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# Generalized Space-Time Fractional Stochastic Kinetic Equation

Junfeng Liu \*, Zhigang Yao and Bin Zhang

School of Statistics and Data Science, Nanjing Audit University, Nanjing 211815, China

\* Correspondence: junfengliu@nau.edu.cn

**Abstract:** In this paper, we study a class of nonlinear space-time fractional stochastic kinetic equations in  $\mathbb{R}^d$  with Gaussian noise which is white in time and homogeneous in space. This type of equation constitutes an extension of the nonlinear stochastic heat equation involving fractional derivatives in time and fractional Laplacian in space. We firstly give a necessary condition on the spatial covariance for the existence and uniqueness of the solution. Furthermore, we also study various properties of the solution, such as Hölder regularity, the upper bound of second moment, and the stationarity with respect to the spatial variable in the case of linear additive noise.

**Keywords:** space-time fractional stochastic kinetic equations; caputo derivatives; gaussian index; hölder continuity

**MSC:** 60H05; 60H07; 60H15



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## 1. Introduction

Fractional stochastic partial differential equations (SPDEs for short) constitute a subclass of stochastic partial differential equations. The main characteristic of this class of stochastic equations is that they involve fractional derivatives and integrals, which replace the usual derivatives and integrals. The fractional stochastic partial differential equations received particular attention in the last several decades because they emerge in anomalous diffusion models in physics, among other areas of applications (see, for example, [1–8] and references therein).

The aim of the present article is to study the following space-time fractional stochastic kinetic equations, for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{cases} \left( \frac{\partial^\beta}{\partial t^\beta} + \nu(I - \Delta)^{\gamma/2}(-\Delta)^{\alpha/2} \right) u(t, x) = I_t^{1-\beta}(\lambda\sigma(u(t, x))\dot{W}(t, x)), \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \quad (1)$$

where  $\beta \in (0, 1]$ ,  $\gamma \geq 0$ ,  $\alpha > 0$  are some fractional parameters and  $\nu$  and  $\lambda$  are two positive parameters, with  $\lambda$  being called the intensity of the noise. The coefficient  $\sigma(\cdot)$  is a measurable function, and  $\dot{W}$  is a Gaussian noise, white in time and correlated in space. Here,  $\Delta$  is the  $d$ -dimensional Laplace operator and the operators  $(I - \Delta)^{\gamma/2}$ ,  $\gamma \geq 0$  and  $(-\Delta)^{\alpha/2}$ ,  $\alpha > 0$  are interpreted as the inverses of the Bessel and Riesz potentials, respectively. They are defined as follows. For a function  $f$  which is sufficiently smooth and small at infinity, the Riesz potential  $(-\Delta)^{-\alpha/2}(f)$ ,  $0 < \alpha < d$  is defined by

$$(-\Delta)^{-\alpha/2}(f)(x) := \frac{1}{\nu(\alpha)} \int_{\mathbb{R}^d} |x - y|^{-d+\alpha} f(y) dy,$$

with  $\nu(\alpha) = \pi^{d/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma(\frac{d-\alpha}{2})}$ . The Bessel potential  $(I - \Delta)^{-\gamma/2}$ ,  $\gamma \geq 0$  on  $\mathbb{R}^d$  can be represented by

$$(I - \Delta)^{-\gamma/2}(f)(x) := \int_{\mathbb{R}^d} H_s(x - y) f(y) dy,$$

where  $H_s(\cdot)$  is defined for  $x \in \mathbb{R}^d / \{0\}$  by the formula  $H_s(x) = \frac{1}{(4\pi)^{\gamma/2}\Gamma(\gamma/2)} \int_0^\infty \frac{1}{y^{1+\frac{d-\gamma}{2}}} e^{-\frac{\pi|x|^2}{y} - \frac{y}{4\pi}} dy$ . For more details, one can consult Chapter V in [9] for the definitions about the Bessel and Riesz potentials. Furthermore, the composition of the Bessel and Riesz potentials plays an important role in describing the behaviour of the process at the spatial macro and microscales. These integral operators and their inverses can be defined as bounded operators on the fractional Sobolev spaces  $\{H^\theta(\mathbb{R}^d); \theta \in \mathbb{R}\}$ .

We will specify later the required conditions on the function  $\sigma(\cdot)$  and the Gaussian noise  $\dot{W}$ . In Equation (1), the time derivative operator  $\frac{\partial^\beta}{\partial t^\beta}$  with order  $\beta \in (0, 1]$  is defined in the Caputo–Djrbashian sense (for example, Caputo [3], Anh, and Leonenko [2]):

$$\frac{\partial^\beta}{\partial t^\beta} u(t, x) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \left[ \frac{\partial}{\partial t} \int_0^t \frac{u(s, x)}{(t-s)^\beta} ds - \frac{u(0, x)}{t^\beta} \right], & \text{if } \beta \in (0, 1), \\ \frac{\partial}{\partial t} u(t, x), & \text{if } \beta = 1. \end{cases} \tag{2}$$

The deterministic counterparts of Equation (1) have received a lot of attention. This is because they appear to be very useful for modeling, being introduced to describe physical phenomena such as diffusion in porous media with fractal geometry, kinematics in viscoelastic media, relaxation processes in complex systems (including viscoelastic materials, glassy materials, synthetic polymers, biopolymers), propagation of seismic waves, anomalous diffusion and turbulence (see, for example, Anh and Leonenko [2], Caputo [3], Chen [10], Chen [11], Chen et al. [12], Meerschaert et al. [13], Nane [14], and references therein). Such equations are obtained from the classical diffusion equation by replacing the first or second-order derivative by a fractional derivative.

In this work, we mainly follow the studies in [1,2,15,16] and references therein, In particular, in [1], the authors showed a connection between the solution to the deterministic counterparts of Equation (1) and the theory of continuous-time random walks (CTRWs for short). In fact, they showed the existence of the stochastic processes which are the limits, in the weak sense, of sequences of CTRWs whose probability density function  $p(t, x)$  are governed by general equations of the form

$$A_n \frac{\partial^{\beta_n}}{\partial t^{\beta_n}} p(t, x) + \dots + A_0 \frac{\partial^{\beta_0}}{\partial t^{\beta_0}} p(t, x) = \mathcal{A}p(t, x),$$

where  $\beta_n, \dots, \beta_0 \in (0, 1]$  and  $\mathcal{A}$  is the infinitesimal generator of a Lévy process. The Riesz–Bessel operator  $(I - \Delta)^{\frac{\alpha}{2}} (-\Delta)^{\frac{\alpha}{2}}$  is a special case of  $\mathcal{A}$ . Hence, this motivates us considering equations of the form (1) containing the Caputo–Djrbashian derivative in this work. On the other hand, it might come natural to add just a additive Gaussian space-time white noise  $\dot{W}(t, x)$  to the deterministic counterparts of Equation (1) and study the equation

$$\begin{cases} \left( \frac{\partial^\beta}{\partial t^\beta} + v(I - \Delta)^{\gamma/2} (-\Delta)^{\alpha/2} \right) u(t, x) = \dot{W}(t, x), \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3}$$

Hence, if we use time fractional Duhamel’s principle (see, for example, [17]), we will get the mild (integral) solution of (1) to be of the form (informally):

$$u(t, x) = (\mathcal{G}u_0)_t(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) \frac{\partial^{1-\beta}}{\partial s^{1-\beta}} (\dot{W}(s, y)) dy dr, \tag{4}$$

where

$$(\mathcal{G}u_0)_t(x) = \int_{\mathbb{R}^d} G_t(x-y) u_0(y) dy.$$

It is not clear what the fractional derivative  $\frac{\partial^{1-\beta}}{\partial s^{1-\beta}}(\dot{W}(s, y))$  means. As explained in [18,19] and etc., one can remove the fractional derivative of the noise term in (4) in the following way. For  $\beta \in (0, 1)$ , define the fractional integral operator  $I_t^{1-\beta}$  as follows:

$$I_t^{1-\beta} u(t, x) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u(s, x)}{(t-s)^\beta} ds, \quad \beta \in (0, 1).$$

Note that (see, for example, [18] and etc.), for every  $\beta \in (0, 1)$  and  $g \in L^\infty(\mathbb{R}_+)$  or  $g \in C(\mathbb{R}_+)$ ,  $\partial_t^\beta I_t^\beta g(t) = g(t)$ . Then, by using the fractional Duhamel's principle, mentioned above, the mild (integral) solution of Equation (3) will be (informally)

$$u(t, x) = (\mathcal{G}u_0)_t(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) W(ds, dy). \quad (5)$$

The time-fractional SPDEs (1) studied in this paper with  $\gamma = 0$  may arise naturally by considering the heat equation in a material with thermal memory; see, for example, [12,18,19], etc.

The fractional SPDEs represent a combination of the deterministic fractional equations and the stochastic integration theory developed by Walsh (see [20], see also Dalang's seminal paper [21]). Several types of fractional SPDEs have been considered in Chen [10], Chen et al. [11], Chen et al. [22], Kim and Kim [12], Chen et al. [23], Foondun and Nane [24], Hu and Hu [25], Liu and Yan [26], Márquez-Carreras [15,16], Mijena and Nane [18,19], and references therein.

In this work, we are interested in space-time fractional SPDEs (1). It includes some widely studied particular cases. We refer, for example, to the classical stochastic heat equation with  $\beta = 1$ ,  $\gamma = 0$  and  $\alpha = 2$  (see, e.g., Dalang [21], Khoshnevisan [27]), the fractional stochastic heat equation with  $\beta = 1$ ,  $\gamma = 0$  and  $\alpha > 0$  (see examples Chen and Dalang [28,29], Foondun and Nane [24], Márquez-Carreras [16], Tudor [30]), the generalized fractional kinetic equation with  $\beta = 0$ ,  $\gamma \geq 0$  and  $\alpha > 0$  (see [15]), the space-time fractional stochastic partial differential equation with  $0 < \beta < 1$ ,  $\gamma = 0$  and  $0 < \alpha \leq 2$  (see [18,19]).

Our paper is motivated by the works of Anh and Leonenko [2], Márquez-Carreras [16], and Mijena and Nane [18,19]. We generalize the results of Márquez-Carreras [16] to the fractional-in-time diffusion equation and of Mijena and Nane [18] to fractional operator including Bessel operator  $(I - \Delta)^{\gamma/2}$ , which is essential for a study of (asymptotically) stationary solutions of Equation (1) (see Anh and Leonenko [2] for some details).

To be more precise, the novelty of this paper is that we extend the result in [15,18,31] by including in the model the Bessel operator  $(I - \Delta)^{\frac{\gamma}{2}}$  with  $\gamma \geq 0$  and by generalizing the stochastic noise, in the sense that we allow a more general structure for the spatial covariance of the Gaussian noise  $W$  in (1) (which is taken to be space-time white noise in [18] and colored by a Riesz kernel in space in [31]). The presence of this Bessel operator brings more flexibility to the model, by including for  $\gamma = 0$  the situation treated in [15,18,31]. From the technical point of view, the appearance of the Bessel operator leads to a new expression of the fundamental solution associated with Equation (1). Indeed, we need new technical estimates for this kernel, which are obtained in Section 2.2. The Bessel operator is also essential in order to get an asymptotically stationary solution, as discussed in Section 4 of our work. Concretely, we study the existence and uniqueness of the solution to Equation (1) under global Lipschitz conditions on diffusion coefficient  $\sigma$  by using the random field approach of Walsh [20] and time fractional Duhamel's principle (see, e.g., [17,18]). Moreover, we study some new properties for the solution to time-space fractional SPDE (1), including an upper bound of the second moment, the Hölder regularity in time and space variables, and the (asymptotically) stationarity of the solution with respect to time and space variables in some particular case.

We organize this paper as follows: In Section 2, we introduce the Gaussian noise  $\dot{W}(t, x)$ , and we prove some properties of Green function  $G_t(x)$  associated with the fractional heat type Equation (13). In Section 3, we give our main result about existence and

uniqueness of the solution and some properties of the solution, including the Hölder regularity and the behavior of the second moment. In Section 4, we study the linear additive case, with zero initial condition, i.e.,  $u_0(x) \equiv 0$  and  $\sigma(x) \equiv 1$ . We see that the solution of (1) is a Gaussian field with zero mean, with stationary increments, and a continuous covariance function in space, while it is not stationary in time but tends to a stationary process when the time goes to infinity.

## 2. Preliminaries

In this section, we recall some basic properties of the stochastic integral with respect to the Gaussian noise  $\dot{W}$  appearing in Equation (1) and some basic facts on the solution to the fractional heat Equation (13).

### 2.1. Gaussian Noise

We denote by  $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$  the space of infinitely differentiable functions on  $\mathbb{R}_+ \times \mathbb{R}^d$  with compact support and by  $\mathcal{S}(\mathbb{R}^d)$  the Schwartz space of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^d$  and let  $\mathcal{S}'(\mathbb{R}^d)$  denote its dual space of rapidly decreasing infinitely differentiable functions on  $\mathbb{R}^d$ . For  $\varphi \in L^1(\mathbb{R}^d)$ , we let  $\mathcal{F}\varphi$  be the Fourier transform of  $\varphi$  defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x) dx, \quad \xi \in \mathbb{R}^d. \tag{6}$$

We begin by introducing the framework in [21]. Let  $\mu$  be a non-negative tempered measure on  $\mathbb{R}^d$ , i.e., a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^m \mu(d\xi) < \infty, \tag{7}$$

for some  $m > 0$ . Since the integrand is non-increasing in  $m$ , we may assume that  $m \geq 1$  is an integer. Note that  $1 + |\xi|^2$  behaves like a constant around 0, and like  $|\xi|^2$  at  $\infty$ , and hence (7) is equivalent to

$$\int_{|\xi| \leq 1} \mu(d\xi) < \infty \quad \text{and} \quad \int_{|\xi| \geq 1} \frac{1}{|\xi|^{2m}} \mu(d\xi) < \infty,$$

for some integer  $m \geq 1$ .

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$  be the Fourier transform of a non-negative tempered measure  $\mu$  in  $\mathcal{S}'(\mathbb{R}^d)$ , which is

$$\int_{\mathbb{R}^d} f(x) \varphi(x) dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where  $\mathcal{F}$  denotes the Fourier transform given by (6). Simple properties of the Fourier transform yield that, for any  $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) f(x - y) \psi(y) dx dy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(d\xi), \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^d). \tag{8}$$

An approximation argument shows that the previous equality also holds for indicator functions  $\varphi = 1_A$  and  $\psi = 1_B$  with  $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ , where  $\mathcal{B}_b(\mathbb{R}^d)$  denotes the class of bounded Borel sets of  $\mathbb{R}^d$ , which is

$$\int_A \int_B f(x - y) dx dy = \int_{\mathbb{R}^d} \mathcal{F}1_A(\xi) \overline{\mathcal{F}1_B(\xi)} \mu(d\xi). \tag{9}$$

In this article, we consider a zero-mean Gaussian process  $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$  with covariance

$$\mathbf{E}(W(t, A)W(s, B)) = (t \wedge s) \int_A \int_B f(x - y) dx dy,$$

on a complete probability space  $(\Omega, \mathcal{F}, P)$ .

Let  $\mathcal{E}$  be the set of linear combinations of elementary functions  $\{1_{[0,t] \times A}, t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ . With the Gaussian process  $W$ , we can associate a canonical Hilbert space  $\mathcal{H}$  which is defined as the closure of  $\mathcal{E}$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  defined by

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t, x) f(x - y) \phi(t, y) dx dy dt.$$

Alternatively,  $\mathcal{H}$  can be defined as the completion of  $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ .

We denote by  $W(\varphi)$  the random field indexed by functions  $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$  and for all  $\varphi, \phi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ , we have

$$\begin{aligned} \mathbf{E}(W(\varphi)W(\phi)) &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t, x) f(x - y) \phi(t, y) dx dy dt \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathcal{F}\varphi(t, \cdot)(\xi) \overline{\mathcal{F}\phi(t, \cdot)(\xi)} \mu(d\xi) dt, \end{aligned} \tag{10}$$

where  $\mathcal{F}\varphi(t, \cdot)(\xi)$  denotes the Fourier transform with respect to the space variable of  $\varphi(t, x)$  only. Hence,  $W(\varphi)$  can be represented as

$$W(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \varphi(t, x) W(dx, dt).$$

Note that  $W(\varphi)$  is  $\mathcal{F}_t$ -measurable whenever  $\varphi$  is supported on  $[0, t] \times \mathbb{R}^d$ .

**Remark 1.** Since the spectral measure  $\mu$  is non-trivial positive tempered measure, we can ensure that there exist positive constants  $c_1, c_2$  and  $k$  such that

$$c_1 < \int_{\{|\xi| < k\}} \mu(d\xi) < c_2. \tag{11}$$

As usual, the Gaussian process  $W$  can be extended to a *worthy martingale measure*, in the sense given by Walsh [20]. Dalang [21] presented an extension of Walsh’s stochastic integral that requires the following integrability condition in terms of the Fourier transform of  $G$

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_t(\cdot)(\xi)|^2 < \infty, \tag{12}$$

where  $G$  is the fundamental solution of

$$\left( \frac{\partial^\beta}{\partial t^\beta} + \nu(I - \Delta)^{\gamma/2} (-\Delta)^{\alpha/2} \right) G_t(x) = 0. \tag{13}$$

Provided that (12) is satisfied and assuming conditions on  $\sigma(\cdot)$  that will be described later, following Walsh [20], we will understand a solution of (1) to be a jointly measurable adapted process  $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$  satisfying the integral equation

$$u(t, x) = (\mathcal{G}u_0)_t(x) + \lambda \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y) \sigma(u(s, y)) W(ds, dy), \tag{14}$$

where

$$(\mathcal{G}u_0)_t(x) = \int_{\mathbb{R}^d} G_t(x - y) u_0(y) dy,$$

and the stochastic integral in (14) is defined with respect to the  $\mathcal{F}$ -martingale measure  $W(t, A)$ . Next, we give the meaning of Walsh–Dalang integrals that is used in (14). (For the details, we refer the readers to Dalang [21]).

1. We say that  $(t, x) \rightarrow \Phi_t(x)$  is an elementary random field when there exist  $0 \leq a < b$ , a  $\mathcal{F}_a$ -measurable random variable  $X \in L^2(\Omega)$  and a deterministic function  $\phi \in L^2(\mathbb{R}^d)$  such that

$$\Phi_t(x) = X1_{[a,b)}(t)\phi(x), \quad t > 0, x \in \mathbb{R}^d.$$

2. If  $h = h_t(x)$  is non-random and  $\Phi$  is elementary as above, then we set

$$\int h\Phi dW := X \int_{[a,b) \times \mathbb{R}^d} h_t(x)\phi(x)W(dt, dx). \tag{15}$$

3. The stochastic integral in (15) is a Wiener integral, and it is well defined if and only if  $h_t(x)\phi(x) \in L^2([a, b) \times \mathbb{R}^d)$ .
4. Under the above notation, we have the Walsh isometry

$$\mathbf{E} \left( \left| \int h\Phi dW \right|^2 \right) = \int_0^T ds \int_{\mathbb{R}^d} dy h_s(y)^2 \mathbf{E}(|\Phi_s(y)|^2).$$

### 2.2. Some Properties of the Fundamental Solution

We will give some estimates for the fundamental solution associated with Equation (1). The properties of this fundamental solution will play an important role in the sequel.

Let  $G_t(x)$  be the fundamental solution of the fractional kinetic Equation (13) with  $\beta \in (0, 1], \nu > 0$ , and  $\gamma \geq 0, \alpha > 0$ . Anh and Leonenko [2] showed that Equation (13) is equivalent to the Cauchy problem:

$$(\mathcal{D}_t^\beta \mathcal{F}G_t(\cdot))(\xi) + \nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2} \mathcal{F}G_t(\cdot)(\xi) = 0, \quad \mathcal{F}G_0(\cdot)(\xi) = 1 \tag{16}$$

and they also have proved that Equation (16) has a unique solution given by

$$\mathcal{F}G_t(\cdot)(\xi) = E_\beta(-\nu t^\beta |\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}), \quad \beta > 0, \tag{17}$$

where

$$E_\beta(x) = \sum_{j=0}^\infty \frac{x^j}{\Gamma(1 + \beta j)}, \quad x > 0, \tag{18}$$

is the Mittag–Leffler function of order  $\beta$ . The inverse Fourier transform yields that

$$G_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(\xi,x)} E_\beta(-\nu t^\beta |\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}) d\xi. \tag{19}$$

We know that

$$E_\beta(-\nu t^\beta |\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}) \in L^1(\mathbb{R}^d), \tag{20}$$

for every  $0 < \beta \leq 1$  if  $\alpha + \gamma > d$ . From this range, we see the role played by the parameter  $\gamma$  in Equation (17).

Moreover, one has the uniform estimates of the Mittag–Leffler function (e.g., Theorem 4 in Simon [32])

$$\frac{1}{1 + \Gamma(1 - \beta)x} \leq E_\beta(-x) \leq \frac{1}{1 + \Gamma(1 + \beta)^{-1}x}, \quad \text{for } x > 0. \tag{21}$$

The following lemma gives a sharp estimate for the  $L^2$ -norm (in time) of the Green kernel. It extends Lemma 1 in [18] and Lemma 2.1 in [31].

**Lemma 1.** For  $0 < \beta < 1$  and  $d < 2(\alpha + \gamma)$ , we have the following

$$\int_{\mathbb{R}^d} G_t(x)^2 dx \leq C_2 t^{-\frac{\beta d}{\alpha + \gamma}}. \tag{22}$$

where  $B\left(\frac{d}{\alpha+\gamma}, 2 - \frac{d}{\alpha+\gamma}\right)$  is a Beta function. The (strictly) positive constant is given by  $C_2 = \frac{B\left(\frac{d}{\alpha+\gamma}, 2 - \frac{d}{\alpha+\gamma}\right)}{\alpha+\gamma} \left(\frac{\Gamma(1-\beta)}{\nu}\right)^{\frac{d}{\alpha+\gamma}} \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d}$ .

**Proof.** Using the Plancherel’s identity and the equality (17), we can write

$$\begin{aligned} \int_{\mathbb{R}^d} G_t(x)^2 dx &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}G_t(\cdot)(\xi)|^2 d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| E_\beta\left(-\nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}\right) \right|^2 d\xi \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \int_0^{+\infty} r^{d-1} \left( E_\beta\left(-\nu t^\beta r^\alpha (1 + r^2)^{\gamma/2}\right) \right)^2 dr, \end{aligned}$$

where we have used the integration in polar coordinates in the last equation above and the positive constant resulting from the integration over the angular spherical coordinates. Now using the upper bound in (21) and the fact  $r^\alpha (1 + r^2)^{\gamma/2} \geq r^{\alpha+\gamma}$  with  $r > 0$ , we obtain, with the change of variable formula  $z = \Gamma(1 + \beta)^{-1} \nu t^\beta r^{\alpha+\gamma}$ ,

$$\begin{aligned} \int_0^{+\infty} r^{d-1} \left( E_\beta\left(-\nu t^\beta r^\alpha (1 + r^2)^{\gamma/2}\right) \right)^2 dr &\leq \int_0^{+\infty} r^{d-1} \frac{1}{(1 + \Gamma(1 + \beta)^{-1} \nu t^\beta r^{\alpha+\gamma})^2} dr \\ &= \frac{1}{\alpha + \gamma} \left( \frac{\Gamma(1 + \beta)}{\nu} \right)^{\frac{d}{\alpha+\gamma}} t^{-\frac{\beta d}{\alpha+\gamma}} \int_0^\infty z^{\frac{d}{\alpha+\gamma}-1} (1 + z)^{-2} dz. \end{aligned}$$

Hence,  $\int_0^\infty z^{\frac{d}{\alpha+\gamma}-1} (1 + z)^{-2} dz < \infty$  if and only if  $d < 2(\alpha + \gamma)$ . In this case, we have

$$\int_0^\infty z^{\frac{d}{\alpha+\gamma}-1} (1 + z)^{-2} dz = B\left(\frac{d}{\alpha + \gamma}, 2 - \frac{d}{\alpha + \gamma}\right),$$

where  $B\left(\frac{d}{\alpha+\gamma}, 2 - \frac{d}{\alpha+\gamma}\right)$  is a Beta function. Then, we can conclude the proof of upper bound in (22).  $\square$

We now prove (12) under an integrability condition on the spectral measure  $\mu$  given as follows, which is also known as the Dalang’s condition (see, for example, [21]).

**Hypothesis 1.** Assume that the spectral measure  $\mu$  associated with the Gaussian noise  $\dot{W}$  satisfies

$$\int_{\mathbb{R}^d} \left( \frac{1}{1 + |\xi|^2} \right)^q \mu(d\xi) < \infty, \tag{23}$$

with the parameter  $q$  satisfying

$$q = \begin{cases} \alpha + \gamma, & \text{if } 0 < \beta < \frac{1}{2}, \\ \frac{\alpha + \gamma}{2}, & \text{if } \beta = \frac{1}{2}, \\ \frac{\alpha + \gamma}{2\beta}, & \text{if } \frac{1}{2} < \beta < 1. \end{cases} \tag{24}$$

**Remark 2.** If the parameter  $\beta = 1$ , Equation (1) reduces to the SPDE (1.1) studied in Márquez-Carreras [15], in which it is assumed (23) with  $q = \frac{\alpha+\gamma}{2}$ . Thus, when  $\beta$  is close to one, the exponent  $q$  in (24) coincides with the exponent studied in Lemma 2.1 in Márquez-Carreras [15]. On the other hand, our assumption (23) is weaker when  $\beta$  is close to zero.

Let us now recall some of the main examples of spatial covariances for the noise which will be our guiding examples in the remainder of the present paper. Below, we denote by  $|x|$  the Euclidean norm of  $x \in \mathbb{R}^d$ .

**Example 1.** Let  $f(x) = \prod_{i=1}^d H_i(2H_i - 1)|x_i|^{2H_i-2}$  with  $1/2 < H_i < 1$  for  $i = 1, \dots, d$ . Then,  $\mu(d\xi) = \prod_{i=1}^d H_i(2H_i - 1)|\xi_i|^{-2H_i+1}d\xi$ . Thus, (23) is equivalent to  $\sum_{i=1}^d (2H_i - 1) > d - 2(\alpha + \gamma)$  if  $0 < \beta < 1/2$ , it is equivalent to  $\sum_{i=1}^d (2H_i - 1) > d - (\alpha + \gamma)$  if  $\beta = 1/2$  and when  $1/2 < \beta < 1$ , the condition (23) is equivalent to  $\sum_{i=1}^d (2H_i - 1) > d - \frac{\alpha+\gamma}{\beta}$ .

**Example 2.** Let  $f(x) = \gamma_{\delta,d} = |x|^{-(d-\delta)}$  be the Riesz kernel of order  $\delta \in (0, d)$ , then  $\mu(d\xi) = |\xi|^{-\delta}d\xi$  and (23) is equivalent to  $2(\alpha + \gamma) + \delta > d$  if  $0 < \beta < 1/2$ , (23) is equivalent to  $(\alpha + \gamma) + \delta > d$  if  $\beta = 1/2$  and (23) is equivalent to  $\frac{\alpha+\gamma}{\beta} + \delta > d$  if  $1/2 < \beta < 1$ . This example is also considered in [31]. Their condition (used in Theorem 1.3 in this reference) reads  $\frac{\alpha}{\beta} + \delta > d$ . Our assumption (23) gives more flexibility when  $\beta$  is close to zero but as well as for  $\beta$  close to 1 (because of the new parameter  $\gamma$  in the expression of the Bessel operator  $(I - \Delta)^{\frac{\gamma}{2}}$ ).

**Example 3.** For the Bessel kernel of order  $\tau > 0$  given by  $f(x) = \gamma_\tau \int_0^\infty \omega^{\frac{\tau-d}{2}-1} e^{-\omega} e^{-\frac{|x|^2}{4\omega}} d\omega$ . Then,  $\mu(d\xi) = (1 + |\xi|^2)^{-\frac{\tau}{2}}d\xi$ . Thus, (23) is equivalent to  $2(\alpha + \gamma) + \tau > d$  if  $0 < \beta < 1/2$ , (23) is equivalent to  $(\alpha + \gamma) + \tau > d$  if  $\beta = 1/2$  and condition (23) is equivalent to  $\frac{\alpha+\gamma}{\beta} + \tau > d$  if  $1/2 < \beta < 1$ .

**Example 4.** Let  $f(0) < \infty$  (i.e.,  $\mu$  is a finite measure). It corresponds to a spatially smooth noise  $\dot{W}$ .

**Example 5.** Suppose  $d = 1$  and  $f = \delta_0$  (i.e.,  $\mu$  is the Lebesgue measure). This corresponds to a (rougher) noise  $\dot{W}$ , which is white in the spatial variable.

For any  $t \in \mathbb{R}_+$ , denote by

$$N_t(\xi) = \int_0^t |\mathcal{F}G_u(\cdot)(\xi)|^2 du. \tag{25}$$

Then, we have the following

**Proposition 1.** Assuming that  $t \in \mathbb{R}_+$  and  $\xi \in \mathbb{R}^d$ , there exist (strictly) positive constants  $C_{2,i}(t), i = 1, 2, 3, 4$  (depending on  $t$ ) such that

$$N_t(\xi) \leq C_{2.2}(t) \left( \frac{1}{1 + |\xi|^2} \right)^{\alpha+\gamma}, \quad \text{if } 0 < \beta < 1/2, \tag{26}$$

$$N_t(\xi) \leq C_{2.3}(t) \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{\alpha+\gamma}{2}}, \quad \text{if } \beta = 1/2, \tag{27}$$

and

$$N_t(\xi) \leq C_{2.4}(t) \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{\alpha+\gamma}{2\beta}}, \quad \text{if } 1/2 < \beta < 1. \tag{28}$$

The “constants” are defined as follows:

$$\begin{aligned}
 C_{2,2}(t) &= t + \frac{2^{\alpha+\gamma}\Gamma(1+\beta)^2}{\nu^2(1-2\beta)}t^{1-2\beta}, \\
 C_{2,3}(t) &= t + 2\nu^{-1}\Gamma(3/2)2^{\frac{\alpha+\gamma}{2}}t^{1/2}, \\
 C_{2,4}(t) &= t + \frac{1}{2\beta-1}\Gamma(1+\beta)^{1/\beta}\nu^{-1/\beta}2^{\frac{\alpha+\gamma}{2\beta}}.
 \end{aligned}$$

**Proof.** For any  $t \in \mathbb{R}_+$ , from Equations (17) and (25), we can rewrite  $N_t(\xi)$  defined by (25) as

$$N_t(\xi) = \int_0^t \left| E_\beta(-\nu u^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) \right|^2 du.$$

We firstly prove the upper bound for  $N_t(\xi)$ . By using the upper bound in (21) and change of variable  $x = \nu u^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}$ , one obtains

$$N_t(\xi) = \frac{1}{\beta} \left( \frac{1}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-1} E_\beta^2(-x) dx.$$

We will divide into two cases to estimate it according to the value of  $|\xi|$ . If  $|\xi| \leq 1$ , and we claim that

$$\begin{aligned}
 N_t(\xi) 1_{|\xi| \leq 1} &\leq \frac{1}{\beta} \left( \frac{1}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-1} \left( \frac{1}{1 + \Gamma(1+\beta)^{-1}x} \right)^2 dx \\
 &= t.
 \end{aligned}$$

If  $|\xi| > 1$  and  $1/2 < \beta < 1$ , we have

$$\begin{aligned}
 N_t(\xi) 1_{|\xi| > 1} &\leq \frac{1}{\beta} \left( \frac{1}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-1} \left( \frac{1}{1 + \Gamma(1+\beta)^{-1}x} \right)^2 dx \\
 &= \frac{1}{\beta} \left( \frac{\Gamma(1+\beta)}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu \Gamma(1+\beta)^{-1} t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-1} (1+x)^{-2} dx \\
 &\leq \frac{1}{\beta} \left( \frac{\Gamma(1+\beta)}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu \Gamma(1+\beta)^{-1} t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} (1+x)^{\frac{1}{\beta}-3} dx \\
 &\leq \frac{1}{2\beta-1} \left( \frac{\Gamma(1+\beta)}{\nu} \right)^{\frac{1}{\beta}} \left( \frac{1}{|\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \\
 &\leq \frac{1}{2\beta-1} \left( \frac{\Gamma(1+\beta)}{\nu} \right)^{\frac{1}{\beta}} 2^{\frac{\alpha+\gamma}{2\beta}} \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{\alpha+\gamma}{2\beta}}.
 \end{aligned}$$

On the other hand, with  $|\xi| > 1$  and  $0 < \beta < 1/2$ , one obtains

$$\begin{aligned}
 N_t(\xi) 1_{|\xi| > 1} &\leq \frac{1}{\beta} \left( \frac{\Gamma(1+\beta)}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\nu \Gamma(1+\beta)^{-1} t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-3} dx \\
 &= \frac{\Gamma(1+\beta)^2}{\nu^2(1-2\beta)} t^{1-2\beta} \left( \frac{1}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^2 \\
 &\leq \frac{2^{\alpha+\gamma}\Gamma(1+\beta)^2}{\nu^2(1-2\beta)} t^{1-2\beta} \left( \frac{1}{1 + |\xi|^2} \right)^{\alpha+\gamma}.
 \end{aligned}$$

For the critical case  $\beta = 1/2$ , one obtains that

$$\begin{aligned} N_t(\xi)1_{|\xi|>1} &\leq 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \int_0^{\nu\Gamma(3/2)^{-1}t^{1/2}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} x(1+x)^{-2} dx \\ &\leq 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \int_0^{\nu\Gamma(3/2)^{-1}t^{1/2}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} (1+x)^{-1} dx \\ &= 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \ln \left( 1 + \nu\Gamma(3/2)^{-1}t^{1/2}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2} \right) \\ &\leq 2\nu^{-1}\Gamma(3/2)t^{1/2}2^{\frac{\alpha+\gamma}{2}} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2}}. \end{aligned}$$

Then, combining the above estimates for  $N_t(\xi)$  with  $|\xi| \leq 1$  and  $|\xi| > 1$ , respectively, we can conclude the proof of bounds (26)–(28).  $\square$

**Remark 3.** From the above result, we see that Hypothesis 1 implies condition (12). In particular, the estimates (26)–(28) give the existence of the solution in the linear additive noise case ( $\sigma = 1$ ).

### 3. Existence and Uniqueness

In this section, we will prove the existence and uniqueness of the mild solution to Equation (14). We first introduce a stronger integrability condition on the spectral measure  $\mu$  than Hypothesis 1. While the existence and uniqueness of the solution can be obtained under Hypothesis 1, the new assumption presented below will be needed in order to prove certain properties of the solution.

**Hypothesis 2.** Assume that the spectral measure  $\mu$  associated with  $\dot{W}$  satisfies

$$\int_{\mathbb{R}^d} \left( \frac{1}{1+|\xi|^2} \right)^\eta \mu(d\xi) < \infty, \tag{29}$$

with some parameter  $\eta$  satisfying

$$\eta \in \begin{cases} (0, \alpha + \gamma), & \text{if } 0 < \beta < \frac{1}{2}, \\ \left(0, \frac{\alpha + \gamma}{2}\right), & \text{if } \beta = \frac{1}{2}, \\ \left(0, \frac{\alpha + \gamma}{2\beta}\right), & \text{if } \frac{1}{2} < \beta < 1. \end{cases} \tag{30}$$

We will need the following estimates for the Green function given by (19) (their proof is given in Appendix A).

**Proposition 2.** Supposing  $\beta \in (0, 1)$ , then we have the following estimates for the temporal and spatial increments of the Green function  $G_t(x)$  given by (19).

1. Under Hypothesis 1, for any  $t, t' \in \mathbb{R}_+$  such that  $t' < t$  and  $x \in \mathbb{R}^d$ , we have

$$\int_0^{t'} ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi) - \mathcal{F}G_{t'-s}(x - \cdot)(\xi)|^2 \leq C_{3.1}|t - t'|^{2\beta}, \tag{31}$$

with  $C_{3.1} = t^{1-2\beta} \int_{|\xi| \leq 1} \mu(d\xi) + t^{-2\beta} \int_{|\xi| > 1} N_{t'}(\xi) \mu(d\xi)$ .

2. Under Hypothesis 2, for any  $t, t' \in \mathbb{R}_+$  such that  $t' < t$  and  $x \in \mathbb{R}^d$ , we have

$$\int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi)|^2 \leq \begin{cases} C_{3.2}|t - t'|^{1-2\beta}, & \text{if } 0 < \beta < \frac{1}{2}, \\ C_{3.3}|t - t'|^{\frac{1}{2}}, & \text{if } \beta = \frac{1}{2}, \\ C_{3.4}|t - t'|, & \text{if } \frac{1}{2} < \beta < 1. \end{cases} \tag{32}$$

with

$$\begin{aligned} C_{3.2} &= |t - t'|^{2\beta} \int_{|\xi| \leq 1} \mu(d\xi) + \frac{\Gamma(1 + \beta)2^{2\alpha+\gamma}}{\nu^2(1 - 2\beta)} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\alpha+\gamma} \mu(d\xi), \\ C_{3.3} &= \int_{|\xi| \leq 1} \mu(d\xi) |t - t'|^{\frac{1}{2}} + 2^{1+\frac{\alpha+\gamma}{2}} \frac{\Gamma(3/2)}{\nu} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\frac{\alpha+\gamma}{2}} \mu(d\xi), \\ C_{3.4} &= \int_{|\xi| \leq 1} \mu(d\xi) + \frac{c}{2\beta - 1} \left(\frac{\Gamma(1 + \beta)}{\nu}\right)^{\frac{1}{\beta}} 2^{\frac{\alpha+\gamma}{2\beta}} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\frac{\alpha+\gamma}{2\beta}} \mu(d\xi). \end{aligned}$$

3. Under Hypothesis 2, for any  $t \in \mathbb{R}_+$  and  $x, x' \in \mathbb{R}^d$ ,  $\rho_1 \in (0, \alpha + \gamma - \eta)$ ,  $\rho_2 \in (0, \frac{\alpha+\gamma}{2} - \eta)$  and  $\rho_3 \in (0, \frac{\alpha+\gamma}{2\beta} - \eta)$ , we have

$$\begin{aligned} &\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi) - \mathcal{F}G_{t-s}(x' - \cdot)(\xi)|^2 \\ &\leq \begin{cases} C_{3.5}|x' - x|^{2\rho_1}, & \text{if } 0 < \beta < \frac{1}{2}, \\ C_{3.6}|x' - x|^{2\rho_2}, & \text{if } \beta = \frac{1}{2}, \\ C_{3.7}|x' - x|^{2\rho_3}, & \text{if } \frac{1}{2} < \beta < 1. \end{cases} \end{aligned} \tag{33}$$

with

$$\begin{aligned} C_{3.5} &= Ct \int_{|\xi| \leq 1} \mu(d\xi) + C \frac{t^{1-2\beta}}{1 - 2\beta} \left(\frac{\Gamma(1 + \beta)}{\nu}\right)^2 2^{\alpha+\gamma-\rho_1} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\alpha+\gamma-\rho_1} \mu(d\xi), \\ C_{3.6} &= Ct \int_{|\xi| \leq 1} \mu(d\xi) + C 2^{1+\frac{\alpha}{2}-\rho_2} t^{\frac{1}{2}} \frac{\Gamma(3/2)}{\nu} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\frac{\alpha+\gamma}{2}-\rho_2} \mu(d\xi), \\ C_{3.7} &= Ct \int_{|\xi| \leq 1} \mu(d\xi) + C \frac{1}{2\beta - 1} \left(\frac{\Gamma(1 + \beta)}{\nu}\right)^{\frac{1}{\beta}} 2^{\frac{\alpha+\gamma}{2\beta}-\rho_3} \int_{|\xi| > 1} \left(\frac{1}{1 + |\xi|^2}\right)^{\frac{\alpha+\gamma}{2\beta}-\rho_3} \mu(d\xi). \end{aligned}$$

Notice that all the constants depend on  $t$  although we omit it in the notation.

**Remark 4.**

1. Our results of Proposition 2 extend the results in Mijena and Nane [18] to the space-time fractional SPDE with colored Gaussian noises and Khoshnevisan [27] to space-time fractional SPDE, respectively.
2. The above Proposition 2 also extends the results in Márquez-Carreras [15,16] to space-time fractional kinetic equation with spatially homogeneous Gaussian noise.

Let us introduce some additional conditions that we need in order to prove our main results. The first condition is required for the existence-uniqueness result as well as for the upper bound on the second moment of the solution.

**Assumption 1.**

1. We assume that the initial condition is a non-random bounded non-negative function  $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ .
2. We assume that  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is Lipschitz continuous satisfying  $|\sigma(x)| \leq L_\sigma|x|$  with  $L_\sigma$  being a positive constant. Moreover, for all  $x, y \in \mathbb{R}^d$ ,

$$|\sigma(x) - \sigma(y)| \leq L_\sigma|x - y|. \tag{34}$$

We may assume, with loss of generality, that  $L_\sigma$  is also greater than  $\sigma(0)$ . Since  $|\sigma(x)| \leq |\sigma(0)| + L_\sigma|x|$ , it follows that  $|\sigma(x)| \leq L_\sigma(1 + |x|)$  for all  $x \in \mathbb{R}^d$ .

Now, we can prove the existence and uniqueness of mild solution of Equation (1) given by (14).

**Theorem 1.** Under Assumption 1 and assuming that the spectral measure  $\mu$  satisfies Hypothesis 1, then Equation (14) has a unique adapted solution and for any  $t \in \mathbb{R}_+$  and  $p \geq 1$ ,

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \mathbf{E}(|u(t, x)|^p) < \infty.$$

Moreover, this unique solution is mean-square continuous.

**Proof.** The proof of existence and uniqueness is standard based on Picard’s iterations. For more information, see, e.g., Walsh [20], Dalang [21]. We give a sketch of the proof. Define

$$\begin{aligned} u^{(0)}(t, x) &= (\mathcal{G}u_0)_t(x), \\ u^{(n+1)}(t, x) &= (\mathcal{G}u_0)_t(x) + \lambda \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(u^{(n)}(s, y))W(ds, dy), \quad n \geq 0. \end{aligned} \tag{35}$$

We could easily prove that the sequence  $\{u^{(n+1)}(t, x), n \geq 0\}$  is well-defined and then using Burkholder’s inequality, we can show that, for any  $n \geq 0$  and  $t \in \mathbb{R}_+$ ,

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \mathbf{E}(|u^{(n+1)}(t, x)|^2) < \infty. \tag{36}$$

Moreover, by using an extension of Gronwall’s lemma (for example, see Lemma 15 in Dalang [21]),

$$\sup_{n \geq 0} \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \mathbf{E}(|u^{(n+1)}(t, x)|^2) < \infty. \tag{37}$$

The same kind of arguments allow us to check (36) and (37), changing the power 2 for  $p > 2$ . Moreover, we can also prove that  $\{u^{(n+1)}(t, x), n \geq 0\}$  converges uniformly in  $L^p$ , denoting this limit by  $u(t, x)$ . We can check that  $u(t, x)$  satisfies Equation (14). Then, it is adapted and satisfies

$$\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \mathbf{E}(|u(t, x)|^p) < \infty.$$

The uniqueness can be accomplished by a similar argument.

The key to the continuity is to show that these Picard iterations are mean-square continuous. Then, it can be easily extended to  $u(t, x)$ . In order to show the ideas of the mean-square continuity, we give some steps of the proof for  $\{u^{(n+1)}(t, x), n \geq 0\}$ . As for the time increments, we have, for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$  and  $\delta > 0$  such that  $t + \delta \in \mathbb{R}_+$ ,

$$\begin{aligned}
 & \mathbf{E} \left[ |u^{(n+1)}(t + \delta, x) - u^{(n+1)}(t, x)|^2 \right] \\
 & \leq \lambda^2 \mathbf{E} \left[ \left| \int_0^t \int_{\mathbb{R}^d} [G_{t+\delta-u}(x-y) - G_{t-u}(x-y)] \sigma(u^{(n)}(u, y)) W(ds, dy) \right|^2 \right] \\
 & + \lambda^2 \mathbf{E} \left[ \left| \int_t^{t+\delta} \int_{\mathbb{R}^d} G_{t+\delta-u}(x-y) \sigma(u^{(n)}(u, y)) W(ds, dy) \right|^2 \right].
 \end{aligned} \tag{38}$$

Using the conditions imposed on  $\sigma$  and (36), we can bound the first term in (38) by

$$C \int_0^t du \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t+\delta-u}(\cdot)(\xi) - \mathcal{F}G_{t-u}(\cdot)(\xi)|^2,$$

which converges to zero as  $\delta \downarrow 0$  according to (31). The second term in (38) can be proved by using the similar arguments by using (32). This proves the right continuity. The left continuity can be proved in the same way.

Concerning the spatial increment, we have, for any  $(t, x), (t, z) \in \mathbb{R}_+ \times \mathbb{R}^d$ ,

$$\begin{aligned}
 & \mathbf{E} \left[ |u^{(n+1)}(t, x) - u^{(n+1)}(t, z)|^2 \right] \\
 & \leq C \lambda^2 \int_0^t du \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-u}(x - \cdot)(\xi) - \mathcal{F}G_{t-u}(z - \cdot)(\xi)|^2 \\
 & \leq C \lambda^2 \int_0^t du \int_{\mathbb{R}^d} \mu(d\xi) |e^{i(x-z, \xi)} - 1|^2 |\mathcal{F}G_{t-u}(z - \cdot)(\xi)|^2.
 \end{aligned} \tag{39}$$

Then, thanks to (33), we can prove that the right hand of (39) converges to zero as  $|x - z| \downarrow 0$ .  $\square$

**Remark 5.** Let us recall that Equation (1) with  $\beta = 1$  (fractional in space stochastic kinetic equation with factorization of the Laplacian) has been studied by Márquez-Carreras [15]. In this case, the Mittag-Leffler function reduces to  $E_1(-x) = e^{-x}, x \geq 0$ .

When  $\gamma = 0$  and spatial kernel  $f(\cdot)$  is the Riesz kernel, then the Equation (1) reduces to the SPDEs studied in Mijena and Nane [18,19]. In this reference, the authors studied the existence, uniqueness, and intermittence of the mild solution for the space-time fractional stochastic partial differential Equations (1).

For  $\gamma = 0$  and  $\alpha = 2$ , the SPDE (1) reduces to the classical stochastic heat equation studied by many authors; see, for example, Dalang [21] and references therein.

Now, let us make the following assumption on the spectral measure  $\mu$  in order to obtain a precise estimate for the upper bound of the second moment of the mild solution of (1).

**Assumption 2.** We assume that the spectral measure  $\mu$  satisfies

$$\mu(d\xi) \asymp |\xi|^{-\delta} d\xi, \quad \text{with } 0 < \delta < d. \tag{40}$$

The symbol “ $\asymp$ ” means that, for every non-negative function  $h$  such that the integral in (41) are finite, there exist two positive and finite constants  $C$  and  $C'$  which may depend on  $h$  such that

$$C' \int_{\mathbb{R}^d} h(\xi) |\xi|^{-\delta} d\xi \leq \int_{\mathbb{R}^d} h(\xi) \mu(d\xi) \leq C \int_{\mathbb{R}^d} h(\xi) |\xi|^{-\delta} d\xi. \tag{41}$$

**Remark 6.** The Riesz kernel of order  $\delta \in (0, d)$  given in Example 2 obviously satisfies (40). The Bessel kernel given in Example 3 satisfies (40) and the constants in (41) are  $C = 1$  and  $C' > 0$  depending on  $\delta$  and  $d$  (see [33]).

We have the following results concerning the upper bound on the second moment of the mild solution to Equation (1).

**Theorem 2.** Suppose  $0 < d - \delta < (\alpha + \gamma)$  and  $0 < \beta < 1$ , if the spectral measure  $\mu$  associated with the noise  $\dot{W}$  satisfies Assumption 2, then, under the Assumption 1, there exist two positive and finite constants  $c$  and  $c'$  such that

$$\sup_{x \in \mathbb{R}^d} \mathbf{E} \left( |u(t, x)|^2 \right) \leq c \exp \left\{ c' \lambda^{\frac{2(\alpha+\gamma)}{(\alpha+\gamma)-\beta(d-\delta)}} t \right\}, \tag{42}$$

for all  $t > 0$ .

**Remark 7.** This theorem implies that, under some conditions, there exists some positive constant  $C$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbf{E} |u(t, x)|^2 \leq C \lambda^{\frac{2(\alpha+\gamma)}{(\alpha+\gamma)-\beta(d-\delta)}},$$

for any fixed  $x \in \mathbb{R}^d$ .

Before giving the proof of Theorem 2, we state an important lemma needed in the proof of this theorem.

**Lemma 2.** Supposing  $0 < d - \delta < (\alpha + \gamma)$  and  $0 < \beta < 1$ , then there exists a positive constant  $C$  such that, for all  $x, y \in \mathbb{R}^d$ , we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - z_1) G_t(y - z_2) f(z_1 - z_2) dz_1 dz_2 \leq C t^{-\frac{\beta(d-\delta)}{\alpha+\gamma}}.$$

**Proof.** If we fix  $t \in \mathbb{R}_+$ , for any  $x, y \in \mathbb{R}^d$ , then, by using (8), we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - z_1) G_t(y - z_2) f(z_1 - z_2) dz_1 dz_2 = \int_{\mathbb{R}^d} \mathcal{F}G_t(x - \cdot)(\xi) \overline{\mathcal{F}G_t(y - \cdot)(\xi)} \mu(d\xi).$$

Recall that the spectral measure  $\mu$  satisfies (40) (i.e., (41)) in Assumption 2. Thus, according to (17), we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - z_1) G_t(y - z_2) f(z_1 - z_2) dz_1 dz_2 \leq C_{\delta,d} \int_{\mathbb{R}^d} E_{\beta}^2(-vt^{\beta} |\xi|^{\alpha} (1 + |\xi|^2)^{\gamma/2}) |\xi|^{-\delta} d\xi. \tag{43}$$

Then, by the similar arguments in the proof of Lemma 1, based on the estimate on the Mittag-Leffler Function (21), we can conclude the proof.  $\square$

Now, we are ready to give the proof of Theorem 2. The idea used here is essentially due to [24].

**Proof of Theorem 2.** Recall the iterated sequences  $\{u^{(n)}(t, x), n \geq 0, (t, x) \in [0, T] \times \mathbb{R}^d\}$  given by (35). Define

$$\begin{aligned} D_n(t, x) &:= \mathbf{E} \left| u^{(n+1)}(t, x) - u^{(n)}(t, x) \right|^2, \\ H_n(t) &= \sup_{x \in \mathbb{R}^d} D_n(t, x), \\ \Xi(t, y, n) &= \left| \sigma(u^{(n)}(t, y)) - \sigma(u^{(n-1)}(t, y)) \right|. \end{aligned}$$

We will prove the result for  $t \in [0, T]$ , where  $T > 0$  is some fixed number. We now use this notation together with the covariance formula (10) and the Assumption 1 on  $\sigma$  to write

$$D_n(t, x) = \lambda^2 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{t-s}(x - y) G_{t-s}(x - z) \mathbf{E}(\Xi(s, y, n) \Xi(s, z, n)) f(y - z) dy dz ds.$$

Now, we estimate the expectation on the right hand side using Cauchy–Schwartz inequality:

$$\begin{aligned} \mathbf{E}(\Xi(s, y, n)\Xi(s, z, n)) &\leq L_\sigma \mathbf{E}\left(|u^{(n)}(s, y) - u^{(n-1)}(s, y)| |u^{(n)}(s, z) - u^{(n-1)}(s, z)|\right) \\ &\leq L_\sigma^2 \left(\mathbf{E}|u^{(n)}(s, y) - u^{(n-1)}(s, y)|^2\right)^{\frac{1}{2}} \left(\mathbf{E}|u^{(n)}(s, z) - u^{(n-1)}(s, z)|^2\right)^{\frac{1}{2}} \\ &\leq L_\sigma^2 (D_{n-1}(s, y)D_{n-1}(s, z))^{\frac{1}{2}} \\ &\leq L_\sigma^2 H_{n-1}(s). \end{aligned}$$

Hence, we have for  $0 < d - \delta < \alpha + \gamma$  by using Lemma 2

$$\begin{aligned} D_n(t, x) &\leq \lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{t-s}(x - y)G_{t-s}(x - z)f(y - z)dydzds \\ &\leq C\lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s)(t - s)^{-\frac{\beta(d-\delta)}{\alpha+\gamma}} ds. \end{aligned}$$

We therefore have

$$H_n(t) \leq C\lambda^2 L_\sigma^2 \int_0^t H_{n-1}(s)(t - s)^{-\frac{\beta(d-\delta)}{\alpha+\gamma}} ds.$$

We now note that the integral appearing on the right-hand side of the above inequality is finite when  $d - \delta < \frac{\alpha+\gamma}{\beta}$ . Hence, by Lemma 3.3 in Walsh [20], the series  $\sum_{n=0}^\infty H_n^{\frac{1}{2}}(t)$  converges uniformly on  $[0, T]$ . Therefore, the sequence  $\{u^{(n)}(t, x), n \geq 0\}$  converges in  $L^2$  and uniformly on  $[0, T] \times \mathbb{R}^d$  and the limit satisfies (14). We can prove uniqueness in a similar way.

We now turn to the proof of the exponential bound. Set

$$A(t) := \sup_{x \in \mathbb{R}^d} \mathbf{E}|u(t, x)|^2.$$

We claim that there exist constants  $c$  and  $c'$  such that, for all  $t > 0$ , we have

$$A(t) \leq c + c'\lambda^2 L_\sigma^2 \int_0^t A(s)(t - s)^{-\frac{\beta(d-\delta)}{\alpha+\gamma}} ds.$$

Recall the renewal inequality in Proposition 2.5 in Foondun, Liu, and Omaba [34] with  $\rho = 1 - \frac{\beta(d-\delta)}{\alpha+\gamma}$ ; then, one can prove the exponential upper bound. To prove this claim, we start with the mild formulation given by (14); then, take the second moment to obtain the following

$$\begin{aligned} \mathbf{E}|u(t, x)|^2 &= |(\mathcal{G}u_0)_t(x)|^2 \\ &+ \lambda^2 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{t-s}(x - y)G_{t-s}(x - z)f(y - z)\mathbf{E}(\sigma(u(s, y))\sigma(u(s, z)))dydzds \\ &:= I_1 + I_2. \end{aligned} \tag{44}$$

Since  $u_0$  is bounded, we have  $I_1 \leq c$  with some positive constant  $c$ . Next, we use the Assumption 1 on the coefficient  $\sigma$  together with Hölder’s inequality to see that

$$\begin{aligned} \mathbf{E}(\sigma(u(s, y))\sigma(u(s, z))) &\leq L_\sigma^2 \mathbf{E}(u(s, y)u(s, z)) \\ &\leq L_\sigma^2 [\mathbf{E}|u(s, y)|^2]^{\frac{1}{2}} [\mathbf{E}|u(s, z)|^2]^{\frac{1}{2}} \\ &\leq L_\sigma^2 \sup_{x \in \mathbb{R}^d} \mathbf{E}(|u(s, x)|^2). \end{aligned} \tag{45}$$

Therefore, using Lemma 2, the second term  $I_2$  is thus bounded as follows:

$$I_2 \leq c\lambda^2 L_\sigma^2 \int_0^t A(s)(t-s)^{-\frac{\beta(d-\delta)}{\alpha+\gamma}} ds.$$

Combining the above estimates, we obtain the desired result.  $\square$

Next, we analyze the Hölder regularity of the solution with respect to time and space variables. The next Theorem 3 extends and improves similar results known for (fractional) stochastic heat equation (e.g., Mijena and Nane [18] with  $\gamma = 0$  in Equation (1), Chen and Dalang [28], corresponding to the case  $0 < \alpha \leq 2, \gamma = 0$  and  $\beta = 1$ , Márquez-Carreras [15] with  $\beta = 1$  in Equation (1)), and also extends some results for (1) with Gaussian white noise (e.g., Dalang [21]). We use a direct method to prove our regularity results in which the Fourier transform and the representation of the Green function (i.e., (17) and (19)) play a crucial role. We state the result as follows.

**Theorem 3.** Under Assumption 1, assuming that the spectral measure  $\mu$  satisfies Hypothesis 2, then, for every  $t, s \in [0, T], T > 0, x, y \in \mathbb{R}^d, p \geq 2$ , the solution  $u(t, x)$  to Equation (1) satisfies

$$\mathbf{E}(|u(t, x) - u(s, y)|^p) \leq C(|t - s|^{p\chi_1} + |x - y|^{p\chi_2}), \tag{46}$$

with  $0 < \chi_1 < \min(\beta, \frac{1}{2} - \beta)$  and  $0 < \chi_2 < \alpha + \gamma - \eta$  if  $0 < \beta < \frac{1}{2}, 0 < \chi_1 < \frac{1}{4}$  and  $0 < \chi_2 < \frac{\alpha+\gamma}{2} - \eta$  if  $\beta = \frac{1}{2}$ , and  $0 < \chi_1 < \beta - \frac{1}{2}$  and  $0 < \chi_2 < \frac{\alpha+\gamma}{2\beta} - \eta$  if  $\frac{1}{2} < \beta < 1$ .

In particular, the random field  $u$  is  $(\chi_1, \chi_2)$ -Hölder continuous with respect to the time and space variables.

**Proof.** Since the function  $(\mathcal{G}u_0)_t(x) = \int_{\mathbb{R}^d} G_t(x - y)u_0(y)dy$  is smooth for any  $t > 0$ , then, by Proposition 2, (38) and (39), we see that, for every  $p \geq 2$  and any  $0 < T < \infty$ , there exists a finite constant  $A_{p,T}$  such that

$$\mathbf{E}\left(|u^{(n)}(t, x) - u^{(n)}(s, y)|^p\right) \leq \begin{cases} A_{p,T}\left(|t - s|^{\min(2\beta, 1-2\beta)\frac{p}{2}} + |x - y|^{p\chi_2}\right), & \text{if } 0 < \beta < \frac{1}{2}, \\ A_{p,T}\left(|t - s|^{\frac{p}{4}} + |x - y|^{p\chi_2}\right), & \text{if } \beta = \frac{1}{2} \\ A_{p,T}\left(|t - s|^{(2\beta-1)\frac{p}{2}} + |x - y|^{p\chi_2}\right), & \text{if } \frac{1}{2} < \beta < 1. \end{cases} \tag{47}$$

with  $\chi_2 \in (0, \alpha + \gamma - \eta)$  if  $0 < \beta < \frac{1}{2}, \chi_2 \in (0, \frac{\alpha+\gamma}{2} - \eta)$  if  $\beta = \frac{1}{2}$  and  $\chi_2 \in (0, \frac{\alpha+\gamma}{2\beta} - \eta)$  if  $\frac{1}{2} < \beta < 1$  simultaneously for all  $t, s \in [0, T]$  and  $x, y \in \mathbb{R}^d$ . The right-hand side of this inequality does not depend on  $n$ . Hence, using Fatou’s lemma, as  $n$  tends to infinity, we obtain the similar estimates for  $u$ , which also satisfies (47). Then, the conclusion of Theorem 3 is a consequence of the Kolmogorov continuity criterion for stochastic processes.  $\square$

Let us also make some discussion about the above regularity results.

**Remark 8.** For  $\beta$  close to 1, the order of Hölder regularity of  $u(t, x)$  in space is  $(\alpha + \gamma)$ -times the order of Hölder continuity to in time. This is coherent with the case in Márquez-Carreras [15]. When  $\gamma = 0$  and  $\alpha \in (0, 2]$ , this happens always in the case of the solution of the (fractional) heat equation (with white noise), see Walsh [20].

**Remark 9.** If  $d = 1, \alpha = 2$  and  $\gamma = 0$  (so, somehow, the operator  $(I - \Delta)^{\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}$  reduces to the Laplacian operator  $\Delta$  and, moreover, we assume that  $\eta$  is close to one-half and  $\beta$  is close to 1, we obtain the well-known regularity of the solution to the heat equation with time-space white noise (which is Hölder continuous of order  $\frac{1}{4}$  in time and of order  $\frac{1}{2}$  in space).

#### 4. The Linear Additive Noise

In the last part of this work, we focus on the solution of (14) with  $u_0(x) = 0$  and  $\sigma = 1$ . This is the additive noise case and in this situation the solution is Gaussian. We will study the stationarity of the solution, both in time and in space. The solution is stationary in space, but not in time.

In the following, we consider

$$U(t, x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)W(ds, dy), \tag{48}$$

which is the mild solution of (1) when the initial condition  $u_0(x) = 0, x \in \mathbb{R}^d$  and  $\sigma(x) \equiv 1, x \in \mathbb{R}$ .

**Remark 10.**

1. As mentioned in the introduction, Anh and Leonenko [2] showed that the presence of the Bessel operator  $-(I - \Delta)^{\gamma/2}$  with  $\gamma \geq 0$  is essential to have an (asymptotically) stationary solution of SPDE (1). In fact, the linear case requires the condition  $0 < \alpha < d/2$  and  $\alpha + \gamma > d/2$  that is to say the parameter  $\gamma > 0$  is necessary.
2. On the other hand, the parameter  $\gamma > 0$  of the Bessel operator is also useful in determining suitable conditions for the spectral density of the solutions of fractional kinetic equations belonging to  $L^1(\mathbb{R}^d)$ .

**Theorem 4.** Under Hypothesis 2 on the spectral measure  $\mu$  associated with  $\dot{W}$ , then, for fixed  $t \in \mathbb{R}_+$ , the spatial covariance function of  $U(t, x)$  given by (48) is

$$R_t(x - z) = \int_{\mathbb{R}^d} \mu(d\xi) e^{i\langle x-z, \xi \rangle} \int_0^t ds E_{\beta}^2(-\nu(t-s)^{\beta} |\xi|^{\alpha} (1 + |\xi|^2)^{\gamma/2}).$$

In particular, for every  $t \in \mathbb{R}_+$ , the process  $\{U(t, x), x \in \mathbb{R}^d\}$  is stationary.

**Proof.** We first calculate the spatial covariance for a fixed  $t \in \mathbb{R}_+$ . By means of the definition of Fourier transform, change of variable and Fubini’s theorem, we obtain, for any  $x, z \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathbf{E}(U(t, x)\overline{U(t, z)}) &= \int_0^t \int_{\mathbb{R}^d} \mathcal{F}G_{t-s}(x - \cdot)(\xi) \overline{\mathcal{F}G_{t-s}(z - \cdot)(\xi)} \mu(d\xi) ds \\ &= \int_0^t \int_{\mathbb{R}^d} e^{-i\langle x-z, \xi \rangle} |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2 \mu(d\xi) ds \\ &= \int_{\mathbb{R}^d} e^{-i\langle x-z, \xi \rangle} \int_0^t E_{\beta}^2(-\nu(t-s)^{\beta} |\xi|^{\alpha} (1 + |\xi|^2)^{\gamma/2}) ds \mu(d\xi) \\ &= R_t(x - z), \end{aligned}$$

where we have used the fact that

$$\begin{aligned} &\int_0^t E_{\beta}^2(-\nu(t-s)^{\beta} |\xi|^{\alpha} (1 + |\xi|^2)^{\gamma/2}) ds \\ &\leq \int_0^t \frac{1}{(1 + \Gamma(1 + \beta))^{-1} (t-s)^{\beta} |\xi|^{\alpha} (1 + |\xi|^2)^{\gamma/2}} ds \\ &\leq t. \end{aligned}$$

Hence, for any fixed  $t \in \mathbb{R}_+$ , the process  $\{U(t, x), x \in \mathbb{R}^d\}$  is a Gaussian field that has zero mean, stationary increments, and a continuous covariance function.  $\square$

**Remark 11.** From the above result, one can obtain the spectral density of the process  $x \rightarrow U(t, x)$ . Indeed, its spectral density  $f_t(\xi)$  is given by

$$f_t(\xi) = \int_0^t ds E_\beta^2(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) g(\xi).$$

where  $g(\cdot)$  is the density of  $\mu(\cdot)$  with respect to the Lebesgue measure.

The next result shows that the process (48) is not stationary in time, but, when  $t$  tends to infinity, it converges to a stationary process.

**Theorem 5.** Under Hypothesis 2 on the spectral measure  $\mu$  associated with  $W$ , assuming  $1/2 < \beta < 1$ , then, for  $t \in \mathbb{R}_+$ ,  $\tau \in \mathbb{R}$  such that  $t + \tau \in \mathbb{R}_+$ ,  $x, z \in \mathbb{R}^d$ , the asymptotic homogeneous spatial-temporal covariance function of  $U(t + \tau, x)$  and  $U(t, z)$  is

$$R(\tau, x - z) = \int_{\mathbb{R}^d} \frac{e^{i(x-z, \xi)}}{\beta(\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^{1/\beta}} \int_{\nu\tau^\beta|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}^\infty x^{1/\beta-1} E_\beta(-x) \cdot E_\beta\left(-\nu\left(\left(\frac{x}{\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}}\right)^{1/\beta} - \tau\right)^\beta |\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}\right) dx \mu(d\xi).$$

Moreover,  $U(\cdot, x)$  is asymptotically in time an index- $(\beta - \frac{1}{2})$  Gaussian field.

**Proof.** For  $t, \tau \in \mathbb{R}_+$  (for  $\tau \in \mathbb{R}_-$  such that  $t + \tau \in \mathbb{R}_+$ , we argue similarly), and  $x, z \in \mathbb{R}^d$ , we have

$$\begin{aligned} & \mathbb{E}\left(U(t + \tau, x)\overline{U(t, z)}\right) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}G_{t+\tau-s}(x - \cdot)(\xi) \overline{\mathcal{F}G_{t-s}(z - \cdot)(\xi)} \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) e^{i(x-z, \xi)} \mathcal{F}G_{t+\tau-s}(x - \cdot)(\xi) \mathcal{F}G_{t-s}(z - \cdot)(\xi) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) e^{i(x-z, \xi)} E_\beta(-\nu(t + \tau - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) E_\beta(-\nu(t - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) \\ &= \int_{\mathbb{R}^d} e^{i(x-z, \xi)} \int_0^t E_\beta(-\nu(t + \tau - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) E_\beta(-\nu(t - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) ds \mu(d\xi). \end{aligned}$$

Next, let us calculate the above integral with respect to  $s$ . In fact, with the change of variable  $x = \nu(t + \tau - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}$ , we have

$$\begin{aligned} & \int_0^t E_\beta(-\nu(t + \tau - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) E_\beta(-\nu(t - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) ds \\ &= \frac{1}{\beta(\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^{1/\beta}} \int_{\nu\tau^\beta|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}^{\nu(t+\tau)^\beta|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} E_\beta(-x) x^{1/\beta-1} \\ & \quad \cdot E_\beta\left(-\nu\left(\left(\frac{x}{\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}}\right)^{1/\beta} - \tau\right)^\beta |\xi|^\alpha(1 + |\xi|^2)^{\gamma/2}\right) dx. \end{aligned}$$

Moreover, as  $t \rightarrow \infty$ , we obtain

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_0^t E_\beta(-\nu(t + \tau - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) E_\beta(-\nu(t - s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}) ds \\ &= \frac{1}{\beta (\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})^{1/\beta}} \int_{\nu \tau^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}}^\infty x^{1/\beta - 1} E_\beta(-x) \\ & \quad \cdot E_\beta \left( -\nu \left( \left( \frac{x}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{1/\beta} - \tau \right)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2} \right) dx, \end{aligned}$$

which is finite because we have

$$\begin{aligned} & \int_{\nu \tau^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}}^\infty x^{1/\beta - 1} E_\beta(-x) E_\beta \left( -\nu \left( \left( \frac{x}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{1/\beta} - \tau \right)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2} \right) dx \\ & \leq C_{\xi, \nu, \beta, \alpha, \gamma, \tau} \int_{\nu \tau^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}}^\infty x^{1/\beta - 3} dx, \end{aligned}$$

which is finite when  $1/2 < \beta < 1$  (i.e.,  $\frac{1}{\beta} - 2 < 0$ ).

We now tackle the second part of this theorem. We assume that  $x \in \mathbb{R}^d, t \in \mathbb{R}_+$  and  $\tau \in \mathbb{R}_+$  are small (the negative case is similar). Then, from (31) and (32), when  $1/2 < \beta < 1$ , we have

$$\begin{aligned} & \mathbf{E} \left( |U(t + \tau, x) - U(t, x)|^2 \right) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t+\tau-s}(x - \cdot)(\xi) - \mathcal{F}G_{t-s}(x - \cdot)(\xi)|^2 \\ & \quad + \int_t^{t+\tau} ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t+\tau-s}(x - \cdot)(\xi)|^2 \\ & \leq C \left( |\tau|^{2\beta} + |\tau|^{2\beta-1} \right). \end{aligned}$$

Then, we can complete the proof of the second part.  $\square$

### 5. Conclusions

In this article, we have studied the space-time fractional stochastic kinetic Equation (1) driven by spatially homogeneous Gaussian noise. The time fractional derivative  $\frac{\partial^\beta}{\partial t^\beta}, 0 < \beta \leq 1$  is defined in the Caputo–Djrbashian sense given by (2). The inverses of the Bessel and Riesz potentials are also included in Equation (1). First, the existence and uniqueness of solutions for the proposed fractional SPDEs (1) were obtained. In particular, when the covariance function of the Gaussian noise is given by the Riesz kernel, we have proved the upper bound for the second moment of the mild solution to Equation (1). Moreover, the main results have been proven based on the classical Picard’s iterations and some estimates about the Fourier transform of the Green function given by (17). Next, we analyze the Hölder regularity of the mild solution to Equation (1) with respect to the time and space variables. Finally, in some special cases (i.e.,  $u_0(x) \equiv 0$  and  $\sigma(u) \equiv 1$ ), we have studied the stationarity of the mild solution, both in time and space variables. We proved that the mild solution is stationary in space, but not in time.

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### Appendix A

We will prove Proposition 2 in this appendix.

**Proof of Proposition 2.** For any  $t, t' \in \mathbb{R}_+$  such that  $t' < t$  and  $x \in \mathbb{R}^d$ , by using the fact

$$\mathcal{F}G_{t-s}(x - \cdot)(\xi) - \mathcal{F}G_{t'-s}(x - \cdot)(\xi) = e^{i(\xi, x)}(\mathcal{F}G_{t-s}(\cdot)(\xi) - \mathcal{F}G_{t'-s}(\cdot)(\xi)),$$

then from (17) and the absolute convergence of the series in (18), one obtains

$$\begin{aligned} \mathcal{F}G_{t-s}(\cdot)(\xi) - \mathcal{F}G_{t'-s}(\cdot)(\xi) &= \sum_{k=0}^{\infty} \frac{(-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^k}{\Gamma(1 + k\beta)} \left( |t - s|^{k\beta} - |t' - s|^{k\beta} \right) \\ &\leq \sum_{k=0}^{\infty} \frac{(-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^k}{\Gamma(1 + k\beta)} \left( |t - s|^k - |t' - s|^k \right)^\beta \quad (A1) \\ &\leq |t - t'|^\beta \sum_{k=1}^{\infty} \frac{k^\beta (-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^k}{\Gamma(1 + k\beta)} t^{(k-1)\beta}, \end{aligned}$$

for all  $t, t' \in \mathbb{R}_+$ , where the last inequality follows from the mean value theorem. Furthermore, since the series in (18) is absolutely convergent, then the series in the last inequality in (A1) can be bounded as follows with  $0 < \beta < 1$ :

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k^\beta (-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2})^k}{\Gamma(1 + k\beta)} t^{(k-1)\beta} &= t^{-\beta} \sum_{k=1}^{\infty} \frac{k^\beta (-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2} t^\beta)^k}{\Gamma(1 + k\beta)} \\ &\leq -\nu t^{-\beta} |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2} \sum_{k=1}^{\infty} \frac{k (-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2} t^\beta)^{k-1}}{\Gamma(1 + k\beta)} \quad (A2) \\ &\leq -\nu t^{-\beta} |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2} \sum_{k=1}^{\infty} \frac{(-\nu|\xi|^\alpha(1 + |\xi|^2)^{\gamma/2} t^\beta)^{k-1}}{\Gamma(1 + k\beta)} \\ &\leq \frac{t^{-\beta}}{1 + \Gamma(1 + \beta)^{-1} \nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\int_0^{t'} ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi) - \mathcal{F}G_{t'-s}(x - \cdot)(\xi)|^2 \\ &\leq |t - t'|^{2\beta} \int_0^{t'} ds \int_{\mathbb{R}^d} \mu(d\xi) \frac{t^{-2\beta}}{(1 + \Gamma(1 + \beta)^{-1} \nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})^2} \quad (A3) \\ &:= A_1 + A_2, \end{aligned}$$

with

$$A_1 = |t - t'|^{2\beta} \int_0^{t'} ds \int_{|\xi| \leq 1} \mu(d\xi) \frac{t^{-2\beta}}{(1 + \Gamma(1 + \beta)^{-1} \nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})^2},$$

and

$$A_2 = |t - t'|^{2\beta} \int_0^{t'} ds \int_{|\xi| > 1} \mu(d\xi) \frac{t^{-2\beta}}{(1 + \Gamma(1 + \beta)^{-1} \nu t^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})^2}.$$

Thus, with (11), we have

$$A_1 \leq t^{1-2\beta} \int_{|\xi| \leq 1} \mu(d\xi) |t - t'|^{2\beta},$$

and

$$A_2 \leq t^{-2\beta} \int_{|\xi| > 1} N_{t'}(\xi) \mu(d\xi) |t - t'|^{2\beta}.$$

Then, combining the above estimates for  $A_1, A_2$  and Proposition 1, we can conclude the proof of (31).

Next, we can follow the similar arguments to prove (32) which will be divided into three cases. Firstly, with  $0 < \beta < \frac{1}{2}$ , we have

$$\begin{aligned} & \int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi)|^2 \\ &= \int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |e^{i\langle \xi, x \rangle}|^2 |E_\beta(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})|^2 \\ &\leq \int_{\mathbb{R}^d} \mu(d\xi) \int_{t'}^t ds \left( \frac{1}{1 + \Gamma(1 + \beta)^{-1} \nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^2 \\ &\leq \int_{|\xi| \leq 1} \mu(d\xi) |t - t'| \\ &\quad + \frac{\Gamma(1 + \beta)^2 2^{\alpha+\gamma}}{\nu^2(1 - 2\beta)} \int_{|\xi| > 1} \left( \frac{1}{1 + |\xi|^2} \right)^{\alpha+\gamma} \mu(d\xi) |t - t'|^{1-2\beta} \\ &= C_{3.2} |t - t'|^{1-2\beta}. \end{aligned}$$

If  $\frac{1}{2} < \beta < 1$ , one obtains with Fubini’s theorem

$$\begin{aligned} & \int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x - \cdot)(\xi)|^2 \\ &= \int_{\mathbb{R}^d} \mu(d\xi) \int_{t'}^t |E_\beta(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})|^2 ds \\ &\leq \int_{\mathbb{R}^d} \mu(d\xi) \int_{t'}^t \left( \frac{1}{1 + \Gamma(1 + \beta)^{-1} \nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^2 ds. \end{aligned}$$

Denote by

$$\begin{aligned} M_{t,t'}(\xi) &:= \int_{t'}^t \left( \frac{1}{1 + \Gamma(1 + \beta)^{-1} \nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^2 ds \\ &= \int_0^{t-t'} \left( \frac{1}{1 + \Gamma(1 + \beta)^{-1} \nu u^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^2 du. \end{aligned}$$

Then, with the change of variable  $x = \Gamma(1 + \beta)^{-1} \nu u^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}$ , we have

$$M_{t,t'}(\xi) = \frac{1}{\beta} \left( \frac{\Gamma(1 + \beta)}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\Gamma(1 + \beta)^{-1} \nu (t-t')^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} x^{\frac{1}{\beta}-1} (1 + x)^{-2} dx.$$

For  $|\xi| \leq 1$ , we have the following:

$$\begin{aligned} M_{t,t'}(\xi) 1_{\{|\xi| \leq 1\}} &\leq \frac{1}{\beta} \left( \frac{\Gamma(1 + \beta)}{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\frac{\nu |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}}{\Gamma(1 + \beta)} (t-t')^\beta} x^{\frac{1}{\beta}-1} dx \\ &= |t - t'|. \end{aligned} \tag{A4}$$

Now we will estimate  $M_{t,t'}(\xi)$  when  $|\xi| > 1$ . In fact, with  $0 < \eta < \frac{\alpha+\gamma}{2\beta}$ , we have

$$\begin{aligned} M_{t,t'}(\xi)1_{\{|\xi|>1\}} &\leq \frac{1}{\beta} \left( \frac{\Gamma(1+\beta)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \int_0^{\frac{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}}{\Gamma(1+\beta)}(t-t')^\beta} (1+x)^{\frac{1}{\beta}-3} dx \\ &\leq \frac{1}{2\beta-1} \left( \frac{\Gamma(1+\beta)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^{\frac{1}{\beta}} \\ &\leq \frac{1}{2\beta-1} \left( \frac{\Gamma(1+\beta)}{\nu} \right)^{\frac{1}{\beta}} 2^{\frac{\alpha+\gamma}{2\beta}} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2\beta}}. \end{aligned} \tag{A5}$$

Then, combining the estimates (A4) and (A5), we can obtain (32).

If  $\beta = \frac{1}{2}$ , by using the similar argument above, we have

$$\begin{aligned} &\int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 \\ &= \int_{\mathbb{R}^d} \mu(d\xi) \int_{t'}^t |E_\beta(-\nu(t-s)^\beta|\xi|^\alpha(1+|\xi|^2)^{\gamma/2})|^2 ds \\ &\leq \int_{|\xi|\leq 1} \mu(d\xi) |t-t'| \\ &+ 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \int_{|\xi|>1} \mu(d\xi) \int_0^{\Gamma(3/2)^{-1}\nu|t-t'|^{\frac{1}{2}}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} x(1+x)^{-2} dx. \end{aligned}$$

and we have

$$\begin{aligned} &2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \int_0^{\Gamma(3/2)^{-1}\nu|t-t'|^{\frac{1}{2}}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} x(1+x)^{-2} dx \\ &\leq 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \int_0^{\Gamma(3/2)^{-1}\nu|t-t'|^{\frac{1}{2}}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} (1+x)^{-1} dx \\ &= 2 \left( \frac{\Gamma(3/2)}{\nu|\xi|^\alpha(1+|\xi|^2)^{\gamma/2}} \right)^2 \ln \left( 1 + \Gamma(3/2)^{-1}\nu|t-t'|^{\frac{1}{2}}|\xi|^\alpha(1+|\xi|^2)^{\gamma/2} \right) \\ &\leq 2^{1+\frac{\alpha+\gamma}{2}} \frac{\Gamma(3/2)}{\nu} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2}} |t-t'|^{\frac{1}{2}}. \end{aligned}$$

Then, we have that

$$\begin{aligned} &\int_{t'}^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x-\cdot)(\xi)|^2 \\ &\leq \left( \int_{|\xi|\leq 1} \mu(d\xi) |t-t'|^{\frac{1}{2}} + 2^{1+\frac{\alpha+\gamma}{2}} \frac{\Gamma(3/2)}{\nu} \int_{|\xi|>1} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2}} \mu(d\xi) \right) |t-t'|^{\frac{1}{2}}. \end{aligned}$$

Since

$$\mathcal{F}G_{t-s}(x-\cdot)(\xi) - \mathcal{F}G_{t-s}(x'-\cdot)(\xi) = (e^{i\langle \xi, x \rangle} - e^{i\langle \xi, x' \rangle}) \mathcal{F}G_{t-s}(\cdot)(\xi),$$

then we have that

$$\begin{aligned} &\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{t-s}(x-\cdot)(\xi) - \mathcal{F}G_{t-s}(x'-\cdot)(\xi)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |e^{i\langle \xi, x \rangle} - e^{i\langle \xi, x' \rangle}|^2 |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2 \\ &:= B_1 + B_2, \end{aligned}$$

with

$$\begin{aligned}
 B_1 &= \int_0^t ds \int_{|\xi| \leq 1} \mu(d\xi) \left| e^{i(\xi, x)} - e^{i(\xi, x')} \right|^2 |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2, \\
 B_2 &= \int_0^t ds \int_{|\xi| > 1} \mu(d\xi) \left| e^{i(\xi, x)} - e^{i(\xi, x')} \right|^2 |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2.
 \end{aligned}$$

The first term  $B_1$  is easy and can be studied in the same way for any  $0 < \beta < 1$ . Indeed, the fact that the Fourier transform of Green function  $G$  given by (17) is bounded by 1, the mean value theorem, and property (11) imply that

$$\begin{aligned}
 B_1 &\leq C \int_0^t ds \int_{|\xi| \leq 1} \mu(d\xi) |\langle x - x', \xi \rangle|^2 \\
 &\leq Ct \int_{|\xi| \leq 1} \mu(d\xi) |x - x'|^2.
 \end{aligned} \tag{A6}$$

The other term  $B_2$  is a little involved. We distinguish three cases depending on the values of  $\beta$ . We first study the case  $0 < \beta < \frac{1}{2}$ . Let  $0 < \rho_1 < \alpha + \gamma - \eta$ . Applying the mean theorem, Fubini’s theorem, the fact  $1 - e^{-x} \leq 1$  for all  $x > 0$  and Hypothesis 2, then we have

$$\begin{aligned}
 B_2 &= \int_0^t ds \int_{|\xi| > 1} \mu(d\xi) \left| e^{i(\xi, x)} - e^{i(\xi, x')} \right|^2 |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2 \\
 &\leq 4 \int_0^t ds \int_{|\xi| > 1} \mu(d\xi) \left| \frac{1}{2} (e^{i(\xi, x)} - e^{i(\xi, x')}) \right|^{2\rho_3} |\mathcal{F}G_{t-s}(\cdot)(\xi)|^2 \\
 &\leq C \int_0^t ds \int_{|\xi| > 1} \mu(d\xi) |\xi|^{2\rho_1} |x - x'|^{2\rho_1} |E_\beta(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})|^2 \\
 &\leq Ct^{1-2\beta} \frac{\Gamma(1+\beta)^2}{(1-2\beta)\nu^2} 2^{\alpha+\gamma-\rho_1} \int_{|\xi| > 1} \left( \frac{1}{1+|\xi|^2} \right)^{\alpha+\gamma-\rho_1} \mu(d\xi) |x - x'|^{2\rho_1}.
 \end{aligned} \tag{A7}$$

For the critical case  $\beta = \frac{1}{2}$ , by choosing  $0 < \rho_2 < \frac{\alpha+\gamma}{2} - \eta$ , then we have that

$$\begin{aligned}
 B_2 &\leq C|x - x'|^{2\rho_2} \int_{|\xi| > 1} \mu(d\xi) |\xi|^{2\rho_2} \int_0^t |E_\beta(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})|^2 ds \\
 &\leq 2Ct^{\frac{1}{2}} |x - x'|^{2\rho_2} \Gamma(3/2)\nu^{-1} \int_{|\xi| > 1} \mu(d\xi) \frac{|\xi|^{2\rho_2}}{|\xi|^\alpha (1 + |\xi|^2)^{\gamma/2}} \\
 &\leq C2^{1+\frac{\alpha+\gamma}{2}-\rho_2} t^{\frac{1}{2}} |x - x'|^{2\rho_2} \Gamma(3/2)\nu^{-1} \int_{|\xi| > 1} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2}-\rho_2} \mu(d\xi).
 \end{aligned} \tag{A8}$$

On the other hand, when  $\frac{1}{2} < \beta < 1$ , let  $0 < \rho_3 < \frac{\alpha+\gamma}{2\beta} - \eta$ , then the similar arguments yield that

$$\begin{aligned}
 B_2 &\leq C \int_0^t ds \int_{|\xi| > 1} \mu(d\xi) |\xi|^{2\rho_3} |x - x'|^{2\rho_3} |E_\beta(-\nu(t-s)^\beta |\xi|^\alpha (1 + |\xi|^2)^{\gamma/2})|^2 \\
 &\leq \frac{C}{(2\beta - 1)} \left( \frac{\Gamma(1+\beta)}{\nu} \right)^{\frac{1}{\beta}} 2^{\frac{\alpha+\gamma}{2\beta}-\rho_3} \int_{|\xi| > 1} \left( \frac{1}{1+|\xi|^2} \right)^{\frac{\alpha+\gamma}{2\beta}-\rho_3} \mu(d\xi) \cdot |x - x'|^{2\rho_3}.
 \end{aligned} \tag{A9}$$

Then, we can conclude the proof of (33) by combining (A6)–(A9). □

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