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Finite-Time Synchronization of Uncertain Fractional-Order Delayed Memristive Neural Networks via Adaptive Sliding Mode Control and Its Application

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Abstract: Finite-time synchronization (FTS) of uncertain fractional-order memristive neural networks (FMNNs) with leakage and discrete delays is studied in this paper, in which the impacts of uncertain parameters as well as external disturbances are considered. First, the fractional-order adaptive terminal sliding mode control scheme (FATSMC) is designed, which can effectively estimate the upper bounds of unknown external disturbances. Second, the FTS of the master–slave FMNNs is realized and the corresponding synchronization criteria and the explicit expression of the settling time (ST) are obtained. Finally, a numerical example and a secure communication application are provided to demonstrate the validity of the obtained results.

Keywords: finite-time synchronization; fractional-order memristive neural networks; adaptive terminal sliding mode control; secure communication



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1. Introduction

Based on the relationship of basic circuit variables, Professor Chua of Berkeley University proposed the existence of another circuit element, the memristor, in 1971 [1]. The memristor entity, based on titanium dioxide film, was implemented by HP Labs in 2008 [2]. In addition, the memristor with switching characteristics which have the same hysteresis properties as neurons in the brain [3]. By introducing the memristor into the neural networks (NNs), the memristive neural networks (MNNs) are obtained. In addition, various studies on MNNs have followed [4,5]. However, these studies are all about integer-order MNNs.

Different from integer-order calculus, the fractional calculus has heritability and memorability, and which is often used to describe some materials and processes [6]. What's more, fractional calculus gives a more accurate description of the model and it is often used to practical engineering application [7]. That has set off a wave of research on the fractional-order system [8–11]. In [9,11], the stability of fractional-order system was demonstrated and many important theoretical results were yielded. Furthermore, as a kind of important fractional-order system, it is very meaningful to study the FMNNs or fractional-order neural networks (FNNs).

Many researches on FMNNs or FNNs have attracted much attention. For example, the stability problem was investigated in [12–14]. In [12], the stability of fractional-order competitive neural networks with impulsive effects was further studied and some stability criteria are yielded. However, these results did not carry out in-depth research on the synchronization problem of FMNNs. In addition, many researchers had also made profound research on the synchronization [15–17]. Pratap et al. [16] first discussed the Riemann–Liouville sense for global robust synchronization of fractional-order complex-valued neural networks and the mixed delays and impulses were considered. Based on the switching feature of memristor, the adaptive switching controller was given to study the

synchronization of fractional-order memristor based BAM neural networks [17]. However, these papers were not based on FMNNs and did not consider the impact of uncertainties. For FMNNs, many results should be noted. Huang et al. [18] obtained synchronization criteria of FMNNs via the interval matrix method and given a new upper bound on the norm of interval matrix. The generalized finite-time stability and stabilization of FMNNs were studied in [19] and some stability criteria based on the LMI technique were given. However, [18,19] did not consider the parameter uncertainty and the influence of external disturbance on the system. The above results had made some progress in the research of deterministic FMNNs or FNNs. However, in fact, due to the errors of system modeling and the influence of parameters fluctuation, the system will produce many uncertain factors. In addition, because of the change of the external environment, the system is often affected by external disturbances. Unfortunately, there are only a few studies on uncertain FMNNs [20]. Due to the existence of uncertainty, the stability of the system will be affected, and the system will produce chaos or even oscillation phenomenon. Therefore, it is necessary to adopt effective control methods to restrain the influence of uncertainty.

Sliding mode control (SMC) and disturbance observer are effective methods to deal with the uncertainty in the system, where [21,22] proposed the stability of quadcopter dynamics model by utilizing the disturbance observer-based altitude controller and this control scheme also improved reliability and accuracy of the landing mission. However, the disturbance observer method used in [21,22] made the system errors converge to a small neighborhood of the origin, but does not make the error converge to 0. Different from them, when there is disturbance in the system, our ATSMC can ensure that the system errors converge to 0 in finite time. Recently, the SMC is used in many studies to realize synchronization of systems with disturbances [23,24]. Zhao et al. [24] used the observer-based SMC to solve the synchronization problem of chaotic system with disturbance and gave the synchronization criteria. However, the upper bounds of disturbances mentioned in [23,24] needed be known in advance. If the disturbance without definite upper bound, the SMC cannot effectively suppress the disturbance. In this case, adaptive sliding mode control (ASMC) is a feasible control technology [25–27]. We know that traditional SMC implements asymptotic synchronization of the system, that is to say, the synchronization time is very long or even infinite. In practice, the FTS of the system is desirable, which can not only improve the response speed of the systems but also prevent the controller from failure due to long time working. For instance, Pratap et al. [28] studied the FTS of FMNNs and Pratap et al. [29] studied the finite-time Mittag-Leffler stability of fractional-order quaternion-valued memristive neural networks. Nonetheless, when the uncertainty exists, the control method of [28,29] cannot achieve the FTS of the system. It is worth mentioning that terminal sliding mode control (TSMC) can help systems attain FTS and Aghababa et al. [30] designed a new fractal-order integral sliding mode surface and investigated the FTS of the system. In addition, the adaptive terminal sliding mode control (ATSMC) was used by [31] to estimate the disturbance in fractional chaotic system and some FTS criteria are acquired. In fact, it is not hard to see that most of the studies using TSMC and ATSMC were about chaotic systems, but there were very few about FMNNs, let alone uncertain FMNNs.

For FMNNs, its synchronization had been mentioned in many studies [18,32]. In addition, there is no denying that discrete delay and leakage delay [32] often exist in the process of neurons transmitting information, so the effect of time delays in the study of FMNNs synchronization is indispensable. In addition, MNNs have irregular time evolution and it reflects the characteristics of real random process, so its synchronization widely used in secure communication. Liu et al. [33] applied anti-synchronization of MNNs with probabilistic mixed time-varying delays to signal encryption. In [34,35], a new quantized event-triggered security algorithm based on chaotic NNs synchronization was proposed, and the image encryption and decryption are successfully realized based on this algorithm. However, according to the above discussion, there are few results about the application

of FMNNs synchronization in secure communication. Therefore, we applied FMNNs synchronization to secure communication.

Inspired by the above analysis, we propose the fractal-order adaptive terminal sliding mode control (FATSMC) strategy and study the FTS problem of FMNNs. In addition, a secure communication scheme is presented, and the signal security is realized by means of FMNNs synchronization. The major contributions are as follows.

- (1) In this paper, the influences of discrete delay, leakage delay, unknown parameters and external disturbances on FMNNs synchronization are considered. Therefore, compared with the previous studies that only consider a single delay or assume that the system is a deterministic system, the theoretical results obtained by us are more comprehensive.
- (2) A FATSMC scheme is proposed to deal with the disturbances with unknown upper bound. With this control scheme, the FTS of uncertain FMNNs is realized. In addition, the corresponding synchronization criteria and the explicit expression of ST are given.
- (3) FMNNs show satisfactory chaos, which makes them have unique advantages in signals masking. Based on the knowledge of secure communication, a signal encryption and decryption scheme is designed, and the FTS of FMNNs is successfully applied to signal encryption.

2. Preliminaries and Model Description

2.1. Preliminaries

To ensure the smooth derivation of the theory, some definitions and lemmas are given below.

Definition 1 ([36]). The α th-order Riemann-Liouville fractional integral of the integrable function $\ell(t)$ is denoted as

$${}_{t_0}I_t^\alpha \ell(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-l)^{\alpha-1} \ell(l) dl$$

where $t \geq t_0, \alpha > 0$. And gamma function $\Gamma(\cdot)$ is denoted as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Definition 2 ([36]). The α th-order Caputo derivative of the function $\ell \in C^b([t_0, +\infty), \mathbb{R})$ is expressed as

$${}_{t_0}D_t^\alpha \ell(t) = \begin{cases} \frac{1}{\Gamma(b-\alpha)} \int_{t_0}^t \frac{\ell^{(b)}(l)}{(t-l)^{\alpha-b+1}} dl, & b-1 < \alpha < b \\ \frac{d^b \ell(t)}{dt^b}, & \alpha = b \end{cases}$$

$$\text{In particular, when } 0 < \alpha < 1, {}_{t_0}D_t^\alpha \ell(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{\ell'(l)}{(t-l)^\alpha} dl.$$

Due to the physical interpretation of its initial conditions, the Caputo derivative is used by us, and for simplicity, $D^\alpha \ell(t)$ is adopted to represent ${}_{t_0}D_t^\alpha \ell(t)$.

Some properties of Caputo fractional derivative should be noted here.

Property 1. The linear condition for Caputo fractional derivative is described as

$$D_t^\alpha [\Theta_1 \ell(t) + \Theta_2 f(t)] = \Theta_1 D_t^\alpha \ell(t) + \Theta_2 D_t^\alpha f(t)$$

where Θ_1, Θ_2 are any constants.

Property 2. The product of Caputo fractional derivative satisfies

$$D_t^\alpha (D_t^\beta \ell(t)) = D_t^{\alpha+\beta} \ell(t)$$

where $\alpha > 0, \beta > 0$ and $\alpha + \beta < 1$. In particular, if $\beta = -\alpha$, $D_t^\alpha (D_t^{-\alpha} \ell(t)) = D_t^0 \ell(t) = \ell(t)$.

Lemma 1 ([37]). If $l = 0$ is the equilibrium point of system $D^\alpha l(t) = \aleph(l, t)$, where $\alpha \in (0, 1)$ and $\aleph(l, t)$ is Lipschitz continuous with Lipschitz coefficient $\hbar > 0$. The equilibrium of the above system is asymptotically stable, if a suitable function $V(t, l(t))$ is constructed for the above system and satisfies the following results

$$\lambda_1 \|l\|^{\tilde{\lambda}} \leq V(t, l(t)) \leq \lambda_2 \|l\|, \dot{V}(t, l(t)) \leq \lambda_3 \|l\|$$

where $\lambda_1, \lambda_2, \lambda_3$ and $\tilde{\lambda}$ are positive constants, $\|\cdot\|$ is the optional norm.

Lemma 2 ([38]). If $f_j(\pm R_j) = 0, g_j(\pm R_j) = 0, j = 1, \dots, n$, then

$$\begin{aligned} |K[p_{ij}(y_j(t))]f_j(y_j(t)) - K[p_{ij}(x_j(t))]f_j(x_j(t))| &\leq \bar{p}_{ij}k_j|y_j(t) - x_j(t)|, \\ |K[q_{ij}(y_j(t - \omega(t)))g_j(y_j(t - \omega(t))) - K[q_{ij}(x_j(t - \omega(t)))g_j(x_j(t - \omega(t)))]| &\leq \bar{q}_{ij}z_j|y_j(t - \omega(t)) - x_j(t - \omega(t))| \end{aligned}$$

where $\bar{p}_{ij} = \max\{|p'_{ij}|, |p''_{ij}|\}$, $\bar{q}_{ij} = \max\{|q'_{ij}|, |q''_{ij}|\}$, k_j and z_j are positive constants.

2.2. Model Description

The master system of FMNNs is denoted as

$$\begin{aligned} D^\alpha x_i(t) = & -r_i x_i(t - \lambda) + \sum_{j=1}^n (p_{ij}(x_j(t)) + \Delta p_{ij}(t))f_j(x_j(t)) \\ & + \sum_{j=1}^n (q_{ij}(x_j(t - \omega(t))) + \Delta q_{ij}(t))g_j(x_j(t - \omega(t))) + I_i \end{aligned} \quad (1)$$

where $i, j = 1, 2, 3, \dots, n$, $\alpha \in (0, 1)$ and the state of i th neuron is represented by $x_i(t) \in \mathbb{R}^n$. r_i represents self-regulation coefficient and $r_i > 0$. $\Delta p_{ij}(t)$ and $\Delta q_{ij}(t)$ stand for unknown parameters. $\lambda > 0$ and $\omega(t)$ denote leakage delay and discrete delay respectively, where $0 \leq \omega(t) \leq \tau$. $f_j(\cdot)$ and $g_j(\cdot)$ are activation functions and they satisfy the Lipschitz condition. I_i means external input. $x_i(s) = \phi_i(s) \in C([-\omega, 0], \mathbb{R}^n)$ stands for the initial value of (1) and $\omega = \max\{\lambda, \tau\}$. The memristive connection weights $p_{ij}(\cdot)$ and $q_{ij}(\cdot)$ are described as $p_{ij}(x_j(t)) = \frac{A_{ij}}{Q_i} \times \bar{p}_{ij}$, $q_{ij}(x_j(t - \omega(t))) = \frac{B_{ij}}{Q_i} \times \bar{p}_{ij}$, where A_{ij}, B_{ij} are memductances of voltage-controlled memristors M_{ij}, N_{ij} respectively. $\bar{p}_{ij} = 1$ when $i = j$ and $\bar{p}_{ij} = -1$ when $i \neq j$. M_{ij} is memristor between $f_j(x_j(t))$ and $x_i(t)$, N_{ij} is memristor between $g_j(x_j(t - \omega(t)))$ and $x_i(t)$. Q_i means fractional capacitor and its voltage is $x_i(t)$.

Due to the characteristics of the voltage threshold type memristor, the memristor connection weights are expressed as

$$p_{ij}(x_j(t)) = \begin{cases} p'_{ij}, & |x_j(t)| \leq R_j, \\ p''_{ij}, & |x_j(t)| > R_j, \end{cases} \quad q_{ij}(x_j(t - \omega(t))) = \begin{cases} q'_{ij}, & |x_j(t - \omega(t))| \leq R_j, \\ q''_{ij}, & |x_j(t - \omega(t))| > R_j, \end{cases}$$

where $p'_{ij}, p''_{ij}, q'_{ij}, q''_{ij}$ are constants and $R_j > 0$ indicates switching jump. Due to the FMNNs are discontinuous with right-hand sides, the classical definition of solution for differential equation is no longer applicable, so the solution of (1) needs to be considered in the mathematical framework of Filippov solutions [39]. The set-valued mapping for (1) is expressed as

$$K[p_{ij}(x_j(t))] = \begin{cases} p'_{ij}, & |x_j(t)| < R_j, \\ co\{p'_{ij}, p''_{ij}\}, & |x_j(t)| = R_j, \\ p''_{ij}, & |x_j(t)| > R_j, \end{cases}$$

$$K[q_{ij}(x_j(t - \omega(t)))] = \begin{cases} q'_{ij}, & |x_j(t - \omega(t))| \leq R_j, \\ co\{q'_{ij}, q''_{ij}\}, & |x_j(t - \omega(t))| = R_j, \\ q''_{ij}, & |x_j(t - \omega(t))| > R_j, \end{cases}$$

where co means convex closure of a set, and we define $\tilde{p}_{ij} = \max\{p'_{ij}, p''_{ij}\}$, $\hat{p}_{ij} = \min\{p'_{ij}, p''_{ij}\}$, $\tilde{q}_{ij} = \max\{q'_{ij}, q''_{ij}\}$, $\hat{q}_{ij} = \min\{q'_{ij}, q''_{ij}\}$, $\bar{p}_{ij} = \max\{|p'_{ij}|, |p''_{ij}|\}$, $\bar{q}_{ij} = \max\{|q'_{ij}|, |q''_{ij}|\}$. With the aid of differential inclusion theory, (1) can be expressed to

$$D^\alpha x_i(t) \in -r_i x_i(t - \lambda) + \sum_{j=1}^n (K[p_{ij}(x_j(t))] + \Delta p_{ij}(t)) f_j(x_j(t)) + \sum_{j=1}^n (K[q_{ij}(x_j(t - \omega(t)))] + \Delta q_{ij}(t)) g_j(x_j(t - \omega(t))) + I_i \quad (2)$$

On the strength of the measurable selection theorem, there exist $\tilde{p}_{ij}(x_j(t)) \in K[p_{ij}(x_j(t))]$, $\tilde{q}_{ij}(x_j(t - \omega(t))) \in K[q_{ij}(x_j(t - \omega(t)))]$. Accordingly, the (2) is equivalent to

$$D^\alpha x_i(t) = -r_i x_i(t - \lambda) + \sum_{j=1}^n (\tilde{p}_{ij}(x_j(t)) + \Delta p_{ij}(t)) f_j(x_j(t)) + \sum_{j=1}^n (\tilde{q}_{ij}(x_j(t - \omega(t))) + \Delta q_{ij}(t)) g_j(x_j(t - \omega(t))) + I_i \quad (3)$$

For slave system of FMNNs, it is denoted as

$$D^\alpha y_i(t) = -r_i y_i(t - \lambda) + \sum_{j=1}^n (p_{ij}(y_j(t)) + \Delta p_{ij}(t)) f_j(y_j(t)) + \sum_{j=1}^n (q_{ij}(y_j(t - \omega(t))) + \Delta q_{ij}(t)) g_j(y_j(t - \omega(t))) + I_i + d_i(t) + u_i(t) \quad (4)$$

where $d_i(t)$ is external disturbance and $u_i(t)$ is controller. Under the identical theoretical analysis method as system (1), the following form of (4) is obtained

$$D^\alpha y_i(t) = -r_i y_i(t - \lambda) + \sum_{j=1}^n (\tilde{p}_{ij}(y_j(t)) + \Delta p_{ij}(t)) f_j(y_j(t)) + \sum_{j=1}^n (\tilde{q}_{ij}(y_j(t - \omega(t))) + \Delta q_{ij}(t)) g_j(y_j(t - \omega(t))) + I_i + d_i(t) + u_i(t) \quad (5)$$

The $e_i(t) = y_i(t) - x_i(t)$ delegates synchronization error and it is calculated as

$$D^\alpha e_i(t) = -r_i e_i(t - \lambda) + \sum_{j=1}^n [\tilde{p}_{ij}(y_j(t)) f_j(y_j(t)) - \tilde{p}_{ij}(x_j(t)) f_j(x_j(t))] + \sum_{j=1}^n [\tilde{q}_{ij}(y_j(t - \omega(t))) g_j(y_j(t - \omega(t))) - \tilde{q}_{ij}(x_j(t - \omega(t))) g_j(x_j(t - \omega(t)))] + \sum_{j=1}^n \Delta p_{ij}(t) (f_j(y_j(t)) - f_j(x_j(t))) + \sum_{j=1}^n \Delta q_{ij}(t) (g_j(y_j(t - \omega(t))) - g_j(x_j(t - \omega(t)))) + d_i(t) + u_i(t) \quad (6)$$

Assumption 1. The uncertainties $\Delta p_{ij}(t)$, $\Delta q_{ij}(t)$ are bounded and satisfy $\Delta p_{ij}(t) \leq \hbar_{ij}$, $\Delta q_{ij}(t) \leq h_{ij}$, where $\hbar_{ij}$ and h_{ij} are positive constants.

Assumption 2. Suppose the disturbance $d_i(t)$ satisfies $d_i(t) \leq \mathfrak{S}_i |s_i(t)|$, where $\mathfrak{S}_i$ is adaptive unknown parameter.

Our main goal is to realize FTS of uncertain FMNNs by using FATSMC. In addition, some synchronization criteria are derived and the explicit expression of ST is given after calculation.

3. Main Results

Consider the following sliding surface

$$s_i(t) = D^{\alpha-1} e_i(t) + D^{\alpha-2} \left(\zeta_i e_i(t) + \delta_i \text{sign}(e_i(t)) |e_i(t)|^\xi \right) \quad (7)$$

where $i = 1, 2, \dots, n$, $0 < \xi < 1$, ζ_i and δ_i are positive constants. According to the relevant principle of SMC, one has

$$s_i(t) = 0, \dot{s}_i(t) = 0 \quad (8)$$

Take the derivative of (7), one obtains

$$\dot{s}_i(t) = D^\alpha e_i(t) + D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \text{sign}(e_i(t)) |e_i(t)|^\xi \right) \quad (9)$$

Combine (8) and (9), one obtains

$$D^\alpha e_i(t) = -D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \text{sign}(e_i(t)) |e_i(t)|^\xi \right) \quad (10)$$

The following FATSMC is proposed to guarantee the synchronization error to reach the sliding surface in finite time. The following FATSMC is given.

$$\begin{aligned} u_i(t) = & -D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \text{sign}(e_i(t)) |e_i(t)|^\xi \right) - \text{sign}(s_i(t)) (\tilde{v}_i |e_i(t-\lambda)| \\ & + \rho_i |e_i(t)| + \varsigma_i \sum_{j=1}^n |e_j(t-\omega(t))| + \eta_i |s_i(t)|^\varepsilon) - s_i(t) (\vartheta_i(t) + \theta_i) \end{aligned} \quad (11)$$

where $0 < \varepsilon < 1$ and $\tilde{v}_i, \rho_i, \varsigma_i, \eta_i, \theta_i$ are positive constants. The adaptive law of $\vartheta_i(t)$ is $\dot{\vartheta}_i(t) = \beta |s_i(t)|$ and β is a constant.

Theorem 1. Assume above assumptions and following inequalities hold, the error system (6) with control tactic (11) will reach sliding surface in finite-time t_c

$$\begin{cases} \tilde{v}_i \geq r_i \\ \rho_i \geq \sum_{j=1}^n (\bar{p}_{ij} + \bar{h}_{ij}) k_j \\ \varsigma_i \geq \sum_{j=1}^n (\bar{q}_{ij} + h_{ij}) z_j \end{cases} \quad (12)$$

where $t_c \leq \frac{1}{w(1-\varepsilon)} \|s(0)\|_1^{1-\varepsilon} - \frac{\beta \|s(0)\|_1^{2-\varepsilon}}{w(3-\varepsilon)}$, $w = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$.

Proof. Provide the Lyapunov function as

$$V_1(t) = \sum_{i=1}^n |s_i(t)| + \sum_{i=1}^n \frac{1}{2\beta} (\vartheta_i(t) - \mathfrak{S}_i)^2 \quad (13)$$

Differentiating (13), it obtains

$$\begin{aligned}
 \dot{V}_1(t) &= \sum_{i=1}^n \operatorname{sign}(s_i(t)) \dot{s}_i(t) + \sum_{i=1}^n (\vartheta_i(t) - \mathfrak{S}_i) |s_i(t)| \\
 &= \sum_{i=1}^n \operatorname{sign}(s_i(t)) \left(D^\alpha e_i(t) + D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \operatorname{sign}(e_i(t)) |e_i(t)|^\xi \right) \right) \\
 &\quad + \sum_{i=1}^n (\vartheta_i(t) - \mathfrak{S}_i) |s_i(t)| \\
 &= \sum_{i=1}^n \operatorname{sign}(s_i(t)) \left\{ -r_i e_i(t - \lambda) + \sum_{j=1}^n \left[\vec{p}_{ij}(y_j(t)) f_j(y_j(t)) - \vec{p}_{ij}(x_j(t)) f_j(x_j(t)) \right] \right. \\
 &\quad + \sum_{j=1}^n \left[\vec{q}_{ij}(y_j(t - \omega(t))) g_j(y_j(t - \omega(t))) - \vec{q}_{ij}(x_j(t - \omega(t))) g_j(x_j(t - \omega(t))) \right] \\
 &\quad + \sum_{j=1}^n \Delta p_{ij}(t) (f_j(y_j(t)) - f_j(x_j(t))) + d_i(t) + u_i(t) + \sum_{j=1}^n \Delta q_{ij}(t) (g_j(y_j(t - \omega(t))) \\
 &\quad \left. - g_j(x_j(t - \omega(t)))) + D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \operatorname{sign}(e_i(t)) |e_i(t)|^\xi \right) \right\} \\
 &\quad + \sum_{i=1}^n (\vartheta_i(t) - \mathfrak{S}_i) |s_i(t)|
 \end{aligned} \tag{14}$$

Based on Lemma 2, it has

$$\begin{aligned}
 \sum_{j=1}^n \left[\vec{p}_{ij}(y_j(t)) f_j(y_j(t)) - \vec{p}_{ij}(x_j(t)) f_j(x_j(t)) \right] &\leq \sum_{j=1}^n \bar{p}_{ij} k_j |e_j(t)|, \\
 \sum_{j=1}^n \left[\vec{q}_{ij}(y_j(t - \omega(t))) g_j(y_j(t - \omega(t))) - \vec{q}_{ij}(x_j(t - \omega(t))) g_j(x_j(t - \omega(t))) \right] \\
 &\leq \sum_{j=1}^n \bar{q}_{ij} z_j |e_j(t - \omega(t))|
 \end{aligned} \tag{15}$$

where k_j and z_j are positive Lipschitz constants.

Under Assumption 1, it acquires

$$\begin{aligned}
 \sum_{j=1}^n \Delta p_{ij}(t) (f_j(y_j(t)) - f_j(x_j(t))) &\leq \sum_{j=1}^n \bar{h}_{ij} k_j |e_j(t)|, \\
 \sum_{j=1}^n \Delta q_{ij}(t) (g_j(y_j(t - \omega(t))) - g_j(x_j(t - \omega(t)))) &\leq \sum_{j=1}^n \bar{h}_{ij} z_j |e_j(t - \omega(t))|
 \end{aligned} \tag{16}$$

Substituting (11), (15) and (16) into (14) and then based on Assumption 2, the (14) is transformed as

$$\begin{aligned}
 \dot{V}_1(t) &\leq \sum_{i=1}^n \{ (r_i - \tilde{v}_i) |e_i(t - \lambda)| + \left(\sum_{j=1}^n (\bar{p}_{ij} + \bar{h}_{ij}) k_j - \rho_i \right) |e_i(t)| \\
 &\quad + \left(\sum_{j=1}^n (\bar{q}_{ij} + \bar{h}_{ij}) z_j - \varsigma_i \right) |e_j(t - \omega(t))| + (\vartheta_i(t) - \mathfrak{S}_i) |s_i(t)| \\
 &\quad - \vartheta_i(t) |s_i(t)| + \mathfrak{S}_i |s_i(t)| - \theta_i |s_i(t)| - \eta_i |s_i(t)|^\varepsilon \}
 \end{aligned} \tag{17}$$

Taking (12) into (17), it yields

$$\begin{aligned}
 \dot{V}_1(t) &\leq - \sum_{i=1}^n \theta_i |s_i(t)| - \sum_{i=1}^n \eta_i |s_i(t)|^\varepsilon \\
 &\leq - \sum_{i=1}^n w |s_i(t)|
 \end{aligned} \tag{18}$$

In addition, (18) is also equivalent to

$$\dot{V}_1(t) \leq -w \|s(t)\|_1 \tag{19}$$

where $w = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$. Then, finite-time t_c will be calculated in the following.

Taking time-derivative of (14) acquires

$$\dot{V}_1(t) = \frac{d\|s(t)\|_1}{dt} + \frac{\sum_{i=1}^n \frac{1}{2\beta} d\kappa^2}{dt} \quad (20)$$

where $\kappa = (\vartheta_i(t) - \mathfrak{S}_i)$.

In terms of to (18) and (19), it yields

$$\dot{V}_1(t) \leq -w \sum_{i=1}^n |s_i(t)|^\varepsilon \quad (21)$$

From (20) and (21), one achieves

$$\frac{d\|s(t)\|_1}{dt} + \frac{\sum_{i=1}^n \frac{1}{2\beta} d\kappa^2}{dt} \leq -w \sum_{i=1}^n |s_i(t)|^\varepsilon \quad (22)$$

and

$$dt \leq -\frac{d\|s(t)\|_1}{w\|s(t)\|_1^\varepsilon} - \frac{\sum_{i=1}^n \frac{1}{2\beta} d\kappa^2}{w\|s(t)\|_1^\varepsilon} \quad (23)$$

Take further calculation of (23) with adaptive law, it achieves

$$dt \leq -\frac{d\|s(t)\|_1}{w\|s(t)\|_1^\varepsilon} - \frac{\beta \sum_{i=1}^n |s_i(t)| \int_0^{t_c} |s_i(t)| dt}{w \|s(t)\|_1^\varepsilon} \quad (24)$$

After calculation, we have

$$dt \leq -\frac{d\|s(t)\|_1}{w\|s(t)\|_1^\varepsilon} - \frac{\beta}{w} \int_0^{t_c} \frac{\|s(t)\|_1^2}{\|s(t)\|_1^\varepsilon} dt \quad (25)$$

Rewrite (25) as follows

$$dt \leq -\frac{1}{w(1-\varepsilon)} d\|s(t)\|_1^{1-\varepsilon} - \frac{\beta}{w} \frac{1}{d\|s(t)\|_1} \int_0^{t_c} \|s(t)\|_1^{2-\varepsilon} d\|s(t)\|_1 dt \quad (26)$$

which is equivalent to

$$dt \leq -\frac{1}{w(1-\varepsilon)} d\|s(t)\|_1^{1-\varepsilon} - \frac{\beta}{w} \frac{1}{3-\varepsilon} \|s(t)\|_1^{3-\varepsilon} \Big|_0^{t_c} d\|s(t)\|_1^{-1} \quad (27)$$

and then

$$dt \leq -\frac{1}{w(1-\varepsilon)} d\|s(t)\|_1^{1-\varepsilon} + \frac{\beta}{w} \frac{1}{3-\varepsilon} \|s(0)\|_1^{3-\varepsilon} d\|s(t)\|_1^{-1} \quad (28)$$

Integrating both sides of (28) from 0 to t_c , thus

$$\int_0^{t_c} dt \leq -\frac{1}{w(1-\varepsilon)} \int_{s(0)}^0 d\|s(t)\|_1^{1-\varepsilon} + \frac{\beta \|s(0)\|_1^{3-\varepsilon}}{w(3-\varepsilon)} \int_{s(0)}^0 d\|s(t)\|_1^{-1} \quad (29)$$

Hence, the finite-time t_c is calculated as

$$t_c \leq \frac{1}{w(1-\varepsilon)} \|s(0)\|_1^{1-\varepsilon} - \frac{\beta \|s(0)\|_1^{2-\varepsilon}}{w(3-\varepsilon)} \quad (30)$$

To guarantee $t_c > 0$, the β satisfies

$$\beta \leq \frac{(3-\varepsilon)\|s(0)\|_1^{-1}}{(1-\varepsilon)} \quad (31)$$

where $0 < \varepsilon < 1$. $\square$

Remark 1. Due to the existence of memristor, FMNNs are right-hand discontinuous systems that depend on the states for switching. To deal with this discontinuity, the set-valued mapping and differential inclusion theory are used. In addition, leakage time delay, discrete time delay, unknown parameters and external disturbances with unknown upper bounds are all considered in the FMNNs. In fact, when these factors exist in the system, it is challenging to realize the FTS of FMNNs. However, these difficulties are overcome and the FTS of FMNNs can be achieved under the proposed FATSMC by us.

Remark 2. ASMC and disturbance observer technique are often used to deal with external disturbance. In [21,22], the uniformly ultimately bounded of the quadcopter was achieved by using the disturbance observer technique, but they assumed that the disturbance satisfies $\|d\| \leq \omega$, $\|\dot{d}\| \leq \delta$, where d represents external disturbance and ω, δ are constants. In fact, when the upper bound of the disturbance is unknown, the method in the above literatures cannot deal with the disturbance. Different from [21,22], the disturbance in this paper does not need to satisfy such an assumption and its upper bound can be unknown. In addition, unlike the [21,22], which achieve the uniformly ultimately bounded of the system, our paper achieves the complete synchronization of the error system in finite time. Therefore, the results obtained in this paper are less conservative.

Remark 3. In the above, FATSMC is used to verify that the sliding surface can be reached in finite time for the error system. Notice that the t_c should be bigger than 0, so the constant β in the adaptive law needs to satisfy $\beta \leq \frac{(3-\varepsilon)\|s(0)\|_1^{-1}}{(1-\varepsilon)}$. Next, we will further study the finite-time stability of the sliding surface (7).

Theorem 2. The sliding mode dynamics (10) is asymptotically stable and its trajectory converges to $e(t) = 0$ in finite-time t_d , where

$$t_d = \frac{1}{\chi(1-\xi)} \ln \left(\|e(t_c)\|_1^{1-\xi} + 1 \right) + t_c$$

where t_c is initial time, $0 < \xi < 1$ and $\chi = \min\{\zeta_i, \delta_i\}$.

Proof. Given the following Lyapunov function

$$V_2(t) = \|e(t)\|_1 = \sum_{i=1}^n |e_i(t)| \quad (32)$$

It is easy to know

$$\begin{aligned} \dot{V}_2(t) &= \sum_{i=1}^n \text{sign}(e_i(t)) \dot{e}_i(t) \\ &= \sum_{i=1}^n \text{sign}(e_i(t)) (D^{1-\alpha} (D^\alpha e_i(t))) \\ &= - \sum_{i=1}^n \left(\zeta_i |e_i(t)| + \delta_i |e_i(t)|^\xi \right) \end{aligned} \quad (33)$$

Select $\chi = \min\{\zeta_i, \delta_i\}$ and it gets

$$\begin{aligned}\dot{V}_2(t) &\leq -\chi \sum_{i=1}^n \left(|e_i(t)| + |e_i(t)|^\zeta \right) \\ &\leq -\chi \sum_{i=1}^n |e_i(t)|\end{aligned}\quad (34)$$

In view of lemma 1, the sliding mode dynamics (10) achieves asymptotic stability. Hence, (1) and (4) attain asymptotic synchronization. Next, the finite-time t_d will be acquired.

From (32) and (34), it gets

$$\dot{V}_2(t) = \frac{d\|e(t)\|_1}{dt} \leq -\chi \left(\|e(t)\|_1 + \|e(t)\|_1^\zeta \right) \quad (35)$$

After transformation, we get the following form

$$dt \leq -\frac{d\|e(t)\|_1}{\chi \left(\|e(t)\|_1 + \|e(t)\|_1^\zeta \right)} = -\frac{\|e(t)\|_1^{-\zeta} d\|e(t)\|_1}{\chi \left(\|e(t)\|_1^{1-\zeta} + 1 \right)} \quad (36)$$

In addition, one obtains

$$dt \leq -\frac{1}{\chi(1-\zeta)} \frac{d\|e(t)\|_1^{1-\zeta}}{\left(\|e(t)\|_1^{1-\zeta} + 1 \right)} \quad (37)$$

For (37), its integral from t_c to t_d with $e(t_d) = 0$ can be calculated as

$$\begin{aligned}t_d - t_c &\leq -\frac{1}{\chi(1-\zeta)} \int_{e(t_c)}^{e(t_d)} \frac{d\|e(t)\|_1^{1-\zeta}}{\|e(t)\|_1^{1-\zeta} + 1} \\ &= -\frac{1}{\chi(1-\zeta)} \ln \left(\|e(t)\|_1^{1-\zeta} + 1 \right) \Big|_{e(t_c)}^{e(t_d)} \\ &= \frac{1}{\chi(1-\zeta)} \ln \left(\|e(t_c)\|_1^{1-\zeta} + 1 \right)\end{aligned}\quad (38)$$

Thus, the t_d can be obtained

$$t_d = \frac{1}{\chi(1-\zeta)} \ln \left(\|e(t_c)\|_1^{1-\zeta} + 1 \right) + t_c \quad (39)$$

In addition, let $t_e = \frac{1}{\chi(1-\zeta)} \ln \left(\|e(t_c)\|_1^{1-\zeta} + 1 \right)$. The t_e stands for the finite-time for the errors to converge to 0 on the sliding mode surface. $\square$

Remark 4. In order to obtain the explicit expression of the ST, we need to perform further calculations on the two stages of SMC. By means of formula transformation, integration and other related calculations, the explicit expression of the time t_c when the errors reach the sliding mode surface and the time t_e when the errors converge on the sliding mode surface are obtained. Therefore, the explicit expression of the ST is $t_d = t_c + t_e$ and which is the total time for the FMNNs to realize FTS under FATSMC. Based on this, the approximate time to realize the FTS of the error system can be estimated effectively.

Remark 5. The synchronization of FMNNs and the finite-time boundedness were addressed by [18] and [19] respectively. Compared with [18] and [19], a more general FMNNs model with time delays, unknown parameters and external disturbance is studied in this paper. Different from the asymptotical synchronization in [18], this paper implements FTS of FMNNs, which means that our synchronization time is faster. The finite-time boundedness of FMNNs with disturbance was considered in [19], but the disturbance in which was required to have known upper bound.

Unlike [19], the disturbance considered in this paper do not need to satisfy this assumption. In addition, both [18] and [19] used feedback control method to study the FMNNs. However, when the upper bound of the disturbance is unknown, this control method is invalid, while the FATSMC proposed in this paper can still achieve the FTS of the FMNNs. In addition, the explicit expression of the ST is given by us and which can be used to estimated the synchronization time. However, this point is not reflected in [18] and [19], that is to say, they cannot estimate the ST.

In above discussion, we study the disturbance with unknown upper bound and realize the FTS of FMNNs with the help of FATSMC. Furthermore, the FTS of FMNNs can also be realized by using FTSMC when the upper bound of disturbance is known or the uncertainty is not considered. For both cases, the following corollaries are given.

Corollary 1. Assume the external disturbance $d_i(t)$ satisfies $|d_i(t)| \leq \mu_i$. The error system (6) with following control strategy and conditions will reach to sliding surface (7) in finite-time t_c . Design the control strategy as

$$u_i(t) = -D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \text{sign}(e_i(t)) |e_i(t)|^\varepsilon \right) - \text{sign}(s_i(t)) (\tilde{v}_i |e_i(t-\lambda)| + \rho_i |e_i(t)| + \varsigma_i \sum_{j=1}^n |e_j(t-\omega(t))| + \eta_i |s_i(t)|^\varepsilon + v_i) - \theta_i s_i(t) \quad (40)$$

where $0 < \varepsilon < 1$, and $\tilde{v}_i, \rho_i, \varsigma_i, \eta_i, v_i, \theta_i$ are positive constants. In addition, the controller parameters satisfy

$$\begin{cases} \tilde{v}_i \geq r_i \\ \rho_i \geq \sum_{j=1}^n (\bar{p}_{ij} + \bar{h}_{ij}) k_j \\ \varsigma_i \geq \sum_{j=1}^n (\bar{q}_{ij} + h_{ij}) z_j \\ v_i \geq \mu_i \end{cases} \quad (41)$$

where $t_c \leq \frac{1}{\tilde{\theta}(1-\varepsilon)} \ln(\|s(0)\|_1^{1-\varepsilon} + 1)$, $\tilde{\theta} = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$.

Proof. Adopt following Lyapunov function

$$V_3(t) = \|s(t)\|_1 = \sum_{i=1}^n |s_i(t)| \quad (42)$$

Through calculation, it yields that

$$\begin{aligned} \dot{V}_3(t) &\leq - \sum_{i=1}^n \theta_i |s_i(t)| - \sum_{i=1}^n \eta_i |s_i(t)|^\varepsilon \\ &\leq - \sum_{i=1}^n \tilde{\theta} |s_i(t)| \end{aligned} \quad (43)$$

where $\tilde{\theta} = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$. And the (43) is equivalent to

$$\dot{V}_3(t) \leq -\tilde{\theta} \|s(t)\|_1 \quad (44)$$

Based on Lemma1 and (44), the error system (6) asymptotically reaches the sliding surface. In order to obtain the finite-time t_c about (6) reaches to sliding surface, we perform the following analysis.

With (42) and (43), it yields

$$\dot{V}_3(t) = \frac{d\|s(t)\|_1}{dt} \leq - \sum_{i=1}^n (\theta_i |s_i(t)| + \eta_i |s_i(t)|^\varepsilon) \quad (45)$$

Due to $\tilde{\theta} = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$, therefore

$$\frac{d\|s(t)\|_1}{dt} \leq -\tilde{\theta}(\|s(t)\|_1 + \|s(t)\|_1^\varepsilon) \quad (46)$$

By the formula transformation, it acquires

$$dt \leq -\frac{d\|s(t)\|_1}{\tilde{\theta}(\|s(t)\|_1 + \|s(t)\|_1^\varepsilon)} = -\frac{\|s(t)\|_1^{-\varepsilon} d\|s(t)\|_1}{\tilde{\theta}(\|s(t)\|_1^{1-\varepsilon} + 1)} \quad (47)$$

Equivalently,

$$dt \leq -\frac{1}{\tilde{\theta}(1-\varepsilon)} \frac{d\|s(t)\|_1^{1-\varepsilon}}{\|s(t)\|_1^{1-\varepsilon} + 1} \quad (48)$$

The integration of (48) from 0 to t_c with $s(t_c) = 0$ can be calculated as

$$\int_0^{t_c} dt \leq -\frac{1}{\tilde{\theta}(1-\varepsilon)} \int_{s(0)}^0 \frac{1}{\|s(t)\|_1^{1-\varepsilon} + 1} d\|s(t)\|_1^{1-\varepsilon} \quad (49)$$

After calculation, finite-time t_c is

$$t_c \leq \frac{1}{\tilde{\theta}(1-\varepsilon)} \ln(\|s(0)\|_1^{1-\varepsilon} + 1) \quad (50)$$

□

As a consequence, the sliding surface (7) can be reached in finite-time t_c for error system (6).

Corollary 2. *If uncertainties are not considered, the following deterministic master–slave system of FMNNs are obtained*

$$D^\alpha x_i(t) = -r_i x_i(t - \lambda) + \sum_{j=1}^n p_{ij}(x_j(t)) f_j(x_j(t)) + \sum_{j=1}^n q_{ij}(x_j(t - \omega(t))) g_j(x_j(t - \omega(t))) + I_i \quad (51)$$

$$D^\alpha y_i(t) = -r_i y_i(t - \lambda) + \sum_{j=1}^n p_{ij}(y_j(t)) f_j(y_j(t)) + \sum_{j=1}^n q_{ij}(y_j(t - \omega(t))) g_j(y_j(t - \omega(t))) + I_i + u_i(t) \quad (52)$$

Based on the following conditions and controller, (51) and (52) can be synchronized in finite-time t_m .

$$\begin{cases} \tilde{v}_i \geq r_i \\ \rho_i \geq \sum_{j=1}^n \tilde{p}_{ij} k_j \\ \varsigma_i \geq \sum_{j=1}^n \tilde{q}_{ij} z_j \end{cases} \quad (53)$$

and the control strategy is given as

$$u_i(t) = -D^{\alpha-1} \left(\zeta_i e_i(t) + \delta_i \operatorname{sign}(e_i(t)) |e_i(t)|^\xi \right) - \operatorname{sign}(s_i(t)) (\tilde{v}_i |e_i(t - \lambda)| + \rho_i |e_i(t)| + \varsigma_i \sum_{j=1}^n |e_j(t - \omega(t))| + \eta_i |s_i(t)|^\varepsilon) - \theta_i s_i(t) \quad (54)$$

where $0 < \xi < 1, 0 < \varepsilon < 1$, and $\zeta_i, \delta_i, \tilde{v}_i, \rho_i, \varsigma_i, \eta_i, \theta_i$ are positive constants. In addition, the finite-time t_m is

$$t_m \leq \frac{1}{\chi(1-\xi)} \ln \left(\left\| e \left(\frac{1}{\tilde{\theta}(1-\varepsilon)} \ln \left(\|s(0)\|_1^{1-\varepsilon} + 1 \right) \right) \right\|_1^{1-\xi} + 1 \right) + \frac{1}{\tilde{\theta}(1-\varepsilon)} \ln \left(\|s(0)\|_1^{1-\varepsilon} + 1 \right) \quad (55)$$

where $\tilde{\theta} = \min_{1 \leq i \leq n} \{\theta_i, \eta_i\}$, $\chi = \min\{\zeta_i, \delta_i\}$.

4. Simulation Example and Application Analysis

Example 1. In this example, the viability of the obtained theoretical results is illustrated. The master system of FMNNs with three neurons is described as

$$D^\alpha x_i(t) = -r_i x_i(t - \lambda) + \sum_{j=1}^n (p_{ij}(x_j(t)) + \Delta p_{ij}(t)) f_j(x_j(t)) + \sum_{j=1}^n (q_{ij}(x_j(t - \omega(t))) + \Delta q_{ij}(t)) g_j(x_j(t - \omega(t))) + I_i \quad (56)$$

where $\alpha = 0.98$, $n = 3$, $r_1 = r_2 = r_3 = 1$, $\lambda = 0.2$, $\omega(t) = 0.2 \sin(t)$, $\Delta p_{ij}(t) = 0.1 \sin(t)$, $\Delta q_{ij}(t) = 0.1 \cos(t)$, $I_i = 0$. The initial values of (60) are given as

$$x_0 = \begin{bmatrix} 1.4 & 0.4 & -0.3 \\ 0.35 & -1.1 & 1.3 \\ -0.25 & 1.2 & 0.6 \end{bmatrix}$$

The activation functions are set as

$$f_j(x_j(t)) = \tanh(x_j(t)), g_j(x_j(t - \omega(t))) = \frac{|x_j(t - \omega(t)) + 1| - |x_j(t - \omega(t)) - 1|}{2}$$

In addition, the memristor connection weights are set to

$$\begin{aligned} p_{11}(x_1(t)) &= \begin{cases} 1.3, & |x_1(t)| \leq 1 \\ 0.9, & |x_1(t)| > 1 \end{cases}, p_{21}(x_1(t)) = \begin{cases} 1.5, & |x_1(t)| \leq 1 \\ 1.2, & |x_1(t)| > 1 \end{cases}, \\ p_{31}(x_1(t)) &= \begin{cases} 1.3, & |x_1(t)| \leq 1 \\ 1.1, & |x_1(t)| > 1 \end{cases}, p_{12}(x_2(t)) = \begin{cases} 2.1, & |x_2(t)| \leq 1 \\ 2.7, & |x_2(t)| > 1 \end{cases}, \\ p_{22}(x_2(t)) &= \begin{cases} 1.7, & |x_2(t)| \leq 1 \\ 1.4, & |x_2(t)| > 1 \end{cases}, p_{32}(x_2(t)) = \begin{cases} 2.5, & |x_2(t)| \leq 1 \\ 3.2, & |x_2(t)| > 1 \end{cases}, \\ p_{13}(x_3(t)) &= \begin{cases} 3.0, & |x_3(t)| \leq 1 \\ 2.6, & |x_3(t)| > 1 \end{cases}, p_{23}(x_3(t)) = \begin{cases} -2.8, & |x_3(t)| \leq 1 \\ -1.2, & |x_3(t)| > 1 \end{cases}, \\ p_{33}(x_3(t)) &= \begin{cases} 3.7, & |x_3(t)| \leq 1 \\ 3.5, & |x_3(t)| > 1 \end{cases}, \end{aligned}$$

$$\begin{aligned} q_{11}(x_1(t - \omega(t))) &= \begin{cases} -1.6, & |x_1(t - \omega(t))| \leq 1 \\ -2.2, & |x_1(t - \omega(t))| > 1 \end{cases}, \\ q_{21}(x_1(t - \omega(t))) &= \begin{cases} -5.4, & |x_1(t - \omega(t))| \leq 1 \\ -5.7, & |x_1(t - \omega(t))| > 1 \end{cases}, \\ q_{31}(x_1(t - \omega(t))) &= \begin{cases} -2.8, & |x_1(t - \omega(t))| \leq 1 \\ -2.1, & |x_1(t - \omega(t))| > 1 \end{cases}, \\ q_{12}(x_2(t - \omega(t))) &= \begin{cases} -1.8, & |x_2(t - \omega(t))| \leq 1 \\ -0.9, & |x_2(t - \omega(t))| > 1 \end{cases}, \\ q_{22}(x_2(t - \omega(t))) &= \begin{cases} -1.4, & |x_2(t - \omega(t))| \leq 1 \\ -2.1, & |x_2(t - \omega(t))| > 1 \end{cases}, \\ q_{32}(x_2(t - \omega(t))) &= \begin{cases} -1.8, & |x_2(t - \omega(t))| \leq 1 \\ -2.0, & |x_2(t - \omega(t))| > 1 \end{cases}, \end{aligned}$$

$$\begin{aligned} q_{13}(x_3(t - \omega(t))) &= \begin{cases} -4.2, |x_3(t - \omega(t))| \leq 1 \\ -3.6, |x_3(t - \omega(t))| > 1 \end{cases}, \\ q_{23}(x_3(t - \omega(t))) &= \begin{cases} -2.9, |x_3(t - \omega(t))| \leq 1 \\ -2.3, |x_3(t - \omega(t))| > 1 \end{cases}, \\ q_{33}(x_3(t - \omega(t))) &= \begin{cases} -4.7, |x_3(t - \omega(t))| \leq 1 \\ -4.1, |x_3(t - \omega(t))| > 1 \end{cases} \end{aligned}$$

The slave system is described as

$$\begin{aligned} D^\alpha y_i(t) = & -r_i y_i(t - \lambda) + \sum_{j=1}^n (p_{ij}(y_j(t)) + \Delta p_{ij}(t)) f_j(y_j(t)) \\ & + \sum_{j=1}^n (q_{ij}(y_j(t - \omega(t))) + \Delta q_{ij}(t)) g_j(y_j(t - \omega(t))) + I_i + d_i(t) + u_i(t) \end{aligned} \quad (57)$$

where $d_i(t) = [0.2 \sin(t) \ 0.2 \cos(2t) \ 0.2 \cos(2t)]^T$ and $I_i = 0$. The initial values of (61) are

$$y_0 = \begin{bmatrix} -0.8 & 0.9 & -1.5 \\ 1.35 & 0.2 & 0.5 \\ 0.4 & -1 & 1.2 \end{bmatrix}$$

For the sliding surface, the relevant parameters are set as $\zeta_1 = \zeta_2 = \zeta_3 = 1, \delta_1 = \delta_2 = \delta_3 = 1, \xi_1 = \xi_2 = \xi_3 = 0.95$. In line accordance with Theorem 1, the correlation parameters of (11) are chosen as $\tilde{v}_1 = \tilde{v}_2 = \tilde{v}_3 = 1.2, \rho_1 = 7.3, \rho_2 = 6.3, \rho_3 = 8.5, \varsigma_1 = 8.5, \varsigma_2 = 11, \varsigma_3 = 9.8, \eta_1 = 1.5, \eta_2 = 1.5, \eta_3 = 2, \varepsilon = 0.95, v_1 = v_2 = v_3 = 0.2, \theta_1 = 0.5, \theta_2 = 1.5, \theta_3 = 2$. Based on the adaptive law of controller (11), the relevant parameter is selected as $\beta = 2$.

For master system (57), we set the $x_{1i}(t)$ as x-axis, $x_{2i}(t)$ as y-axis and $x_{3i}(t)$ as z-axis. Then the chaotic curves of states $x_{1i}(t)$, $x_{2i}(t)$ and $x_{3i}(t)$ are exhibited in Figures 1–3. As can be seen from the pictures, the system presents chaos. The existence of this chaos characteristic can realize the chaos masking of signals, which makes FMNNs have a good application in secure communication. The error curves of $e_{i1}(t)$, $e_{i2}(t)$ and $e_{i3}(t)$ under the controller (11) are shown in Figures 4–6. It can be clearly seen that under the action of controller (11), the errors converge to 0 in finite time. According to (30), the reaching time is calculated as $t_c \leq 35.2224$ s. Based on (39), the synchronization time of the errors is calculated as $t_d \leq 49.7905$ s. The constant β is calculated as $\beta \leq 11.5214$. On the basis of Figures 4–6, the synchronization time is $t \approx 0.6$ s, and it is far less than $t_d = 49.7905$ s.

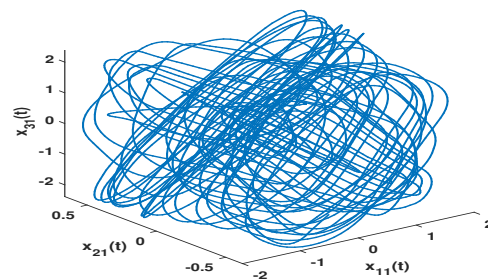


Figure 1. The chaotic trajectories of $x_{i1}(t)$.

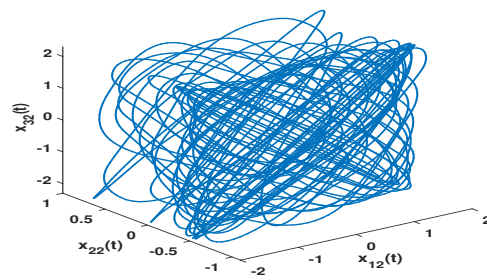


Figure 2. The chaotic trajectories of $x_{i2}(t)$.

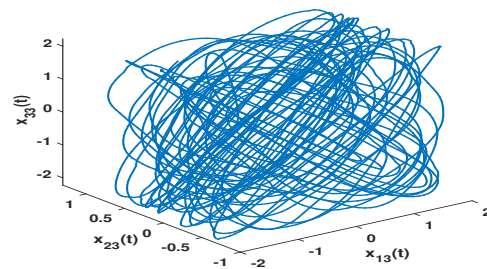


Figure 3. The chaotic trajectories of $x_{i3}(t)$.

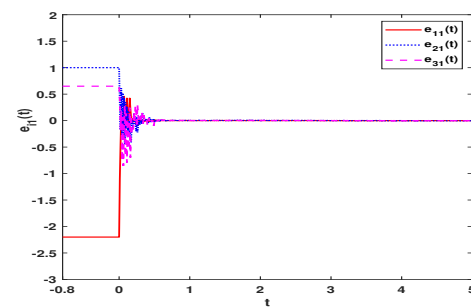


Figure 4. The synchronization errors $e_{i1}(t)$.

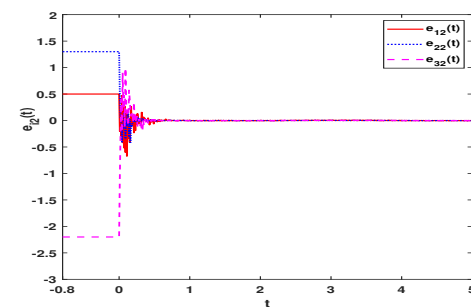


Figure 5. The synchronization errors $e_{i2}(t)$.

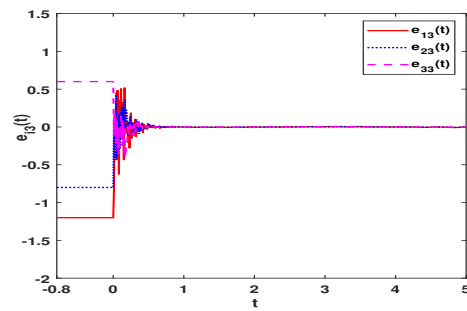


Figure 6. The synchronization errors $e_{i3}(t)$.

The adaptive laws $\vartheta_{i1}(t)$, $\vartheta_{i2}(t)$ and $\vartheta_{i3}(t)$ are demonstrated in Figures 7–9. The adaptive laws finally tend to be constants, which indicates that the FATSMC achieves the estimation of the external disturbances. As a consequence, the theoretical result in this paper is feasible.

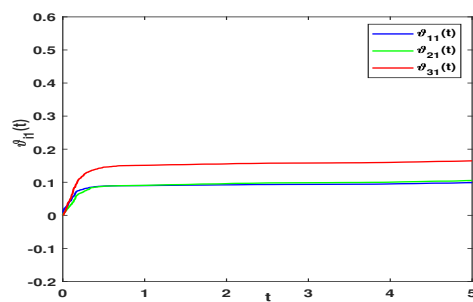


Figure 7. The adaptive laws $\vartheta_{i1}(t)$.

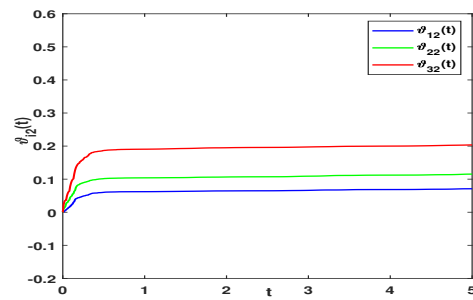


Figure 8. The adaptive laws $\vartheta_{i2}(t)$.

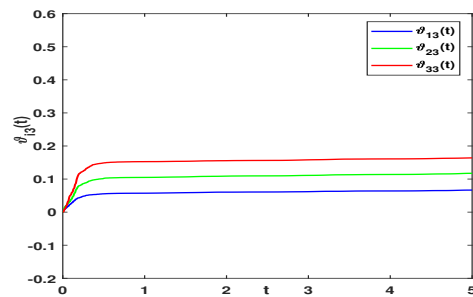


Figure 9. The adaptive laws $\vartheta_{i3}(t)$.

In order to illustrate the effectiveness of the proposed method in this paper, the proposed FTSMC in Corollary 1 is compared with the method in [18] when the upper

bound of the disturbance is known. On the basis of the model and related parameters in this paper, the comparison results are shown in Figures 10 and 11. As can be seen from the comparison figures between 0.5 s to 1.5 s, the proposed FTSMC in this paper can make the FMNNs realize synchronization faster than the method in [18]. Therefore, the proposed method in this paper has more advantages on the synchronization speed.

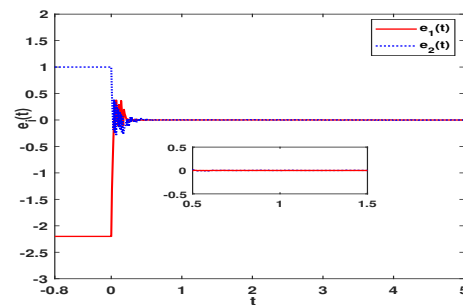


Figure 10. The synchronization errors $e_i(t)$ under the FTSMC.

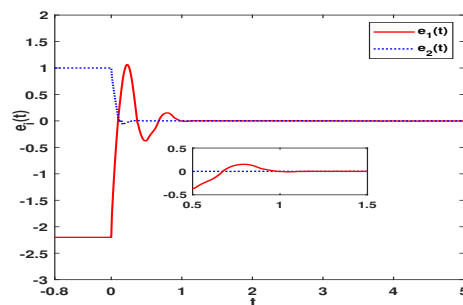


Figure 11. The synchronization errors $e_i(t)$ under the method in [18].

The Application on Signal Encryption. The secure communication of signals is divided into two processes: encryption and decryption. We consider the communication network as chaotic FMNNs. The master system is regarded as the sender in the secure communication, and the slave system is the receiver. The encryption process uses the chaotic characteristics of the master system to realize the chaos masking of the plaintext signals. In addition, the encrypted plaintext signals are called ciphertext. Then the sender sends the ciphertext to the receiver. Without knowing the secret keys, the ciphertext cannot be decrypted by a third party. The decryption process is that after receiving the ciphertext, the receiver uses the known decryption keys to decrypt it and then obtains the plaintext signals. By using the FTSMC in Theorem 1, the receiver can accurately track the sender within finite-time. Therefore, when the master–slave FTS of FMNNs are realized, the encryption and decryption of secure communication can be completed on the basis of known secret keys.

To explain the whole process of secure communication better, the signal masking method is used to encrypt plaintext signals here. Let $G_i(t)$ be the transmitted signal, in fact $G_i(t)$ is also the ciphertext signal. The $G_i(t) = x_i(t) + \gamma m_i(t)$ is generated by the sender and $m_i(t)$ is the plaintext signal. It should be noted that in order to avoid γ affecting the chaotic characteristics of the main system, the γ should be small enough. The ciphertext signals are sent by the sender and received by the receiver. With the aid of the FTSMC, master–slave synchronization is realized and plaintext signals are finally recovered. In addition, Figure 12 exhibits the secure communication policy. In above numerical examples, the FMNNs with three nodes are discussed, and the first node of the FMNNs is used for secure communication.

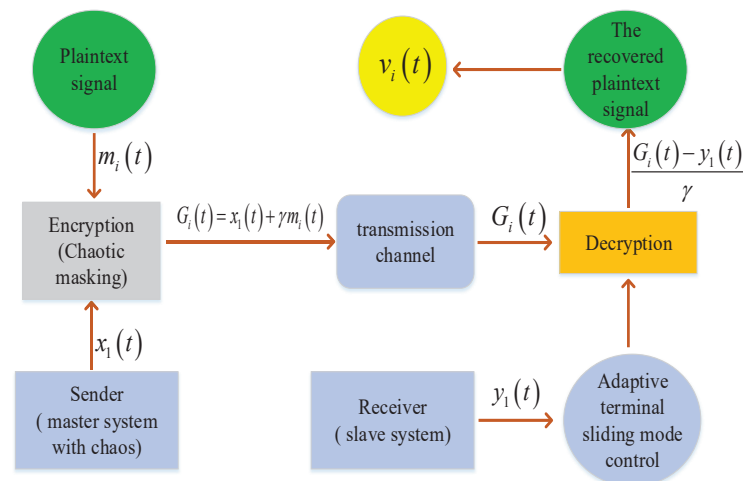


Figure 12. Secure communication scheme.

The plaintext signals are chosen as $m_1(t) = \sin(5t) + 0.5 \cos(0.3t)$, $m_2(t) = 2 \sin(7t) + 0.3 \cos(t)$, $m_3(t) = -1.6 \sin(3t) - 0.4 \cos \sin(0.5t)$. Without loss of generality, the secret keys are chosen as $r_i, \lambda, \omega(t), p_{ij}(x_j(t)), q_{ij}(x_j(t - \omega(t))), f_j(x_j(t)), g_j(x_j(t - \omega(t)))$. The chaotic masking signals $x_1(t)$ and the the plaintext signals $m_i(t)$ are shown in Figures 13 and 14. The ciphertext signals $G_i(t)$ are displayed in Figure 15. By comparison, it can be found that the curves of plaintext signals and ciphertext signals $G_i(t)$ are quite different, which also shows that the encryption scheme is effective.

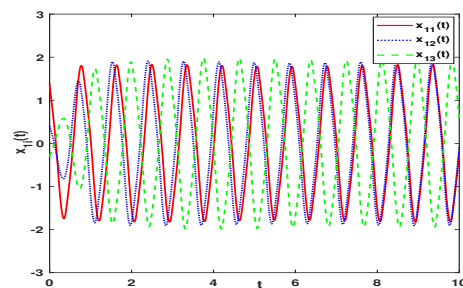


Figure 13. The chaotic masking signals $x_{1i}(t)$.

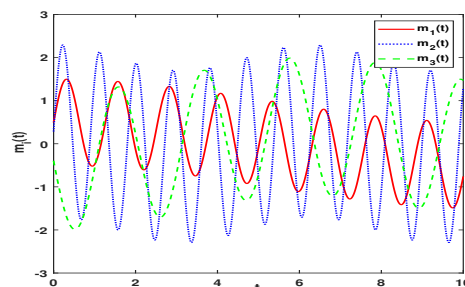


Figure 14. The plaintext signals $m_i(t)$.

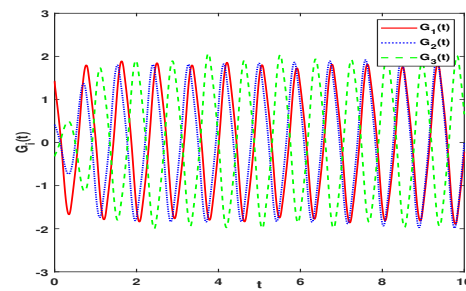


Figure 15. The transmitted signals $G_i(t)$.

When the receiver obtains the ciphertext signals, the plaintext signals $v_i(t)$ can be obtained by calculating $v_i(t) = \frac{G_i(t) - y_1(t)}{\gamma}$, where $\gamma = 0.05$. And the decrypted signals $v_i(t)$ are shown in Figure 16. It can be known that the plaintext signals are decrypted successfully. The errors between the recovered and the plaintext signals $\omega_i(t)$ are described in Figure 17, and the error eventually converge to 0. This indicates that the recovered signals are consistent with the plaintext signals, so the decryption process is effective.

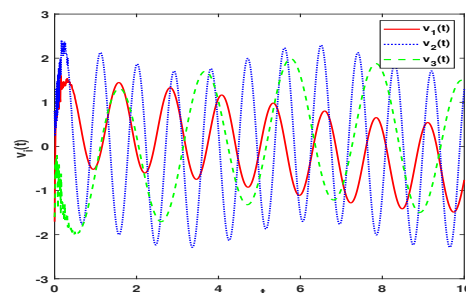


Figure 16. The recovered signals $v_i(t)$.

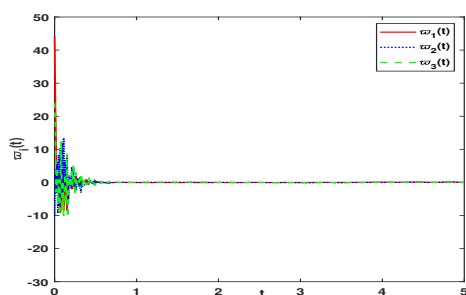


Figure 17. The error curves $\omega_i(t)$ between the recovered signals and plaintext signals.

5. Conclusions

The FTS problem of delayed FMNNs with unknown parameters and external disturbances is studied in this paper. In addition, both the influence of leakage and discrete delays on FMNNs are taken into consideration. On the basis of the proposed FATSMC, the synchronization criteria and the explicit expression for the settling time are given by calculation. Finally, the theoretical result of FMNNs synchronization is applied to signal encryption. It is undeniable that the transmission capacity of the channel is limited, when a large number of signals suddenly appear, it will cause the congestion of the channel. Quantized control can reduce the amount of transmission information and channel blocking, so as to realize the purpose of saving channel resources. In addition, the control cost is often a problem to be considered in the control process. Different from the continuous control, the event-triggered control makes the controller trigger when the trigger condition is met. This

control scheme can greatly save the control cost. Therefore, our future work is to study the quantized synchronization problem of MNNs based on event-triggered control.

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