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On Implicit Coupled Hadamard Fractional Differential Equations with Generalized Hadamard Fractional Integro-Differential Boundary Conditions

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Abstract: This study is devoted to studying the existence and uniqueness of solutions for Hadamard implicit fractional differential equations with generalized Hadamard fractional integro-differential boundary conditions by utilizing the contraction principle of the Banach and Leray–Schauder fixed point theorems. Moreover, with two different approaches, the Hyers–Ulam stabilities are also discussed. Different ordinary differential equations of the third order with different boundary conditions (e.g., initial, anti periodic and integro-differential) can be obtained as a special case for our proposed model. Finally, for verification, an example is presented, and some graphs for the particular variables and particular functions are drawn using MATLAB.

Keywords: Hadamard fractional derivative; coupled systems; boundary value problem; existence of solutions; Ulam stability

MSC: 26A33; 34B27; 45M10

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1. Introduction

The notion of fractional derivatives was first introduced in 1695, when Leibniz used the notation $\frac{d^n}{d\chi^n}$ for the nth derivative and de L'Hospital questioned what this may imply when n is $\frac{1}{2}$ (see [1]). The generalization of integer-order derivatives and integration is actually the fractional order derivatives and integration. Riemann, Liouville, Caputo, Hadamard, Hilfer, Riesz, Erdelyi–Kober, Fourier and Laplace worked on non-integer-order fractional derivatives and made the fractional calculus more informative for mathematicians. An operator in the form of a fractional order is a global one which applies as a tool for many complicated applied phenomena (e.g., biological mathematical models [2], signal as well as image processes [3], mathematical chemistry [4], control theory [5] and processes involving dynamics [6]). For more details about fractional order differential equations ($\mathcal{FDE}$s), see [7–10].

Among the qualitative properties of the nonlinear $\mathcal{FDE}$s, the existence of solutions is the most important one and the first priority of researchers. As a result, $\mathcal{FDE}$s involving conditions on the boundary such as initial, anti-periodic ($\mathcal{AP}$) and periodic, integro-differential multi-points, have been investigated [11–13]. The idea of Hyers–Ulam stability ($\mathcal{HUS}$) began in 1940 [14]. Among other types, it is an important type of stability which guarantees an exact solution for each approximate solution within a specific boundary. Therefore, it can be required in a number of applications such as optimization, numerical analysis, error analysis, biology and economics [15–17]. Alqifiary et al. [18] investigated the generalized Hyers–Ulam stability ($\mathcal{GHUS}$) for linear

$\mathcal{DE}$s. Razaei et al. [19] studied $\mathcal{HUS}$ by utilizing the approach of a Laplace transform for linear $\mathcal{DE}$s. Wang et al. [20] obtained the $\mathcal{HUS}$ of two different kinds of $\mathcal{FDE}$s. Liu et al. [21] achieved the $\mathcal{HUS}$ of linear Caputo–Fabrizio $\mathcal{FDE}$s. Liu et al. [22] gave the $\mathcal{HUS}$ of linear Caputo–Fabrizio $\mathcal{FDE}$s by utilizing the property of a Mittag–Leffler kernel and using the method of Laplace transformation.

Luo et al. [23] discussed the existence, uniqueness ($\mathcal{EU}$) and $\mathcal{HUS}$ of

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi)+\mu(\chi,\mathrm{v}(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi))=0;\ \chi\in\mathcal{M},\\ \mathrm{v}(\chi)\,|_{\chi=1}={}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathrm{v}(\chi)\,|_{\chi=1}=0,\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathrm{v}(\chi)\,|_{\chi=2}=\alpha[\mathrm{v}]+\int_1^{\zeta}\varphi(\chi){}_{\mathrm{H}}\mathfrak{D}^{\beta}\mathrm{v}(\chi)\mathrm{d}\chi, \end{cases}$$

They also extended their results to a system of the form

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi)+\mu(\chi,\mathrm{v}(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\chi))=0;\ \chi\in\mathcal{M},\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\chi)+\nu(\chi,\mathrm{w}(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi))=0;\ \chi\in\mathcal{M},\\ \mathrm{v}(\chi)\,|_{\chi=1}={}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathrm{v}(\chi)\,|_{\chi=1}=0,\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathrm{v}(\chi)\,|_{\chi=2}=\alpha[\mathrm{v}]+\int_1^{\zeta}\varphi_{\kappa}(\chi){}_{\mathrm{H}}\mathfrak{D}^{\beta}\mathrm{v}(\chi)\mathrm{d}\chi,\\ \mathrm{w}(\chi)\,|_{\chi=1}={}_{\mathrm{H}}\mathfrak{D}^{\sigma-2}\mathrm{w}(\chi)\,|_{\chi=1}=0,\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-1}\mathrm{w}(\chi)\,|_{\chi=2}=\gamma[\mathrm{w}]+\int_1^{\xi}\varphi_{\sigma}(\chi){}_{\mathrm{H}}\mathfrak{D}^{\delta}\mathrm{w}(\chi)\mathrm{d}\chi, \end{cases}$$

where $\kappa,\sigma\in(2,3]$ and ${}_{\mathrm{H}}\mathfrak{D}^{(\cdot)}$ are the Hadamard fractional derivative of the order $(\cdot)$ with $\kappa-1-\beta>0$ and $\sigma-1-\delta>0$, $\varphi_{\kappa},\varphi_{\sigma}\in \mathrm{L}^1[1,2]$ is nonnegative and $\alpha[\mathrm{v}]$ and $\gamma[\mathrm{w}]$ are linear functions.

The above study motivated us to study the $\mathcal{EU}$ and at least one solution of the following coupled system ($\mathbb{CS}$) of $\mathcal{FDE}$s in the sense of Hadamard derivatives:

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi)-\mu(\chi,\mathrm{w}(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\chi))=0;\ \chi\in\mathcal{M},\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\chi)-\nu(\chi,\mathrm{v}(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\chi))=0;\ \chi\in\mathcal{M}, \end{cases} \tag{1}$$

having the generalized Hadamard fractional integro-differential boundary conditions

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathrm{v}(1)=\mathrm{a}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathrm{v}(\mathrm{T})+\eta_1{}_{\mathrm{H}}\mathfrak{I}^{\kappa-1}\mathrm{v}(\mathrm{T}),\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathrm{v}(1)=\mathrm{b}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathrm{v}(\mathrm{T})+\eta_2{}_{\mathrm{H}}\mathfrak{I}^{\kappa-2}\mathrm{v}(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-3}\mathrm{v}(1)=\mathrm{c}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-3}\mathrm{v}(\mathrm{T})+\eta_3{}_{\mathrm{H}}\mathfrak{I}^{\kappa-3}\mathrm{v}(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-1}\mathrm{w}(1)=\mathrm{a}_1{}_{\mathrm{H}}\mathfrak{D}^{\sigma-1}\mathrm{w}(\mathrm{T})+\eta_4{}_{\mathrm{H}}\mathfrak{I}^{\sigma-1}\mathrm{w}(\mathrm{T}),\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-2}\mathrm{w}(1)=\mathrm{b}_1{}_{\mathrm{H}}\mathfrak{D}^{\sigma-2}\mathrm{w}(\mathrm{T})+\eta_5{}_{\mathrm{H}}\mathfrak{I}^{\sigma-2}\mathrm{w}(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-3}\mathrm{w}(1)=\mathrm{c}_1{}_{\mathrm{H}}\mathfrak{D}^{\sigma-3}\mathrm{w}(\mathrm{T})+\eta_6{}_{\mathrm{H}}\mathfrak{I}^{\sigma-3}\mathrm{w}(\mathrm{T}), \end{cases} \tag{2}$$

where $2<\kappa,\sigma\leq 3$, $\mathcal{M}=[1,\mathrm{T}]$, $\mathrm{T}>1$, $\mathrm{a_i},\mathrm{b_i},\mathrm{c_i}\neq 1$ for $\mathrm{i}=0,1$ and $\eta_{\mathrm{j}}(\mathrm{j}=1,2,\ldots,6)$ are coefficients of the integrals in boundary conditions (BCs). $\mu,\nu:\mathcal{M}\times\mathcal{R}^2\to\mathcal{R}$ are continuous functions, and ${}_{\mathrm{H}}\mathfrak{D}^{\kappa}/{}_{\mathrm{H}}\mathfrak{I}^{\kappa}$ and ${}_{\mathrm{H}}\mathfrak{D}^{\sigma}/{}_{\mathrm{H}}\mathfrak{I}^{\sigma}$ are the Hadamard derivatives and integrals of the fractional orders κ and σ, respectively.

The said system of Hadamard $\mathcal{FDE}$s (Equation (1)) is the generalization of third-order ordinary differential equations ($\mathcal{ODE}$s), and according to our information, the proposed $\mathcal{ODE}$s have several uses in different engineering fields and the area of applied sciences [24–27]. It is attractive that some mathematical modeling of numerous physical phenomena gives systems in a form coupled with the aforementioned third-order $\mathcal{ODE}$s. Such models are linked jerk-type equations, which are mainly used in the processes of manufacturing [28–30]. Additionally, for $\eta_{\mathrm{i}}=0(\mathrm{i}=1,2,\ldots,6)$ and $\mathrm{a_i}=\mathrm{b_i}=\mathrm{c_i}=-1(\mathrm{i}=0,1)$, we can acquire BCs in the form of $\mathcal{AP}$, which often occur in the models of different physical phenomena (e.g., in the partial, ordinary and $\mathcal{DE}$s with impulses, $\mathcal{AP}$ trigonometric polynomials in the investigation of the interpolation issues, $\mathcal{AP}$ wavelets and equations in the discrete form) [31–34].

2. Background Materials

Let $\mathcal{N}(\mathcal{M})$ be the space of all continuous functions on $\mathcal{M}$, which is a Banach space with $\|\mathsf{v}\| = \max_{\chi\in\mathcal{M}} |\mathsf{v}(\chi)|$. Let $\mathcal{W} = \mathcal{N}_{\mathrm{r}}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$ be the space of all functions v such that for $\chi \in \mathcal{M}$ and $\mathrm{r} \geq 0$, we have $\mathsf{v}_{\mathrm{r}}(\chi) = (\log\chi)^{\mathrm{r}}\mathsf{v}(\chi)$. Obviously, $\mathcal{W}$ is a Banach space with

$$\|\mathsf{v}\|_{\mathcal{W}} = \max\{\sup_{\chi\in\mathcal{M}} (\log\chi)^{\mathrm{r}}|\mathsf{v}(\chi)|, \sup_{\chi\in\mathcal{M}} (\log\chi)^{\mathrm{r}}|{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi)|\}.$$

Additionally, $(\mathcal{W}\times\mathcal{W}, \|(\mathsf{v},\mathsf{w})\|_{\mathcal{W}\times\mathcal{W}})$ is a Banach space with norm $\|(\mathsf{v},\mathsf{w})\|_{\mathcal{W}\times\mathcal{W}} = \|\mathsf{v}\|_{\mathcal{W}} + \|\mathsf{w}\|_{\mathcal{W}}$:

Definition 1 ([35]). *The $\kappa > 0$ order Hadamard integral for $\mathsf{v} : \mathcal{R}^+ \to \mathcal{R}$ (continuous) is represented by*

$${}_{\mathrm{H}}\mathfrak{I}^{\kappa}\mathsf{v}(\chi) = \frac{1}{\Gamma(\kappa)}\int_1^{\chi}(\log\chi - \log s)^{\kappa-1}\mathsf{v}(s)\frac{ds}{s}.$$

Definition 2 ([35]). *The Hadamard derivative of the order $\kappa > 0$ for $\mathsf{v} : \mathcal{R}^+ \to \mathcal{R}$ (continuous) is given by*

$${}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) = \frac{1}{\Gamma(\mathrm{n}-\kappa)}\left(\chi\frac{d}{d\chi}\right)^{\mathrm{n}}\int_1^{\chi}(\log\chi - \log s)^{\mathrm{n}-\kappa+1}\mathsf{v}(s)\frac{ds}{s},$$

where $\mathrm{n} = [\kappa] + 1$ and $[\kappa]$ is the integer value of κ.

It can be observed that for $\varrho > -1$, $\varrho \neq \kappa-1, \kappa-2, \ldots, \kappa-\mathrm{n}$, we have

$${}_{\mathrm{H}}\mathfrak{D}^{\kappa}\log(\chi)^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho-\kappa+1)}\log(\chi)^{\varrho-\kappa}$$

and

$${}_{\mathrm{H}}\mathfrak{D}^{\kappa}\log(\chi)^{\kappa-\mathrm{i}} = 0, \quad \mathrm{i} = 1, 2, \ldots, \mathrm{n}.$$

In addition, we have

$${}_{\mathrm{H}}\mathfrak{I}^{\kappa}\log(\chi)^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+\kappa+1)}\log(\chi)^{\varrho+\kappa}.$$

Lemma 1 ([35]). *For a Hadamard $\mathcal{FDE}$ of the order $\kappa > 0$, we have*

$${}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) = \mu(\chi),$$

which is

$${}_{\mathrm{H}}\mathfrak{I}^{\kappa}{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) = {}_{\mathrm{H}}\mathfrak{I}^{\kappa}\mu(\chi) + \mathrm{k}_1(\log\chi)^{\kappa-1} + \mathrm{k}_2(\log\chi)^{\kappa-2} + \cdots + \mathrm{k}_{\mathrm{n}-1}(\log\chi)^{\kappa-(\mathrm{n}-1)} + \mathrm{k}_{\mathrm{n}}(\log\chi)^{\kappa-\mathrm{n}},$$

where $\mathrm{k}_1, \mathrm{k}_2, \ldots, \mathrm{k}_{\mathrm{n}}$ are real constants.

3. Existence of Solutions

Lemma 2. *Assume that $\mu \in \mathcal{N}(\mathcal{M})$ and $\kappa \in (2, 3]$. Then, the solution of*

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) = \mu(\chi);\ \chi \in \mathcal{M}, \\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathsf{v}(1) = \mathsf{a}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\mathsf{v}(\mathrm{T}) + \eta_1{}_{\mathrm{H}}\mathfrak{I}^{\kappa-1}\mathsf{v}(\mathrm{T}), \\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathsf{v}(1) = \mathsf{b}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\mathsf{v}(\mathrm{T}) + \eta_2{}_{\mathrm{H}}\mathfrak{I}^{\kappa-2}\mathsf{v}(\mathrm{T}), \\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-3}\mathsf{v}(1) = \mathsf{c}_0{}_{\mathrm{H}}\mathfrak{D}^{\kappa-3}\mathsf{v}(\mathrm{T}) + \eta_3{}_{\mathrm{H}}\mathfrak{I}^{\kappa-3}\mathsf{v}(\mathrm{T}), \end{cases}$$

is represented by

$$\mathsf{v}(\chi) = \int_1^{\mathrm{T}}\mathbf{G}_{\kappa}(\chi, s)\mu(s)\mathrm{d}s,$$

where

$$\mathbf{G}_{\kappa}(\chi,s)=\begin{cases}\frac{(\log\frac{\chi}{s})^{\kappa-1}}{\Gamma(\kappa)}-\left[\frac{(\log\chi)^{\kappa-1}a_0\delta_{11}}{\delta}+\frac{(\log\chi)^{\kappa-2}a_0\delta_{21}}{\delta}+\frac{(\log\chi)^{\kappa-3}a_0\delta_{31}}{\delta}\right]\\ -\left[\frac{(\log\chi)^{\kappa-1}b_0\delta_{12}}{\delta}+\frac{(\log\chi)^{\kappa-2}b_0\delta_{22}}{\delta}+\frac{(\log\chi)^{\kappa-3}b_0\delta_{32}}{\delta}\right]\log\frac{T}{s}\\ -\frac{1}{2}\left[\frac{(\log\chi)^{\kappa-1}c_0\delta_{13}}{\delta}+\frac{(\log\chi)^{\kappa-2}c_0\delta_{23}}{\delta}+\frac{(\log\chi)^{\kappa-3}c_0\delta_{33}}{\delta}\right](\log\frac{T}{s})^2\\ -\frac{1}{\Gamma(2\kappa-1)}\left[\frac{(\log\chi)^{\kappa-1}\eta_1\delta_{11}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_1\delta_{21}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_1\delta_{31}}{\delta}\right](\log\frac{T}{s})^{2\kappa-2}\\ -\frac{1}{\Gamma(2\kappa-2)}\left[\frac{(\log\chi)^{\kappa-1}\eta_2\delta_{12}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_2\delta_{22}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_2\delta_{32}}{\delta}\right](\log\frac{T}{s})^{2\kappa-3}\\ -\frac{1}{\Gamma(2\kappa-3)}\left[\frac{(\log\chi)^{\kappa-1}\eta_3\delta_{13}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_3\delta_{23}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_3\delta_{33}}{\delta}\right](\log\frac{T}{s})^{2\kappa-4}, \quad 1\le s<\chi\le T,\\ -\left[\frac{(\log\chi)^{\kappa-1}a_0\delta_{11}}{\delta}+\frac{(\log\chi)^{\kappa-2}a_0\delta_{21}}{\delta}+\frac{(\log\chi)^{\kappa-3}a_0\delta_{31}}{\delta}\right]\\ -\left[\frac{(\log\chi)^{\kappa-1}b_0\delta_{12}}{\delta}+\frac{(\log\chi)^{\kappa-2}b_0\delta_{22}}{\delta}+\frac{(\log\chi)^{\kappa-3}b_0\delta_{32}}{\delta}\right]\log\frac{T}{s}\\ -\frac{1}{2}\left[\frac{(\log\chi)^{\kappa-1}c_0\delta_{13}}{\delta}+\frac{(\log\chi)^{\kappa-2}c_0\delta_{23}}{\delta}+\frac{(\log\chi)^{\kappa-3}c_0\delta_{33}}{\delta}\right](\log\frac{T}{s})^2\\ -\frac{1}{\Gamma(2\kappa-1)}\left[\frac{(\log\chi)^{\kappa-1}\eta_1\delta_{11}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_1\delta_{21}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_1\delta_{31}}{\delta}\right](\log\frac{T}{s})^{2\kappa-2}\\ -\frac{1}{\Gamma(2\kappa-2)}\left[\frac{(\log\chi)^{\kappa-1}\eta_2\delta_{12}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_2\delta_{22}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_2\delta_{32}}{\delta}\right](\log\frac{T}{s})^{2\kappa-3}\\ -\frac{1}{\Gamma(2\kappa-3)}\left[\frac{(\log\chi)^{\kappa-1}\eta_3\delta_{13}}{\delta}+\frac{(\log\chi)^{\kappa-2}\eta_3\delta_{23}}{\delta}+\frac{(\log\chi)^{\kappa-3}\eta_3\delta_{33}}{\delta}\right](\log\frac{T}{s})^{2\kappa-4}, \quad 1\le\chi<s\le T.\end{cases}\tag{3}$$

Proof. By applying Lemma 1, we obtain

$$v(\chi)={}_{H}\mathfrak{I}^{\kappa}\mu(\chi)+k_1(\log\chi)^{\kappa-1}+k_2(\log\chi)^{\kappa-2}+k_3(\log\chi)^{\kappa-3}.\tag{4}$$

By utilizing the BCs, Equation (4) implies

$$k_1\Gamma(\kappa)\left(a_0-1+\frac{\eta_1(\log T)^{2\kappa-2}}{\Gamma(2\kappa-1)}\right)+k_2\frac{\eta_1\Gamma(\kappa-1)(\log T)^{2\kappa-3}}{\Gamma(2\kappa-2)}+k_3\frac{\eta_1\Gamma(\kappa-2)(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}$$
$$=-a_{0H}\mathfrak{I}\mu(T)-\eta_{1H}\mathfrak{I}^{2\kappa-1}\mu(T),\tag{5}$$
$$k_1\Gamma(\kappa)\left(b_0\log T+\frac{\eta_2(\log T)^{2\kappa-3}}{\Gamma(2\kappa-2)}\right)+k_2\Gamma(\kappa-1)\left(b_0-1+\frac{\eta_2(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}\right)$$
$$+k_3\frac{\eta_2\Gamma(\kappa-2)(\log T)^{2\kappa-5}}{\Gamma(2\kappa-4)}=-b_{0H}\mathfrak{I}^2\mu(T)-\eta_{2H}\mathfrak{I}^{2\kappa-2}\mu(T),\tag{6}$$
$$k_1\Gamma(\kappa)\left(\frac{c_0(\log T)^2}{2}+\frac{\eta_3(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}\right)+k_2\Gamma(\kappa-1)\left(c_0\log T+\frac{\eta_3(\log T)^{2\kappa-5}}{\Gamma(2\kappa-4)}\right)$$
$$+k_3\Gamma(\kappa-2)\left(c_0-1+\frac{\eta_3(\log T)^{2\kappa-6}}{\Gamma(2\kappa-5)}\right)=-c_{0H}\mathfrak{I}^3\mu(T)-\eta_{3H}\mathfrak{I}^{2\kappa-3}\mu(T).\tag{7}$$

We can write Equations (5)–(7) in matrix form:

$$AK=I,\quad \text{implies}\quad K=A^{-1}I$$

where

$$A=\begin{bmatrix} \Gamma(\kappa)\left(a_0-1+\frac{\eta_1(\log T)^{2\kappa-2}}{\Gamma(2\kappa-1)}\right) & \frac{\eta_1\Gamma(\kappa-1)(\log T)^{2\kappa-3}}{\Gamma(2\kappa-2)} & \frac{\eta_1\Gamma(\kappa-2)(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)} \\ \Gamma(\kappa)\left(b_0\log T+\frac{\eta_2(\log T)^{2\kappa-3}}{\Gamma(2\kappa-2)}\right) & \Gamma(\kappa-1)\left(b_0-1+\frac{\eta_2(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}\right) & \frac{\eta_2\Gamma(\kappa-2)(\log T)^{2\kappa-5}}{\Gamma(2\kappa-4)} \\ \Gamma(\kappa)\left(\frac{c_0(\log T)^2}{2}+\frac{\eta_3(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}\right) & \Gamma(\kappa-1)\left(c_0\log T+\frac{\eta_3(\log T)^{2\kappa-5}}{\Gamma(2\kappa-4)}\right) & \Gamma(\kappa-2)\left(c_0-1+\frac{\eta_3(\log T)^{2\kappa-6}}{\Gamma(2\kappa-5)}\right) \end{bmatrix},$$

$$K=\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix},\quad I=\begin{bmatrix} -a_{0H}\Im\mu(T)-\eta_{1H}\Im^{2\kappa-1}\mu(T) \\ -b_{0H}\Im^2\mu(T)-\eta_{2H}\Im^{2\kappa-2}\mu(T) \\ -c_{0H}\Im^3\mu(T)-\eta_{3H}\Im^{2\kappa-3}\mu(T) \end{bmatrix} \text{ and } A^{-1}=\frac{1}{\det(A)}\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}.$$

Here, the determinant of matrix A is

$$\begin{aligned} \det(A) =&\Gamma(\kappa)\Gamma(\kappa-1)\Gamma(\kappa-2)\left(a_0-1+\frac{\eta_1(\log T)^{2\kappa-2}}{\Gamma(2\kappa-1)}\right)\left[(b_0-1)\left(c_0-1+\frac{\eta_3(\log T)^{2\kappa-6}}{\Gamma(2\kappa-5)}\right)\right. \\ &\left.+\frac{(c_0(2\kappa-3)-1)\eta_2(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}+\frac{\eta_2\eta_3(\log T)^{4\kappa-10}}{\Gamma(2\kappa-3)\Gamma(2\kappa-5)}+\frac{\eta_2\eta_3(\log T)^{4\kappa-10}}{\Gamma(2\kappa-4)\Gamma(2\kappa-4)}\right] \\ &-\Gamma(\kappa)\Gamma(\kappa-1)\Gamma(\kappa-2)\frac{\eta_1(\log T)^{2\kappa-3}}{\Gamma(2\kappa-2)}\left[(c_0-1)b_0\log T+b_0\log T\frac{\eta_3(\log T)^{2\kappa-6}}{\Gamma(2\kappa-5)}\right. \\ &\left.-\frac{[c_0(4\kappa^2-14\kappa+10)+2]\eta_2(\log T)^{2\kappa-3}}{2\Gamma(2\kappa-2)}+\frac{\eta_2\eta_3(\log T)^{4\kappa-9}}{\Gamma(2\kappa-2)\Gamma(2\kappa-5)}-\frac{\eta_2\eta_3(\log T)^{4\kappa-9}}{\Gamma(2\kappa-3)\Gamma(2\kappa-4)}\right] \\ &+\Gamma(\kappa)\Gamma(\kappa-1)\Gamma(\kappa-2)\frac{\eta_1(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}\left[\frac{(b_0+1)c_0(\log T)^2}{2}-\frac{(c_0(2\kappa-5))\eta_2(\log T)^{2\kappa-2}}{\Gamma(2\kappa-2)}\right. \\ &\left.+\frac{(b_0(2\kappa-3)+1)\eta_3(\log T)^{2\kappa-4}}{\Gamma(2\kappa-3)}+\frac{\eta_2\eta_3(\log T)^{4\kappa-8}}{\Gamma(2\kappa-2)\Gamma(2\kappa-4)}-\frac{\eta_2\eta_3(\log T)^{4\kappa-8}}{\Gamma(2\kappa-3)\Gamma(2\kappa-3)}\right]=\delta\neq 0 \end{aligned}$$

In addition, the co-factors are

$$\begin{aligned}
\delta_{11} =&\Gamma(\kappa-1)\Gamma(\kappa-2)\left[(\mathrm{b}_0-1)\left(\mathrm{c}_0-1+\frac{\eta_3(\log \mathrm{T})^{2\kappa-6}}{\Gamma(2\kappa-5)}\right)+\frac{(\mathrm{c}_0(2\kappa-3)-1)\eta_2(\log \mathrm{T})^{2\kappa-4}}{\Gamma(2\kappa-3)}\right.\\
&\left.+\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-10}}{\Gamma(2\kappa-3)\Gamma(2\kappa-5)}+\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-10}}{\Gamma(2\kappa-4)\Gamma(2\kappa-4)}\right],\\
\delta_{12} =&-\Gamma(\kappa)\Gamma(\kappa-2)\left[(\mathrm{c}_0-1)\mathrm{b}_0\log \mathrm{T}+\mathrm{b}_0\log \mathrm{T}\frac{\eta_3(\log \mathrm{T})^{2\kappa-6}}{\Gamma(2\kappa-5)}+\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-9}}{\Gamma(2\kappa-2)\Gamma(2\kappa-5)}\right.\\
&\left.+\frac{[\mathrm{c}_0(-4\kappa^2+14\kappa-10)-2]\eta_2(\log \mathrm{T})^{2\kappa-3}}{2\Gamma(2\kappa-2)}-\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-9}}{\Gamma(2\kappa-3)\Gamma(2\kappa-4)}\right],\\
\delta_{13} =&\Gamma(\kappa)\Gamma(\kappa-1)\left[\frac{(\mathrm{b}_0+1)\mathrm{c}_0(\log \mathrm{T})^2}{2}-\frac{(\mathrm{c}_0(2\kappa-5))\eta_2(\log \mathrm{T})^{2\kappa-2}}{\Gamma(2\kappa-2)}+\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-2)\Gamma(2\kappa-4)}\right.\\
&\left.+\frac{(\mathrm{b}_0(2\kappa-3)+1)\eta_3(\log \mathrm{T})^{2\kappa-4}}{\Gamma(2\kappa-3)}-\frac{\eta_2\eta_3(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-3)\Gamma(2\kappa-3)}\right],\\
\delta_{21} =&-\Gamma(\kappa-1)\Gamma(\kappa-2)\left[\frac{(\mathrm{c}_0(4-2\kappa)-1)\eta_1(\log \mathrm{T})^{2\kappa-3}}{\Gamma(2\kappa-2)}+\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-9}}{\Gamma(2\kappa-2)\Gamma(2\kappa-5)}-\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-9}}{\Gamma(2\kappa-3)\Gamma(2\kappa-4)}\right],\\
\delta_{22} =&\Gamma(\kappa)\Gamma(\kappa-2)\left[(\mathrm{c}_0-1)(\mathrm{a}_0-1)+\frac{(\mathrm{a}_0-1)\eta_3(\log \mathrm{T})^{2\kappa-6}}{\Gamma(2\kappa-5)}\right.\\
&\left.-\frac{(\mathrm{c}_0(4\kappa^2-10\kappa-4)+2)\eta_1(\log \mathrm{T})^{2\kappa-2}}{2\Gamma(2\kappa-3)}+\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-1)\Gamma(2\kappa-5)}-\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-3)\Gamma(2\kappa-3)}\right],\\
\delta_{23} =&-\Gamma(\kappa)\Gamma(\kappa-1)\left[(\mathrm{a}_0-1)\mathrm{c}_0\log \mathrm{T}-\frac{\mathrm{c}_0\kappa\eta_1(\log \mathrm{T})^{2\kappa-1}}{\Gamma(2\kappa-1)}+\frac{(\mathrm{a}_0-1)\eta_3(\log \mathrm{T})^{2\kappa-5}}{\Gamma(2\kappa-4)}\right.\\
&\left.+\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-7}}{\Gamma(2\kappa-1)\Gamma(2\kappa-4)}-\frac{\eta_1\eta_3(\log \mathrm{T})^{4\kappa-7}}{\Gamma(2\kappa-2)\Gamma(2\kappa-3)}\right],\\
\delta_{31} =&-\Gamma(\kappa-1)\Gamma(\kappa-2)\left[\frac{(\mathrm{b}_0-1)\eta_1(\log \mathrm{T})^{2\kappa-4}}{\Gamma(2\kappa-3)}-\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-2)\Gamma(2\kappa-4)}+\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-8}}{\Gamma(2\kappa-3)\Gamma(2\kappa-3)}\right],\\
\delta_{32} =&-\Gamma(\kappa)\Gamma(\kappa-2)\left[\frac{(\mathrm{a}_0-1)\eta_2(\log \mathrm{T})^{2\kappa-5}}{\Gamma(2\kappa-4)}-\frac{\mathrm{b}_0\eta_1(\log \mathrm{T})^{2\kappa-3}}{\Gamma(2\kappa-3)}+\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-7}}{\Gamma(2\kappa-1)\Gamma(2\kappa-4)}\right.\\
&\left.-\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-7}}{\Gamma(2\kappa-2)\Gamma(2\kappa-3)}\right],\\
\delta_{33} =&\Gamma(\kappa)\Gamma(\kappa-1)\left[(\mathrm{a}_0-1)(\mathrm{b}_0-1)+\frac{(\mathrm{a}_0-1)\eta_2(\log \mathrm{T})^{2\kappa-4}}{\Gamma(2\kappa-3)}-\frac{(\mathrm{b}_0(2\kappa-3)+1)\eta_1(\log \mathrm{T})^{2\kappa-2}}{\Gamma(2\kappa-1)}\right.\\
&\left.+\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-6}}{\Gamma(2\kappa-1)\Gamma(2\kappa-3)}-\frac{\eta_1\eta_2(\log \mathrm{T})^{4\kappa-6}}{\Gamma(2\kappa-2)\Gamma(2\kappa-2)}\right].
\end{aligned}$$

Hence, the unknowns are

$$\begin{aligned}
\mathrm{k}_1 =&\frac{\delta_{11}}{\delta}(-\mathrm{a}_{0\mathrm{H}}\mathfrak{I}\mu(\mathrm{T})-\eta_{1\mathrm{H}}\mathfrak{I}^{2\kappa-1}\mu(\mathrm{T}))+\frac{\delta_{12}}{\delta}(-\mathrm{b}_{0\mathrm{H}}\mathfrak{I}^2\mu(\mathrm{T})-\eta_{2\mathrm{H}}\mathfrak{I}^{2\kappa-2}\mu(\mathrm{T}))\\
&+\frac{\delta_{13}}{\delta}(-\mathrm{c}_{0\mathrm{H}}\mathfrak{I}^3\mu(\mathrm{T})-\eta_{3\mathrm{H}}\mathfrak{I}^{2\kappa-3}\mu(\mathrm{T})),\\
\mathrm{k}_2 =&\frac{\delta_{21}}{\delta}(-\mathrm{a}_{0\mathrm{H}}\mathfrak{I}\mu(\mathrm{T})-\eta_{1\mathrm{H}}\mathfrak{I}^{2\kappa-1}\mu(\mathrm{T}))+\frac{\delta_{22}}{\delta}(-\mathrm{b}_{0\mathrm{H}}\mathfrak{I}^2\mu(\mathrm{T})-\eta_{2\mathrm{H}}\mathfrak{I}^{2\kappa-2}\mu(\mathrm{T}))\\
&+\frac{\delta_{23}}{\delta}(-\mathrm{c}_{0\mathrm{H}}\mathfrak{I}^3\mu(\mathrm{T})-\eta_{3\mathrm{H}}\mathfrak{I}^{2\kappa-3}\mu(\mathrm{T})),\\
\mathrm{k}_3 =&\frac{\delta_{31}}{\delta}(-\mathrm{a}_{0\mathrm{H}}\mathfrak{I}\mu(\mathrm{T})-\eta_{1\mathrm{H}}\mathfrak{I}^{2\kappa-1}\mu(\mathrm{T}))+\frac{\delta_{32}}{\delta}(-\mathrm{b}_{0\mathrm{H}}\mathfrak{I}^2\mu(\mathrm{T})-\eta_{2\mathrm{H}}\mathfrak{I}^{2\kappa-2}\mu(\mathrm{T}))\\
&+\frac{\delta_{33}}{\delta}(-\mathrm{c}_{0\mathrm{H}}\mathfrak{I}^3\mu(\mathrm{T})-\eta_{3\mathrm{H}}\mathfrak{I}^{2\kappa-3}\mu(\mathrm{T})).
\end{aligned}$$

By putting the values of k_0, k_1 and k_2 into Equation (4), we obtain

$$\begin{aligned}
\nu(\chi) =& \frac{1}{\Gamma(\kappa)}\int_1^{\chi}(\log\frac{\chi}{s})^{\kappa-1}\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}a_0\delta_{11}}{\delta}+\frac{(\log\chi)^{\kappa-2}a_0\delta_{21}}{\delta}+\frac{(\log\chi)^{\kappa-3}a_0\delta_{31}}{\delta}\right]\int_1^{T}\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}b_0\delta_{12}}{\delta}+\frac{(\log\chi)^{\kappa-2}b_0\delta_{22}}{\delta}+\frac{(\log\chi)^{\kappa-3}b_0\delta_{32}}{\delta}\right]\int_1^{T}\log\frac{T}{s}\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}c_0\delta_{13}}{2\delta}+\frac{(\log\chi)^{\kappa-2}c_0\delta_{23}}{2\delta}+\frac{(\log\chi)^{\kappa-3}c_0\delta_{33}}{2\delta}\right]\int_1^{T}(\log\frac{T}{s})^2\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}\eta_1\delta_{11}}{\delta\Gamma(2\kappa-1)}+\frac{(\log\chi)^{\kappa-2}\eta_1\delta_{21}}{\delta\Gamma(2\kappa-1)}+\frac{(\log\chi)^{\kappa-3}\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)}\right]\int_1^{T}(\log\frac{T}{s})^{2\kappa-2}\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}\eta_2\delta_{12}}{\delta\Gamma(2\kappa-2)}+\frac{(\log\chi)^{\kappa-2}\eta_2\delta_{22}}{\delta\Gamma(2\kappa-2)}+\frac{(\log\chi)^{\kappa-3}\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right]\int_1^{T}(\log\frac{T}{s})^{2\kappa-3}\mu(s)\frac{ds}{s}\\
&-\left[\frac{(\log\chi)^{\kappa-1}\eta_3\delta_{13}}{\delta\Gamma(2\kappa-3)}+\frac{(\log\chi)^{\kappa-2}\eta_3\delta_{23}}{\delta\Gamma(2\kappa-3)}+\frac{(\log\chi)^{\kappa-3}\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\right]\int_1^{T}(\log\frac{T}{s})^{2\kappa-4}\mu(s)\frac{ds}{s}\\
=&\int_1^{T}\mathbf{G}_{\kappa}(\chi,s)\mu(s)\frac{ds}{s},
\end{aligned}$$

where $\mathbf{G}_{\kappa}(\chi, s)$ is given by Equation (3). □

Remark 1. *If in Equation (3), we place $\kappa = 3$ and $a_0 = b_0 = c_0 = -1$, then the Green function $\mathbf{G}_{\kappa}(\chi, s)$ for third-order $\mathcal{ODE}$s with $\mathcal{AP}$ BCs will be derived.*

Remark 2. *If in Equation (3), we place $\kappa = 3$ and $a_0 = b_0 = c_0 = 0$, then the solution for the third-order $\mathcal{ODE}$s with the initial BCs can be derived.*

Lemma 3. *Let $\nu \in \mathcal{N}(\mathcal{M})$, $\sigma \in (2, 3]$. Then, the following*

$$\begin{cases}
{}_{H}\mathfrak{D}^{\sigma}w(\chi) = \nu(\chi);\ \chi \in \mathcal{M},\\
{}_{H}\mathfrak{D}^{\sigma-1}w(1) = a_1{}_{H}\mathfrak{D}^{\sigma-1}w(T) + \eta_4{}_{H}\mathfrak{I}^{\sigma-1}w(T),\\
{}_{H}\mathfrak{D}^{\sigma-2}w(1) = b_1{}_{H}\mathfrak{D}^{\sigma-2}w(T) + \eta_5{}_{H}\mathfrak{I}^{\sigma-2}w(T),\\
{}_{H}\mathfrak{D}^{\sigma-3}w(1) = c_1{}_{H}\mathfrak{D}^{\sigma-3}w(T) + \eta_6{}_{H}\mathfrak{I}^{\sigma-3}w(T),
\end{cases}$$

has a (unique) solution presented as

$$w(\chi) = \int_1^{T}\mathbf{G}_{\sigma}(\chi, s)\nu(s)\frac{ds}{s},$$

where $\mathbf{G}_{\sigma}(\chi, s)$ is given by

$$
\mathbf{G}_{\sigma}(\chi, s) = \begin{cases}
\frac{(\log \frac{\chi}{s})^{\sigma-1}}{\Gamma(\sigma)} - \left[\frac{(\log \chi)^{\sigma-1}\mathsf{a}_1\delta_{11}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{a}_1\delta_{21}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{a}_1\delta_{31}^*}{\delta^*}\right] \\
-\left[\frac{(\log \chi)^{\sigma-1}\mathsf{b}_1\delta_{12}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{b}_1\delta_{22}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{b}_1\delta_{32}^*}{\delta^*}\right]\log \frac{\mathrm{T}}{s} \\
-\frac{1}{2}\left[\frac{(\log \chi)^{\sigma-1}\mathsf{c}_1\delta_{13}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{c}_1\delta_{23}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{c}_1\delta_{33}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^2 \\
-\frac{1}{\Gamma(2\sigma-1)}\left[\frac{(\log \chi)^{\sigma-1}\eta_4\delta_{11}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_4\delta_{21}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_4\delta_{31}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-2} \\
-\frac{1}{\Gamma(2\sigma-2)}\left[\frac{(\log \chi)^{\sigma-1}\eta_5\delta_{12}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_5\delta_{22}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_5\delta_{32}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-3} \\
-\frac{1}{\Gamma(2\sigma-3)}\left[\frac{(\log \chi)^{\sigma-1}\eta_6\delta_{13}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_6\delta_{23}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_6\delta_{33}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-4}, \qquad 1 \le s < \chi \le \mathrm{T}, \\
-\left[\frac{(\log \chi)^{\sigma-1}\mathsf{a}_1\delta_{11}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{a}_1\delta_{21}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{a}_1\delta_{31}^*}{\delta^*}\right] \\
-\left[\frac{(\log \chi)^{\sigma-1}\mathsf{b}_1\delta_{12}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{b}_1\delta_{22}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{b}_1\delta_{32}^*}{\delta^*}\right]\log \frac{\mathrm{T}}{s} \\
-\frac{1}{2}\left[\frac{(\log \chi)^{\sigma-1}\mathsf{c}_1\delta_{13}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\mathsf{c}_1\delta_{23}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\mathsf{c}_1\delta_{33}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^2 \\
-\frac{1}{\Gamma(2\sigma-1)}\left[\frac{(\log \chi)^{\sigma-1}\eta_4\delta_{11}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_4\delta_{21}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_4\delta_{31}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-2} \\
-\frac{1}{\Gamma(2\sigma-2)}\left[\frac{(\log \chi)^{\sigma-1}\eta_5\delta_{12}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_5\delta_{22}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_5\delta_{32}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-3} \\
-\frac{1}{\Gamma(2\sigma-3)}\left[\frac{(\log \chi)^{\sigma-1}\eta_6\delta_{13}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-2}\eta_6\delta_{23}^*}{\delta^*} + \frac{(\log \chi)^{\sigma-3}\eta_6\delta_{33}^*}{\delta^*}\right](\log \frac{\mathrm{T}}{s})^{2\sigma-4}, \qquad 1 \le \chi < s \le \mathrm{T}.
\end{cases}
$$

Proof. The proof can be derived as in Lemma 2. □

For simplicity, let

$$
\begin{aligned}
\mathfrak{R}_{\kappa} = \max\Bigg\{ & \left|\frac{\mathsf{a}_0\delta_{31}\log \mathrm{T}}{\delta}\right| + \left|\frac{(2\mathsf{a}_0\delta_{21} + \mathsf{b}_0\delta_{32})(\log \mathrm{T})^2}{2\delta}\right| + \frac{(\log \mathrm{T})^3}{\Gamma(\kappa)} \\
& + \left|\frac{(6\mathsf{a}_0\delta_{11} + 3\mathsf{b}_0\delta_{22} + \mathsf{c}_0\delta_{33})(\log \mathrm{T})^3}{6\delta}\right| + \left|\frac{(3\mathsf{b}_0\delta_{12} + \mathsf{c}_0\delta_{23})(\log \mathrm{T})^4}{6\delta}\right| + \left|\frac{\mathsf{c}_0\delta_{13}(\log \mathrm{T})^5}{6\delta}\right| \\
& + \left|\frac{\eta_1\delta_{11}(\log \mathrm{T})^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right| + \left|\frac{(\eta_1\delta_{21} + (2\kappa-1)\eta_2\delta_{12})(\log \mathrm{T})^{2\kappa}}{\delta\Gamma(2\kappa)}\right| \\
& + \left|\frac{(\eta_1\delta_{31} + (2\kappa-1)\eta_2\delta_{22} + (2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log \mathrm{T})^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right| \\
& + \left|\frac{(\eta_2\delta_{32} + (2\kappa-2)\eta_3\delta_{23})(\log \mathrm{T})^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right| + \left|\frac{\eta_3\delta_{33}(\log \mathrm{T})^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right|\Bigg\}
\end{aligned} \tag{8}
$$

and

$$\begin{aligned}\mathfrak{R}_\sigma = \max\Bigg\{ &\left|\frac{a_1\delta_{31}^*\log T}{\delta^*}\right| + \left|\frac{(2a_1\delta_{21}^* + b_1\delta_{32}^*)(\log T)^2}{2\delta^*}\right| + \frac{(\log T)^3}{\Gamma(\sigma)} \\ &+ \left|\frac{(6a_1\delta_{11}^* + 3b_1\delta_{22}^* + c_1\delta_{33}^*)(\log T)^3}{6\delta^*}\right| + \left|\frac{(3b_1\delta_{12}^* + c_1\delta_{23}^*)(\log T)^4}{6\delta^*}\right| + \left|\frac{c_1\delta_{13}^*(\log T)^5}{6\delta^*}\right| \\ &+ \left|\frac{\eta_4\delta_{11}^*(\log T)^{2\sigma+1}}{\delta^*\Gamma(2\sigma)}\right| + \left|\frac{(\eta_4\delta_{21}^* + (2\sigma-1)\eta_5\delta_{12}^*)(\log T)^{2\sigma}}{\delta^*\Gamma(2\sigma)}\right| \\ &+ \left|\frac{(\eta_4\delta_{31}^* + (2\sigma-1)\eta_5\delta_{22}^* + (2\sigma-1)(2\sigma-2)\eta_6\delta_{13}^*)(\log T)^{2\sigma-1}}{\delta^*\Gamma(2\sigma)}\right| \\ &+ \left|\frac{(\eta_5\delta_{32}^* + (2\sigma-2)\eta_6\delta_{23}^*)(\log T)^{2\sigma-2}}{\delta^*\Gamma(2\sigma-1)}\right| + \left|\frac{\eta_6\delta_{33}^*(\log T)^{2\sigma-3}}{\delta^*\Gamma(2\sigma-2)}\right|\Bigg\}. \end{aligned} \tag{9}$$

By letting v, w be the solutions to Equation (1) and $\chi \in \mathcal{M}$, then

$$\begin{aligned} v(\chi) = &\frac{1}{\Gamma(\kappa)}\int_1^\chi (\log\frac{\chi}{s})^{\kappa-1}\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{a_0\delta_{11}(\log\chi)^{\kappa-1}}{\delta} + \frac{a_0\delta_{21}(\log\chi)^{\kappa-2}}{\delta} + \frac{a_0\delta_{31}(\log\chi)^{\kappa-3}}{\delta}\right]\int_1^T \mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{b_0\delta_{12}(\log\chi)^{\kappa-1}}{\delta} + \frac{b_0\delta_{22}(\log\chi)^{\kappa-2}}{\delta} + \frac{b_0\delta_{32}(\log\chi)^{\kappa-3}}{\delta}\right]\int_1^T \log\frac{T}{s}\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{c_0\delta_{13}(\log\chi)^{\kappa-1}}{2\delta} + \frac{c_0\delta_{23}(\log\chi)^{\kappa-2}}{2\delta} + \frac{c_0\delta_{33}(\log\chi)^{\kappa-3}}{2\delta}\right]\int_1^T (\log\frac{T}{s})^2\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{\eta_1\delta_{11}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-1)} + \frac{\eta_1\delta_{21}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-1)} + \frac{\eta_1\delta_{31}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-1)}\right]\int_1^T (\log\frac{T}{s})^{2\kappa-2}\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{\eta_2\delta_{12}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-2)} + \frac{\eta_2\delta_{22}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-2)} + \frac{\eta_2\delta_{32}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-2)}\right]\int_1^T (\log\frac{T}{s})^{2\kappa-3}\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ &- \left[\frac{\eta_3\delta_{13}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-3)} + \frac{\eta_3\delta_{23}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-3)} + \frac{\eta_3\delta_{33}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-3)}\right]\int_1^T (\log\frac{T}{s})^{2\kappa-4}\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \\ = &\int_1^T \mathbf{G}_\kappa(\chi, s)\mu(s, w(s), {}_H\mathfrak{D}^\kappa v(s))\frac{ds}{s} \end{aligned}$$

and

$$
\begin{aligned}
\mathsf{w}(\chi) =& \frac{1}{\Gamma(\sigma)} \int_1^{\chi} (\log \frac{\chi}{\mathsf{s}})^{\sigma-1} \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{a}_1 \delta_{11}^* (\log \chi)^{\sigma-1}}{\delta^*} + \frac{\mathsf{a}_1 \delta_{21}^* (\log \chi)^{\sigma-2}}{\delta^*} + \frac{\mathsf{a}_1 \delta_{31}^* (\log \chi)^{\sigma-3}}{\delta^*} \right] \int_1^{\mathrm{T}} \nu(\mathsf{s}, \nu(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{b}_1 \delta_{12}^* (\log \chi)^{\sigma-1}}{\delta^*} + \frac{\mathsf{b}_1 \delta_{22}^* (\log \chi)^{\sigma-2}}{\delta^*} + \frac{\mathsf{b}_1 \delta_{32}^* (\log \chi)^{\sigma-3}}{\delta^*} \right] \int_1^{\mathrm{T}} \log \frac{\mathrm{T}}{\mathsf{s}} \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{c}_1 \delta_{13}^* (\log \chi)^{\sigma-1}}{2\delta^*} + \frac{\mathsf{c}_1 \delta_{23}^* (\log \chi)^{\sigma-2}}{2\delta^*} + \frac{\mathsf{c}_1 \delta_{33}^* (\log \chi)^{\sigma-3}}{2\delta^*} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^2 \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_4 \delta_{11}^* (\log \chi)^{\sigma-1}}{\delta^* \Gamma(2\sigma-1)} + \frac{\eta_4 \delta_{21}^* (\log \chi)^{\sigma-2}}{\delta^* \Gamma(2\sigma-1)} + \frac{\eta_4 \delta_{31}^* (\log \chi)^{\sigma-3}}{\delta^* \Gamma(2\sigma-1)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-2} \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_5 \delta_{12}^* (\log \chi)^{\sigma-1}}{\delta^* \Gamma(2\sigma-2)} + \frac{\eta_5 \delta_{22}^* (\log \chi)^{\sigma-2}}{\delta^* \Gamma(2\sigma-2)} + \frac{\eta_5 \delta_{32}^* (\log \chi)^{\sigma-3}}{\delta^* \Gamma(2\sigma-2)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-3} \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_6 \delta_{13}^* (\log \chi)^{\sigma-1}}{\delta^* \Gamma(2\sigma-3)} + \frac{\eta_6 \delta_{23}^* (\log \chi)^{\sigma-2}}{\delta^* \Gamma(2\sigma-3)} + \frac{\eta_6 \delta_{33}^* (\log \chi)^{\sigma-3}}{\delta^* \Gamma(2\sigma-3)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-4} \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
=& \int_1^{\mathrm{T}} \mathbf{G}_{\sigma}(\chi, \mathsf{s}) \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), {}_{\mathrm{H}}\mathfrak{D}^{\sigma} \mathsf{w}(\mathsf{s})) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}.
\end{aligned}
$$

For simplicity, we have

$$
\begin{aligned}
\mathsf{u}(\chi) &= \mu(\chi, \mathsf{w}(\chi), \mathfrak{D}^{\kappa} \mathsf{v}(\chi)) = \mu(\chi, \mathsf{w}(\chi), \mathsf{u}(\chi)), \\
\mathsf{x}(\chi) &= \nu(\chi, \mathsf{v}(\chi), \mathfrak{D}^{\sigma} \mathsf{w}(\chi)) = \nu(\chi, \mathsf{v}(\chi), \mathsf{x}(\chi)).
\end{aligned}
$$

To deal with Equation (1) as a problem of a fixed point, let $\mathfrak{U} : \mathcal{W} \times \mathcal{W} \to \mathcal{W} \times \mathcal{W}$ be an operator represented by

$$
\mathfrak{U}(\mathsf{v}, \mathsf{w})(\chi) = \begin{pmatrix} \int_1^{\mathrm{T}} \mathbf{G}_{\kappa}(\chi, \mathsf{s}) \mu(\mathsf{s}, \mathsf{w}(\mathsf{s}), \mathsf{u}(\mathsf{s})) \mathrm{d}\mathsf{s} \\ \int_1^{\mathrm{T}} \mathbf{G}_{\sigma}(\chi, \mathsf{s}) \nu(\mathsf{s}, \mathsf{v}(\mathsf{s}), \mathsf{x}(\mathsf{s})) \mathrm{d}\mathsf{s} \end{pmatrix} = \begin{pmatrix} \mathfrak{U}_{\kappa}(\mathsf{w}, \mathsf{u})(\chi) \\ \mathfrak{U}_{\sigma}(\mathsf{v}, \mathsf{x})(\chi) \end{pmatrix} = \begin{pmatrix} \mathfrak{U}_{\kappa}(\mathsf{v})(\chi) \\ \mathfrak{U}_{\sigma}(\mathsf{w})(\chi) \end{pmatrix}. \tag{10}
$$

Then, obviously, the solution to Equation (1) is the fixed point of $\mathfrak{U}$, where

$$
\begin{aligned}
\mathfrak{U}_{\kappa}(\mathsf{v})(\chi) =& \frac{1}{\Gamma(\kappa)} \int_1^{\chi} (\log \frac{\chi}{\mathsf{s}})^{\kappa-1} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{a}_0 \delta_{11} (\log \chi)^{\kappa-1}}{\delta} + \frac{\mathsf{a}_0 \delta_{21} (\log \chi)^{\kappa-2}}{\delta} + \frac{\mathsf{a}_0 \delta_{31} (\log \chi)^{\kappa-3}}{\delta} \right] \int_1^{\mathrm{T}} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{b}_0 \delta_{12} (\log \chi)^{\kappa-1}}{\delta} + \frac{\mathsf{b}_0 \delta_{22} (\log \chi)^{\kappa-2}}{\delta} + \frac{\mathsf{b}_0 \delta_{32} (\log \chi)^{\kappa-3}}{\delta} \right] \int_1^{\mathrm{T}} \log \frac{\mathrm{T}}{\mathsf{s}} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\mathsf{c}_0 \delta_{13} (\log \chi)^{\kappa-1}}{2\delta} + \frac{\mathsf{c}_0 \delta_{23} (\log \chi)^{\kappa-2}}{2\delta} + \frac{\mathsf{c}_0 \delta_{33} (\log \chi)^{\kappa-3}}{2\delta} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^2 \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_1 \delta_{11} (\log \chi)^{\kappa-1}}{\delta \Gamma(2\kappa-1)} + \frac{\eta_1 \delta_{21} (\log \chi)^{\kappa-2}}{\delta \Gamma(2\kappa-1)} + \frac{\eta_1 \delta_{31} (\log \chi)^{\kappa-3}}{\delta \Gamma(2\kappa-1)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-2} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_2 \delta_{12} (\log \chi)^{\kappa-1}}{\delta \Gamma(2\kappa-2)} + \frac{\eta_2 \delta_{22} (\log \chi)^{\kappa-2}}{\delta \Gamma(2\kappa-2)} + \frac{\eta_2 \delta_{32} (\log \chi)^{\kappa-3}}{\delta \Gamma(2\kappa-2)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-3} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&- \left[\frac{\eta_3 \delta_{13} (\log \chi)^{\kappa-1}}{\delta \Gamma(2\kappa-3)} + \frac{\eta_3 \delta_{23} (\log \chi)^{\kappa-2}}{\delta \Gamma(2\kappa-3)} + \frac{\eta_3 \delta_{33} (\log \chi)^{\kappa-3}}{\delta \Gamma(2\kappa-3)} \right] \int_1^{\mathrm{T}} (\log \frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-4} \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
=& \int_1^{\mathrm{T}} \mathbf{G}_{\kappa}(\chi, \mathsf{s}) \mathsf{u}(\mathsf{s}) \frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}
\end{aligned}
$$

and

$$\begin{aligned}\mathfrak{U}_\sigma(\mathsf{w})(\chi) =& \frac{1}{\Gamma(\sigma)}\int_1^{\chi}(\log\frac{\chi}{\mathsf{s}})^{\sigma-1}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\mathsf{a}_1\delta_{11}^*(\log\chi)^{\sigma-1}}{\delta^*}+\frac{\mathsf{a}_1\delta_{21}^*(\log\chi)^{\sigma-2}}{\delta^*}+\frac{\mathsf{a}_1\delta_{31}^*(\log\chi)^{\sigma-3}}{\delta^*}\right]\int_1^{\mathrm{T}}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\mathsf{b}_1\delta_{12}^*(\log\chi)^{\sigma-1}}{\delta^*}+\frac{\mathsf{b}_1\delta_{22}^*(\log\chi)^{\sigma-2}}{\delta^*}+\frac{\mathsf{b}_1\delta_{32}^*(\log\chi)^{\sigma-3}}{\delta^*}\right]\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathsf{s}}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\mathsf{c}_1\delta_{13}^*(\log\chi)^{\sigma-1}}{2\delta^*}+\frac{\mathsf{c}_1\delta_{23}^*(\log\chi)^{\sigma-2}}{2\delta^*}+\frac{\mathsf{c}_1\delta_{33}^*(\log\chi)^{\sigma-3}}{2\delta^*}\right]\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^2\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\eta_4\delta_{11}^*(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-1)}+\frac{\eta_4\delta_{21}^*(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-1)}+\frac{\eta_4\delta_{31}^*(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-1)}\right]\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-2}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\eta_5\delta_{12}^*(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-2)}+\frac{\eta_5\delta_{22}^*(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-2)}+\frac{\eta_5\delta_{32}^*(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-2)}\right]\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-3}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ &-\left[\frac{\eta_6\delta_{13}^*(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-3)}+\frac{\eta_6\delta_{23}^*(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-3)}+\frac{\eta_6\delta_{33}^*(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-3)}\right]\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\sigma-4}\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\\ =&\int_1^{\mathrm{T}}\mathbf{G}_\sigma(\chi,\mathsf{s})\mathsf{x}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}.\end{aligned}$$

In the coming section, we utilize the Banach principle to obtain our first result:

Theorem 1. *Let $\mu,\nu:\mathcal{M}\times\mathcal{R}^2\to\mathcal{R}$ be continuous such that the following condition holds true:*

$[\mathsf{H}_1]$ *For every $\chi\in\mathcal{M}$ and $\mathsf{w},\mathsf{v},\mathsf{u},\mathsf{x},\overline{\mathsf{w}},\overline{\mathsf{v}},\overline{\mathsf{x}},\overline{\mathsf{u}}:\mathcal{W}\to\mathcal{R}$, there are $\mathcal{L}_\mu,\mathcal{L}_\nu>0$ and $0\le\overline{\mathcal{L}}_\mu,\overline{\mathcal{L}}_\nu<1$ such that*

$$\left|\mu(\chi,\mathsf{w}(\chi),\mathsf{u}(\chi))-\mu(\chi,\overline{\mathsf{w}}(\chi),\overline{\mathsf{u}}(\chi))\right|\le\mathcal{L}_\mu\left|\mathsf{w}(\chi)-\overline{\mathsf{w}}(\chi)\right|+\overline{\mathcal{L}}_\mu\left|\mathsf{u}(\chi)-\overline{\mathsf{u}}(\chi)\right|,$$

$$\left|\nu(\chi,\mathsf{v}(\chi),\mathsf{x}(\chi))-\nu(\chi,\overline{\mathsf{v}}(\chi),\overline{\mathsf{x}}(\chi))\right|\le\mathcal{L}_\nu\left|\mathsf{v}(\chi)-\overline{\mathsf{v}}(\chi)\right|+\overline{\mathcal{L}}_\nu\left|\mathsf{x}(\chi)-\overline{\mathsf{x}}(\chi)\right|.$$

Furthermore, suppose that

$$\frac{\mathfrak{R}_\kappa\mathcal{L}_\mu(1-\overline{\mathcal{L}}_\nu)+\mathfrak{R}_\sigma\mathcal{L}_\nu(1-\overline{\mathcal{L}}_\mu)}{(1-\overline{\mathcal{L}}_\nu)(1-\overline{\mathcal{L}}_\mu)}<1,$$

where $\mathfrak{R}_\kappa$ and $\mathfrak{R}_\sigma$ are presented in Equations (8) and (9), respectively. Then, Equation (1) has a (unique) solution.

Proof. Let $\sup_{\chi\in\mathcal{M}}\mu(\chi,0,0)=\Phi^*<\infty$ and $\sup_{\chi\in\mathcal{M}}\nu(\chi,0,0)=\Psi^*<\infty$ such that

$$\mathrm{r}\ge\frac{2\mathfrak{R}_\kappa\Phi^*(1-\overline{\mathcal{L}}_\nu)+2\mathfrak{R}_\sigma\Psi^*(1-\overline{\mathcal{L}}_\mu)}{2(1-\overline{\mathcal{L}}_\mu)(1-\overline{\mathcal{L}}_\nu)-\mathfrak{R}_\kappa\mathcal{L}_\mu-\mathfrak{R}_\sigma\mathcal{L}_\nu}.$$

We need to prove that $\mathfrak{U}(\mathscr{B}_\mathrm{r})\subset\mathscr{B}_\mathrm{r}$, where

$$\mathscr{B}_\mathrm{r}=\left\{(\mathsf{v},\mathsf{w})\in\mathcal{W}\times\mathcal{W}:\|(\mathsf{v},\mathsf{w})\|_{\mathcal{W}\times\mathcal{W}}\le\mathrm{r},\ \|\mathsf{v}\|_\mathcal{W}\le\frac{\mathrm{r}}{2},\ \|\mathsf{w}\|_\mathcal{W}\le\frac{\mathrm{r}}{2}\right\}.$$

For $(\mathsf{v},\mathsf{w})\in\mathscr{B}_\mathrm{r}$, we have

$$
\begin{aligned}
&(\log\chi)^{3-\kappa}\left|\mathfrak{U}_{\kappa}(\mathsf{v})(\chi)\right| \\
&\leq \frac{(\log\chi)^{3-\kappa}}{\Gamma(\kappa)}\int_1^{\chi}(\log\frac{\chi}{\mathsf{s}})^{\kappa-1}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad+\left|\frac{\mathsf{a}_0\delta_{11}(\log\chi)^2}{\delta}+\frac{\mathsf{a}_0\delta_{21}\log(\chi)}{\delta}+\frac{\mathsf{a}_0\delta_{31}}{\delta}\right|\int_1^{\mathrm{T}}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad+\left|\frac{\mathsf{b}_0\delta_{12}(\log\chi)^2}{\delta}+\frac{\mathsf{b}_0\delta_{22}\log(\chi)}{\delta}+\frac{\mathsf{b}_0\delta_{32}}{\delta}\right|\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathsf{s}}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad+\left|\frac{\mathsf{c}_0\delta_{13}(\log\chi)^2}{2\delta}+\frac{\mathsf{c}_0\delta_{23}\log(\chi)}{2\delta}+\frac{\mathsf{c}_0\delta_{33}}{2\delta}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^2[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad+\left|\frac{\eta_1\delta_{11}(\log\chi)^2}{\delta\Gamma(2\kappa-1)}+\frac{\eta_1\delta_{21}\log(\chi)}{\delta\Gamma(2\kappa-1)}+\frac{\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)}\right| \\
&\quad\times\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-2}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\eta_2\delta_{12}(\log\chi)^2}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{22}\log(\chi)}{\delta\Gamma(2\kappa-2)}\right. \\
&\quad\left.+\frac{\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-3}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\eta_3\delta_{13}(\log\chi)^2}{\delta\Gamma(2\kappa-3)}\right. \\
&\quad\left.+\frac{\eta_3\delta_{23}\log(\chi)}{\delta\Gamma(2\kappa-3)}+\frac{\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-4}[\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),\mathsf{u}(\mathsf{s}))-\mu(0,0,0)\,|\,+\,|\,\mu(0,0,0)\,|\,]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}. \qquad (11)
\end{aligned}
$$

Consider

$$
\begin{aligned}
|\,\mathsf{u}(\chi)\,| &\leq |\,\mu(\chi,\mathsf{w}(\chi),\mathsf{u}(\chi))-\mu(\chi,0,0)\,|+|\,\mu(\chi,0,0)\,| \\
&\leq |\,\mu(\chi,0,0)\,|+\mathcal{L}_{\mu}\,|\,\mathsf{w}(\chi)\,|+\overline{\mathcal{L}}_{\mu}\,|\,\mathsf{u}(\chi)\,| \\
&\leq \frac{|\,\mu(\chi,0,0)\,|+\mathcal{L}_{\mu}\,|\,\mathsf{w}(\chi)\,|}{1-\overline{\mathcal{L}}_{\mu}}. \qquad (12)
\end{aligned}
$$

By substituting Equation (12) into Equation (11), we obtain

$$
\begin{aligned}
\|\mathfrak{U}_{\kappa}(\mathsf{v})\|_{\mathcal{W}} \leq \Bigg[&\left|\frac{\mathsf{a}_0\delta_{31}\log\mathrm{T}}{\delta}\right|+\left|\frac{(2\mathsf{a}_0\delta_{21}+\mathsf{b}_0\delta_{32})(\log\mathrm{T})^2}{2\delta}\right|+\left|\frac{(6\mathsf{a}_0\delta_{11}+3\mathsf{b}_0\delta_{22}+\mathsf{c}_0\delta_{33})(\log\mathrm{T})^3}{6\delta}\right| \\
&+\left|\frac{(3\mathsf{b}_0\delta_{12}+\mathsf{c}_0\delta_{23})(\log\mathrm{T})^4}{6\delta}\right|+\left|\frac{\mathsf{c}_0\delta_{13}(\log\mathrm{T})^5}{6\delta}\right|+\frac{(\log\mathrm{T})^3}{\Gamma(\kappa)}+\left|\frac{\eta_1\delta_{11}(\log\mathrm{T})^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right| \\
&+\left|\frac{(\eta_1\delta_{31}+(2\kappa-1)\eta_2\delta_{22}+(2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log\mathrm{T})^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{\eta_3\delta_{33}(\log\mathrm{T})^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right| \\
&+\left|\frac{(\eta_1\delta_{21}+(2\kappa-1)\eta_2\delta_{12})(\log\mathrm{T})^{2\kappa}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{(\eta_2\delta_{32}+(2\kappa-2)\eta_3\delta_{23})(\log\mathrm{T})^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right|\Bigg]\frac{2\Phi^*+\mathcal{L}_{\mu}\mathsf{r}}{2(1-\overline{\mathcal{L}}_{\mu})}.
\end{aligned}
$$

Hence, we have

$$
\|\mathfrak{U}_{\kappa}(\mathsf{v})\|_{\mathcal{W}} \leq \mathfrak{R}_{\kappa}\frac{2\Phi^*+\mathcal{L}_{\mu}\mathsf{r}}{2(1-\overline{\mathcal{L}}_{\mu})}. \qquad (13)
$$

Equivalently, we have

$$
\|\mathfrak{U}_{\sigma}(\mathsf{w})\|_{\mathcal{W}} \leq \mathfrak{R}_{\sigma}\frac{2\Psi^*+\mathcal{L}_{\nu}\mathsf{r}}{2(1-\overline{\mathcal{L}}_{\nu})}. \qquad (14)
$$

Equations (13) and (14) imply that

$$
\|\mathfrak{U}(\mathsf{v},\mathsf{w})\|_{\mathcal{W}\times\mathcal{W}} \leq \mathsf{r}.
$$

For $(v_1, w_1), (v_2, w_2) \in \mathcal{W} \times \mathcal{W}$ and for any $\chi \in \mathcal{M}$, we obtain

$$
\begin{aligned}
&(\log\chi)^{3-\kappa}\left|\mathfrak{U}_\kappa(v_1)(\chi)-\mathfrak{U}_\kappa(v_2)(\chi)\right| \\
&\le \frac{(\log\chi)^{3-\kappa}}{\Gamma(\kappa)}\int_1^\chi (\log\frac{\chi}{s})^{\kappa-1}\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}+\left|\frac{a_0\delta_{11}(\log\chi)^2}{\delta}\right. \\
&\quad+\frac{a_0\delta_{21}\log(\chi)}{\delta}+\frac{a_0\delta_{31}}{\delta}\left|\int_1^T |\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}+\right|\frac{b_0\delta_{12}(\log\chi)^2}{\delta} \\
&\quad+\frac{b_0\delta_{22}\log(\chi)}{\delta}+\frac{b_0\delta_{32}}{\delta}\left|\int_1^T \log\frac{T}{s}\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}+\right|\frac{c_0\delta_{13}(\log\chi)^2}{2\delta} \\
&\quad+\frac{c_0\delta_{23}\log(\chi)}{2\delta}+\frac{c_0\delta_{33}}{2\delta}\left|\int_1^T (\log\frac{T}{s})^2\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}+\right|\frac{\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)} \\
&\quad+\frac{\eta_1\delta_{21}\log(\chi)}{\delta\Gamma(2\kappa-1)}+\frac{\eta_1\delta_{11}(\log\chi)^2}{\delta\Gamma(2\kappa-1)}\left|\int_1^T (\log\frac{T}{s})^{2\kappa-2}\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}\right. \\
&\quad+\left|\frac{\eta_2\delta_{12}(\log\chi)^2}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{22}\log(\chi)}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right| \\
&\quad\times\int_1^T (\log\frac{T}{s})^{2\kappa-3}\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}+\left|\frac{\eta_3\delta_{13}(\log\chi)^2}{\delta\Gamma(2\kappa-3)}\right. \\
&\quad+\frac{\eta_3\delta_{23}\log(\chi)}{\delta\Gamma(2\kappa-3)}+\frac{\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\left|\int_1^T (\log\frac{T}{s})^{2\kappa-4}\,|\,\mu(s,w_1(s),u_1(s))-\mu(s,w_2(s),u_2(s))\,|\,\frac{ds}{s}\right. \\
&\le\left[\left|\frac{a_0\delta_{31}\log T}{\delta}\right|+\left|\frac{(2a_0\delta_{21}+b_0\delta_{32})(\log T)^2}{2\delta}\right|+\left|\frac{(6a_0\delta_{11}+3b_0\delta_{22}+c_0\delta_{33})(\log T)^3}{6\delta}\right|\right. \\
&\quad+\left|\frac{(3b_0\delta_{12}+c_0\delta_{23})(\log T)^4}{6\delta}\right|+\left|\frac{c_0\delta_{13}(\log T)^5}{6\delta}\right|+\frac{(\log T)^3}{\Gamma(\kappa)}+\left|\frac{\eta_1\delta_{11}(\log T)^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right| \\
&\quad+\left|\frac{(\eta_1\delta_{31}+(2\kappa-1)\eta_2\delta_{22}+(2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log T)^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{\eta_3\delta_{33}(\log T)^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right| \\
&\quad+\left.\left|\frac{(\eta_1\delta_{21}+(2\kappa-1)\eta_2\delta_{12})(\log T)^{2\kappa}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{(\eta_2\delta_{32}+(2\kappa-2)\eta_3\delta_{23})(\log T)^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right|\right]\frac{\mathcal{L}_\mu}{1-\overline{\mathcal{L}}_\mu}\|w_1-w_2\|_{\mathcal{W}}
\end{aligned}
$$

Hence, we have

$$\|\mathfrak{U}_\kappa(v_1)-\mathfrak{U}_\kappa(v_2)\|_{\mathcal{W}} \le \frac{\mathfrak{R}_\kappa\mathcal{L}_\mu}{1-\overline{\mathcal{L}}_\mu}\|w_1-w_2\|_{\mathcal{W}}. \tag{15}$$

Similarly, we have

$$\|\mathfrak{U}_\sigma(w_1)-\mathfrak{U}_\sigma(w_2)\|_{\mathcal{W}} \le \frac{\mathfrak{R}_\sigma\mathcal{L}_\nu}{1-\overline{\mathcal{L}}_\nu}\|v_1-v_2\|_{\mathcal{W}}. \tag{16}$$

Equations (15) and (16) imply that

$$\|\mathfrak{U}(v_1,w_1)-\mathfrak{U}(v_2,w_2)\|_{\mathcal{W}\times\mathcal{W}} \le \frac{\mathfrak{R}_\kappa\mathcal{L}_\mu(1-\overline{\mathcal{L}}_\nu)+\mathfrak{R}_\sigma\mathcal{L}_\nu(1-\overline{\mathcal{L}}_\mu)}{(1-\overline{\mathcal{L}}_\nu)(1-\overline{\mathcal{L}}_\mu)}\|(v_1,w_1)-(v_2,w_2)\|_{\mathcal{W}\times\mathcal{W}}.$$

Therefore, $\mathfrak{U}$ is a contraction. Thus, Equation (1) has a (unique) solution. □

In the sequel, we need the following theorem:

Theorem 2 ([36]). *Let $\mathfrak{U} : \mathcal{W} \to \mathcal{W}$ be a completely continuous operator, and let*

$$\mathcal{B}(\mathfrak{U}) = \left\{v \in \mathcal{W} : v = \lambda\mathfrak{U}(v),\ \lambda \in [0,1]\right\}.$$

Then either the set $\mathcal{B}(\mathfrak{U})$ is unbounded or the operator $\mathfrak{U}$ has at least one fixed point.

Theorem 3. *Suppose the functions $\mu, \nu : \mathcal{M} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are continuous such that the following are true:*

[H_2] For any $\chi \in \mathcal{M}$ and $w, u : \mathcal{M} \to \mathcal{R}$, $\exists\ \phi_i(i = 1, 2, 3) : \mathcal{M} \to \mathcal{R}$, they satisfy

$$\left|\mu(\chi, w(\chi), u(\chi))\right| \leq \phi_1(\chi) + \phi_2(\chi) \mid w(\chi) \mid + \phi_3(\chi) \mid u(\chi) \mid .$$

In addition, for $\chi \in \mathcal{M}$ and $v, x : \mathcal{M} \to \mathcal{R}$, $\exists\ \varphi_i(i = 1, 2, 3) : \mathcal{M} \to \mathcal{R}$, they satisfy

$$\left|\nu(\chi, v(\chi), x(\chi))\right| \leq \varphi_1(\chi) + \varphi_2(\chi) \mid v(\chi) \mid + \varphi_3(\chi) \mid x(\chi) \mid ,$$

with $\sup_{\chi \in \mathcal{M}} \phi_i(\chi) = \phi_i^*, \sup_{\chi \in \mathcal{M}} \varphi_i(\chi) = \varphi_i^*(i = 1, 2, 3)$.
Furthermore, the following is true:

$$\mathfrak{R}_0 = \max\left\{\frac{\mathfrak{R}_\sigma \varphi_2^*}{1-\varphi_3^*}, \frac{\mathfrak{R}_\kappa \phi_2^*}{1-\phi_3^*}\right\} < 1 \ \text{ and } \ 0 \leq \phi_3^*, \varphi_3^* < 1. \tag{17}$$

Then, Equation (1) has at least one solution.

Proof. First, we prove that $\mathfrak{U}$ is completely continuous. As μ and ν are continuous, the operator $\mathfrak{U}$ is thus also continuous. Now, for $(v, w) \in \mathcal{B}_r$, we obtain

$$\begin{aligned}
&(\log\chi)^{3-\kappa}\left|\mathfrak{U}_\kappa(v)(\chi)\right| \\
&\leq \frac{(\log\chi)^{3-\kappa}}{\Gamma(\kappa)}\int_1^\chi (\log\frac{\chi}{s})^{\kappa-1} \mid u(s) \mid \frac{ds}{s} + \left|\frac{a_0\delta_{11}(\log\chi)^2}{\delta} + \frac{a_0\delta_{21}\log(\chi)}{\delta} + \frac{a_0\delta_{31}}{\delta}\right| \int_1^T \mid u(s) \mid \frac{ds}{s} \\
&\quad + \left|\frac{b_0\delta_{12}(\log\chi)^2}{\delta} + \frac{b_0\delta_{22}\log(\chi)}{\delta} + \frac{b_0\delta_{32}}{\delta}\right| \int_1^T \log\frac{T}{s} \mid u(s) \mid \frac{ds}{s} \\
&\quad + \left|\frac{c_0\delta_{13}(\log\chi)^2}{2\delta} + \frac{c_0\delta_{23}\log(\chi)}{2\delta} + \frac{c_0\delta_{33}}{2\delta}\right| \int_1^T (\log\frac{T}{s})^2 \mid u(s) \mid \frac{ds}{s} \\
&\quad + \left|\frac{\eta_1\delta_{11}(\log\chi)^2}{\delta\Gamma(2\kappa-1)} + \frac{\eta_1\delta_{21}\log(\chi)}{\delta\Gamma(2\kappa-1)} + \frac{\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)}\right| \int_1^T (\log\frac{T}{s})^{2\kappa-2} \mid u(s) \mid \frac{ds}{s} \\
&\quad + \left|\frac{\eta_2\delta_{12}(\log\chi)^2}{\delta\Gamma(2\kappa-2)} + \frac{\eta_2\delta_{22}\log(\chi)}{\delta\Gamma(2\kappa-2)} + \frac{\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right| \int_1^T (\log\frac{T}{s})^{2\kappa-3} \mid u(s) \mid \frac{ds}{s} \\
&\quad + \left|\frac{\eta_3\delta_{13}(\log\chi)^2}{\delta\Gamma(2\kappa-3)} + \frac{\eta_3\delta_{23}\log(\chi)}{\delta\Gamma(2\kappa-3)} + \frac{\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\right| \int_1^T (\log\frac{T}{s})^{2\kappa-4} \mid u(s) \mid \frac{ds}{s}.
\end{aligned} \tag{18}$$

By applying $[H_2]$, we have

$$\begin{aligned}
\mid u(\chi) \mid &= \mid \mu(\chi, w(\chi), u(\chi)) \mid \\
&\leq \phi_1(\chi) + \phi_2(\chi) \mid w(\chi) \mid + \phi_3(\chi) \mid u(\chi) \mid \\
&\leq \frac{\phi_1(\chi) + \phi_2(\chi) \mid w(\chi) \mid}{1-\phi_3(\chi)}.
\end{aligned}$$

Thus, from Equation (18), we have

$$\begin{aligned}
\|\mathfrak{U}_\kappa(v)\|_{\mathcal{W}} \leq \Bigg[&\left|\frac{a_0\delta_{31}\log T}{\delta}\right| + \left|\frac{(2a_0\delta_{21} + b_0\delta_{32})(\log T)^2}{2\delta}\right| + \left|\frac{(6a_0\delta_{11} + 3b_0\delta_{22} + c_0\delta_{33})(\log T)^3}{6\delta}\right| \\
&+ \left|\frac{(3b_0\delta_{12} + c_0\delta_{23})(\log T)^4}{6\delta}\right| + \left|\frac{c_0\delta_{13}(\log T)^5}{6\delta}\right| + \frac{(\log T)^3}{\Gamma(\kappa)} + \left|\frac{\eta_1\delta_{11}(\log T)^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right| \\
&+ \left|\frac{(\eta_1\delta_{31} + (2\kappa-1)\eta_2\delta_{22} + (2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log T)^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right| + \left|\frac{\eta_3\delta_{33}(\log T)^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right| \\
&+ \left|\frac{(\eta_1\delta_{21} + (2\kappa-1)\eta_2\delta_{12})(\log T)^{2\kappa}}{\delta\Gamma(2\kappa)}\right| + \left|\frac{(\eta_2\delta_{32} + (2\kappa-2)\eta_3\delta_{23})(\log T)^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right|\Bigg] \frac{2\phi_1^* + \phi_2^* r}{2(1-\phi_3^*)},
\end{aligned} \tag{19}$$

which implies that

$$\|\mathfrak{U}_{\kappa}(\mathsf{v})\|_{\mathcal{W}} \leq \mathfrak{R}_{\kappa}\frac{2\phi_1^* + \phi_2^*\mathsf{r}}{2(1-\phi_3^*)}. \tag{20}$$

Similarly, we obtain

$$\|\mathfrak{U}_{\sigma}(\mathsf{w})\|_{\mathcal{W}} \leq \mathfrak{R}_{\sigma}\frac{2\varphi_1^* + \varphi_2^*\mathsf{r}}{2(1-\varphi_3^*)}. \tag{21}$$

Therefore, Equations (20) and (21) imply that $\mathfrak{U}$ is uniformly bounded. To prove that $\mathfrak{U}$ is equicontinuous, let $0 \leq \chi_2 \leq \chi_1 \leq \chi$. Then, we have

$$\begin{aligned}
&\left|(\log\chi_1)^{3-\kappa}\mathfrak{U}_{\kappa}(\mathsf{v})(\chi_1) - (\log\chi_2)^{3-\kappa}\mathfrak{U}_{\kappa}(\mathsf{v})(\chi_2)\right| \\
&\quad\leq \left|\frac{1}{\Gamma(\kappa)}\int_1^{\chi_1}\left[(\log\chi_1)^{3-\kappa}(\log\frac{\chi_1}{\mathsf{s}})^{\kappa-1} - (\log\chi_2)^{3-\kappa}(\log\frac{\chi_2}{\mathsf{s}})^{\kappa-1}\right]\mathsf{u}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right. \\
&\quad\quad\left. - \frac{1}{\Gamma(\kappa)}\int_{\chi_1}^{\chi_2}(\log\chi_2)^{3-\kappa}(\log\frac{\chi_2}{\mathsf{s}})^{\kappa-1}\mathsf{u}(\mathsf{s})\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right| \\
&\quad\quad + \left|\frac{\mathsf{a}_0\delta_{11}[(\log\chi_1)^2 - (\log\chi_2)^2]}{\delta} + \frac{\mathsf{a}_0\delta_{21}[\log(\chi_1) - \log(\chi_2)]}{\delta}\right|\int_1^{\mathsf{T}} |\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad\quad + \left|\frac{\mathsf{b}_0\delta_{12}[(\log\chi_1)^2 - (\log\chi_2)^2]}{\delta} + \frac{\mathsf{b}_0\delta_{22}[\log(\chi_1) - \log(\chi_2)]}{\delta}\right|\int_1^{\mathsf{T}} \log\frac{\mathsf{T}}{\mathsf{s}}\,|\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad\quad + \left|\frac{\mathsf{c}_0\delta_{13}[(\log\chi_1)^2 - (\log\chi_2)^2]}{2\delta} + \frac{\mathsf{c}_0\delta_{23}[\log(\chi_1) - \log(\chi_2)]}{2\delta}\right|\int_1^{\mathsf{T}} (\log\frac{\mathsf{T}}{\mathsf{s}})^2\,|\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad\quad + \left|\frac{\eta_1\delta_{11}[(\log\chi_1)^2 - (\log\chi_2)^2]}{\delta\Gamma(2\kappa-1)} + \frac{\eta_1\delta_{21}[\log(\chi_1) - \log(\chi_2)]}{\delta\Gamma(2\kappa-1)}\right|\int_1^{\mathsf{T}} (\log\frac{\mathsf{T}}{\mathsf{s}})^{2\kappa-2}\,|\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad\quad + \left|\frac{\eta_2\delta_{12}[(\log\chi_1)^2 - (\log\chi_2)^2]}{\delta\Gamma(2\kappa-2)} + \frac{\eta_2\delta_{22}[\log(\chi_1) - \log(\chi_2)]}{\delta\Gamma(2\kappa-2)}\right|\int_1^{\mathsf{T}} (\log\frac{\mathsf{T}}{\mathsf{s}})^{2\kappa-3}\,|\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad\quad + \left|\frac{\eta_3\delta_{13}[(\log\chi_1)^2 - (\log\chi_2)^2]}{\delta\Gamma(2\kappa-3)} + \frac{\eta_3\delta_{23}[\log(\chi_1) - \log(\chi_2)]}{\delta\Gamma(2\kappa-3)}\right|\int_1^{\mathsf{T}} (\log\frac{\mathsf{T}}{\mathsf{s}})^{2\kappa-4}\,|\,\mathsf{u}(\mathsf{s})\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}.
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
&\left|(\log\chi_1)^{3-\kappa}\mathfrak{U}_\kappa(\mathsf{v})(\chi_1)-(\log\chi_2)^{3-\kappa}\mathfrak{U}_\kappa(\mathsf{v})(\chi_2)\right|\\
&\quad\le\Bigg[\left|\frac{1}{\Gamma(\kappa)}\int_1^{\chi_1}\left[(\log\chi_1)^{3-\kappa}(\log\frac{\chi_1}{\mathsf{s}})^{\kappa-1}-(\log\chi_2)^{3-\kappa}(\log\frac{\chi_2}{\mathsf{s}})^{\kappa-1}\right]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right.\\
&\qquad\left.-\frac{1}{\Gamma(\kappa)}\int_{\chi_1}^{\chi_2}(\log\chi_2)^{3-\kappa}(\log\frac{\chi_2}{\mathsf{s}})^{\kappa-1}\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right|+\left|\frac{\mathsf{a}_0\delta_{11}\log\mathrm{T}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta}\right|\\
&\qquad+\left|\frac{\mathsf{a}_0\delta_{21}\log\mathrm{T}[\log(\chi_1)-\log(\chi_2)]}{\delta}\right|+\left|\frac{\mathsf{b}_0\delta_{12}(\log\mathrm{T})^2[(\log\chi_1)^2-(\log\chi_2)^2]}{2\delta}\right|\\
&\qquad+\left|\frac{\mathsf{b}_0\delta_{22}(\log\mathrm{T})^2[\log(\chi_1)-\log(\chi_2)]}{2\delta}\right|+\left|\frac{\mathsf{c}_0\delta_{13}(\log\mathrm{T})^3[(\log\chi_1)^2-(\log\chi_2)^2]}{6\delta}\right|\\
&\qquad+\left|\frac{\mathsf{c}_0\delta_{23}(\log\mathrm{T})^3[\log(\chi_1)-\log(\chi_2)]}{6\delta}\right|+\left|\frac{\eta_1\delta_{11}(\log\mathrm{T})^{2\kappa-1}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta\Gamma(2\kappa)}\right|\\
&\qquad+\left|\frac{\eta_1\delta_{21}(\log\mathrm{T})^{2\kappa-1}[\log(\chi_1)-\log(\chi_2)]}{\delta\Gamma(2\kappa)}\right|+\left|\frac{\eta_2\delta_{12}(\log\mathrm{T})^{2\kappa-2}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta\Gamma(2\kappa-1)}\right|\\
&\qquad+\left|\frac{\eta_2\delta_{22}(\log\mathrm{T})^{2\kappa-2}[\log(\chi_1)-\log(\chi_2)]}{\delta\Gamma(2\kappa-1)}\right|+\left|\frac{\eta_3\delta_{13}(\log\mathrm{T})^{2\kappa-3}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta\Gamma(2\kappa-2)}\right|\\
&\qquad+\left|\frac{\eta_3\delta_{23}(\log\mathrm{T})^{2\kappa-3}[\log(\chi_1)-\log(\chi_2)]}{\delta\Gamma(2\kappa-2)}\right|\Bigg]\frac{\phi_1^*+\phi_2^*\,|\,\mathsf{w}\,|}{1-\phi_3^*}\longrightarrow 0\quad\text{as}\quad\chi_1\longrightarrow\chi_2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left|(\log\chi_1)^{3-\sigma}\mathfrak{U}_\sigma(\mathsf{w})(\chi_1)-(\log\chi_2)^{3-\sigma}\mathfrak{U}_\sigma(\mathsf{w})(\chi_2)\right|\\
&\quad\le\Bigg[\left|\frac{1}{\Gamma(\sigma)}\int_1^{\chi_1}\left[(\log\chi_1)^{3-\sigma}(\log\frac{\chi_1}{\mathsf{s}})^{\sigma-1}-(\log\chi_2)^{3-\sigma}(\log\frac{\chi_2}{\mathsf{s}})^{\sigma-1}\right]\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right.\\
&\qquad\left.-\frac{1}{\Gamma(\sigma)}\int_{\chi_1}^{\chi_2}(\log\chi_2)^{3-\sigma}(\log\frac{\chi_2}{\mathsf{s}})^{\sigma-1}\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}\right|+\left|\frac{\mathsf{a}_1\delta_{11}^*\log\mathrm{T}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta^*}\right|\\
&\qquad+\left|\frac{\mathsf{a}_1\delta_{21}^*\log\mathrm{T}[\log(\chi_1)-\log(\chi_2)]}{\delta^*}\right|+\left|\frac{\mathsf{b}_1\delta_{12}^*(\log\mathrm{T})^2[(\log\chi_1)^2-(\log\chi_2)^2]}{2\delta^*}\right|\\
&\qquad+\left|\frac{\mathsf{b}_1\delta_{22}^*(\log\mathrm{T})^2[\log(\chi_1)-\log(\chi_2)]}{2\delta^*}\right|+\left|\frac{\mathsf{c}_1\delta_{13}^*(\log\mathrm{T})^3[(\log\chi_1)^2-(\log\chi_2)^2]}{6\delta^*}\right|\\
&\qquad+\left|\frac{\mathsf{c}_1\delta_{23}^*(\log\mathrm{T})^3[\log(\chi_1)-\log(\chi_2)]}{6\delta^*}\right|+\left|\frac{\eta_4\delta_{11}^*(\log\mathrm{T})^{2\sigma-1}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta^*\Gamma(2\sigma)}\right|\\
&\qquad+\left|\frac{\eta_4\delta_{21}^*(\log\mathrm{T})^{2\sigma-1}[\log(\chi_1)-\log(\chi_2)]}{\delta^*\Gamma(2\sigma)}\right|+\left|\frac{\eta_5\delta_{12}^*(\log\mathrm{T})^{2\sigma-2}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta^*\Gamma(2\sigma-1)}\right|\\
&\qquad+\left|\frac{\eta_5\delta_{22}^*(\log\mathrm{T})^{2\sigma-2}[\log(\chi_1)-\log(\chi_2)]}{\delta^*\Gamma(2\sigma-1)}\right|+\left|\frac{\eta_6\delta_{13}^*(\log\mathrm{T})^{2\sigma-3}[(\log\chi_1)^2-(\log\chi_2)^2]}{\delta^*\Gamma(2\sigma-2)}\right|\\
&\qquad+\left|\frac{\eta_6\delta_{23}^*(\log\mathrm{T})^{2\sigma-3}[\log(\chi_1)-\log(\chi_2)]}{\delta^*\Gamma(2\sigma-2)}\right|\Bigg]\frac{\varphi_1^*+\varphi_2^*\,|\,\mathsf{v}\,|}{1-\varphi_3^*}\longrightarrow 0\quad\text{as}\quad\chi_1\longrightarrow\chi_2.
\end{aligned}
$$

Thus, $\mathfrak{U}(\mathsf{v},\mathsf{w})$ is equicontinuous, and consequently, $\mathfrak{U}(\mathsf{v},\mathsf{w})$ is completely continuous.

Now, we prove that $\mathcal{B}=\left\{(\mathsf{v},\mathsf{w})\in\mathcal{W}\times\mathcal{W}\middle|(\mathsf{v},\mathsf{w})=\lambda\mathfrak{U}(\mathsf{v},\mathsf{w}),\ \lambda\in[0,1]\right\}$ is bounded. Let $(\mathsf{v},\mathsf{w})\in\mathcal{B}$. Then, $(\mathsf{v},\mathsf{w})=\lambda\mathfrak{U}(\mathsf{v},\mathsf{w})$. For $\chi\in\mathcal{M}$, we have

$$\mathsf{v}(\chi)=\lambda\mathfrak{U}_\kappa(\mathsf{v})(\chi),\quad \mathsf{w}(\chi)=\lambda\mathfrak{U}_\sigma(\mathsf{w})(\chi).$$

Then, we have

$$
\begin{aligned}
(\log\chi)^{3-\kappa}|\mathsf{v}(\chi)| \leq \Bigg[& \left|\frac{\mathsf{a}_0\delta_{31}\log \mathrm{T}}{\delta}\right| + \left|\frac{(2\mathsf{a}_0\delta_{21}+\mathsf{b}_0\delta_{32})(\log \mathrm{T})^2}{2\delta}\right| + \frac{(\log \mathrm{T})^3}{\Gamma(\kappa)} \\
& + \left|\frac{(6\mathsf{a}_0\delta_{11}+3\mathsf{b}_0\delta_{22}+\mathsf{c}_0\delta_{33})(\log \mathrm{T})^3}{6\delta}\right| + \left|\frac{(3\mathsf{b}_0\delta_{12}+\mathsf{c}_0\delta_{23})(\log \mathrm{T})^4}{6\delta}\right| + \left|\frac{\mathsf{c}_0\delta_{13}(\log \mathrm{T})^5}{6\delta}\right| \\
& + \left|\frac{\eta_1\delta_{11}(\log \mathrm{T})^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right| + \left|\frac{(\eta_1\delta_{31}+(2\kappa-1)\eta_2\delta_{22}+(2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log \mathrm{T})^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right| \\
& + \left|\frac{\eta_3\delta_{33}(\log \mathrm{T})^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right| + \left|\frac{(\eta_1\delta_{21}+(2\kappa-1)\eta_2\delta_{12})(\log \mathrm{T})^{2\kappa}}{\delta\Gamma(2\kappa)}\right| \\
& + \left|\frac{(\eta_2\delta_{32}+(2\kappa-2)\eta_3\delta_{23})(\log \mathrm{T})^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right|\Bigg] \frac{\phi_1(\chi)+\phi_2(\chi)\,|\,\mathsf{w}(\chi)\,|}{1-\phi_3(\chi)},
\end{aligned}
$$

which implies that

$$
\|\mathsf{v}\|_{\mathcal{W}} \leq \mathfrak{R}_\kappa \frac{\phi_1^* + \phi_2^*\|\mathsf{w}\|_{\mathcal{W}}}{1-\phi_3^*} \tag{22}
$$

In addition, we have

$$
\begin{aligned}
(\log\chi)^{3-\sigma}|\mathsf{w}(\chi)| \leq \Bigg[& \left|\frac{\mathsf{a}_1\delta_{31}^*\log \mathrm{T}}{\delta^*}\right| + \left|\frac{(2\mathsf{a}_1\delta_{21}^*+\mathsf{b}_1\delta_{32}^*)(\log \mathrm{T})^2}{2\delta^*}\right| + \frac{(\log \mathrm{T})^3}{\Gamma(\sigma)} \\
& + \left|\frac{(6\mathsf{a}_1\delta_{11}^*+3\mathsf{b}_1\delta_{22}^*+\mathsf{c}_1\delta_{33}^*)(\log \mathrm{T})^3}{6\delta^*}\right| + \left|\frac{(3\mathsf{b}_1\delta_{12}^*+\mathsf{c}_1\delta_{23}^*)(\log \mathrm{T})^4}{6\delta^*}\right| + \left|\frac{\mathsf{c}_1\delta_{13}^*(\log \mathrm{T})^5}{6\delta^*}\right| \\
& + \left|\frac{\eta_4\delta_{11}^*(\log \mathrm{T})^{2\sigma+1}}{\delta^*\Gamma(2\sigma)}\right| + \left|\frac{(\eta_4\delta_{31}^*+(2\sigma-1)\eta_5\delta_{22}^*+(2\sigma-1)(2\sigma-2)\eta_6\delta_{13}^*)(\log \mathrm{T})^{2\sigma-1}}{\delta^*\Gamma(2\sigma)}\right| \\
& + \left|\frac{\eta_6\delta_{33}^*(\log \mathrm{T})^{2\sigma-3}}{\delta^*\Gamma(2\sigma-2)}\right| + \left|\frac{(\eta_4\delta_{21}^*+(2\sigma-1)\eta_5\delta_{12}^*)(\log \mathrm{T})^{2\sigma}}{\delta^*\Gamma(2\sigma)}\right| \\
& + \left|\frac{(\eta_5\delta_{32}^*+(2\sigma-2)\eta_6\delta_{23}^*)(\log \mathrm{T})^{2\sigma-2}}{\delta^*\Gamma(2\sigma-1)}\right|\Bigg] \frac{\varphi_1(\chi)+\varphi_2(\chi)\,|\,\mathsf{v}(\chi)\,|}{1-\varphi_3(\chi)},
\end{aligned}
$$

which implies that

$$
\|\mathsf{w}\|_{\mathcal{W}} \leq \mathfrak{R}_\sigma \frac{\varphi_1^* + \varphi_2^*\|\mathsf{v}\|_{\mathcal{W}}}{1-\varphi_3^*}. \tag{23}
$$

Equations (22) and (23) imply that

$$
\|\mathsf{v}\|_{\mathcal{W}} + \|\mathsf{w}\|_{\mathcal{W}} = \frac{\mathfrak{R}_\kappa\phi_1^*}{1-\phi_3^*} + \frac{\mathfrak{R}_\sigma\varphi_1^*}{1-\varphi_3^*} + \frac{\mathfrak{R}_\sigma\varphi_2^*\|\mathsf{v}\|_{\mathcal{W}}}{1-\varphi_3^*} + \frac{\mathfrak{R}_\kappa\phi_2^*\|\mathsf{w}\|_{\mathcal{W}}}{1-\phi_3^*}.
$$

Thus, we have

$$
\|(\mathsf{v},\mathsf{w})\|_{\mathcal{W}\times\mathcal{W}} \leq \frac{\mathfrak{R}_\kappa\phi_1^* + \mathfrak{R}_\sigma\varphi_1^*}{(1-\phi_3^*)(1-\varphi_3^*)(1-\mathfrak{R}_0)},
$$

for any $\chi \in \mathcal{M}$, where $\mathfrak{R}_0$ is presented in Equation (17), and thus $\mathcal{B}$ is bounded. Consequently, Equation (1) has at least one solution. □

4. Stability

Let us introduce the definitions of the Ulam stabilities for Equation (1).

Let $\Theta_\mu, \Theta_\nu : \mathcal{M} \to \mathcal{R}^+$ be nondecreasing functions and $\epsilon_\mu, \epsilon_\nu > 0$. Additionally, let

$$\begin{cases} \left|{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) - \mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi))\right| \leq \epsilon_\mu, \ \chi \in \mathcal{M}, \\ \left|{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi) - \nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi))\right| \leq \epsilon_\nu, \ \chi \in \mathcal{M}, \end{cases} \tag{24}$$

$$\begin{cases} \left|{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) - \mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi))\right| \leq \Theta_\mu(\chi)\epsilon_\mu, \ \chi \in \mathcal{M}, \\ \left|{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi) - \nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi))\right| \leq \Theta_\nu(\chi)\epsilon_\nu, \ \chi \in \mathcal{M}, \end{cases} \tag{25}$$

$$\begin{cases} \left|{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) - \mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi))\right| \leq \Theta_\mu(\chi), \ \chi \in \mathcal{M}, \\ \left|{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi) - \nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi))\right| \leq \Theta_\nu(\chi), \ \chi \in \mathcal{M}. \end{cases} \tag{26}$$

Definition 3 ([37]). *Equation* (1) *has* $\mathcal{HUS}$ *if* $\exists\ \mathbf{C}_{\mu,\nu} = (\mathbf{C}_\mu, \mathbf{C}_\nu) > 0$ *such that for some* $\epsilon = (\epsilon_\mu, \epsilon_\nu) > 0$ *and for any* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *satisfying Equation* (24), $\exists$ *a solution* $(\omega, v) \in \mathcal{W} \times \mathcal{W}$ *to Equation* (1) *with*

$$\left|(\mathsf{v}, \mathsf{w})(\chi) - (\omega, v)(\chi)\right| \leq \mathbf{C}_{\mu,\nu}\epsilon, \ \chi \in \mathcal{M}. \tag{27}$$

Definition 4 ([37]). *Equation* (1) *has generalized* $\mathcal{HUS}$ *if* $\exists\ \Phi_{\mu,\nu} \in \mathcal{N}(\mathcal{R}^+, \mathcal{R}^+)$ *with* $\Phi_{\mu,\nu}(0) = 0$ *such that for any* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *satisfying Equation* (24), $\exists$ *a solution* $(\omega, v) \in \mathcal{W} \times \mathcal{W}$ *to Equation* (1) *satisfying*

$$\left|(\mathsf{v}, \mathsf{w})(\chi) - (\omega, v)(\chi)\right| \leq \Phi_{\mu,\nu}(\epsilon), \ \chi \in \mathcal{M}. \tag{28}$$

Definition 5 ([37]). *Equation* (1) *has* $\mathcal{HURS}$ *corresponding to* $\Theta_{\mu,\nu} = (\Theta_\mu, \Theta_\nu) \in \mathcal{N}(\mathcal{M}, \mathcal{R})$ *if* $\exists$ *constants* $\mathbf{C}_{\Theta_\mu,\Theta_\nu} = (\mathbf{C}_{\Theta_\mu}, \mathbf{C}_{\Theta_\nu}) > 0$ *such that for some* $\epsilon = (\epsilon_\mu, \epsilon_\nu) > 0$ *and for any* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *satisfying Equation* (25), $\exists$ *a solution* $(\omega, v) \in \mathcal{W} \times \mathcal{W}$ *with*

$$\left|(\mathsf{v}, \mathsf{w})(\chi) - (\omega, v)(\chi)\right| \leq \mathbf{C}_{\Theta_\mu,\Theta_\nu}\Theta_{\mu,\nu}(\chi)\epsilon, \ \chi \in \mathcal{M}. \tag{29}$$

Definition 6 ([37]). *Equation* (1) *has generalized* $\mathcal{HURS}$ *with respect to* $\Theta_{\mu,\nu} = (\Theta_\mu, \Theta_\nu) \in \mathcal{N}(\mathcal{M}, \mathcal{R})$ *if* $\exists$ *a constant* $\mathbf{C}_{\Theta_\mu,\Theta_\nu} = (\mathbf{C}_{\Theta_\mu}, \mathbf{C}_{\Theta_\nu}) > 0$ *such that for any* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *satisfying Equation* (26), $\exists$ *a solution* $(\omega, v) \in \mathcal{W} \times \mathcal{W}$ *to Equation* (1) *satisfying*

$$\left|(\mathsf{v}, \mathsf{w})(\chi) - (\omega, v)(\chi)\right| \leq \mathbf{C}_{\Theta_\mu,\Theta_\nu}\Theta_{\mu,\nu}(\chi), \ \chi \in \mathcal{M}. \tag{30}$$

Remark 3. *Note that* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *is a solution to Equation* (24) *if* $\exists\ \Psi_\mu, \Psi_\nu \in \mathcal{N}(\mathcal{M}, \mathcal{R})$, *which depend on* v *and* w *respectively, in such a way that the following are true:*

(A_1) $\left|\Psi_\mu(\chi)\right| \leq \epsilon_\mu, \ \left|\Psi_\nu(\chi)\right| \leq \epsilon_\nu, \ \chi \in \mathcal{M}$;

(A_2)

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) = \mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi)) + \Psi_\mu(\chi), \ \chi \in \mathcal{M}, \\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi) = \nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi)) + \Psi_\nu(\chi), \ \chi \in \mathcal{M}. \end{cases}$$

Lemma 4. *Let* $(\mathsf{v}, \mathsf{w}) \in \mathcal{W} \times \mathcal{W}$ *be a solution to Equation* (24). *Then, we have*

$$\begin{cases} \|\mathsf{v} - \mathrm{m}_1\|_{\mathcal{W}} \leq \mathfrak{R}_\kappa\epsilon_\mu, \ \chi \in \mathcal{M}, \\ \|\mathsf{w} - \mathrm{m}_2\|_{\mathcal{W}} \leq \mathfrak{R}_\sigma\epsilon_\nu, \ \chi \in \mathcal{M}. \end{cases}$$

Proof. By applying (A_2) of Remark 3 and letting $\chi \in \mathcal{M}$, then we obtain

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi) - \mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi)) = \Psi_\mu(\chi); \ \chi \in \mathcal{M}, \\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi) - \nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi)) = \Psi_\nu(\chi); \ \chi \in \mathcal{M}, \end{cases} \tag{31}$$

with the generalized integro-differential BCs in Equation (2). Thus, utilizing Lemma 1, the solution to Equation (31) is

$$
\begin{cases}
\mathrm{v}(\chi) = \dfrac{1}{\Gamma(\kappa)}\displaystyle\int_1^{\chi}(\log\frac{\chi}{\mathrm{s}})^{\kappa-1}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{a}_0\delta_{11}(\log\chi)^{\kappa-1}}{\delta}+\dfrac{\mathrm{a}_0\delta_{21}(\log\chi)^{\kappa-2}}{\delta}+\dfrac{\mathrm{a}_0\delta_{31}(\log\chi)^{\kappa-3}}{\delta}\right]\displaystyle\int_1^{\mathrm{T}}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{b}_0\delta_{12}(\log\chi)^{\kappa-1}}{\delta}+\dfrac{\mathrm{b}_0\delta_{22}(\log\chi)^{\kappa-2}}{\delta}+\dfrac{\mathrm{b}_0\delta_{32}(\log\chi)^{\kappa-3}}{\delta}\right]\displaystyle\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathrm{s}}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{c}_0\delta_{13}(\log\chi)^{\kappa-1}}{2\delta}+\dfrac{\mathrm{c}_0\delta_{23}(\log\chi)^{\kappa-2}}{2\delta}+\dfrac{\mathrm{c}_0\delta_{33}(\log\chi)^{\kappa-3}}{2\delta}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^2[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_1\delta_{11}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-1)}+\dfrac{\eta_1\delta_{21}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-1)}+\dfrac{\eta_1\delta_{31}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-1)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-2}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_2\delta_{12}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-2)}+\dfrac{\eta_2\delta_{22}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-2)}+\dfrac{\eta_2\delta_{32}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-2)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-3}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_3\delta_{13}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-3)}+\dfrac{\eta_3\delta_{23}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-3)}+\dfrac{\eta_3\delta_{33}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-3)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-4}[\mu(\mathrm{s},\mathrm{w}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathrm{v}(\mathrm{s}))+\Psi_{\mu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}},\\
\mathrm{w}(\chi) = \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_1^{\chi}(\log\frac{\chi}{\mathrm{s}})^{\sigma-1}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{a}_1\delta^*_{11}(\log\chi)^{\sigma-1}}{\delta^*}+\dfrac{\mathrm{a}_1\delta^*_{21}(\log\chi)^{\sigma-2}}{\delta^*}+\dfrac{\mathrm{a}_1\delta^*_{31}(\log\chi)^{\sigma-3}}{\delta^*}\right]\displaystyle\int_1^{\mathrm{T}}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{b}_1\delta^*_{12}(\log\chi)^{\sigma-1}}{\delta^*}+\dfrac{\mathrm{b}_1\delta^*_{22}(\log\chi)^{\sigma-2}}{\delta^*}+\dfrac{\mathrm{b}_1\delta^*_{32}(\log\chi)^{\sigma-3}}{\delta^*}\right]\displaystyle\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathrm{s}}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{c}_1\delta^*_{13}(\log\chi)^{\sigma-1}}{2\delta^*}+\dfrac{\mathrm{c}_1\delta^*_{23}(\log\chi)^{\sigma-2}}{2\delta^*}+\dfrac{\mathrm{c}_1\delta^*_{33}(\log\chi)^{\sigma-3}}{2\delta^*}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^2[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_4\delta^*_{11}(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-1)}+\dfrac{\eta_4\delta^*_{21}(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-1)}+\dfrac{\eta_4\delta^*_{31}(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-1)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-2}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_5\delta^*_{12}(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-2)}+\dfrac{\eta_5\delta^*_{22}(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-2)}+\dfrac{\eta_5\delta^*_{32}(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-2)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-3}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_6\delta^*_{13}(\log\chi)^{\sigma-1}}{\delta^*\Gamma(2\sigma-3)}+\dfrac{\eta_6\delta^*_{23}(\log\chi)^{\sigma-2}}{\delta^*\Gamma(2\sigma-3)}+\dfrac{\eta_6\delta^*_{33}(\log\chi)^{\sigma-3}}{\delta^*\Gamma(2\sigma-3)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-4}[\nu(\mathrm{s},\mathrm{v}(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathrm{w}(\mathrm{s}))+\Psi_{\nu}(\mathrm{s})]\frac{\mathrm{ds}}{\mathrm{s}}.
\end{cases}
\tag{32}
$$

By using the first part of Equation (32), we have

$$
\begin{aligned}
(\log\chi)^{3-\kappa}\left|\mathrm{v}(\chi)-\mathrm{m}_1(\chi)\right| \le{}& \frac{1}{\Gamma(\kappa)}\int_1^{\chi}(\log\frac{\chi}{\mathrm{s}})^{\kappa-1}\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\mathrm{a}_0\delta_{11}(\log\chi)^2}{\delta}+\frac{\mathrm{a}_0\delta_{21}\log\chi}{\delta}+\frac{\mathrm{a}_0\delta_{31}}{\delta}\right|\int_1^{\mathrm{T}}|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\mathrm{b}_0\delta_{12}(\log\chi)^2}{\delta}+\frac{\mathrm{b}_0\delta_{22}\log(\chi)}{\delta}+\frac{\mathrm{b}_0\delta_{32}}{\delta}\right|\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathrm{s}}\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\mathrm{c}_0\delta_{13}(\log\chi)^2}{2\delta}+\frac{\mathrm{c}_0\delta_{23}\log\chi}{2\delta}+\frac{\mathrm{c}_0\delta_{33}}{2\delta}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^2\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\eta_1\delta_{11}(\log\chi)^2}{\delta\Gamma(2\kappa-1)}+\frac{\eta_1\delta_{21}\log(\chi)}{\delta\Gamma(2\kappa-1)}+\frac{\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-2}\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\eta_2\delta_{12}(\log\chi)^2}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{22}\log\chi}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-3}\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}}\\
&+\left|\frac{\eta_3\delta_{13}(\log\chi)^2}{\delta\Gamma(2\kappa-3)}+\frac{\eta_3\delta_{23}\log\chi}{\delta\Gamma(2\kappa-3)}+\frac{\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-4}\,|\,\Psi_{\mu}(\mathrm{s})\,|\,\frac{\mathrm{ds}}{\mathrm{s}},
\end{aligned}
\tag{33}
$$

where $\mathsf{m}_1(\chi)$ are those terms which are free of $\Psi_{\mathfrak{U}}$. By using Equation (8) and (A_1) of Remark 3, Equation (33) becomes

$$\|\mathsf{v}-\mathsf{m}_1\|_{\mathcal{W}} \leq \mathfrak{R}_{\kappa}\epsilon_{\mu}.$$

By following similar steps for Equation (32), we have

$$\|\mathsf{w}-\mathsf{m}_2\|_{\mathcal{W}} \leq \mathfrak{R}_{\sigma}\epsilon_{\nu}.$$

□

4.1. First Method

Theorem 4. *Let* $[\mathsf{H}_1]$ *hold, and*

$$\Lambda = 1-\frac{\mathfrak{R}_{\kappa}\mathfrak{R}_{\sigma}\mathcal{L}_{\mathfrak{U}}\mathcal{L}_{\mathsf{g}}}{(1-\mathfrak{R}_{\kappa}\overline{\mathcal{L}}_{\mathfrak{U}})(1-\mathfrak{R}_{\sigma}\overline{\mathcal{L}}_{\mathsf{g}})} > 0. \tag{34}$$

Then, Equation (1) *has* $\mathcal{HUS}$.

Proof. Let $(\mathsf{v},\mathsf{w}) \in \mathcal{W}\times\mathcal{W}$ satisfy Equation (24) and $(\omega,\upsilon) \in \mathcal{W}\times\mathcal{W}$ satisfy

$$\begin{cases} {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\chi)-\mathfrak{U}(\chi,\upsilon(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\chi))=0,\ \chi\in\mathcal{M},\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\upsilon(\chi)-\mathsf{g}(\chi,\omega(\chi),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}\upsilon(\chi))=0,\ \chi\in\mathcal{M},\\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-1}\omega(1)=\mathsf{a}_{0\mathrm{H}}\mathfrak{D}^{\kappa-1}\omega(\mathrm{T})+\eta_{1\mathrm{H}}\mathfrak{I}^{\kappa-1}\omega(\mathrm{T}),\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-2}\omega(1)=\mathsf{b}_{0\mathrm{H}}\mathfrak{D}^{\kappa-2}\omega(\mathrm{T})+\eta_{2\mathrm{H}}\mathfrak{I}^{\kappa-2}\omega(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\kappa-3}\omega(1)=\mathsf{c}_{0\mathrm{H}}\mathfrak{D}^{\kappa-3}\omega(\mathrm{T})+\eta_{3\mathrm{H}}\mathfrak{I}^{\kappa-3}\omega(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-1}\upsilon(1)=\mathsf{a}_{1\mathrm{H}}\mathfrak{D}^{\sigma-1}\upsilon(\mathrm{T})+\eta_{4\mathrm{H}}\mathfrak{I}^{\sigma-1}\upsilon(\mathrm{T}),\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-2}\upsilon(1)=\mathsf{b}_{1\mathrm{H}}\mathfrak{D}^{\sigma-2}\upsilon(\mathrm{T})+\eta_{5\mathrm{H}}\mathfrak{I}^{\sigma-2}\upsilon(\mathrm{T}),\\ {}_{\mathrm{H}}\mathfrak{D}^{\sigma-3}\upsilon(1)=\mathsf{c}_{1\mathrm{H}}\mathfrak{D}^{\sigma-3}\upsilon(\mathrm{T})+\eta_{6\mathrm{H}}\mathfrak{I}^{\sigma-3}\upsilon(\mathrm{T}). \end{cases} \tag{35}$$

Now, by using Lemma 1, for $\chi\in\mathcal{M}$, we find that Equation (35) has the following solution:

$$
\begin{cases}
\omega(\chi) = \dfrac{1}{\Gamma(\kappa)}\displaystyle\int_1^{\chi}(\log\frac{\chi}{\mathrm{s}})^{\kappa-1}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{a}_0\delta_{11}(\log\chi)^{\kappa-1}}{\delta}+\dfrac{\mathrm{a}_0\delta_{21}(\log\chi)^{\kappa-2}}{\delta}+\dfrac{\mathrm{a}_0\delta_{31}(\log\chi)^{\kappa-3}}{\delta}\right]\displaystyle\int_1^{\mathrm{T}}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{b}_0\delta_{12}(\log\chi)^{\kappa-1}}{\delta}+\dfrac{\mathrm{b}_0\delta_{22}(\log\chi)^{\kappa-2}}{\delta}+\dfrac{\mathrm{b}_0\delta_{32}(\log\chi)^{\kappa-3}}{\delta}\right]\displaystyle\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathrm{s}}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{c}_0\delta_{13}(\log\chi)^{\kappa-1}}{2\delta}+\dfrac{\mathrm{c}_0\delta_{23}(\log\chi)^{\kappa-2}}{2\delta}+\dfrac{\mathrm{c}_0\delta_{33}(\log\chi)^{\kappa-3}}{2\delta}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_1\delta_{11}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-1)}+\dfrac{\eta_1\delta_{21}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-1)}+\dfrac{\eta_1\delta_{31}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-1)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-2}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_2\delta_{12}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-2)}+\dfrac{\eta_2\delta_{22}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-2)}+\dfrac{\eta_2\delta_{32}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-2)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-3}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_3\delta_{13}(\log\chi)^{\kappa-1}}{\delta\Gamma(2\kappa-3)}+\dfrac{\eta_3\delta_{23}(\log\chi)^{\kappa-2}}{\delta\Gamma(2\kappa-3)}+\dfrac{\eta_3\delta_{33}(\log\chi)^{\kappa-3}}{\delta\Gamma(2\kappa-3)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\kappa-4}\mu(\mathrm{s},v(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}},\\
v(\chi) = \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_1^{\chi}(\log\frac{\chi}{\mathrm{s}})^{\sigma-1}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{a}_1\delta^{*}_{11}(\log\chi)^{\sigma-1}}{\delta^{*}}+\dfrac{\mathrm{a}_1\delta^{*}_{21}(\log\chi)^{\sigma-2}}{\delta^{*}}+\dfrac{\mathrm{a}_1\delta^{*}_{31}(\log\chi)^{\sigma-3}}{\delta^{*}}\right]\displaystyle\int_1^{\mathrm{T}}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{b}_1\delta^{*}_{12}(\log\chi)^{\sigma-1}}{\delta^{*}}+\dfrac{\mathrm{b}_1\delta^{*}_{22}(\log\chi)^{\sigma-2}}{\delta^{*}}+\dfrac{\mathrm{b}_1\delta^{*}_{32}(\log\chi)^{\sigma-3}}{\delta^{*}}\right]\displaystyle\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathrm{s}}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\mathrm{c}_1\delta^{*}_{13}(\log\chi)^{\sigma-1}}{2\delta^{*}}+\dfrac{\mathrm{c}_1\delta^{*}_{23}(\log\chi)^{\sigma-2}}{2\delta^{*}}+\dfrac{\mathrm{c}_1\delta^{*}_{33}(\log\chi)^{\sigma-3}}{2\delta^{*}}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_4\delta^{*}_{11}(\log\chi)^{\sigma-1}}{\delta^{*}\Gamma(2\sigma-1)}+\dfrac{\eta_4\delta^{*}_{21}(\log\chi)^{\sigma-2}}{\delta^{*}\Gamma(2\sigma-1)}+\dfrac{\eta_4\delta^{*}_{31}(\log\chi)^{\sigma-3}}{\delta^{*}\Gamma(2\sigma-1)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-2}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_5\delta^{*}_{12}(\log\chi)^{\sigma-1}}{\delta^{*}\Gamma(2\sigma-2)}+\dfrac{\eta_5\delta^{*}_{22}(\log\chi)^{\sigma-2}}{\delta^{*}\Gamma(2\sigma-2)}+\dfrac{\eta_5\delta^{*}_{32}(\log\chi)^{\sigma-3}}{\delta^{*}\Gamma(2\sigma-2)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-3}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}\\
-\left[\dfrac{\eta_6\delta^{*}_{13}(\log\chi)^{\sigma-1}}{\delta^{*}\Gamma(2\sigma-3)}+\dfrac{\eta_6\delta^{*}_{23}(\log\chi)^{\sigma-2}}{\delta^{*}\Gamma(2\sigma-3)}+\dfrac{\eta_6\delta^{*}_{33}(\log\chi)^{\sigma-3}}{\delta^{*}\Gamma(2\sigma-3)}\right]\displaystyle\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathrm{s}})^{2\sigma-4}\nu(\mathrm{s},\omega(\mathrm{s}),{}_{\mathrm{H}}\mathfrak{D}^{\sigma}v(\mathrm{s}))\frac{\mathrm{ds}}{\mathrm{s}}.
\end{cases}
\tag{36}
$$

Consider

$$
(\log\chi)^{3-\kappa}\left|\nu(\chi)-\omega(\chi)\right| \le (\log\chi)^{3-\kappa}\left|\nu(\chi)-\mathrm{m}_1(\chi)\right| + (\log\chi)^{3-\kappa}\left|\mathrm{m}_1(\chi)-\omega(\chi)\right|. \tag{37}
$$

By applying Lemma 4 in Equation (37), we have

$$
\begin{aligned}
&(\log\chi)^{3-\kappa}\left|\mathsf{v}(\chi)-\omega(\chi)\right| \\
&\leq \mathfrak{R}_\kappa\epsilon_\mu+\frac{(\log\chi)^{3-\kappa}}{\Gamma(\kappa)}\int_1^{\chi}(\log\frac{\chi}{\mathsf{s}})^{\kappa-1}\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\mathsf{a}_0\delta_{11}(\log\chi)^2}{\delta}\right. \\
&\quad+\frac{\mathsf{a}_0\delta_{21}\log(\chi)}{\delta}+\left.\frac{\mathsf{a}_0\delta_{31}}{\delta}\right|\int_1^{\mathrm{T}}|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\mathsf{b}_0\delta_{12}(\log\chi)^2}{\delta}\right. \\
&\quad+\frac{\mathsf{b}_0\delta_{22}\log(\chi)}{\delta}+\left.\frac{\mathsf{b}_0\delta_{32}}{\delta}\right|\int_1^{\mathrm{T}}\log\frac{\mathrm{T}}{\mathsf{s}}\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\mathsf{c}_0\delta_{13}(\log\chi)^2}{2\delta}\right. \\
&\quad+\frac{\mathsf{c}_0\delta_{23}\log(\chi)}{2\delta}+\left.\frac{\mathsf{c}_0\delta_{33}}{2\delta}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^2\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\eta_1\delta_{31}}{\delta\Gamma(2\kappa-1)}\right. \\
&\quad+\frac{\eta_1\delta_{21}\log(\chi)}{\delta\Gamma(2\kappa-1)}+\left.\frac{\eta_1\delta_{11}(\log\chi)^2}{\delta\Gamma(2\kappa-1)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-2}\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\quad+\left|\frac{\eta_2\delta_{12}(\log\chi)^2}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{22}\log(\chi)}{\delta\Gamma(2\kappa-2)}+\frac{\eta_2\delta_{32}}{\delta\Gamma(2\kappa-2)}\right| \\
&\quad\times\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-3}\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}}+\left|\frac{\eta_3\delta_{13}(\log\chi)^2}{\delta\Gamma(2\kappa-3)}\right. \\
&\quad+\frac{\eta_3\delta_{23}\log(\chi)}{\delta\Gamma(2\kappa-3)}+\left.\frac{\eta_3\delta_{33}}{\delta\Gamma(2\kappa-3)}\right|\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\mathsf{s}})^{2\kappa-4}\,|\,\mu(\mathsf{s},\mathsf{w}(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\mathsf{s}))-\mu(\mathsf{s},v(\mathsf{s}),{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega(\mathsf{s}))\,|\,\frac{\mathrm{d}\mathsf{s}}{\mathsf{s}} \\
&\leq \mathfrak{R}_\kappa\epsilon_\mu+\left[\left|\frac{\mathsf{a}_0\delta_{31}\log(\mathrm{T})}{\delta}\right|+\left|\frac{(2\mathsf{a}_0\delta_{21}+\mathsf{b}_0\delta_{32})(\log\mathrm{T})^2}{2\delta}\right|+\left|\frac{(6\mathsf{a}_0\delta_{11}+3\mathsf{b}_0\delta_{22}+\mathsf{c}_0\delta_{33})(\log\mathrm{T})^3}{6\delta}\right|\right. \\
&\quad+\left|\frac{(3\mathsf{b}_0\delta_{12}+\mathsf{c}_0\delta_{23})(\log\mathrm{T})^4}{6\delta}\right|+\left|\frac{\mathsf{c}_0\delta_{13}(\log\mathrm{T})^5}{6\delta}\right|+\frac{(\log\mathrm{T})^3}{\Gamma(\kappa)}+\left|\frac{(\eta_1\delta_{21}+(2\kappa-1)\eta_2\delta_{12})(\log\mathrm{T})^{2\kappa}}{\delta\Gamma(2\kappa)}\right| \\
&\quad+\left|\frac{(\eta_1\delta_{31}+(2\kappa-1)\eta_2\delta_{22}+(2\kappa-1)(2\kappa-2)\eta_3\delta_{13})(\log\mathrm{T})^{2\kappa-1}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{\eta_3\delta_{33}(\log\mathrm{T})^{2\kappa-3}}{\delta\Gamma(2\kappa-2)}\right| \\
&\quad+\left.\left|\frac{\eta_1\delta_{11}(\log\mathrm{T})^{2\kappa+1}}{\delta\Gamma(2\kappa)}\right|+\left|\frac{(\eta_2\delta_{32}+(2\kappa-2)\eta_3\delta_{23})(\log\mathrm{T})^{2\kappa-2}}{\delta\Gamma(2\kappa-1)}\right|\right]\Big(\mathcal{L}_\mu\|\mathsf{v}-\omega\|_{\mathcal{W}}+\overline{\mathcal{L}}_\mu\|{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}-{}_{\mathrm{H}}\mathfrak{D}^{\kappa}\omega\|_{\mathcal{W}}\Big). \qquad (38)
\end{aligned}
$$

By utilizing [H_1] of Theorem 1 and Equation (8) in Equation (38), we obtain

$$\|\mathsf{v}-\omega\|_{\mathcal{W}}\leq\frac{\mathfrak{R}_\kappa\epsilon_\mu}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu}+\frac{\mathfrak{R}_\kappa\mathcal{L}_\mu}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu}\|\mathsf{w}-v\|_{\mathcal{W}}. \qquad (39)$$

Equivalently, we have

$$\|\mathsf{w}-v\|_{\mathcal{W}}\leq\frac{\mathfrak{R}_\sigma\epsilon_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu}+\frac{\mathfrak{R}_\sigma\mathcal{L}_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu}\|\mathsf{v}-\omega\|_{\mathcal{W}}. \qquad (40)$$

Equations (39) and (40) can be written as

$$
\begin{aligned}
&\|\mathsf{v}-\omega\|_{\mathcal{W}}-\frac{\mathfrak{R}_\kappa\mathcal{L}_{\mathfrak{U}}}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_{\mathfrak{U}}}\|\mathsf{w}-v\|_{\mathcal{W}}\leq\frac{\mathfrak{R}_\kappa\epsilon_\mu}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu}, \\
&\|\mathsf{w}-v\|_{\mathcal{W}}-\frac{\mathfrak{R}_\sigma\mathcal{L}_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu}\|\mathsf{v}-\omega\|_{\mathcal{W}}\leq\frac{\mathfrak{R}_\sigma\epsilon_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu},
\end{aligned}
$$

$$
\Rightarrow\begin{bmatrix}1 & -\frac{\mathfrak{R}_\kappa\mathcal{L}_\mu}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu}\\ -\frac{\mathfrak{R}_\sigma\mathcal{L}_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu} & 1\end{bmatrix}\begin{bmatrix}\|\mathsf{v}-\omega\|_{\mathcal{W}}\\ \|\mathsf{w}-v\|_{\mathcal{W}}\end{bmatrix}\leq\begin{bmatrix}\frac{\mathfrak{R}_\kappa\epsilon_\mu}{1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu}\\ \frac{\mathfrak{R}_\sigma\epsilon_\nu}{1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu}\end{bmatrix}.
$$

Thus, we have

$$\begin{bmatrix} \|\mathsf{v}-\omega\|_{\mathcal{W}} \\ \|\mathsf{w}-v\|_{\mathcal{W}} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Lambda} & \frac{\Re_\kappa \mathcal{L}_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)} \\ \frac{\Re_\sigma \mathcal{L}_\nu}{\Lambda(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} & \frac{1}{\Lambda} \end{bmatrix} \begin{bmatrix} \frac{\Re_\kappa \epsilon_\mu}{1-\Re_\kappa \overline{\mathcal{L}}_\mu} \\ \frac{\Re_\sigma \epsilon_\nu}{1-\Re_\sigma \overline{\mathcal{L}}_\nu} \end{bmatrix},$$

where

$$\Lambda = 1 - \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\mu \mathcal{L}_\nu}{(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} > 0.$$

Consequently, we have

$$\|\mathsf{v}-\omega\|_{\mathcal{W}} \leq \frac{\Re_\kappa \epsilon_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)} + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\mu \epsilon_\nu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)},$$
$$\|\mathsf{w}-v\|_{\mathcal{W}} \leq \frac{\Re_\sigma \epsilon_\nu}{\Lambda(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\nu \epsilon_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)},$$

from which we have

$$\begin{aligned} \|\mathsf{v}-\omega\|_{\mathcal{W}} + \|\mathsf{w}-v\|_{\mathcal{W}} \leq & \frac{\Re_\kappa \epsilon_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)} + \frac{\Re_\sigma \epsilon_\nu}{\Lambda(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\mu \epsilon_\nu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} \\ & + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\nu \epsilon_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)}. \end{aligned} \tag{41}$$

By letting $\epsilon = \max\{\epsilon_\mu, \epsilon_\nu\}$, then from Equation (41), we have

$$\|(\mathsf{v},\mathsf{w})-(\omega,v)\|_{\mathcal{W}\times\mathcal{W}} \leq \mathbf{C}_{\mu,\nu}\epsilon, \tag{42}$$

where

$$\mathbf{C}_{\mu,\nu} = \frac{\Re_\kappa}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)} + \frac{\Re_\sigma}{\Lambda(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\mu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)} + \frac{\Re_\kappa \Re_\sigma \mathcal{L}_\nu}{\Lambda(1-\Re_\kappa \overline{\mathcal{L}}_\mu)(1-\Re_\sigma \overline{\mathcal{L}}_\nu)}.$$

□

Remark 4. *By letting $\Phi_{\mu,\nu}(\epsilon) = \mathbf{C}_{\mu,\nu}\epsilon$, $\Phi_{\mu,\nu}(0) = 0$ in (42), then by Definition 4, Equation (1) is generalized as $\mathcal{HU}$ stable.*

- [(H_3)] Let functions $\Theta_\mu, \Theta_\nu : \mathcal{M} \to \mathcal{R}^+$ be nondecreasing. Then, there are $\mathrm{Y}_{\Theta_\mu}, \mathrm{Y}_{\Theta_\nu} > 0$ such that for every $\chi \in \mathcal{M}$, the following inequalities hold:

$$\mathfrak{I}^\kappa \Theta_\mu(\chi) \leq \mathrm{Y}_{\Theta_\mu} \Theta_\mu(\chi) \quad \text{and} \quad \mathfrak{I}^\sigma \Theta_\nu(\chi) \leq \mathrm{Y}_{\Theta_\nu} \Theta_\nu(\chi)$$

Remark 5. *Similarly, we obtained the $\mathcal{HURS}$ and $\mathcal{GHURS}$ for Equation (1) by utilizing Lemma 4, Theorem 5 and Definitions 5 and 6 with* [H_3] *and $\Lambda^* > 0$.*

4.2. Second Method

Theorem 5. *Let* [H_1] *hold, and if $\Lambda^* = 1 - \left[\frac{\Re_\sigma \mathcal{L}_\nu}{1-\Re_\sigma \overline{\mathcal{L}}_\nu} + \frac{\Re_\kappa \mathcal{L}_\mu}{1-\Re_\kappa \overline{\mathcal{L}}_\mu}\right] > 0$, then Equation (1) is $\mathcal{HU}$ stable.*

Proof. Equations (39) and (40) imply that

$$\begin{aligned} & \|\mathsf{v}-\omega\|_{\mathcal{W}} + \|\mathsf{w}-v\|_{\mathcal{W}} \\ \leq & \frac{\Re_\kappa \epsilon_\mu}{1-\Re_\kappa \overline{\mathcal{L}}_\mu} + \frac{\Re_\sigma \epsilon_\nu}{1-\Re_\sigma \overline{\mathcal{L}}_\nu} + \frac{\Re_\sigma \mathcal{L}_\nu}{1-\Re_\sigma \overline{\mathcal{L}}_\nu}\|\mathsf{v}-\omega\|_{\mathcal{W}} + \frac{\Re_\kappa \mathcal{L}_\mu}{1-\Re_\kappa \overline{\mathcal{L}}_\mu}\|\mathsf{w}-v\|_{\mathcal{W}}. \end{aligned} \tag{43}$$

By letting $\max\{\epsilon_\mu, \epsilon_\nu\} = \epsilon$, then from Equation (43), we have

$$\|(\mathsf{v}, \mathsf{w}) - (\omega, \upsilon)\|_{\mathcal{W}\times\mathcal{W}} \leq \mathbf{C}_{\mu,\nu}\epsilon, \tag{44}$$

where

$$\mathbf{C}_{\mu,\nu} = \left[\frac{\mathfrak{R}_\kappa}{\Lambda^*(1-\mathfrak{R}_\kappa\overline{\mathcal{L}}_\mu)} + \frac{\mathfrak{R}_\sigma}{\Lambda^*(1-\mathfrak{R}_\sigma\overline{\mathcal{L}}_\nu)}\right].$$

□

Remark 6. *Using Remark 4, we have $\mathcal{GHUS}$.*

Remark 7. *1. The $\mathcal{EU}$ and $\mathcal{HUS}$ results for $\mathbb{CS}$s of third-order nonlinear $\mathcal{DE}$s with $\mathcal{AP}$ conditions can be obtained if we set $\kappa = \sigma = 3$, $\mathsf{a}_\mathrm{i} = \mathsf{b}_\mathrm{i} = \mathsf{c}_\mathrm{i} = -1$ $(\mathrm{i} = 0, 1)$ and $\eta_\mathrm{i} = 0$ $(\mathrm{i} = 1, 2, \ldots, 6)$ in Equation (1).*

2. The $\mathcal{EU}$ and $\mathcal{HUS}$ results for $\mathbb{CS}$s of third-order nonlinear $\mathcal{DE}$s with the initial conditions will be obtained if we set $\kappa = \sigma = 3$, $\mathsf{a}_\mathrm{i} = \mathsf{b}_\mathrm{i} = \mathsf{c}_\mathrm{i} = 0$ $(\mathrm{i} = 0, 1)$ and $\eta_\mathrm{i} = 0$ $(\mathrm{i} = 1, 2, \ldots, 6)$ in Equation (1).

3. The $\mathcal{EU}$ and $\mathcal{HUS}$ results for $\mathbb{CS}$s of third-order nonlinear $\mathcal{DE}$s with integro-differential BCs will be obtained if we set $\kappa = \sigma = 3$ and $\mathsf{a}_\mathrm{i} = \mathsf{b}_\mathrm{i} = \mathsf{c}_\mathrm{i} = 0$ $(\mathrm{i} = 0, 1)$ in Equation (1).

5. Example

Example 1. *Let*

$$\begin{cases}
{}_\mathrm{H}\mathfrak{D}^\kappa \mathsf{v}(\chi) - \dfrac{2 + |\,{}_\mathrm{H}\mathfrak{D}^\kappa \mathsf{v}(\chi)\,| + |\,\mathsf{w}(\chi)\,|}{30\mathrm{e}^{\chi+50}(0.5 + |\,{}_\mathrm{H}\mathfrak{D}^\kappa \mathsf{v}(\chi)\,| + |\,\mathsf{w}(\chi)\,|)} = 0, \quad \chi \in [1, \mathrm{e}],\\
{}_\mathrm{H}\mathfrak{D}^\sigma \mathsf{w}(\chi) - \dfrac{\chi\cos \mathsf{v}(\chi) - \mathsf{w}(\chi)\sin\chi}{70} - \dfrac{|\,{}_\mathrm{H}\mathfrak{D}^\sigma \mathsf{w}(\chi)\,|}{50 + |\,{}_\mathrm{H}\mathfrak{D}^\sigma \mathsf{w}(\chi)\,|} = 0, \quad \chi \in [1, \mathrm{e}],\\
{}_\mathrm{H}\mathfrak{D}^{\kappa-1}\mathsf{v}(1) = \mathsf{a}_0\,{}_\mathrm{H}\mathfrak{D}^{\kappa-1}\mathsf{v}(\mathrm{e}) + \eta_1\,{}_\mathrm{H}\mathfrak{I}^{\kappa-1}\mathsf{v}(\mathrm{e}),\ {}_\mathrm{H}\mathfrak{D}^{\kappa-2}\mathsf{v}(1) = \mathsf{b}_0\,{}_\mathrm{H}\mathfrak{D}^{\kappa-2}\mathsf{v}(\mathrm{e}) + \eta_2\,{}_\mathrm{H}\mathfrak{I}^{\kappa-2}\mathsf{v}(\mathrm{e})\\
{}_\mathrm{H}\mathfrak{D}^{\kappa-3}\mathsf{v}(1) = \mathsf{c}_0\,{}_\mathrm{H}\mathfrak{D}^{\kappa-3}\mathsf{v}(\mathrm{e}) + \eta_3\,{}_\mathrm{H}\mathfrak{I}^{\kappa-3}\mathsf{v}(\mathrm{e}),\\
{}_\mathrm{H}\mathfrak{D}^{\sigma-1}\mathsf{w}(1) = \mathsf{a}_1\,{}_\mathrm{H}\mathfrak{D}^{\sigma-1}\mathsf{w}(\mathrm{e}) + \eta_4\,{}_\mathrm{H}\mathfrak{I}^{\sigma-1}\mathsf{w}(\mathrm{e}),\ {}_\mathrm{H}\mathfrak{D}^{\sigma-2}\mathsf{w}(1) = \mathsf{b}_1\,{}_\mathrm{H}\mathfrak{D}^{\sigma-2}\mathsf{w}(\mathrm{e}) + \eta_5\,{}_\mathrm{H}\mathfrak{I}^{\sigma-2}\mathsf{w}(\mathrm{e})\\
{}_\mathrm{H}\mathfrak{D}^{\sigma-3}\mathsf{w}(1) = \mathsf{c}_1\,{}_\mathrm{H}\mathfrak{D}^{\sigma-3}\mathsf{w}(\mathrm{e}) + \eta_6\,{}_\mathrm{H}\mathfrak{I}^{\sigma-3}\mathsf{w}(\mathrm{e}).
\end{cases} \tag{45}$$

Here, we have $\kappa = \sigma = \frac{5}{2}$, $\mathsf{a}_0 = \mathsf{a}_1 = \eta_1 = \eta_2 = \frac{2}{5}$, $\mathsf{b}_0 = \mathsf{b}_1 = \eta_3 = \eta_4 = \frac{3}{5}$ and $\mathsf{c}_0 = \mathsf{c}_1 = \eta_5 = \eta_6 = \frac{1}{5}$. Moreover, we find $\mathcal{L}_\mu = \overline{\mathcal{L}}_\mu = \frac{1}{30\mathrm{e}^{50}}$ and $\mathcal{L}_\nu = \overline{\mathcal{L}}_\nu = \frac{1}{50}$ from the nonlinear terms of Equation (45). Therefore, we have

$$\frac{\mathfrak{R}_\kappa\mathcal{L}_\mu(1-\overline{\mathcal{L}}_\nu) + \mathfrak{R}_\sigma\mathcal{L}_\nu(1-\overline{\mathcal{L}}_\mu)}{(1-\overline{\mathcal{L}}_\nu)(1-\overline{\mathcal{L}}_\mu)} \approx 0.75141 < 1,$$

Thus, Equation (45) has a solution which must be unique.

(i) *If we take $\mu(\chi, \mathsf{w}(\chi), {}_\mathrm{H}\mathfrak{D}^\kappa \mathsf{v}(\chi)) = \dfrac{(\mathsf{v}(\chi))^2 + 1}{50} + \sqrt{\mathsf{w}(\chi)}$, $\nu(\chi, \mathsf{v}(\chi), {}_\mathrm{H}\mathfrak{D}^\sigma \mathsf{w}(\chi)) = \frac{80\mathsf{v}(\chi)+(\mathsf{w}(\chi))^2}{80}$, and $\mathsf{v}(\chi) = \mathsf{w}(\chi) = \chi$, then with the values $\mathsf{a}_0 = \mathsf{a}_1 = \eta_1 = \eta_2 = \frac{2}{5}$, $\mathsf{b}_0 = \mathsf{b}_1 = \eta_3 = \eta_4 = \frac{3}{5}$ and $\mathsf{c}_0 = \mathsf{c}_1 = \eta_5 = \eta_6 = \frac{1}{5}$, the graph of the solution is shown in Figure 1.*

(ii) *If we take $\mu(\chi, \mathsf{w}(\chi), {}_\mathrm{H}\mathfrak{D}^\kappa \mathsf{v}(\chi)) = \dfrac{(\mathsf{v}(\chi))^2 + 1}{50} + \sqrt{\mathsf{w}(\chi)}$, $\nu(\chi, \mathsf{v}(\chi), {}_\mathrm{H}\mathfrak{D}^\sigma \mathsf{w}(\chi)) = \frac{80\mathsf{v}(\chi)+(\mathsf{w}(\chi))^2}{80}$ and $\mathsf{v}(\chi) = \mathsf{w}(\chi) = \chi$, then with the values $\mathsf{a}_0 = \mathsf{a}_1 = \eta_1 = \eta_2 = -\frac{2}{5}$, $\mathsf{b}_0 = \mathsf{b}_1 = \eta_3 = \eta_4 = -\frac{3}{5}$ and $\mathsf{c}_0 = \mathsf{c}_1 = \eta_5 = \eta_6 = -\frac{1}{5}$, the graph of the solution is shown in Figure 2.*

(iii) *If we take* $\mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi)) = \frac{(\mathsf{v}(\chi))^2+1}{50} + \sqrt{\mathsf{w}(\chi)}$, $\nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi)) = \frac{80\mathsf{v}(\chi)+(\mathsf{w}(\chi))^2}{80}$ *and* $\mathsf{v}(\chi) = \mathsf{w}(\chi) = \chi$, *then with the values* $\mathsf{a}_0 = \mathsf{a}_1 = -\eta_1 = -\eta_2 = \frac{2}{5}$, $\mathsf{b}_0 = \mathsf{b}_1 = -\eta_3 = -\eta_4 = \frac{3}{5}$ *and* $\mathsf{c}_0 = \mathsf{c}_1 = -\eta_5 = -\eta_6 = \frac{1}{5}$, *the graph of the solution is shown in Figure 3.*

(iv) *If we take* $\mu(\chi, \mathsf{w}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\kappa}\mathsf{v}(\chi)) = \frac{(\mathsf{v}(\chi))^2+1}{50} + \sqrt{\mathsf{w}(\chi)}$, $\nu(\chi, \mathsf{v}(\chi), {}_{\mathrm{H}}\mathfrak{D}^{\sigma}\mathsf{w}(\chi)) = \frac{80\mathsf{v}(\chi)+(\mathsf{w}(\chi))^2}{80}$ *and* $\mathsf{v}(\chi) = \mathsf{w}(\chi) = \chi$, *then with the values* $-\mathsf{a}_0 = -\mathsf{a}_1 = \eta_1 = \eta_2 = \frac{2}{5}$, $-\mathsf{b}_0 = -\mathsf{b}_1 = \eta_3 = \eta_4 = \frac{3}{5}$ *and* $-\mathsf{c}_0 = -\mathsf{c}_1 = \eta_5 = \eta_6 = \frac{1}{5}$, *the graph of the solution is shown in Figure 4.*

Furthermore, the conditions $\Lambda > 0$ *and* $\Lambda^* > 0$ *of Theorems 4 and 5 are also verified. Therefore, in both approaches, Equation (45) has* $\mathcal{HUS}$, $\mathcal{GHUS}$, $\mathcal{HURS}$ *and* $\mathcal{GHURS}$.

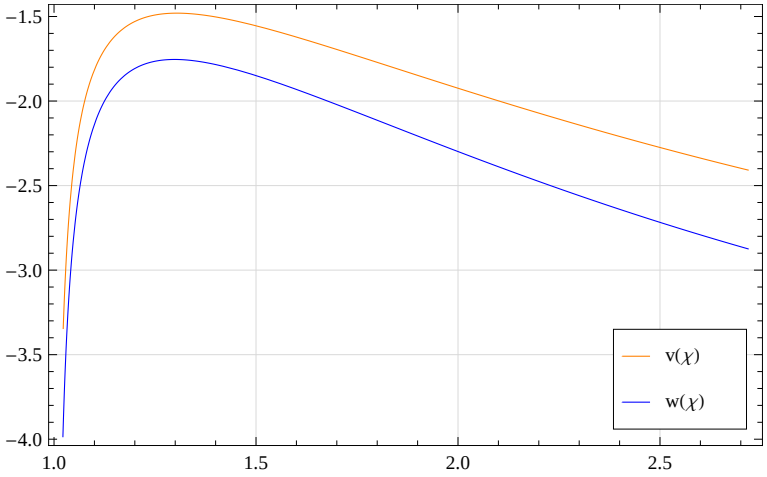

Figure 1. Sketch of solution in case **(i)**.

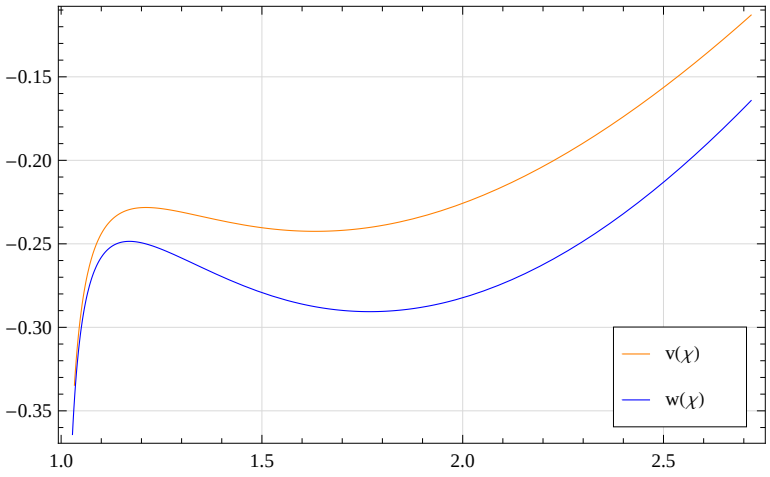

Figure 2. Sketch of solution in case **(ii)**.

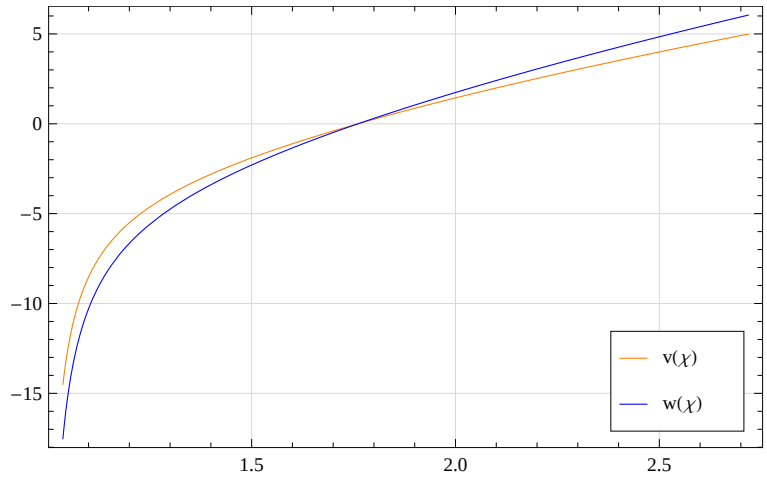

Figure 3. Sketch of solution in case **(iii)**.

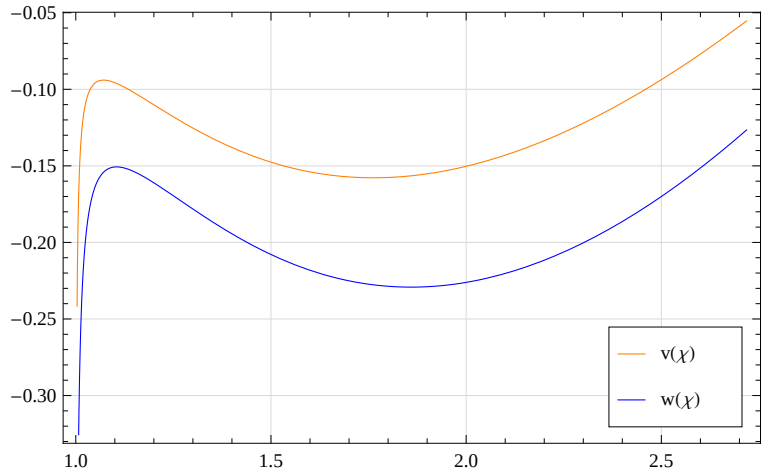

Figure 4. Sketch of solution in case **(iv)**.

6. Conclusions

We utilized the contraction principle and Leray–Schauder fixed point theorem for the existence theory of $\mathcal{FDE}$s (Equation (1)). We concluded that under some specific assumptions, the systems have at least one solution. Furthermore, we presented the $\mathcal{HUS}$, $\mathcal{GHUS}$, $\mathcal{HURS}$ and $\mathcal{GHURS}$ by utilizing the perturbations in Equation (1) and some necessary conditions.

The applicability of the outcomes was checked with an example. The proposed system in Equation (1) is the generalization of third-order $\mathcal{ODE}$s, particularly for $\kappa = \sigma = 3$, which is vastly utilized in applied sciences [2]. See the following for $\kappa = \sigma = 3$:

1. $\eta_i = 0$ $(i = 1, 2, \ldots, 6)$ and $a_j = b_j = c_j = 0$ $(j = 0, 1)$, and then we gain third-order coupled $\mathcal{ODE}$s with the initial conditions.
2. $\eta_i = 0$ $(i = 1, 2, \ldots, 6)$ and $a_j = b_j = c_j = -1$ $(j = 0, 1)$, and then we gain third-order coupled $\mathcal{ODE}$s with BCs in the form of $\mathcal{AP}$.
3. $a_j = b_j = c_j = 0$ $(j = 0, 1)$, and then we gain third-order coupled $\mathcal{ODE}$s with integro-differential BCs. Using MATLAB, we draw graphs for the particular variables and particular functions.

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