by

$$E(\hat{\mathcal{M}}(t)) = \hat{r}_1 t,\ \mathrm{Var}(\hat{\mathcal{M}}(t)) = \hat{r}_2 t,\ \mathrm{Cov}(\hat{\mathcal{M}}(s),\hat{\mathcal{M}}(t)) = \hat{r}_2\min\{s,t\}. \tag{49}$$

Fractional Poisson-Logarithmic Process

Here, we introduce a fractional version of the PLP, namely, the fractional Poisson-logarithmic process (FPLP). We define it as the stochastic process $\{\hat{\mathcal{M}}_\beta(t)\}_{t\geq 0}$, $0<\beta\leq 1$, whose state probabilities $\hat{q}_\beta(n,t) = \Pr\{\hat{\mathcal{M}}_\beta(t) = n\}$ satisfy the following system of differential equations:

$$\begin{aligned}\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\hat{q}_\beta(0,t) &= -\lambda\hat{q}_\beta(0,t),\\ \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\hat{q}_\beta(n,t) &= -\lambda\hat{q}_\beta(n,t) - \frac{\lambda}{\ln p}\sum_{j=1}^{n}\frac{(1-p)^j}{j}\hat{q}_\beta(n-j,t),\ n\geq 1,\end{aligned} \tag{50}$$

with initial conditions $\hat{q}_\beta(0,0) = 1$ and $\hat{q}_\beta(n,0) = 0$, $n\geq 1$.

Note that the system of Equation (50) is obtained by replacing the integer order derivative in (17) with a Caputo fractional derivative.

Remark 10. *On taking* $\lambda_j = -\lambda(1-p)^j/j\ln p$ *for all* $j\geq 1$ *and letting* $k\to\infty$*, the System (1) reduces to System (50). Thus, the FPLP is a limiting case of the GFCP.*

Further, on taking $\lambda = -\ln p$, System (50) reduces to the system of differential equations that governs the state probabilities of fractional negative binomial process (see [23], Equation (66)).

Using (50), it can be shown that the pgf $\hat{G}_\beta(u,t) = \mathbb{E}\left(u^{\hat{\mathcal{M}}_\beta(t)}\right)$, $|u| \leq 1$ of FPLP satisfies

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\hat{G}_\beta(u,t) = -\lambda\left(1 - \frac{\ln(1-(1-p)u)}{\ln p}\right)\hat{G}_\beta(u,t),$$

with $\hat{G}_\beta(u,0) = 1$. On taking the Laplace transform in the above equation and using (3), we get

$$s^\beta\tilde{\hat{G}}_\beta(u,s) - s^{\beta-1}\hat{G}_\beta(u,0) = -\lambda\left(1 - \frac{\ln(1-(1-p)u)}{\ln p}\right)\tilde{\hat{G}}_\beta(u,s),\ s > 0.$$

Thus,

$$\tilde{\hat{G}}_\beta(u,s) = \frac{s^{\beta-1}}{s^\beta + \lambda\left(1 - \frac{\ln(1-(1-p)u)}{\ln p}\right)}.$$

On taking inverse Laplace transform and using (12), we get

$$\hat{G}_\beta(u,t) = E_{\beta,1}\left(-\lambda\left(1 - \frac{\ln(1-(1-p)u)}{\ln p}\right)t^\beta\right). \tag{51}$$

Remark 11. *On taking $\beta = 1$ in (51), we get the pgf of PLP given in (18). Further, from (51) we can verify that the pmf $\hat{q}_\beta(n,t)$ sums up to one; that is, $\sum_{n=0}^{\infty}\hat{q}_\beta(n,t) = \hat{G}_\beta(u,t)|_{u=1} = E_{\beta,1}(0) = 1$.*

The following time-changed relationship between the PLP and FPLP holds, whose proof follows similar lines to that of Theorem 1:

$$\hat{\mathcal{M}}_\beta(t) \overset{d}{=} \hat{\mathcal{M}}(Y_\beta(t)),\ t \geq 0, \tag{52}$$

where the PLP is independent of the inverse stable subordinator $\{Y_\beta(t)\}_{t\geq 0}$, $0 < \beta < 1$. In view of Remark 5, we have

$$\hat{\mathcal{M}}_\beta(t) \overset{d}{=} \hat{\mathcal{M}}(T_{2\beta}(t)),\ t > 0,$$

where $\{T_{2\beta}(t)\}_{t>0}$ is independent of $\{\hat{\mathcal{M}}(t)\}_{t>0}$.

Remark 12. *In view of (10) and (52), we note that the FPLP is equal in distribution to the following compound fractional Poisson process:*

$$\hat{\mathcal{M}}_\beta(t) \overset{d}{=} \sum_{i=1}^{N_\beta(t)} X_i,\ t \geq 0, \tag{53}$$

where $\{N_\beta(t)\}_{t\geq 0}$ is a TFPP with intensity λ independent of the sequence of independent and identically distributed random variables $\{X_i\}_{i\geq 1}$. Therefore, it is neither Markovian nor a Lévy process. In view of (53), System (50) can alternatively be obtained using Proposition 1 of Beghin and Macci [13].

The proof of the next result follows similar lines to that of Theorem 2.

Theorem 4. *The pmf $\hat{q}_\beta(n,t) = \Pr\{\hat{\mathcal{M}}_\beta(t) = n\}$ of FPLP is given by*

$$\hat{q}_\beta(n,t) = \begin{cases} E_{\beta,1}(-\lambda t^\beta),\ n = 0, \\ \displaystyle\sum_{\Omega_n}\prod_{j=1}^{n}\frac{((1-p)^j/j)^{x_j}}{x_j!}z_n!\left(\frac{-\lambda t^\beta}{\ln p}\right)^{z_n}E^{z_n+1}_{\beta,z_n\beta+1}\left(-\lambda t^\beta\right),\ n \geq 1, \end{cases} \tag{54}$$

where $z_n = x_1 + x_2 + \cdots + x_n$ and Ω_n is given in (5).

Remark 13. *If we use (48) in the proof of Theorem 4, then we can obtain the following alternate form of the pmf of FPLP:*

$$\hat{q}_\beta(n,t) = \sum_{\Omega_k^n} \prod_{j=1}^{k} \frac{(1-p)^{x_j}}{x_j} \left(\frac{-\lambda t^\beta}{\ln p} \right)^k E_{\beta,k\beta+1}^{k+1}\left(-\lambda t^\beta\right), \tag{55}$$

where Ω_k^n is given in (7).

From (53), we have

$$\hat{q}_\beta(0,t) = \Pr\{N_\beta(t) = 0\} = E_{\beta,1}(-\lambda t^\beta).$$

For $n \geq 1$, we get

$$\begin{aligned} \hat{q}_\beta(n,t) &= \sum_{k=1}^{n} \Pr\{X_1 + X_2 + \cdots + X_k = n\} \Pr\{N_\beta(t) = k\} && (56) \\ &= \sum_{k=1}^{n} \sum_{\Theta_n^k} k! \prod_{j=1}^{n} \frac{((1-p)^j/j)^{x_j}}{x_j!} \left(\frac{-\lambda t^\beta}{\ln p} \right)^k E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta), && (57) \end{aligned}$$

where Θ_n^k is given in (39).

As X_i's are independent and identically distributed, we have

$$\begin{aligned} \Pr\{X_1 + X_2 + \cdots + X_k = n\} &= \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \prod_{j=1}^{k} \Pr\{X_j = m_j\} \\ &= \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \left(\frac{-1}{\ln p} \right)^k \prod_{j=1}^{k} \frac{(1-p)^{m_j}}{m_j}, && (58) \end{aligned}$$

where we have used (10). Substituting (58) in (56), we get an equivalent expression for the pmf of the FPLP in the following form:

$$\hat{q}_\beta(n,t) = \sum_{k=1}^{n} \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \prod_{j=1}^{k} \frac{(1-p)^{m_j}}{m_j!} \left(\frac{-\lambda t^\beta}{\ln p} \right)^k E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta). \tag{59}$$

The pmf (57) can be written in the following equivalent form by using Lemma 2.4 of Kataria and Vellaisamy [21]:

$$\hat{q}_\beta(n,t) = \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{j=1}^{n-k+1} \frac{((1-p)^j/j)^{x_j}}{x_j!} \left(\frac{-\lambda t^\beta}{\ln p} \right)^k E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta), \tag{60}$$

where Λ_n^k is given in (43).

Thus, we have obtained the five alternate forms of the pmf of FPLP given in (54), (55), (57), (59) and (60). Note that the one-dimensional distributions of FPLP are not infinitely divisible, whose proof follows on similar lines to that of Proposition 4.

Remark 14. *The distribution of the first waiting time $\hat{\mathcal{W}}_1$ of FPLP is given by*

$$\Pr\{\hat{\mathcal{W}}_1 > t\} = \Pr\{\hat{\mathcal{M}}_\beta(t) = 0\} = E_{\beta,1}\left(-\lambda t^\beta\right).$$

By using the same arguments as used in Remark 9, we conclude that the FPLP is not a renewal process.

By using (49) and Theorem 2.1 of Leonenko et al. [14], we obtain the mean, variance and covariance of FPLP as follows:

$$\begin{aligned}\mathbb{E}\big(\hat{\mathcal{M}}_\beta(t)\big) &= \hat{r}_1\mathbb{E}\big(Y_\beta(t)\big),\\ \mathrm{Var}\big(\hat{\mathcal{M}}_\beta(t)\big) &= \hat{r}_2\mathbb{E}\big(Y_\beta(t)\big) + \hat{r}_1^2\,\mathrm{Var}\big(Y_\beta(t)\big),\\ \mathrm{Cov}\big(\hat{\mathcal{M}}_\beta(s), \hat{\mathcal{M}}_\beta(t)\big) &= \hat{r}_2\mathbb{E}\big(Y_\beta(\min\{s,t\})\big) + \hat{r}_1^2\,\mathrm{Cov}\big(Y_\beta(s), Y_\beta(t)\big).\end{aligned}$$

The FPLP exhibits overdispersion as $\mathrm{Var}\big(\hat{\mathcal{M}}_\beta(t)\big) - \mathbb{E}\big(\hat{\mathcal{M}}_\beta(t)\big) > 0$ for all $t > 0$.

Remark 15. *The FPLP exhibits the LRD property and, as in Remark 8, the increment process of FPLP exhibits the SRD property.*

5. Generalized Pólya–Aeppli Process and Its Fractional Version

Here, we introduce a fractional version of the GPAP. First, we give some additional properties of it.

On taking $\beta = 1$, $\lambda_j = \lambda\binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$ for all $j \geq 1$ and letting $k \to \infty$, the governing system of differential Equation (1) for GCP reduces to the governing system of differential Equation (19) of GPAP. Thus, the GPAP is a limiting case of the GCP.

The next result gives a recurrence relation for the pmf of GPAP, whose proof follows from Proposition 1 of Kataria and Khandakar [6].

Proposition 7. *The state probabilities $\bar{q}(n,t) = \Pr\{\bar{\mathcal{M}}(t) = n\}$, $n \geq 1$ of GPAP satisfy*

$$\bar{q}(n,t) = \frac{\lambda t(1-\rho)^r}{n(1-(1-\rho)^r)}\sum_{j=1}^{n} j\rho^j\binom{r+j-1}{j}\bar{q}(n-j,t).$$

The pgf (20) can be expressed as

$$\bar{G}(u,t) = \prod_{j=1}^{\infty}\exp\left(\lambda t\frac{\rho^j(1-\rho)^r}{1-(1-\rho)^r}\binom{r+j-1}{j}(u^j-1)\right).$$

It follows that the GPAP is equal in distribution to a weighted sum of independent Poisson processes; that is,

$$\bar{\mathcal{M}}(t) \stackrel{d}{=} \sum_{j=1}^{\infty} jN_j(t), \tag{61}$$

where $\{N_j(t)\}_{t\geq 0}$ is a Poisson process with intensity $\lambda_j = \lambda\binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$. Thus,

$$\begin{aligned}\lim_{t\to\infty}\frac{\bar{\mathcal{M}}(t)}{t} &\stackrel{d}{=} \sum_{j=1}^{\infty} j\lim_{t\to\infty}\frac{N_j(t)}{t}\\ &= \frac{\lambda r\rho}{(1-\rho)(1-(1-\rho)^r)}, \text{ in probability,}\end{aligned} \tag{62}$$

where we have used $\lim_{t\to\infty} N_j(t)/t = \lambda\binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$, $j \geq 1$ almost surely. On substituting $r = 1$ in (62), we get the corresponding limiting result for the Pólya–Aeppli process (see [6], Section 4.3).

In view of (61), the pmf of GPAP can be obtained from (4), and it is given by

$$\bar{q}(n,t)=\sum_{\Omega_n}\prod_{j=1}^{n}\frac{\left(\rho^j\binom{r+j-1}{j}\right)^{x_j}}{x_j!}\left(\frac{\lambda t(1-\rho)^r}{1-(1-\rho)^r}\right)^{z_n}e^{-\lambda t}, \tag{63}$$

where $z_n=x_1+x_2+\cdots+x_n$ and Ω_n is given in (5).

Using (6), the pmf of GPAP can alternatively be written as

$$\bar{q}(n,t)=\sum_{\Omega_k^n}\prod_{j=1}^{k}\rho^{x_j}\binom{r+x_j-1}{x_j}\left(\frac{\lambda t(1-\rho)^r}{1-(1-\rho)^r}\right)^k\frac{e^{-\lambda t}}{k!}, \tag{64}$$

where Ω_k^n is given in (7).

From (11), it follows that the GPAP is a Lévy process. Its Lévy measure can be obtained by taking $\lambda_j=\lambda\binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$ for all $j\geq 1$ and letting $k\to\infty$ in Equation (13) of Kataria and Khandakar [6]. It is given by

$$\bar{\mu}(\mathrm{d}x)=\frac{\lambda(1-\rho)^r}{1-(1-\rho)^r}\sum_{j=1}^{\infty}\binom{r+j-1}{j}\rho^j\delta_j\mathrm{d}x.$$

Fractional Generalized Pólya–Aeppli Process

Here, we introduce a fractional version of the GPAP, namely, the fractional generalized Pólya–Aeppli process (FGPAP). We define it as the stochastic process $\{\bar{\mathcal{M}}_\beta(t)\}_{t\geq 0}$, $0<\beta\leq 1$, whose state probabilities $\bar{q}_\beta(n,t)=\Pr\{\bar{\mathcal{M}}_\beta(t)=n\}$ satisfy the following system of differential equations:

$$\begin{aligned}&\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\bar{q}_\beta(0,t)=-\lambda\bar{q}_\beta(0,t),\\&\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\bar{q}_\beta(n,t)=-\lambda\bar{q}_\beta(n,t)+\frac{\lambda}{(1-\rho)^{-r}-1}\sum_{j=1}^{n}\binom{r+j-1}{j}\rho^j\bar{q}_\beta(n-j,t),\ n\geq 1,\end{aligned} \tag{65}$$

with initial conditions $\bar{q}_\beta(0,0)=1$ and $\bar{q}_\beta(n,0)=0$, $n\geq 1$.

Note that the system of Equation (65) is obtained by replacing the integer order derivative in (19) by Caputo fractional derivative.

Remark 16. *On taking $\lambda_j=\lambda\binom{r+j-1}{j}\rho^j(1-\rho)^r/(1-(1-\rho)^r)$ for all $j\geq 1$ and letting $k\to\infty$, System (1) reduces to System (65). Thus, the FGPAP is a limiting case of the GFCP.*

Further, on taking $r=1$, System (65) reduces to the system of differential equations that governs the state probabilities of the fractional Pólya–Aeppli process (see [13], Equation (19)).

Using (65), it can be shown that the pgf $\bar{G}_\beta(u,t)=\mathbb{E}\left(u^{\bar{\mathcal{M}}_\beta(t)}\right)$, $|u|\leq 1$ of FGPAP satisfies

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\bar{G}_\beta(u,t)=-\lambda\left(1-\frac{(1-\rho u)^{-r}-1}{(1-\rho)^{-r}-1}\right)\bar{G}_\beta(u,t),$$

with $\bar{G}_\beta(u,0)=1$. On taking the Laplace transform in the above equation and using (3), we get

$$s^\beta\tilde{\bar{G}}_\beta(u,s)-s^{\beta-1}\bar{G}_\beta(u,0)=-\lambda\left(1-\frac{(1-\rho u)^{-r}-1}{(1-\rho)^{-r}-1}\right)\tilde{\bar{G}}_\beta(u,s),\ s>0.$$

Thus,

$$\tilde{\bar{G}}_\beta(u,s) = \frac{s^{\beta-1}}{s^\beta + \lambda\left(1 - \frac{(1-\rho u)^{-r}-1}{(1-\rho)^{-r}-1}\right)}.$$

On taking the inverse Laplace transform and using (12), we get

$$\bar{G}_\beta(u,t) = E_{\beta,1}\left(-\lambda\left(1 - \frac{(1-\rho u)^{-r}-1}{(1-\rho)^{-r}-1}\right)t^\beta\right). \tag{66}$$

Remark 17. *On taking $\beta = 1$ in (66), we get the pgf of GPAP given in (20). Further, from (66) we can verify that the pmf $\bar{q}_\beta(n,t)$ sums up to one; that is, $\sum_{n=0}^{\infty} \bar{q}_\beta(n,t) = \bar{G}_\beta(u,t)|_{u=1} = E_{\beta,1}(0) = 1$.*

The following time-changed relationship holds between the GPAP and FGPAP:

$$\bar{\mathcal{M}}_\beta(t) \overset{d}{=} \bar{\mathcal{M}}(Y_\beta(t)),\ t \geq 0, \tag{67}$$

where the GPAP is independent of $\{Y_\beta(t)\}_{t\geq 0}$, $0 < \beta < 1$. Further,

$$\bar{\mathcal{M}}_\beta(t) \overset{d}{=} \bar{\mathcal{M}}(T_{2\beta}(t)),\ t > 0,$$

where $\{T_{2\beta}(t)\}_{t>0}$ is independent of $\{\bar{\mathcal{M}}(t)\}_{t>0}$.

Remark 18. *In view of (11) and (67), we note that the FGPAP is equal in distribution to the following compound fractional Poisson process:*

$$\bar{\mathcal{M}}_\beta(t) \overset{d}{=} \sum_{i=1}^{N_\beta(t)} X_i,\ t \geq 0, \tag{68}$$

where $\{N_\beta(t)\}_{t\geq 0}$ is a TFPP with intensity λ independent of $\{X_i\}_{i\geq 1}$. Therefore, it is neither Markovian nor a Lévy process. In view of (68), System (65) can alternatively be obtained using Proposition 1 of Beghin and Macci [13].

On using the pmf of GPAP (see [11], Equation (3)), we get the following pmf of FGPAP.

Theorem 5. *The pmf $\bar{q}_\beta(n,t) = \Pr\{\bar{\mathcal{M}}_\beta(t) = n\}$ of FGPAP is given by*

$$\bar{q}_\beta(n,t) = \begin{cases} E_{\beta,1}(-\lambda t^\beta),\ n = 0, \\ \rho^n \displaystyle\sum_{j=1}^{n}\sum_{m=1}^{j}(-1)^m\binom{j}{m}\left(\frac{-\lambda t^\beta}{(1-\rho)^{-r}-1}\right)^j\binom{rm+n-1}{n}E_{\beta,j\beta+1}^{j+1}\left(-\lambda t^\beta\right),\ n \geq 1. \end{cases} \tag{69}$$

Remark 19. *If we use (63) in the proof of Theorem 5, then we can obtain the following alternate form of the pmf of FGPAP:*

$$\bar{q}_\beta(n,t) = \begin{cases} E_{\beta,1}(-\lambda t^\beta),\ n = 0, \\ \displaystyle\sum_{\Omega_n}\prod_{j=1}^{n}\frac{\left(\rho^j\binom{r+j-1}{j}\right)^{x_j}}{x_j!}z_n!\left(\frac{\lambda t^\beta(1-\rho)^r}{1-(1-\rho)^r}\right)^{z_n}E_{\beta,z_n\beta+1}^{z_n+1}\left(-\lambda t^\beta\right),\ n \geq 1. \end{cases} \tag{70}$$

Again, if we use (64) in the proof of Theorem 5, then we can obtain the following:

$$\bar{q}_\beta(n,t) = \sum_{\Omega_k^n}\prod_{j=1}^{k}\rho^{x_j}\binom{r+x_j-1}{x_j}\left(\frac{\lambda t^\beta(1-\rho)^r}{1-(1-\rho)^r}\right)^k E_{\beta,k\beta+1}^{k+1}\left(-\lambda t^\beta\right). \tag{71}$$

From (68), we have

$$\bar{q}_\beta(0,t) = \Pr\{N_\beta(t) = 0\} = E_{\beta,1}(-\lambda t^\beta).$$

For $n \geq 1$, we get

$$\bar{q}_\beta(n,t) = \sum_{k=1}^{n} \Pr\{X_1 + X_2 + \cdots + X_k = n\}\Pr\{N_\beta(t) = k\} \tag{72}$$

$$= \sum_{k=1}^{n} \sum_{\Theta_n^k} k! \prod_{j=1}^{n} \frac{\left(\rho^j \binom{r+j-1}{j}\right)^{x_j}}{x_j!} \left(\frac{\lambda t^\beta (1-\rho)^r}{1-(1-\rho)^r}\right)^k E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta), \tag{73}$$

where Θ_n^k is given in (39).

As X_i's are independent and identically distributed, we have

$$\Pr\{X_1 + X_2 + \cdots + X_k = n\} = \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \prod_{j=1}^{k} \Pr\{X_j = m_j\}$$

$$= \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \left(\frac{(1-\rho)^r}{1-(1-\rho)^r}\right)^k \prod_{j=1}^{k} \rho^{m_j} \binom{r+m_j-1}{m_j}, \tag{74}$$

where we have used (11). Substituting (74) into (72), we get an equivalent expression for the pmf of FGPAP in the following form:

$$\bar{q}_\beta(n,t) = \sum_{k=1}^{n} \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_j \in \mathbb{N}}} \left(\frac{\lambda t^\beta (1-\rho)^r}{1-(1-\rho)^r}\right)^k \prod_{j=1}^{k} \rho^{m_j} \binom{r+m_j-1}{m_j} E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta). \tag{75}$$

The pmf (73) can be written in the following equivalent form by using Lemma 2.4 of Kataria and Vellaisamy [21]:

$$\bar{q}_\beta(n,t) = \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{j=1}^{n-k+1} \frac{\left(\rho^j \binom{r+j-1}{j}\right)^{x_j}}{x_j!} \left(\frac{\lambda t^\beta (1-\rho)^r}{1-(1-\rho)^r}\right)^k E_{\beta,k\beta+1}^{k+1}(-\lambda t^\beta), \tag{76}$$

where Λ_n^k is given in (43).

Thus, we have obtained the six alternate forms of the pmf of FGPAP given in (69)–(71), (73), (75) and (76). On substituting $r = 1$ in these pmfs, we get the equivalent versions of the pmf of fractional Pólya–Aeppli process. Note that the one-dimensional distributions of FGPAP are not infinitely divisible.

Remark 20. *The distribution of the first waiting time $\bar{\mathcal{W}}_1$ of FGPAP is given by*

$$\Pr\{\bar{\mathcal{W}}_1 > t\} = \Pr\{\bar{\mathcal{M}}_\beta(t) = 0\} = E_{\beta,1}\left(-\lambda t^\beta\right).$$

By using the same arguments as used in Remark 9, we conclude that the FGPAP is not a renewal process.

By using (21) and Theorem 2.1 of Leonenko et al. [14], we obtain the mean, variance and covariance of FGPAP as follows:

$$\mathbb{E}\left(\bar{\mathcal{M}}_\beta(t)\right) = \bar{r}_1 \mathbb{E}\left(Y_\beta(t)\right),$$
$$\mathrm{Var}\left(\bar{\mathcal{M}}_\beta(t)\right) = \bar{r}_2 \mathbb{E}\left(Y_\beta(t)\right) + \bar{r}_1^2 \mathrm{Var}\left(Y_\beta(t)\right),$$
$$\mathrm{Cov}\left(\bar{\mathcal{M}}_\beta(s), \bar{\mathcal{M}}_\beta(t)\right) = \bar{r}_2 \mathbb{E}\left(Y_\beta(\min\{s,t\})\right) + \bar{r}_1^2 \mathrm{Cov}\left(Y_\beta(s), Y_\beta(t)\right).$$

The FGPAP exhibits overdispersion as $\mathrm{Var}\big(\bar{\mathcal{M}}_\beta(t)\big) - \mathbb{E}\big(\bar{\mathcal{M}}_\beta(t)\big) > 0$ for all $t > 0$.

Remark 21. *The FGPAP has the LRD property, and the increment process of FGPAP exhibits the SRD property.*

6. Concluding Remarks

In this paper, we obtain some additional results for three recently introduced counting processes: BTP, PLP and GPAP. We study their fractional versions and observe that they are equal in distribution to some particular cases of the compound fractional Poisson process. We obtain several distributional properties for their fractional variants and establish their LRD property. Therefore, these processes have potential applications in modelling high-frequency financial data. Several equivalent forms of the pmf of FBTP, FPLP and FGPAP are obtained. We have shown that these fractional processes are overdispersed and non-renewal. It is observed that their one-dimensional distributions are not infinitely divisible, and their increment process has an SRD property. As an application in risk theory, FBTP, FPLP and FGPAP can be used to model the number of claims received by an insurance company.

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