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Positive Solutions for a System of Riemann–Liouville Type Fractional-Order Integral Boundary Value Problems

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Abstract: In this paper, we use the fixed-point index to establish positive solutions for a system of Riemann–Liouville type fractional-order integral boundary value problems. Some appropriate concave and convex functions are used to characterize coupling behaviors of our nonlinearities.

Keywords: Riemann–Liouville type fractional-order differential equations; integral boundary value problems; positive solutions; fixed-point index



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1. Introduction

In this paper, we use the fixed-point index to investigate the existence of positive solutions for the system of Riemann–Liouville type fractional-order integral boundary value problems:

$$\begin{cases} -D_{0+}^{\alpha}\varphi(t) = f(t, \varphi(t), \phi(t)), & 0 < t < 1, \\ -D_{0+}^{\alpha}\phi(t) = g(t, \varphi(t), \phi(t)), & 0 < t < 1, \\ \varphi(0) = \phi(0) = \varphi'(0) = \phi'(0) = 0, \quad \varphi(1) = \int_0^1 \varphi(t) d\beta(t), \quad \phi(1) = \int_0^1 \phi(t) d\beta(t), \end{cases} \quad (1)$$

where $\alpha \in (2, 3)$ is a real number, and f, g and β satisfy the conditions:

(H1) $f, g \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, and there exist $M > 0, Q_f$ and $Q_g \in C([0, 1], \mathbb{R}^+)$ such that

$$f(t, x, y) \geq -Mx - Q_f(t), \quad g(t, x, y) \geq -My - Q_g(t), \quad \forall t \in [0, 1], x, y \in \mathbb{R}^+.$$

Define a function

$$h(t) = \sum_{k=0}^{+\infty} \frac{(k\alpha + \alpha - 2)(k\alpha + \alpha - 3)t^k}{\Gamma(k\alpha + \alpha)}.$$

As in [1], there exists a unique number $M^* > 0$ such that $h(M^*) = 0$. Then, M in (H1) satisfies the condition:

(H2) $M \in (0, M^*]$.

Define a function

$$H(t) = t^{\alpha-1} E_{\alpha, \alpha}(Mt^{\alpha}),$$

where $E_{\alpha, \alpha}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma((k+1)\alpha)}$ is the Mittag–Leffler function (see [2,3]). Then, β in (1) satisfies the condition:

(H3) β is a nonnegative function of bounded variation, and $\int_0^1 H(t) d\beta(t) \in [0, H(1))$.

Fractional-order equations are widely used in mathematics, physics, engineering and other fields; for example they arise in problems of robotics, signal processing and conversion. There are many papers in the literature establishing the existence of solutions using the Leray–Schauder fixed-point theorem, the coincidence degree theory and the Guo–Krasnoselskii fixed-point theorem in cones; we refer the reader to [1,4–21] and the references cited therein.

In [4], the authors studied positive solutions of an abstract fractional semipositone differential system involving integral boundary conditions arising from the study of HIV infection models

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), D_{0+}^{\beta} u(t), v(t)) = 0, \\ D_{0+}^{\gamma} v(t) + \lambda g(t, u(t)) = 0, & 0 < t < 1, \\ D_{0+}^{\beta} u(0) = D_{0+}^{\beta+1} u(0) = 0, & D_{0+}^{\beta} u(1) = \int_0^1 D_{0+}^{\beta} u(s) dA(s), \\ v(0) = v'(0) = 0, & v(1) = \int_0^1 v(s) dB(s), \end{cases}$$

where $2 < \alpha, \gamma \leq 3, 0 < \beta < 1$, u denotes the number of uninfected $CD4^+$ T cells and v denotes the number of infected cells, and the nonlinearities f, g satisfy:

$$\begin{aligned} f : [0, 1] \times [0, +\infty)^3 &\rightarrow (-\infty, +\infty), g : [0, 1] \times [0, +\infty) \rightarrow (-\infty, +\infty) \text{ are continuous functions and} \\ f(t, z_1, z_2, z_3) &\geq -e(t), \quad (t, z_1, z_2, z_3) \in [0, 1] \times [0, +\infty)^3, \\ g(t, z) &\geq -\bar{e}(t), \quad (t, z) \in [0, 1] \times [0, +\infty), \quad e, \bar{e} : [0, 1] \rightarrow [0, +\infty). \end{aligned} \quad (2)$$

In [5], the authors investigated positive solutions for the nonlinear semipositone fractional q -difference system with coupled integral boundary conditions

$$\begin{cases} D_q^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, & D_q^{\beta} v(t) + \lambda g(t, u(t), v(t)) = 0, \quad t \in (0, 1), \quad \lambda > 0, \\ D_q^j u(0) = D_q^j v(0) = 0, & 0 \leq j \leq n-2, \quad u(1) = \mu \int_0^1 v(s) d_q s, \quad v(1) = v \int_0^1 u(s) d_q s, \end{cases}$$

where $\alpha, \beta \in (n-1, n]$ are two real numbers and $n \geq 3, D_q^{\alpha}, D_q^{\beta}$ are the fractional q -derivative of the Riemann–Liouville type, and the nonlinearities f and g satisfy some similar conditions in (2). In [6], the authors studied the existence and multiplicity of positive solutions for the system of Riemann–Liouville fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1} \left(D_{0+}^{\beta_1} u(t) \right) + \lambda f(t, u(t), v(t)) = 0, & t \in (0, 1), \\ D_{0+}^{\alpha_2} \left(D_{0+}^{\beta_2} v(t) \right) + \mu g(t, u(t), v(t)) = 0, & t \in (0, 1), \end{cases}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, j = 0, \dots, n-2; & D_{0+}^{\beta_1} u(0) = 0, D_{0+}^{\gamma_0} u(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i} v(t) dH_i(t), \\ v^{(j)}(0) = 0, j = 0, \dots, m-2; & D_{0+}^{\beta_2} v(0) = 0, D_{0+}^{\delta_0} v(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i} u(t) dK_i(t), \end{cases}$$

where the nonlinearities f and g satisfy some growth conditions.

Motivated by the above, in this paper we use the fixed-point index to establish positive solutions for the system (1). Some appropriate concave and convex functions are used to characterize the coupling behaviors of our nonlinearities. Moreover, our condition (H1) is more general than (2).

2. Preliminaries

Definition 1 (see [2,3]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} \varphi(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} \varphi(s) ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α , provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 1 (see [1]). Suppose that (H2) holds and $y \in L[0, 1]$. Then, the boundary value problem

$$\begin{cases} -D_{0+}^{\alpha} \varphi(t) + M\varphi(t) = y(t), & 0 < t < 1, \\ \varphi(0) = \varphi'(0) = \varphi(1) = 0, \end{cases}$$

has a unique solution

$$\varphi(t) = \int_0^1 \tilde{K}(t, s) y(s) ds,$$

where

$$\tilde{K}(t, s) = \frac{1}{H(1)} \begin{cases} H(t)H(1-s), & 0 \leq t \leq s \leq 1, \\ H(t)H(1-s) - H(t-s)H(1), & 0 \leq s \leq t \leq 1. \end{cases}$$

Lemma 2. Suppose that (H2)–(H3) hold and $y \in L[0, 1]$. Then, the boundary value problem

$$\begin{cases} -D_{0+}^{\alpha} \varphi(t) + M\varphi(t) = y(t), & 0 < t < 1, \\ \varphi(0) = \varphi'(0) = 0, \varphi(1) = \int_0^1 \varphi(t) d\beta(t), \end{cases}$$

has a unique solution

$$\varphi(t) = \int_0^1 K(t, s) y(s) ds,$$

where

$$K(t, s) = \tilde{K}(t, s) + \frac{H(t)}{H(1) - \int_0^1 H(t) d\beta(t)} \int_0^1 \tilde{K}(t, s) d\beta(t).$$

Proof. From Lemma 2.1 in [1], we have

$$\varphi(t) = - \int_0^t H(t-s) y(s) ds + c_1 H(t) + c_2 H'(t) + c_3 H''(t),$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3$. Since $\varphi(0) = \varphi'(0) = 0$, $c_2 = c_3 = 0$. Therefore,

$$\varphi(t) = - \int_0^t H(t-s) y(s) ds + c_1 H(t).$$

Using $\varphi(1) = \int_0^1 \varphi(t) d\beta(t)$, we have

$$- \int_0^1 H(1-s) y(s) ds + c_1 H(1) = - \int_0^1 \int_0^t H(t-s) y(s) ds d\beta(t) + c_1 \int_0^1 H(t) d\beta(t),$$

and

$$c_1 = \frac{1}{H(1) - \int_0^1 H(t) d\beta(t)} \int_0^1 H(1-s) y(s) ds - \frac{1}{H(1) - \int_0^1 H(t) d\beta(t)} \int_0^1 \int_0^t H(t-s) y(s) ds d\beta(t).$$

Consequently, we obtain

$$\begin{aligned}
\varphi(t) &= - \int_0^t H(t-s)y(s)ds + \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 H(1-s)y(s)ds \\
&\quad - \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 \int_0^t H(t-s)y(s)dsd\beta(t) \\
&= - \int_0^t H(t-s)y(s)ds + \frac{1}{H(1)} \int_0^1 H(t)H(1-s)y(s)ds - \frac{1}{H(1)} \int_0^1 H(t)H(1-s)y(s)ds \\
&\quad + \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 H(1-s)y(s)ds - \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 \int_0^t H(t-s)y(s)dsd\beta(t) \\
&= \int_0^1 \tilde{K}(t,s)y(s)ds + \frac{\int_0^1 H(t)d\beta(t)}{H(1)[H(1) - \int_0^1 H(t)d\beta(t)]} \int_0^1 H(t)H(1-s)y(s)ds \\
&\quad - \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 \int_0^t H(t-s)y(s)dsd\beta(t) \\
&= \int_0^1 \tilde{K}(t,s)y(s)ds + \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \left[\frac{1}{H(1)} \int_0^1 \left(\int_0^1 H(t)H(1-s)d\beta(t) - \int_s^1 H(1)H(t-s)d\beta(t) \right) y(s)ds \right] \\
&= \int_0^1 \tilde{K}(t,s)y(s)ds + \frac{H(t)}{H(1) - \int_0^1 H(t)d\beta(t)} \int_0^1 \int_0^1 \tilde{K}(t,s)d\beta(t)y(s)ds \\
&= \int_0^1 K(t,s)y(s)ds.
\end{aligned}$$

This completes the proof. $\square$

Lemma 3 (see [1]). *The function $\tilde{K}$ has the properties*

- (i) $\tilde{K}(t,s) > 0, \forall t,s \in (0,1)$;
 - (ii) $\tilde{K}(t,s) = \tilde{K}(1-s,1-t), \forall t,s \in [0,1]$;
 - (iii) $\tilde{K}(t,s) \leq M_2 s(1-s)^{\alpha-1}, \forall t,s \in [0,1]$;
 - (iv) $\tilde{K}(t,s) \geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}, \forall t,s \in [0,1]$, where
- $$M_1 = \frac{1}{H(1)[\Gamma(\alpha)]^2}, \quad M_2 = \frac{[H'(1)]^2}{H(1)s^*}, \quad s^* \in (0,1) \text{ is a unique solution for the equation } s = (1-s)^{\alpha-2}.$$

Lemma 4. *The function K has the properties*

- (i) $K(t,s) > 0, \forall t,s \in (0,1)$;
- (ii) $K(t,s) \leq M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] s(1-s)^{\alpha-1}, \forall t,s \in [0,1]$;
- (iii) $K(t,s) \geq M_1 \left[1 + \frac{\int_0^1 (1-t)t^{\alpha-1}d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}, \forall t,s \in [0,1]$;
- (iv) $K(t,s) \geq \frac{M_1 \int_0^1 (1-t)t^{\alpha-1}d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \forall t,s \in [0,1]$.

Proof. From Theorem 3.1 in [21], we have

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leq H(t) \leq t^{\alpha-1}H(1) \leq H(1), t \in [0,1].$$

Then, by Lemma 3 (iii) we obtain

$$\begin{aligned} K(t, s) &\leq M_2 s(1-s)^{\alpha-1} + \frac{H(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \int_0^1 M_2 s(1-s)^{\alpha-1} d\beta(t) \\ &= M_2 s(1-s)^{\alpha-1} \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right]. \end{aligned}$$

On the other hand, from Lemma 3 (iv), we obtain

$$\begin{aligned} K(t, s) &\geq M_1 s(1-s)^{\alpha-1} (1-t)t^{\alpha-1} + \frac{t^{\alpha-1}}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \int_0^1 M_1 s(1-s)^{\alpha-1} (1-t)t^{\alpha-1} d\beta(t) \\ &\geq M_1 s(1-s)^{\alpha-1} (1-t)t^{\alpha-1} \left[1 + \frac{\int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \right]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} K(t, s) &\geq \frac{H(t)}{H(1) - \int_0^1 H(t) d\beta(t)} \int_0^1 \tilde{K}(t, s) d\beta(t) \\ &\geq \frac{t^{\alpha-1}}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \int_0^1 M_1 s(1-s)^{\alpha-1} (1-t)t^{\alpha-1} d\beta(t) \\ &= \frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}. \end{aligned}$$

This completes the proof. $\square$

Lemma 5. Let $\zeta(t) = t(1-t)^{\alpha-1}$, $t, s \in [0, 1]$. Then, there exist positive constants κ_i ($i = 1, 2$) such that

$$\kappa_1 \zeta(s) \leq \int_0^1 K(t, s) \zeta(t) dt \leq \kappa_2 \zeta(s), s \in [0, 1],$$

where

$$\kappa_1 = \frac{M_1 \Gamma^2(\alpha + 1)}{\Gamma(2\alpha + 2)} \left[1 + \frac{\int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \right], \quad \kappa_2 = \frac{M_2 \Gamma(\alpha)}{\Gamma(\alpha + 2)} \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right].$$

Proof. We use Lemma 4 (ii)–(iii). Indeed, we have

$$\begin{aligned} \int_0^1 K(t, s) \zeta(t) dt &\leq \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] \zeta(s) \zeta(t) dt \\ &= \frac{M_2 \Gamma(\alpha)}{\Gamma(\alpha + 2)} \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] \zeta(s), s \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} \int_0^1 K(t, s) \zeta(t) dt &\geq \int_0^1 M_1 \left[1 + \frac{\int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \right] \zeta(s) (1-t)t^{\alpha-1} \zeta(t) dt \\ &= \frac{M_1 \Gamma^2(\alpha + 1)}{\Gamma(2\alpha + 2)} \left[1 + \frac{\int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t) d\beta(t)]\Gamma(\alpha)} \right] \zeta(s), s \in [0, 1]. \end{aligned}$$

This completes the proof. $\square$

Let $E = C[0, 1]$ be endowed with the maximum norm $\|\varphi\| = \max_{0 \leq t \leq 1} |\varphi(t)|$. Define a cone P by

$$P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0, 1]\}.$$

Lemma 6. Let $P_0 = \left\{ \varphi \in P : \varphi(t) \geq \frac{[\frac{\kappa_1 \Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1] \Gamma(\alpha)}{\kappa_2 \Gamma(\alpha+2)} t^{\alpha-1} \|\varphi\| \right\}$. Then, $L(P) \subset P_0$, where

$$(L\varphi)(t) = \int_0^1 K(t, s) \varphi(s) ds.$$

Proof. Let $\varphi \in P$. Then, from Lemma 4 (ii) and (iv) we have

$$(L\varphi)(t) \leq \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] s(1-s)^{\alpha-1} \varphi(s) ds,$$

and

$$\begin{aligned} (L\varphi)(t) &\geq \int_0^1 \frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1} \varphi(s) ds \\ &= \frac{\frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)}}{M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} t^{\alpha-1} \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] s(1-s)^{\alpha-1} \varphi(s) ds \\ &\geq \frac{\frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)}}{M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} t^{\alpha-1} \|L\varphi\| \\ &= \frac{[\frac{\kappa_1 \Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1] \Gamma(\alpha)}{\kappa_2 \Gamma(\alpha+2)} t^{\alpha-1} \|L\varphi\|. \end{aligned}$$

This completes the proof. $\square$

To obtain our main results, we consider the following auxiliary problem

$$\begin{cases} -D_{0+}^\alpha \varphi(t) + M\varphi(t) = f(t, \varphi(t)), & 0 < t < 1, \\ \varphi(0) = \varphi'(0) = 0, \varphi(1) = \int_0^1 \varphi(t) d\beta(t), \end{cases} \quad (3)$$

where f satisfies the condition:

(Hf) $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$, and there exists $Q \in C([0, 1], \mathbb{R}^+)$ such that

$$f(t, y) \geq -Q(t), \quad \forall t \in [0, 1], y \in \mathbb{R}^+.$$

From Lemma 2, (3) is equivalent to the following Hammerstein type integral equation

$$\varphi(t) = \int_0^1 K(t, s) f(s, \varphi(s)) ds. \quad (4)$$

Let $w(t) = \int_0^1 K(t, s) Q(s) ds$ and

$$\tilde{f}(t, y) = \begin{cases} f(t, y) + Q(t), & y \geq 0, \\ f(t, 0) + Q(t), & y < 0. \end{cases}$$

From this, we obtain an integral equation

$$\varphi(t) = \int_0^1 K(t,s) \tilde{f}(s, \varphi(s) - w(s)) ds. \quad (5)$$

Lemma 7. (i) If φ^* is a positive solution for (4), then $\varphi^* + w$ is a positive solution for (5);

(ii) If φ^{**} is a positive solution for (5) and $\varphi^{**}(t) \geq w(t), t \in [0, 1]$, then $\varphi^{**} - w$ is a positive solution for (4).

Proof. Note that

$$\varphi^*(t) = \int_0^1 K(t,s) f(s, \varphi^*(s)) ds.$$

Therefore, we have

$$\begin{aligned} \varphi^*(t) + w(t) &= \int_0^1 K(t,s) f(s, \varphi^*(s)) ds + w(t) \\ &= \int_0^1 K(t,s) [f(s, \varphi^*(s)) + Q(s)] ds \\ &= \int_0^1 K(t,s) \tilde{f}(s, \varphi^*(s) + w(s) - w(s)) ds. \end{aligned}$$

Thus, $\varphi^* + w$ is a positive solution for (5).

On the other hand, we have

$$\begin{aligned} \varphi^{**}(t) &= \int_0^1 K(t,s) \tilde{f}(s, \varphi^{**}(s) - w(s)) ds \\ &= \int_0^1 K(t,s) [f(s, \varphi^{**}(s) - w(s)) + Q(s)] ds \\ &= \int_0^1 K(t,s) f(s, \varphi^{**}(s) - w(s)) ds + w(t), \end{aligned}$$

and thus

$$\varphi^{**}(t) - w(t) = \int_0^1 K(t,s) f(s, \varphi^{**}(s) - w(s)) ds.$$

This implies that $\varphi^{**} - w$ is a positive solution for (4). This completes the proof. $\square$

From Lemma 7, we only study solutions of (5), which are greater than w . For this we define an operator $\tilde{A} : P \rightarrow P$ as follows:

$$(\tilde{A}\varphi)(t) = \int_0^1 K(t,s) \tilde{f}(s, \varphi(s) - w(s)) ds,$$

and we turn to study the fixed points of $\tilde{A}$, which also are required to be greater than w . If there exists $\varphi^* \in P$ such that $\tilde{A}\varphi^* = \varphi^*$ and $\varphi^*(t) \geq w(t), t \in [0, 1]$, then this, together with Lemmas 2 and 6, implies that $\varphi^* \in P_0$, and

$$\begin{aligned} \varphi^*(t) - w(t) &\geq \frac{[\frac{\kappa_1 \Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1] \Gamma(\alpha)}{\kappa_2 \Gamma(\alpha+2)} t^{\alpha-1} \|\varphi^*\| - \int_0^1 K(t,s) Q(s) ds \\ &\geq \frac{[\frac{\kappa_1 \Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1] \Gamma(\alpha)}{\kappa_2 \Gamma(\alpha+2)} t^{\alpha-1} \|\varphi^*\| - \int_0^1 t^{\alpha-1} \left[H(1) + \frac{H^2(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] Q(s) ds \\ &= t^{\alpha-1} \left[\frac{[\frac{\kappa_1 \Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1] \Gamma(\alpha)}{\kappa_2 \Gamma(\alpha+2)} \|\varphi^*\| - \left(H(1) + \frac{H^2(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right) \int_0^1 Q(s) ds \right] \\ &\geq 0, \end{aligned}$$

and

$$\|\varphi^*\| \geq \frac{\kappa_2 \Gamma(\alpha + 2)}{[\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1] \Gamma(\alpha)} \left(H(1) + \frac{H^2(1) \beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right) \int_0^1 Q(s) \zeta(s) ds := \Theta_Q.$$

As a result, there exists $\varphi^* \in P$ such that $\tilde{A}\varphi^* = \varphi^*$ with $\|\varphi^*\| \geq \Theta_Q$, and then $\varphi^* - w$ is a positive solution for (4).

Now, we begin to study (1). Let $\bar{f}(t, \varphi(t), \phi(t)) = f(t, \varphi(t), \phi(t)) + M\varphi(t)$, $\bar{g}(t, \varphi(t), \phi(t)) = g(t, \varphi(t), \phi(t)) + M\phi(t)$, $t \in [0, 1]$. Then, (1) can be transformed into the following system of fractional-order integral boundary value problems

$$\begin{cases} -D_{0+}^\alpha \varphi(t) + M\varphi(t) = \bar{f}(t, \varphi(t), \phi(t)), & 0 < t < 1, \\ -D_{0+}^\alpha \phi(t) + M\phi(t) = \bar{g}(t, \varphi(t), \phi(t)), & 0 < t < 1, \\ \varphi(0) = \phi(0) = \varphi'(0) = \phi'(0) = 0, & \varphi(1) = \int_0^1 \varphi(t) d\beta(t), \phi(1) = \int_0^1 \phi(t) d\beta(t), \end{cases} \quad (6)$$

where $\bar{f}, \bar{g}$ satisfy the condition:

(H1)' $\bar{f}, \bar{g} \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, and

$$\bar{f}(t, x, y) \geq -Q_f(t), \quad \bar{g}(t, x, y) \geq -Q_g(t), \quad \forall t \in [0, 1], x, y \in \mathbb{R}^+.$$

From Lemma 2, (6) is equivalent to the following system of Hammerstein type integral equations

$$\begin{cases} \varphi(t) = \int_0^1 K(t, s) \bar{f}(s, \varphi(s), \phi(s)) ds, \\ \phi(t) = \int_0^1 K(t, s) \bar{g}(s, \varphi(s), \phi(s)) ds. \end{cases} \quad (7)$$

In what follows, we establish an appropriate operator equation for problem (6). Note that $E^2 = E \times E$ is also a Banach space with norm $\|(\varphi, \phi)\| = \|\varphi\| + \|\phi\|$, and $P^2 = P \times P$ a cone on E^2 . Let

$$w_1(t) = \int_0^1 K(t, s) Q_f(s) ds, \quad w_2(t) = \int_0^1 K(t, s) Q_g(s) ds, \quad t \in [0, 1],$$

and

$$\begin{cases} A_1(\varphi, \phi)(t) = \int_0^1 K(t, s) \tilde{f}(s, \varphi(s) - w_1(s), \phi(s) - w_2(s)) ds, \\ A_2(\varphi, \phi)(t) = \int_0^1 K(t, s) \tilde{g}(s, \varphi(s) - w_1(s), \phi(s) - w_2(s)) ds, \\ A(\varphi, \phi)(t) = (A_1, A_2)(\varphi, \phi)(t), \quad t \in [0, 1], \end{cases}$$

where

$$\tilde{f}(t, \varphi, \phi) = \begin{cases} \bar{f}(t, \varphi, \phi) + Q_f(t), & \varphi, \phi \geq 0, \\ \bar{f}(t, 0, \phi) + Q_f(t), & \varphi < 0, \phi \geq 0, \\ \bar{f}(t, \varphi, 0) + Q_f(t), & \varphi \geq 0, \phi < 0, \\ \bar{f}(t, 0, 0) + Q_f(t), & \varphi, \phi < 0, \end{cases}$$

and

$$\tilde{g}(t, \varphi, \phi) = \begin{cases} \bar{g}(t, \varphi, \phi) + Q_g(t), & \varphi, \phi \geq 0, \\ \bar{g}(t, 0, \phi) + Q_g(t), & \varphi < 0, \phi \geq 0, \\ \bar{g}(t, \varphi, 0) + Q_g(t), & \varphi \geq 0, \phi < 0, \\ \bar{g}(t, 0, 0) + Q_g(t), & \varphi, \phi < 0. \end{cases}$$

It is clear that if there exists $(\varphi^*, \phi^*) \in P^2$ such that $A(\varphi^*, \phi^*) = (\varphi^*, \phi^*)$ with

$$\varphi^*(t) \geq w_1(t), \phi^*(t) \geq w_2(t), t \in [0, 1], \quad (8)$$

then $(\varphi^* - w_1, \phi^* - w_2)$ is a positive solution for (6).

Let

$$\Theta_{Q_f} = \frac{\kappa_2 \Gamma(\alpha + 2)}{[\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1] \Gamma(\alpha)} \left(H(1) + \frac{H^2(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right) \int_0^1 Q_f(s)\zeta(s)ds,$$

and

$$\Theta_{Q_g} = \frac{\kappa_2 \Gamma(\alpha + 2)}{[\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1] \Gamma(\alpha)} \left(H(1) + \frac{H^2(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right) \int_0^1 Q_g(s)\zeta(s)ds.$$

Then, if $\|\varphi^*\| \geq \Theta_{Q_f}$ and $\|\phi^*\| \geq \Theta_{Q_g}$, we obtain that (8) holds true.

Lemma 8 (see [22]). Let E be a real Banach space and P a cone on E . Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \rightarrow P$ is a continuous compact operator. If there exists $\omega_0 \in P \setminus \{0\}$ such that

$$\omega - A\omega \neq \lambda\omega_0, \forall \lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P .

Lemma 9 (see [22]). Let E be a real Banach space and P a cone on E . Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \rightarrow P$ is a continuous compact operator. If

$$\omega - \lambda A\omega \neq 0, \forall \lambda \in [0, 1], \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main Results

Let $\mathcal{L}_k = H(1) + \frac{H^2(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)}$. Now, we list the selection of assumptions for our nonlinearities.

(H4) There exist $\xi_1, \eta_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) ξ_1 is a strictly increasing concave function on $\mathbb{R}^+$;

(ii) $\liminf_{\varphi \rightarrow +\infty} \frac{\tilde{f}(t, \varphi, \phi)}{\xi_1(\phi)} \geq 1$, $\liminf_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_1(\varphi)} \geq 1$ uniformly for $t \in [0, 1]$;

(iii) There exists $e_1 > \kappa_1^{-2}$ such that $\liminf_{\varphi \rightarrow +\infty} \frac{\xi_1(\mathcal{L}_k \eta_1(\varphi))}{\varphi} \geq e_1 \mathcal{L}_k$.

(H5) There exist nonnegative functions $\mathcal{O}_i (i = 1, 2)$ on $[0, 1]$ with $0 < \int_0^1 \mathcal{O}_1(s)\zeta(s)ds < \frac{\Theta_{Q_f}}{M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]}$ and $0 < \int_0^1 \mathcal{O}_2(s)\zeta(s)ds < \frac{\Theta_{Q_g}}{M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]}$ such that

$$\tilde{f}(t, \varphi - w_1, \phi - w_2) \leq \mathcal{O}_1(t), \tilde{g}(t, \varphi - w_1, \phi - w_2) \leq \mathcal{O}_2(t), \forall t \in [0, 1], \varphi \in [0, \Theta_{Q_f}], \phi \in [0, \Theta_{Q_g}].$$

(H6) There exist nonnegative functions $\tilde{\mathcal{O}}_i (i = 1, 2)$ on $[0, 1]$ with $\int_0^1 \zeta(s)\tilde{\mathcal{O}}_1(s)ds > \frac{\Theta_{Q_f} [H(1) - \int_0^1 H(t)d\beta(t)] \Gamma(\alpha)}{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}$ and $\int_0^1 \zeta(s)\tilde{\mathcal{O}}_2(s)ds > \frac{\Theta_{Q_g} [H(1) - \int_0^1 H(t)d\beta(t)] \Gamma(\alpha)}{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}$ such that

$$\tilde{f}(t, \varphi - w_1, \phi - w_2) \geq \tilde{\mathcal{O}}_1(t), \tilde{g}(t, \varphi - w_1, \phi - w_2) \geq \tilde{\mathcal{O}}_2(t), \forall t \in [0, 1], \varphi \in [0, \Theta_{Q_f}], \phi \in [0, \Theta_{Q_g}].$$

(H7) There exist $\xi_2, \eta_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) ξ_2 is a strictly increasing convex function on $\mathbb{R}^+$;

(ii) $\limsup_{\varphi \rightarrow +\infty} \frac{\tilde{f}(t, \varphi, \phi)}{\xi_2(\phi)} \leq 1$, $\limsup_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_2(\varphi)} \leq 1$ uniformly for $t \in [0, 1]$;

(iii) There exists $e_2 \in (0, \kappa_2^{-2})$ such that $\limsup_{\varphi \rightarrow +\infty} \frac{\xi_2(\mathcal{L}_k \eta_2(\varphi))}{\varphi} \leq e_2 \mathcal{L}_k$.

Theorem 1. Suppose that (H1)–(H5) hold. Then, (1) has at least one positive solution.

Proof. Step 1. We shall prove that

$$(\varphi, \phi) \neq \lambda A(\varphi, \phi), \varphi \in \partial B_{\Theta_{Q_f}} \cap P, \phi \in \partial B_{\Theta_{Q_g}} \cap P, \lambda \in [0, 1], \quad (9)$$

where $B_\rho = \{\varphi \in E : \|\varphi\| < \rho\}$, $\rho > 0$. Suppose the contrary. Then, there exist $\varphi_1 \in \partial B_{\Theta_{Q_f}} \cap P$, $\phi_1 \in \partial B_{\Theta_{Q_g}} \cap P$ and $\lambda_1 \in [0, 1]$ such that

$$(\varphi_1, \phi_1) = \lambda_1 A(\varphi_1, \phi_1).$$

This implies that

$$\|(\varphi_1, \phi_1)\| = \|\lambda_1 A(\varphi_1, \phi_1)\| \leq \|A(\varphi_1, \phi_1)\|. \quad (10)$$

Note that $\varphi_1 \in \partial B_{\Theta_{Q_f}} \cap P$, $\phi_1 \in \partial B_{\Theta_{Q_g}} \cap P$ (i.e., $\varphi_1(t) \leq \Theta_{Q_f}$, $\phi_1(t) \leq \Theta_{Q_g}$, $t \in [0, 1]$), and from (H5), we have

$$\begin{aligned} A_1(\varphi_1, \phi_1)(t) &\leq \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \mathcal{O}_1(s)\zeta(s)ds \\ &< \Theta_{Q_f}, \end{aligned}$$

and

$$\begin{aligned} A_2(\varphi_1, \phi_1)(t) &\leq \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \mathcal{O}_2(s)\zeta(s)ds \\ &< \Theta_{Q_g}. \end{aligned}$$

Therefore, we have

$$\|A(\varphi_1, \phi_1)\| < \Theta_{Q_f} + \Theta_{Q_g} = \|\varphi_1\| + \|\phi_1\| = \|(\varphi_1, \phi_1)\|.$$

This contradicts (10), and thus (9) holds. Hence, Lemma 9 implies that

$$i(A, (B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}) \cap (P \times P), P \times P) = 1. \quad (11)$$

Step 2. We claim that there exist sufficiently large $R_1 > \Theta_{Q_f}$, $R_2 > \Theta_{Q_g}$ such that

$$(\varphi, \phi) \neq A(\varphi, \phi) + \lambda(\varrho_1, \varrho_2), \varphi \in \partial B_{R_1} \cap P, \phi \in \partial B_{R_2} \cap P, \lambda \geq 0, \quad (12)$$

where ϱ_i ($i = 1, 2$) are fixed elements in P_0 . Assume the contrary. Then, there exist $\varphi_2 \in \partial B_{R_1} \cap P$, $\phi_2 \in \partial B_{R_2} \cap P$, $\lambda_2 \geq 0$ such that

$$(\varphi_2, \phi_2) = A(\varphi_2, \phi_2) + \lambda_2(\varrho_1, \varrho_2).$$

This implies that

$$\varphi_2(t) = A_1(\varphi_2, \phi_2)(t) + \lambda_2 \varrho_1(t), \phi_2(t) = A_2(\varphi_2, \phi_2)(t) + \lambda_2 \varrho_2(t), t \in [0, 1].$$

Note that $\varrho_i \in P_0$ ($i = 1, 2$), and from Lemma 6 we have

$$\varphi_2, \phi_2 \in P_0.$$

From (H4) (ii), we have

$$\liminf_{\varphi \rightarrow +\infty} \frac{\tilde{f}(t, \varphi, \phi)}{\zeta_1(\phi)} \geq 1, \liminf_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_1(\varphi)} \geq 1$$

uniformly for $t \in [0, 1]$, and there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\tilde{f}(t, \varphi, \phi) \geq \xi_1(\phi) - c_1, \quad \tilde{g}(t, \varphi, \phi) \geq \eta_1(\varphi) - c_2, \quad \forall t \in [0, 1], \varphi, \phi \in \mathbb{R}^+.$$

From these inequalities, we have

$$\begin{aligned} \phi_2(t) &\geq A_1(\phi_2, \phi_2)(t) \\ &\geq \int_0^1 K(t, s) [\xi_1(\phi_2(s) - w_2(s)) - c_1] ds \\ &\geq \int_0^1 K(t, s) \xi_1(\phi_2(s) - w_2(s)) ds - c_1 \mathcal{L}_k, \end{aligned} \quad (13)$$

and

$$\begin{aligned} \phi_2(t) &\geq A_2(\phi_2, \phi_2)(t) \\ &\geq \int_0^1 K(t, s) [\eta_1(\phi_2(s) - w_1(s)) - c_2] ds \\ &\geq \int_0^1 K(t, s) \eta_1(\phi_2(s) - w_1(s)) ds - c_2 \mathcal{L}_k. \end{aligned} \quad (14)$$

Consequently, we have

$$\phi_2(t) - w_2(t) \geq \int_0^1 K(t, s) \eta_1(\phi_2(s) - w_1(s)) ds - c_2 \mathcal{L}_k - M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] \int_0^1 \zeta(s) Q_g(s) ds.$$

Let $c_3 = c_2 \mathcal{L}_k + M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] \int_0^1 \zeta(s) Q_g(s) ds$. Then, we have

$$\begin{aligned} \xi_1(\phi_2(t) - w_2(t)) &\geq \xi_1(\phi_2(t) - w_2(t) + c_3) - \xi_1(c_3) \\ &\geq \xi_1 \left(\int_0^1 K(t, s) \eta_1(\phi_2(s) - w_1(s)) ds \right) - \xi_1(c_3) \\ &= \xi_1 \left(\int_0^1 \frac{K(t, s)}{\mathcal{L}_k} \mathcal{L}_k \eta_1(\phi_2(s) - w_1(s)) ds \right) - \xi_1(c_3) \\ &\geq \int_0^1 \frac{K(t, s)}{\mathcal{L}_k} \xi_1(\mathcal{L}_k \eta_1(\phi_2(s) - w_1(s))) ds - \xi_1(c_3). \end{aligned}$$

From (H4) (iii), there exists $c_4 > 0$ such that

$$\xi_1(\mathcal{L}_k \eta_1(\varphi)) \geq e_1 \mathcal{L}_k \varphi - c_4 \mathcal{L}_k, \quad \varphi \in \mathbb{R}^+.$$

Hence, we obtain

$$\begin{aligned} \xi_1(\phi_2(t) - w_2(t)) &\geq \int_0^1 \frac{K(t, s)}{\mathcal{L}_k} [e_1 \mathcal{L}_k (\phi_2(s) - w_1(s)) - c_4 \mathcal{L}_k] ds - \xi_1(c_3) \\ &\geq e_1 \int_0^1 K(t, s) (\phi_2(s) - w_1(s)) ds - \xi_1(c_3) - c_4 \mathcal{L}_k, \end{aligned}$$

and then

$$\begin{aligned} \phi_2(t) &\geq \int_0^1 K(t, s) \left[e_1 \int_0^1 K(s, \tau) (\phi_2(\tau) - w_1(\tau)) d\tau - \xi_1(c_3) - c_4 \mathcal{L}_k \right] ds - c_1 \mathcal{L}_k \\ &\geq e_1 \int_0^1 \int_0^1 K(t, s) K(s, \tau) (\phi_2(\tau) - w_1(\tau)) d\tau ds - \xi_1(c_3) \mathcal{L}_k - c_4 \mathcal{L}_k^2 - c_1 \mathcal{L}_k. \end{aligned}$$

Let $c_5 = M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t) d\beta(t)} \right] \int_0^1 \zeta(s) Q_f(s) ds + \xi_1(c_3) \mathcal{L}_k + c_4 \mathcal{L}_k^2 + c_1 \mathcal{L}_k$. Then, we have

$$\begin{aligned}\varphi_2(t) - w_1(t) &\geq e_1 \int_0^1 \int_0^1 K(t,s)K(s,\tau)(\varphi_2(\tau) - w_1(\tau))d\tau ds - \int_0^1 K(t,s)Q_f(s)ds - \xi_1(c_3)\mathcal{L}_k - c_4\mathcal{L}_k^2 - c_1\mathcal{L}_k \\ &\geq e_1 \int_0^1 \int_0^1 K(t,s)K(s,\tau)(\varphi_2(\tau) - w_1(\tau))d\tau ds - c_5.\end{aligned}$$

Multiply by $\zeta(t)$ on both sides of the above, integrate over $[0, 1]$ and use Lemma 5 to obtain

$$\begin{aligned}\int_0^1 (\varphi_2(t) - w_1(t))\zeta(t)dt &\geq e_1 \int_0^1 \int_0^1 \int_0^1 \zeta(t)K(t,s)K(s,\tau)(\varphi_2(\tau) - w_1(\tau))d\tau ds dt - c_5 \int_0^1 \zeta(t)dt \\ &\geq e_1 \kappa_1^2 \int_0^1 (\varphi_2(t) - w_1(t))\zeta(t)dt - \frac{c_5\Gamma(\alpha)}{\Gamma(\alpha+2)}.\end{aligned}$$

Solving this inequality, we obtain

$$\int_0^1 (\varphi_2(t) - w_1(t))\zeta(t)dt \leq \frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)}.$$

Note that $\varphi_2 \in P_0$, we have

$$\int_0^1 \|\varphi_2\| \frac{[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma(\alpha)}{\kappa_2\Gamma(\alpha+2)} t^{\alpha-1}\zeta(t)dt \leq \int_0^1 \varphi_2(t)\zeta(t)dt \leq \frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)} + \kappa_2 \int_0^1 Q_f(t)\zeta(t)dt,$$

and

$$\|\varphi_2\| \leq \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1) \left[\frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)} + \kappa_2 \int_0^1 Q_f(t)\zeta(t)dt \right]}{[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} := \mathcal{N}_{\varphi_2}.$$

Multiply by $\zeta(t)$ on both sides of (13), integrate over $[0, 1]$ and use Lemma 5 to obtain

$$\begin{aligned}\kappa_1 \int_0^1 \zeta(t)\xi_1(\varphi_2(t) - w_2(t))dt &\leq \int_0^1 \zeta(t)(\varphi_2(t) + c_1\mathcal{L}_k)dt \\ &\leq \frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)} + \kappa_2 \int_0^1 Q_f(t)\zeta(t)dt + \frac{c_1\mathcal{L}_k\Gamma(\alpha)}{\Gamma(\alpha+2)}.\end{aligned}$$

Note that $\varphi_2, w_2 \in P_0$ and $\|\varphi_2\| = R_2 > \Theta_{Q_g}$, then $\varphi_2 - w_2 \in P_0$. By the concavity of ξ_1 , we have

$$\begin{aligned}\|\varphi_2 - w_2\| &\leq \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1)}{[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} \int_0^1 \zeta(t)(\varphi_2(t) - w_2(t))dt \\ &= \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1)}{[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} \frac{\|\varphi_2 - w_2\|}{\xi_1(\|\varphi_2 - w_2\|)} \int_0^1 \frac{\varphi_2(t) - w_2(t)}{\|\varphi_2 - w_2\|} \zeta(t)\xi_1(\|\varphi_2 - w_2\|)dt \\ &\leq \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1)}{[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} \frac{\|\varphi_2 - w_2\|}{\xi_1(\|\varphi_2 - w_2\|)} \int_0^1 \zeta(t)\xi_1\left(\frac{\varphi_2(t) - w_2(t)}{\|\varphi_2 - w_2\|}\|\varphi_2 - w_2\|\right)dt \\ &\leq \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1)}{\kappa_1[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} \frac{\|\varphi_2 - w_2\|}{\xi_1(\|\varphi_2 - w_2\|)} \left[\frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)} + \kappa_2 \int_0^1 Q_f(t)\zeta(t)dt + \frac{c_1\mathcal{L}_k\Gamma(\alpha)}{\Gamma(\alpha+2)} \right],\end{aligned}$$

and thus,

$$\xi_1(\|\varphi_2 - w_2\|) \leq \frac{\kappa_2\Gamma(\alpha+2)\Gamma(2\alpha+1)}{\kappa_1[\frac{\kappa_1\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)} - M_1]\Gamma^2(\alpha)\Gamma(\alpha+1)} \left[\frac{c_5\Gamma(\alpha)}{(e_1\kappa_1^2 - 1)\Gamma(\alpha+2)} + \kappa_2 \int_0^1 Q_f(t)\zeta(t)dt + \frac{c_1\mathcal{L}_k\Gamma(\alpha)}{\Gamma(\alpha+2)} \right].$$

Therefore, we have

$$\begin{aligned}\xi_1(\|\phi_2\|) &= \xi_1(\|\phi_2 - w_2 + w_2\|) \leq \xi_1(\|\phi_2 - w_2\| + \|w_2\|) \leq \xi_1(\|\phi_2 - w_2\|) + \xi_1(\|w_2\|) \\ &\leq \frac{\kappa_2 \Gamma(\alpha + 2) \Gamma(2\alpha + 1)}{\kappa_1 [\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1] \Gamma^2(\alpha) \Gamma(\alpha + 1)} \left[\frac{c_5 \Gamma(\alpha)}{(e_1 \kappa_1^2 - 1) \Gamma(\alpha + 2)} + \kappa_2 \int_0^1 Q_f(t) \zeta(t) dt + \frac{c_1 \mathcal{L}_k \Gamma(\alpha)}{\Gamma(\alpha + 2)} \right] + \xi_1(\|w_2\|) \\ &< +\infty.\end{aligned}$$

Note that ξ_1 is a strictly increasing function, and there exists $\mathcal{N}_{\phi_2} > 0$ such that

$$\|\phi_2\| \leq \mathcal{N}_{\phi_2}.$$

Now, if we choose $R_1 > \max\{\Theta_{Q_f}, \mathcal{N}_{\phi_2}\}$, $R_2 > \max\{\Theta_{Q_g} \text{ and } \mathcal{N}_{\phi_2}\}$, then (12) is satisfied. Hence, Lemma 8 implies that

$$i(A, (B_{R_1} \times B_{R_2}) \cap (P \times P), P \times P) = 0. \quad (15)$$

From (11) and (15), we have

$$\begin{aligned}i(A, (B_{R_1} \times B_{R_2}) \setminus (\overline{B_{\Theta_{Q_f}}} \times \overline{B_{\Theta_{Q_g}}}) \cap (P \times P), P \times P) \\ = i(A, (B_{R_1} \times B_{R_2}) \cap (P \times P), P \times P) - i(A, (B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}) \cap (P \times P), P \times P) = -1.\end{aligned}$$

Therefore, there exists (φ^*, ϕ^*) in $(B_{R_1} \times B_{R_2}) \setminus (\overline{B_{\Theta_{Q_f}}} \times \overline{B_{\Theta_{Q_g}}}) \cap (P \times P)$ such that $A(\varphi^*, \phi^*) = (\varphi^*, \phi^*)$. Note that $\varphi^*(t) \geq w_1(t)$, $\phi^*(t) \geq w_2(t)$, $t \in [0, 1]$, and thus $(\varphi^* - w_1, \phi^* - w_2)$ is a positive solution for (1). Thus, (1) has at least one positive solution. This completes the proof. $\square$

Theorem 2. Suppose that (H1)–(H3) and (H6)–(H7) hold. Then, (1) has at least one positive solution.

Proof. Step 1. We shall verify

$$(\varphi, \phi) \neq A(\varphi, \phi) + \lambda(q_3, q_4), \lambda \geq 0, \varphi \in \partial B_{\Theta_{Q_f}} \cap P, \phi \in \partial B_{\Theta_{Q_g}} \cap P, \quad (16)$$

where q_i ($i = 3, 4$) are given elements in P . Assume the contrary. Suppose there exist $\varphi_3 \in \partial B_{\Theta_{Q_f}} \cap P$, $\phi_3 \in \partial B_{\Theta_{Q_g}} \cap P$ and $\lambda_3 \geq 0$ such that

$$(\varphi_3, \phi_3) = A(\varphi_3, \phi_3) + \lambda_3(q_3, q_4).$$

This implies that

$$\varphi_3(t) = A_1(\varphi_3, \phi_3)(t) + \lambda_3 q_3(t) \geq A_1(\varphi_3, \phi_3)(t),$$

and

$$\phi_3(t) = A_2(\varphi_3, \phi_3)(t) + \lambda_3 q_4(t) \geq A_2(\varphi_3, \phi_3)(t).$$

From these inequalities, we have

$$\|\varphi_3\| \geq \|A_1(\varphi_3, \phi_3)\|, \|\phi_3\| \geq \|A_2(\varphi_3, \phi_3)\|,$$

and then

$$\|(\varphi_3, \phi_3)\| = \|\varphi_3\| + \|\phi_3\| \geq \|A_1(\varphi_3, \phi_3)\| + \|A_2(\varphi_3, \phi_3)\| = \|A(\varphi_3, \phi_3)\|. \quad (17)$$

On the other hand, from (H6), we have

$$\begin{aligned}\|A_1(\varphi_3, \phi_3)\| &= \max_{t \in [0,1]} A_1(\varphi_3, \phi_3)(t) \\ &\geq \max_{t \in [0,1]} \int_0^1 \frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)} t^{\alpha-1} \zeta(s) \tilde{O}_1(s) ds \\ &> \Theta_{Q_f},\end{aligned}$$

and

$$\begin{aligned}\|A_2(\varphi_3, \phi_3)\| &= \max_{t \in [0,1]} A_2(\varphi_3, \phi_3)(t) \\ &\geq \max_{t \in [0,1]} \int_0^1 \frac{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}{[H(1) - \int_0^1 H(t)d\beta(t)]\Gamma(\alpha)} t^{\alpha-1} \zeta(s) \tilde{O}_2(s) ds \\ &> \Theta_{Q_g}.\end{aligned}$$

These two inequalities imply that

$$\|A(\varphi_3, \phi_3)\| = \|A_1(\varphi_3, \phi_3)\| + \|A_2(\varphi_3, \phi_3)\| > \Theta_{Q_f} + \Theta_{Q_g} = \|(\varphi_3, \phi_3)\|.$$

This contradicts (17). Hence, Lemma 8 implies that

$$i(A, (B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}) \cap (P \times P), P \times P) = 0. \quad (18)$$

Step 2. We claim that there exist sufficiently large $R_3 > \Theta_{Q_f}$ and $R_4 > \Theta_{Q_g}$ such that

$$(\varphi, \phi) \neq \lambda A(\varphi, \phi), \lambda \in [0, 1], \varphi \in \partial B_{R_3} \cap P, \phi \in \partial B_{R_4} \cap P. \quad (19)$$

Suppose the contrary. Then, there exist $\varphi_4 \in \partial B_{R_3} \cap P, \phi_4 \in \partial B_{R_4} \cap P$ and $\lambda_4 \in [0, 1]$ such that

$$(\varphi_4, \phi_4) = \lambda_4 A(\varphi_4, \phi_4). \quad (20)$$

Combining with Lemma 6 gives that

$$\varphi_4, \phi_4 \in P_0.$$

From (H7), we have

$$\limsup_{\varphi \rightarrow +\infty} \frac{\tilde{f}(t, \varphi, \phi)}{\xi_2(\phi)} \leq 1, \limsup_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_2(\varphi)} \leq 1$$

uniformly for $t \in [0, 1]$, and there exists $\tilde{\mathcal{M}}_4 > 0$ such that

$$\tilde{f}(t, \varphi, \phi) \leq \xi_2(\phi), \tilde{g}(t, \varphi, \phi) \leq \eta_2(\varphi), \text{ for } \varphi, \phi \geq \tilde{\mathcal{M}}_4.$$

Let $R_3, R_4 > \tilde{\mathcal{M}}_4$. Then, from (20), we have

$$\varphi_4(t) \leq A_1(\varphi_4, \phi_4)(t) \leq \int_0^1 K(t, s) \xi_2(\phi_4(s) - w_2(s)) ds,$$

and

$$\phi_4(t) \leq A_2(\varphi_4, \phi_4)(t) \leq \int_0^1 K(t, s) \eta_2(\varphi_4(s) - w_1(s)) ds.$$

Note that from (H7) (iii), there exists $c_6 > 0$ such that

$$\xi_2(\mathcal{L}_k \eta_2(\varphi)) \leq e_2 \mathcal{L}_k \varphi + c_6 \mathcal{L}_k, \forall \varphi \in \mathbb{R}^+.$$

Therefore, from (H7) (i) we have

$$\begin{aligned}\zeta_2(\phi_4(t) - w_2(t)) &\leq \zeta_2\left(\int_0^1 K(t,s)\eta_2(\phi_4(s) - w_1(s))ds\right) \\ &= \zeta_2\left(\int_0^1 \frac{K(t,s)}{\mathcal{L}_k} \mathcal{L}_k \eta_2(\phi_4(s) - w_1(s))ds\right) \\ &\leq \int_0^1 \frac{K(t,s)}{\mathcal{L}_k} \zeta_2(\mathcal{L}_k \eta_2(\phi_4(s) - w_1(s)))ds \\ &\leq \int_0^1 \frac{K(t,s)}{\mathcal{L}_k} [e_2 \mathcal{L}_k(\phi_4(s) - w_1(s)) + c_6 \mathcal{L}_k]ds \\ &\leq e_2 \int_0^1 K(t,s)\phi_4(s)ds + c_6 \mathcal{L}_k,\end{aligned}$$

and thus

$$\phi_4(t) \leq \int_0^1 K(t,s) \left[e_2 \int_0^1 K(s,\tau) \phi_4(\tau) d\tau + c_6 \mathcal{L}_k \right] ds.$$

Multiply by $\zeta(t)$ on both sides of the above, integrate over $[0, 1]$ and use Lemma 5 to obtain

$$\begin{aligned}\int_0^1 \phi_4(t) \zeta(t) dt &\leq \int_0^1 \zeta(t) \int_0^1 K(t,s) \left[e_2 \int_0^1 K(s,\tau) \phi_4(\tau) d\tau + c_6 \mathcal{L}_k \right] ds dt \\ &\leq e_2 \kappa_2^2 \int_0^1 \phi_4(t) \zeta(t) dt + \frac{c_6 \mathcal{L}_k \kappa_2 \Gamma(\alpha)}{\Gamma(\alpha + 2)}.\end{aligned}$$

Solving this inequality, we obtain

$$\int_0^1 \phi_4(t) \zeta(t) dt \leq \frac{c_6 \mathcal{L}_k \kappa_2 \Gamma(\alpha)}{(1 - e_2 \kappa_2^2) \Gamma(\alpha + 2)}.$$

Note that $\phi_4 \in P_0$ and we have

$$\|\phi_4\| \leq \frac{c_6 \mathcal{L}_k \kappa_2^2 \Gamma(2\alpha + 1)}{(1 - e_2 \kappa_2^2) \left[\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1 \right] \Gamma(\alpha) \Gamma(\alpha + 1)}.$$

On the other hand, from Lemma 4 (ii), we have

$$\begin{aligned}\zeta_2(\phi_4(t) - w_2(t)) &\leq e_2 \int_0^1 M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \zeta(s)\phi_4(s)ds + c_6 \mathcal{L}_k \\ &\leq M_2 e_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \frac{c_6 \mathcal{L}_k \kappa_2 \Gamma(\alpha)}{(1 - e_2 \kappa_2^2) \Gamma(\alpha + 2)} + c_6 \mathcal{L}_k.\end{aligned}$$

This implies that there exists $\mathcal{N}_{\phi_4} > 0$ such that

$$\|\phi_4 - w_2\| \leq \mathcal{N}_{\phi_4},$$

and thus

$$\|\phi_4\| = \|\phi_4 - w_2 + w_2\| \leq \|\phi_4 - w_2\| + \|w_2\| \leq \mathcal{N}_{\phi_4} + M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \int_0^1 \zeta(s) Q_s(s) ds.$$

Now, if we choose

$$R_3 > \max \left\{ \Theta_{Q_f}, \widetilde{\mathcal{M}}_4, \frac{c_6 \mathcal{L}_k \kappa_2^2 \Gamma(2\alpha + 1)}{(1 - e_2 \kappa_2^2) \left[\frac{\kappa_1 \Gamma(2\alpha + 2)}{\Gamma^2(\alpha + 1)} - M_1 \right] \Gamma(\alpha) \Gamma(\alpha + 1)} \right\}$$

and

$$R_4 > \max \left\{ \Theta_{Q_g}, \widetilde{\mathcal{M}}_4, \mathcal{N}_{\phi_4} + M_2 \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right] \int_0^1 \zeta(s)Q_g(s)ds \right\},$$

then (19) holds. Hence, Lemma 9 implies that

$$i(A, (B_{R_3} \times B_{R_4}) \cap (P \times P), P \times P) = 1. \quad (21)$$

From (18) and (21) we have

$$\begin{aligned} & i(A, (B_{R_3} \times B_{R_4}) \setminus (\overline{B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}}) \cap (P \times P), P \times P) \\ &= i(A, (B_{R_3} \times B_{R_4}) \cap (P \times P), P \times P) - i(A, (B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}) \cap (P \times P), P \times P) = 1. \end{aligned}$$

Therefore, there exists (φ^*, ϕ^*) in $(B_{R_3} \times B_{R_4}) \setminus (\overline{B_{\Theta_{Q_f}} \times B_{\Theta_{Q_g}}}) \cap (P \times P)$ such that $A(\varphi^*, \phi^*) = (\varphi^*, \phi^*)$. Note that $\varphi^*(t) \geq w_1(t), \phi^*(t) \geq w_2(t), t \in [0, 1]$, and thus $(\varphi^* - w_1, \phi^* - w_2)$ is a positive solution for (1). Thus, (1) has at least one positive solution. This completes the proof. $\square$

Example 1. Let $\zeta_1(\phi) = \phi^{\frac{4}{5}}, \eta_1(\phi) = \phi^2$ and $\phi, \phi \in \mathbb{R}^+$. Then, we have

$$\liminf_{\phi \rightarrow +\infty} \frac{\zeta_1(\mathcal{L}_k \eta_1(\phi))}{\phi} = \liminf_{\phi \rightarrow +\infty} \frac{\mathcal{L}_k^{\frac{4}{5}} \phi^{\frac{8}{5}}}{\phi} = +\infty.$$

Thus, (H4) (i) and (iii) hold.

Take

$$\widetilde{f}(t, \phi, \phi) = \frac{1}{t|\sin \phi| + 2} \frac{\Theta_{Q_f} \Gamma(\alpha + 2)}{\Theta_{Q_g} M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} \phi, \quad t \in [0, 1], \phi, \phi \in \mathbb{R}^+,$$

and

$$\widetilde{g}(t, \phi, \phi) = \frac{1}{t|\cos \phi| + 3} \frac{\Theta_{Q_g} \Gamma(\alpha + 2)}{M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} \Theta_{Q_f}^3 \phi^3, \quad t \in [0, 1], \phi, \phi \in \mathbb{R}^+.$$

Then, when $t \in [0, 1], \phi \in [0, \Theta_{Q_f}]$ and $\phi \in [0, \Theta_{Q_g}]$, we have

$$\widetilde{f}(t, \phi, \phi) \leq \frac{1}{2} \frac{\Theta_{Q_f} \Gamma(\alpha + 2)}{M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} \equiv \mathcal{O}_1(t), \quad t \in [0, 1],$$

and

$$\widetilde{g}(t, \phi, \phi) \leq \frac{1}{3} \frac{\Theta_{Q_g} \Gamma(\alpha + 2)}{M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} \equiv \mathcal{O}_2(t), \quad t \in [0, 1].$$

On the other hand,

$$\liminf_{\phi \rightarrow +\infty} \frac{\widetilde{f}(t, \phi, \phi)}{\zeta_1(\phi)} = \liminf_{\phi \rightarrow +\infty} \frac{\frac{1}{t|\sin \phi| + 2} \frac{\Theta_{Q_f} \Gamma(\alpha + 2)}{\Theta_{Q_g} M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1) - \int_0^1 H(t)d\beta(t)} \right]} \phi}{\phi^{\frac{4}{5}}} = +\infty,$$

and

$$\liminf_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_1(\varphi)} = \liminf_{\phi \rightarrow +\infty} \frac{\frac{1}{t|\cos \phi|+3} \frac{\Theta_{Q_g} \Gamma(\alpha+2)}{M_2 \Gamma(\alpha) \left[1 + \frac{H(1)\beta(1)}{H(1)-\int_0^1 H(t)d\beta(t)}\right]} \Theta_{Q_f}^3 \varphi^3}{\varphi^2} = +\infty$$

uniformly for $t \in [0, 1]$. Therefore, (H4) (ii) and (H5) hold.

Example 2. Let $\xi_2(\phi) = \phi^2, \eta_2(\varphi) = \varphi^{\frac{2}{5}}$ and $\varphi, \phi \in \mathbb{R}^+$. Then, $\limsup_{\varphi \rightarrow +\infty} \frac{\xi_2(\mathcal{L}_k \eta_2(\varphi))}{\varphi} = \limsup_{\varphi \rightarrow +\infty} \frac{\mathcal{L}_k^2 \varphi^{\frac{4}{5}}}{\varphi} = 0$. Thus, (H7) (i) and (iii) hold. Take

$$\tilde{O}_1(t) \equiv 2 \frac{\Theta_{Q_f} [H(1) - \int_0^1 H(t)d\beta(t)] \Gamma(\alpha+2)}{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}, t \in [0, 1],$$

$$\tilde{O}_2(t) \equiv 3 \frac{\Theta_{Q_g} [H(1) - \int_0^1 H(t)d\beta(t)] \Gamma(\alpha+2)}{M_1 \int_0^1 (1-t)t^{\alpha-1} d\beta(t)}, t \in [0, 1],$$

$$\tilde{f}(t, \varphi, \phi) = \tilde{O}_1(t) + (\phi + t|\cos \varphi|)^{\gamma_1}, t \in [0, 1], \varphi, \phi \in \mathbb{R}^+,$$

and

$$\tilde{g}(t, \varphi, \phi) = \tilde{O}_2(t) + (\varphi + t|\sin \phi|)^{\gamma_2}, t \in [0, 1], \varphi, \phi \in \mathbb{R}^+,$$

where $\gamma_1 \in (0, 2), \gamma_2 \in (0, \frac{2}{5})$. Note that when $t \in [0, 1], \varphi \in [0, \Theta_{Q_f}]$ and $\phi \in [0, \Theta_{Q_g}]$, we have

$$\tilde{f}(t, \varphi, \phi) \geq \tilde{O}_1(t), \tilde{g}(t, \varphi, \phi) \geq \tilde{O}_2(t), t \in [0, 1].$$

Moreover,

$$\limsup_{\varphi \rightarrow +\infty} \frac{\tilde{f}(t, \varphi, \phi)}{\xi_2(\phi)} = \limsup_{\varphi \rightarrow +\infty} \frac{\tilde{O}_1(t) + (\phi + t|\cos \varphi|)^{\gamma_1}}{\varphi^2} = 0,$$

and

$$\limsup_{\phi \rightarrow +\infty} \frac{\tilde{g}(t, \varphi, \phi)}{\eta_2(\varphi)} = \limsup_{\phi \rightarrow +\infty} \frac{\tilde{O}_2(t) + (\varphi + t|\sin \phi|)^{\gamma_2}}{\varphi^{\frac{2}{5}}} = 0$$

uniformly for $t \in [0, 1]$. Therefore, (H7) (ii) and (H6) hold.

4. Conclusions

In this paper, we used the fixed-point index to study the existence of positive solutions for the system (1) of Riemann–Liouville type fractional-order integral boundary value problems. Note our nonlinearities could be sign-changing, and some concave and convex functions were used to characterize their coupling behaviors. The results obtained here improved some existing results in the literature.

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