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An Implicit Difference Scheme for the Fourth-Order Nonlinear Evolution Equation with Multi-Term Riemann–Liouville Fractional Integral Kernels

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Abstract: In this paper, an implicit difference scheme is proposed and analyzed for a class of nonlinear fourth-order equations with the multi-term Riemann–Liouville (R–L) fractional integral kernels. For the nonlinear convection term, we handle implicitly and attain a system of nonlinear algebraic equations by using the Galerkin method based on piecewise linear test functions. The Riemann–Liouville fractional integral terms are treated by convolution quadrature. In order to obtain a fully discrete method, the standard central difference approximation is used to discretize the spatial derivative. The stability and convergence are rigorously proved by the discrete energy method. In addition, the existence and uniqueness of numerical solutions for nonlinear systems are proved strictly. Additionally, we introduce and compare the Besse relaxation algorithm, the Newton iterative method, and the linearized iterative algorithm for solving the nonlinear systems. Numerical results confirm the theoretical analysis and show the effectiveness of the method.



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1. Introduction

Partial integro-differential equations (PIDEs) have been applied widely in physical models, chemistry and biology [1–4]. Additionally, the fractional reaction–subdiffusion equation is believed to provide a powerful tool for the modeling plenty of natural phenomena in physics, biology, and chemistry [5–7]. Many numerical methods have been extensively studied. In [8], Sanz-Serna was the first to propose the difference scheme for nonlinear integro-differential equations; then, Lopez-Marcos [9] made a direct extension and considered the difference method for a class of nonlinear partial integro differential equations. Tang [10] considered a finite difference scheme for nonlinear PIDEs, approximated the differential term using the Crank–Nicolson scheme, and dealt with the integral term with the product trapezoidal method. Fairweather and Pani [11] used the backward Euler–Galerkin method for some partial integral differentials and derived the prior error estimates. Xu [12–14] also completed a series of studies for nonlinear integro-differential equations. A class of fractional convection–diffusion equations with variable coefficients are solved with the Sinc–Legendre collocation method [15], and nonlinear fractional convection–diffusion equations are solved using the homotopy analysis method [16]. For more development of numerical methods and analysis of the fractional reaction–subdiffusion equations, we refer the readers to [17–19].

This paper is devoted to the study of an implicit difference scheme for the nonlinear fourth-order equation with the multi-term Riemann–Liouville fractional integral kernels

$$u_t(x, t) + u(x, t)u_x(x, t) - \mathfrak{L}_1 u(x, t) + \mathfrak{L}_2 u(x, t) = f(x, t), \quad (1)$$

$$0 < t \leq T, \quad 0 < x < L,$$

the initial condition and the boundary value conditions are

$$u(x, 0) = u^0(x), \quad 0 \leq x \leq L, \quad (2)$$

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0, \quad 0 < t \leq T, \quad (3)$$

respectively, where $f(x, t)$ and $u^0(x)$ are the given smooth functions. Additionally, the $\mathfrak{L}_1 u$ and $\mathfrak{L}_2 u$ are defined by

$$\mathfrak{L}_1 u(x, t) = u_{xx}(x, t) + \mathcal{I}^{\alpha_1} u_{xx}(x, t), \quad 0 < \alpha_1 < 1, \quad (4)$$

$$\mathfrak{L}_2 u(x, t) = u_{xxxx}(x, t) + \mathcal{I}^{\alpha_2} u_{xxxx}(x, t), \quad 0 < \alpha_2 < 1, \quad (5)$$

where for $\gamma = \alpha_1, \alpha_2$, $0 < \gamma < 1$, $\mathcal{I}^\gamma$ denote the R–L fractional integral operator [2] defined by

$$\mathcal{I}^\gamma \varphi(t) = \int_0^t \beta(t-s)\varphi(s)ds = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \varphi(s)ds, t > 0. \quad (6)$$

For the fourth-order nonlinear partial differential equations, many scholars have carried out extensive research [9,20–23]. In the paper, we propose the backward Euler scheme and convolution quadrature finite difference method for (1)–(3). The nonlinear convective term in our equation deals with Galerkin method, which attains an advantage over the scheme in [23]. We also introduce and compare three nonlinear iterative methods, including the Besse relaxation algorithm, the Newton iterative method, and the linearized iterative algorithm, to solve the nonlinear systems. We also discuss the advantages and disadvantages of three kinds of methods. The existence and uniqueness of numerical solutions for nonlinear systems are proved strictly. The stability and convergence are rigorously proved by the discrete energy method.

The outline of the paper is as follows. In Section 2, the backward Euler implicit difference scheme is derived. In Section 3, it is proved that the stability of the difference scheme under the L^2 and H^1 norms. In particular, the existence of the backward Euler implicit difference scheme is proved by the Leray–Schauder Theorem. In the Section 4, convergence is proved, and the uniqueness of solution is also proved. The numerical examples are given to check our analysis in Section 5. Finally, this paper ends with a brief conclusion in Section 6.

2. The Construction of the Fully Discrete Scheme

Let J be a positive integer, define the space-step size $h := \frac{L}{J}$, and $x_j := jh$ ($0 \leq j \leq J$) is the mesh points. For a positive integer N , we introduce the time-step size $k := \frac{T}{N}$, the nodes $t_n := nk$ ($0 \leq n \leq N$), and the intermediate nodes $t_{n-\frac{1}{2}} := t_n - \frac{1}{2}k$ ($1 \leq n \leq N$).

Additionally, we define the following grid functions:

$$U_j^n := u(x_j, t_n), f_j^n := f(x_j, t_n), 0 \leq j \leq J, 0 \leq n \leq N.$$

Giving grid function $U = \{U_j^n | 0 \leq j \leq J, 0 \leq n \leq N\}$. Some notations are defined as follows

$$\begin{aligned}
\delta_t U_j^n &= \frac{1}{k}(U_j^n - U_j^{n-1}), \quad \delta_x U_j^n = \frac{1}{h}(U_j^n - U_{j-1}^n); \\
\Delta U_j^n &= U_{j+1}^n - U_{j-1}^n, \quad \Delta_+ U_j^n = U_{j+1}^n - U_j^n; \\
\Delta_- U_j^n &= U_j^n - U_{j-1}^n, \quad \Delta_x U_j^n = \frac{1}{2h}(U_{j+1}^n - U_{j-1}^n); \\
\nabla U_j^n &= \frac{1}{3}(U_{j+1}^n + U_j^n + U_{j-1}^n), \quad \delta_x^2 U_j^n = \frac{1}{h}(\delta_x U_{j+1}^n - \delta_x U_j^n); \\
\delta_x^4 U_j^n &= \frac{1}{h^2}(\delta_x^2 U_{j+1}^n - 2\delta_x^2 U_j^n + \delta_x^2 U_{j-1}^n).
\end{aligned} \tag{7}$$

To construct the scheme fully, we first introduce the first-order quadrature rule [20,21] to approximate the R–L fractional integral $\mathcal{I}^\gamma \varphi(t)$

$$\mathcal{I}^\gamma \varphi(t_n) \approx \hat{q}_n^\gamma(\varphi) = k^\gamma \sum_{p=1}^n \omega_{n-p}^\gamma \varphi^p = k^\gamma \sum_{p=0}^{n-1} \omega_p^\gamma \varphi^{n-p}, \tag{8}$$

by the generating power series $(\delta(\zeta))^{-\gamma} = (1 - \zeta)^{-\gamma}$, the quadrature weights ω_p^γ can be attained by

$$\sum_{p=0}^{\infty} \omega_p^\gamma \zeta^p = (1 - \zeta)^{-\gamma}. \tag{9}$$

Further, the quadrature weights ω_p^γ can be computed by

$$\omega_0^\gamma = 1, \omega_p^\gamma = \frac{\gamma(\gamma+1) \cdots (\gamma+p-1)}{p!}, \quad p = 1, 2, \dots. \tag{10}$$

Let $E(\varphi)(t_n) = \mathcal{I}^\gamma \varphi(t_n) - \hat{q}_n^\gamma(\varphi)$, we can obtain the quadrature error in the next lemma.

Lemma 1 ([3,14]). *Let $\varphi(t)$ be a real and continuously differentiable function in $0 < t \leq T$, and $\varphi_t(t)$ is continuous and integrable for $0 < t \leq T$. Then, based on the Equation (9), the error of the convolution quadrature is bounded by*

$$\begin{aligned}
|E(\varphi)(t_n)| &\leq C k t_n^{\gamma-1} |\varphi(0)| + C k \int_0^{t_{n-1}} (t_n - s)^{\gamma-1} |\varphi_t(s)| ds \\
&\quad + C k^\gamma \int_{t_{n-1}}^{t_n} |\varphi_t(s)| ds,
\end{aligned}$$

where the constant C does not rely on k .

Lemma 2. *Let $u(x, t) \in C_{x,t}^{4,2}([0, L] \times (0, T])$, for $1 \leq j \leq J-1, 1 \leq n \leq N$; it holds that*

$$|(R_1)_j^n| = |\mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) - \hat{q}_n^{\alpha_1}(\delta_x^2 U_j)| \leq C(k^{\alpha_1} n^{\alpha_1-1} + h^2).$$

Proof. By using the Taylor expansion with integral remainder [24–26], we obtain

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x_j, t_n) &= \delta_x^2 U_j^n - \frac{1}{6} h^2 \int_0^1 \left[\frac{\partial^4 u}{\partial x^4}(x_j + sh, t_n) \right. \\
&\quad \left. + \frac{\partial^4 u}{\partial x^4}(x_j - sh, t_n) \right] (1-s)^3 ds, \quad 1 \leq j \leq J-1, 1 \leq n \leq N,
\end{aligned} \tag{11}$$

by triangle inequality, we obtain

$$\begin{aligned} |(R_1)_j^n| &= |\mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) - \hat{q}_n^{\alpha_1}(\delta_x^2 U_j)| \\ &\leq |\mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) - \hat{q}_n^{\alpha_1}(u_{xx}(x_j, \cdot))| \\ &\quad + |\hat{q}_n^{\alpha_1}(u_{xx}(x_j, \cdot)) - \hat{q}_n^{\alpha_1}(\delta_x^2 U_j)|, 1 \leq j \leq J-1, 1 \leq n \leq N. \end{aligned}$$

Since $\hat{q}_n^{\alpha_1}(1) = k^{\alpha_1} \sum_{p=0}^{n-1} \omega_p^{\alpha_1} = \frac{1}{\Gamma(\alpha_1)} \int_0^{t_n} (t_n - s)^{\alpha_1-1} \cdot 1 ds = \frac{t_n^{\alpha_1}}{\Gamma(\alpha_1+1)}$, then

$$\begin{aligned} |(R_1)_j^n| &\leq \hat{q}_n^{\alpha_1}(1) \frac{h^2}{6} \left| \int_0^1 \left[\frac{\partial^4 U}{\partial x^4}(x_j + sh, t_{n-i}) \right. \right. \\ &\quad \left. \left. + \frac{\partial^4 U}{\partial x^4}(x_j - sh, t_{n-i}) \right] (1-s)^3 ds \right| + |\mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) - \hat{q}_n^{\alpha_1}(u_{xx}(x_j, \cdot))|, \\ &\leq Ch^2 \frac{t_n^{\alpha_1}}{\Gamma(\alpha_1+1)}, 1 \leq j \leq J-1, 1 \leq n \leq N. \end{aligned}$$

By Lemma 1, we obtain

$$|\mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) - \hat{q}_n^{\alpha_1}(u_{xx}(x_j, \cdot))| \leq Cn^{\alpha_1-1}k^{\alpha_1}, 1 \leq j \leq J-1, 1 \leq n \leq N.$$

The proof is finished. $\square$

Lemma 3. Let $u(x, t) \in C_{x,t}^{6,2}([0, L] \times (0, T])$, for $1 \leq j \leq J-1, 1 \leq n \leq N$, we know

$$|(R_2)_j^n| = |\mathcal{I}^{\alpha_2} u_{xxxx}(x_i, t_n) - \hat{q}_n^{\alpha_2}(\delta_x^4 U_j)| \leq C(n^{\alpha_2-1}k^{\alpha_2} + h^2).$$

Proof. By using the Taylor expansion with integral remainder, we have

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4}(x_j, t_n) &= \delta_x^4 U_j^n - \frac{1}{6}h^2 \int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_j + sh, t_n) \right. \\ &\quad \left. + \frac{\partial^6 u}{\partial x^6}(x_j - sh, t_n) \right] (1-s)^3 ds, 1 \leq j \leq J-1, 1 \leq n \leq N. \end{aligned} \quad (12)$$

Similarly to Lemma 2, we can complete the proof of the Lemma 3. $\square$

We now derive the backward Euler implicit difference scheme for the problem (1)–(3). Considering (1) at the point (x_j, t_n) , we obtain

$$\begin{aligned} u_t(x_j, t_n) + u(x_j, t_n)u_x(x_j, t_n) - \mathfrak{L}_1 u(x_j, t_n) + \mathfrak{L}_2 u(x_j, t_n) \\ = f(x_j, t_n), \quad 1 \leq j \leq J-1, 1 \leq n \leq N. \end{aligned} \quad (13)$$

Next, we discretize the (13) one by one. First, from Lemmas 2 and 3, we obtain

$$\begin{cases} \mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) = \hat{q}_n^{\alpha_1}(\delta_x^2 U_j) + (R_1)_j^n, & 1 \leq j \leq J-1, 1 \leq n \leq N, \\ \mathcal{I}^{\alpha_2} u_{xxxx}(x_j, t_n) = \hat{q}_n^{\alpha_2}(\delta_x^4 U_j) + (R_2)_j^n, & 1 \leq j \leq J-1, 1 \leq n \leq N, \end{cases} \quad (14)$$

and

$$\begin{cases} u_{xx}(x_j, t_n) = \delta_x^2 U_j^n + (R_3)_j^n, \\ u_{xxxx}(x_j, t_n) = \delta_x^4 U_j^n + (R_4)_j^n, \end{cases}$$

where

$$\begin{cases} (R_3)_j^n = -\frac{1}{6}h^2 \int_0^1 \left[\frac{\partial^4 u}{\partial x^4}(x_j + sh, t_n) + \frac{\partial^4 u}{\partial x^4}(x_j - sh, t_n) \right] (1-s)^3 ds, \\ (R_4)_j^n = -\frac{1}{6}h^2 \int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_j + sh, t_n) + \frac{\partial^6 u}{\partial x^6}(x_j - sh, t_n) \right] (1-s)^3 ds. \end{cases}$$

Thus, we have

$$\begin{aligned}\mathfrak{L}_1 u(x_j, t_n) &= u_{xx}(x_j, t_n) + \mathcal{I}^{\alpha_1} u_{xx}(x_j, t_n) \\ &= \delta_x^2 U_j^n + \hat{q}_n^{\alpha_1} (\delta_x^2 U_j) + (R_1)_j^n + (R_3)_j^n, \\ &\quad 1 \leq j \leq J-1, 1 \leq n \leq N,\end{aligned}\quad (15)$$

and

$$\begin{aligned}\mathfrak{L}_2 u(x_j, t_n) &= u_{xxxx}(x_j, t_n) + \mathcal{I}^{\alpha_2} u_{xxxx}(x_j, t_n) \\ &= \delta_x^4 U_j^n + \hat{q}_n^{\alpha_2} (\delta_x^4 U_j) + (R_2)_j^n + (R_4)_j^n. \\ &\quad 1 \leq j \leq J-1, 1 \leq n \leq N.\end{aligned}\quad (16)$$

Second, for the nonlinear convection term uu_x , we discretize it by the Galerkin method with piecewise linear test functions

$$\begin{aligned}u(x_j, t_n) u_x(x_j, t_n) &= \frac{U_{j+1}^n + U_j^n + U_{j-1}^n}{3} \frac{U_{j+1}^n - U_{j-1}^n}{2h} + (R_5)_j^n \\ &= \frac{1}{6h} (U_j^n \Delta U_j^n + \Delta (U_j^n)^2) + O(h^2), \\ &\quad 1 \leq j \leq J-1, 1 \leq n \leq N.\end{aligned}\quad (17)$$

Third, for $u_t(x_j, t_n)$, we have

$$u_t(x_j, t_n) = \delta_t U_j^n + (R_6)_j^n, \quad 1 \leq j \leq J-1, 1 \leq n \leq N, \quad (18)$$

where

$$(R_6)_j^n = -k \int_0^1 \frac{\partial^2 u}{\partial t^2}(x_j, t_{n-1} + sk) ds.$$

Substituting (14)–(18) into (13), we obtain

$$\begin{aligned}&\delta_t U_j^n + \frac{1}{6h} (U_j^n \Delta U_j^n + \Delta (U_j^n)^2) - k^{\alpha_1} \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \delta_x^2 U_j^p \\ &+ k^{\alpha_2} \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \delta_x^4 U_j^p - \delta_x^2 U_j^n + \delta_x^4 U_j^n = f_j^n + R_j^n, \\ &\quad 1 \leq j \leq J-1, 1 \leq n \leq N,\end{aligned}\quad (19)$$

in which

$$\begin{aligned}R_j^n &= (R_1)_j^n - (R_2)_j^n - (R_3)_j^n - (R_4)_j^n - (R_5)_j^n - (R_6)_j^n, \\ &\quad 1 \leq j \leq J-1, 1 \leq n \leq N.\end{aligned}$$

By Lemmas 1–3, there is a constant C independent of h and k , which satisfies

$$|R_j^n| \leq C(n^{\alpha_1-1} k^{\alpha_1} + n^{\alpha_2-1} k^{\alpha_2} + h^2), \quad 1 \leq j \leq J-1, 1 \leq n \leq N. \quad (20)$$

The following initial and boundary value conditions can be attained

$$\begin{aligned}U_0^n &= U_J^n = 0, \quad \delta_x^2 U_0^n = \delta_x^2 U_J^n = O(h^2), \quad 1 \leq n \leq N. \\ U_j^0 &= u^0(x_j), \quad 0 \leq j \leq J.\end{aligned}\quad (21)$$

Omitting the small terms in (19) and (21), and replacing U_j^n with its numerical approximation u_j^n , $1 \leq j \leq J-1$, $1 \leq n \leq N$, we obtain the backward Euler implicit difference scheme

$$\begin{cases} \delta_t u_j^n + \frac{1}{6h}(u_j^n \Delta u_j^n + \Delta(u_j^n)^2) - \mathfrak{L}_1 u_j^n + \mathfrak{L}_2 u_j^n = f_j^n, \\ u_0^n = u_j^n = 0, \quad \delta_x^2 u_0^n = \delta_x^2 u_j^n = 0, \\ u_j^0 = u^0(x_j), \end{cases} \quad (22)$$

and

$$\mathfrak{L}_1 u_j^n = \delta_x^2 u_j^n + \hat{q}_n^{\alpha_1}(\delta_x^2 u_j), \quad 1 \leq j \leq J-1, 1 \leq n \leq N,$$

$$\mathfrak{L}_2 u_j^n = \delta_x^4 u_j^n + \hat{q}_n^{\alpha_2}(\delta_x^4 u_j), \quad 1 \leq j \leq J-1, 1 \leq n \leq N,$$

where

$$\delta_x^4 u_j^n = \begin{cases} (-2\delta_x^2 u_1^n + \delta_x^2 u_2^n)/h^2, & j=1, 1 \leq n \leq N, \\ (\delta_x^2 u_{j-1}^n - 2\delta_x^2 u_j^n + \delta_x^2 u_{j+1}^n)/h^2, & 2 \leq j \leq J-2, 1 \leq n \leq N, \\ (\delta_x^2 u_{j-2}^n - 2\delta_x^2 u_{j-1}^n)/h^2, & j=J-1, 1 \leq n \leq N. \end{cases}$$

3. Existence and Stability

In this section, we analyze the L^2 stability, L^∞ stability, and existence of the backward Euler implicit difference scheme (22).

Firstly, we shall introduce some notations and lemmas that will be used for the proof of the stability. Let $V_h = \{s | s = (s_0, s_1, \dots, s_J), s_0 = s_J = 0\}$. For any grid functions $s, g \in V_h$, we denote

$$\langle s, g \rangle = h \sum_{j=1}^{J-1} s_j g_j, \quad \|s\|_\infty = \max_{1 \leq j \leq J-1} \{|s_j|\}, \quad \|s\| = \sqrt{\langle s, s \rangle}. \quad (23)$$

Lemma 4 ([27,28]). For any function s defined on V_h , we obtain

$$\|s\|_\infty \leq \frac{\sqrt{L}}{2} \|\delta_x s\|.$$

Lemma 5 ([9,29]). Let $s, g \in V_h$; then

$$\langle \delta_x^2 s, g \rangle = - \sum_{j=0}^{J-1} h(\delta_x s_{j+1})(\delta_x g_{j+1}).$$

Lemma 6 ([24,25,30]). For any $s, g \in V_h$, such that $\delta_x^2 s_0 = \delta_x^2 s_J = 0$; then, we have

$$\langle \delta_x^4 s, g \rangle = \sum_{j=1}^{J-1} h(\delta_x^2 s_j)(\delta_x^2 g_j).$$

Lemma 7 ([31]). Let $\beta(t) = t^{\alpha-1}/\Gamma(\alpha)$ be defined in Equation (6); $\beta(t) \in L^{1,loc}(0, \infty)$ is a positive type if and only if

$$\operatorname{Re}(\hat{\beta}(t)) \geq 0, \quad \text{for } t \in \mathbb{C}, \operatorname{Re}(t) > 0,$$

where Re denotes the real part, $\hat{\beta}$ denotes the Laplace transform of $\beta(t)$.

Lemma 8 ([24,25]). If $\{b_0, b_1, \dots, b_n, \dots\}$ is a real-valued sequence such that $\hat{b}(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $D = \{z \in \mathbb{C} : |z| \leq 1\}$, then for any positive integer N and any $(V^1, V^2, \dots, V^N) \in \mathbb{R}^N$,

$$\sum_{n=1}^N \left(\sum_{p=1}^n b_{n-p} V^p \right) V^n \geq 0,$$

if and only if

$$\text{Re} \hat{b}(z) \geq 0, \text{ for } z \in D.$$

It is noticed that the generating function (9) satisfies the condition of Lemma 8.

3.1. Stability

Theorem 1. (L^2 -stability) Assume that $\{u_j^n | 1 \leq j \leq J-1, 1 \leq n \leq N\}$ is the solution of the backward Euler implicit difference scheme (22). We can obtain

$$\|u^n\| \leq \|u^0\| + 2k \sum_{i=1}^n \|f^i\|, \quad 1 \leq n \leq N.$$

Proof. Taking the inner product of (22) with u^n , for $1 \leq n \leq N$, we obtain the following formula

$$\begin{aligned} \langle \delta_t u^n, u^n \rangle + \frac{1}{6h} \langle u^n \Delta u^n + \Delta(u^n)^2, u^n \rangle - k^{\alpha_1} \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, u^n \rangle \\ - \langle \delta_x^2 u^n, u^n \rangle + k^{\alpha_2} \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, u^n \rangle + \langle \delta_x^4 u^n, u^n \rangle = \langle f^n, u^n \rangle. \end{aligned} \quad (24)$$

From [9,22,32], we have

$$\langle u^n \Delta u^n + \Delta(u^n)^2, u^n \rangle = 0,$$

then for $N \geq 1$, (24) can be rearranged

$$\begin{aligned} 2k \sum_{n=1}^N \langle \delta_t u^n, u^n \rangle - 2k^{\alpha_1+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, u^n \rangle \\ - 2k \sum_{n=1}^N \langle \delta_x^2 u^n, u^n \rangle + 2k \sum_{n=1}^N \langle \delta_x^4 u^n, u^n \rangle \\ + 2k^{\alpha_2+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, u^n \rangle = 2k \sum_{n=1}^N \langle f^n, u^n \rangle, \quad 1 \leq n \leq N. \end{aligned} \quad (25)$$

Next, we estimate the terms in (25) one by one. First, it is clear that

$$\begin{aligned} \langle \delta_t u^n, u^n \rangle &= \frac{1}{2k} \langle u^n - u^{n-1}, u^n - u^{n-1} + u^n + u^{n-1} \rangle \\ &\geq \frac{1}{2k} (\|u^n\|^2 - \|u^{n-1}\|^2), \end{aligned}$$

we arrive at

$$2k \sum_{n=1}^N \langle \delta_t u^n, u^n \rangle \geq \|u^N\|^2 - \|u^0\|^2. \quad (26)$$

Second, utilizing Lemmas 4, 5, 7, and 8, we have

$$\begin{aligned}
 & -2k \sum_{n=1}^N \langle \delta_x^2 u^n, u^n \rangle - 2k^{\alpha_1+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, u^n \rangle \\
 = & 2k \sum_{n=1}^N h \sum_{j=1}^{J-1} (\delta_x u_j^n)(\delta_x u_j^n) + 2k^{\alpha_1+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_1} h \sum_{j=1}^{J-1} (\delta_x u_j^p)(\delta_x u_j^n) \\
 = & 2kh \sum_{n=1}^N \sum_{j=1}^{J-1} (\delta_x u_j^n)(\delta_x u_j^n) + 2k^{\alpha_1+1} h \sum_{j=1}^{J-1} \left[\sum_{n=1}^N \left(\sum_{p=1}^n \omega_{n-p}^{\alpha_1} \delta_x u_j^p \right) \delta_x u_j^n \right] \\
 \geq & 0, \quad 1 \leq n \leq N.
 \end{aligned} \tag{27}$$

Additionally,

$$\begin{aligned}
 & 2k \sum_{n=1}^N \langle \delta_x^4 u^n, u^n \rangle + 2k^{\alpha_2+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, u^n \rangle \\
 = & 2k \sum_{n=1}^N \sum_{j=1}^{J-1} h (\delta_x^2 u_j^n)(\delta_x^2 u_j^n) + 2k^{\alpha_2+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} h \sum_{j=1}^{J-1} (\delta_x^2 u_j^p)(\delta_x^2 u_j^n) \\
 = & 2k \sum_{n=1}^N \sum_{j=1}^{J-1} h (\delta_x^2 u_j^n)(\delta_x^2 u_j^n) + 2k^{\alpha_2+1} h \sum_{j=1}^{J-1} \left[\sum_{n=1}^N \left(\sum_{p=1}^n \omega_{n-p}^{\alpha_2} \delta_x^2 u_j^p \right) \delta_x^2 u_j^n \right] \\
 \geq & 0, \quad 1 \leq n \leq N.
 \end{aligned} \tag{28}$$

Substituting (26)–(28) into (25), and using the Cauchy–Schwarz inequality, we have

$$\|u^N\|^2 \leq \|u^0\|^2 + 2k \sum_{n=1}^N \|f^n\| \|u^n\|. \tag{29}$$

Taking $\|u^M\| = \max_{0 \leq n \leq N} \|u^n\|$, we obtain

$$\|u^N\| \leq \|u^M\| \leq \|u^0\| + 2k \sum_{i=1}^M \|f^i\| \leq \|u^0\| + 2k \sum_{i=1}^N \|f^i\|. \tag{30}$$

The proof of the Theorem 1 is finished.

□

Theorem 2. (H^1 -stability) Assume that $\{u_j^n | 1 \leq j \leq J-1, 1 \leq n \leq N\}$ is the solution of the backward Euler implicit difference scheme (22). Then, it holds that

$$|u^n|_1 \leq |u^0|_1 + 2k \sum_{n=1}^N |f^n|_1, \quad 1 \leq n \leq N.$$

Proof. Taking the inner product of (22) with $-2k\delta_x^2 u^n$, for $1 \leq n \leq N$, we have

$$\begin{aligned}
 & -2k \langle \delta_t u^n, \delta_x^2 u^n \rangle - \frac{k}{3h} \langle u^n \Delta u^n + \Delta (u^n)^2, \delta_x^2 u^n \rangle \\
 & + 2k^{1+\alpha_1} \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, \delta_x^2 u^n \rangle + 2k \langle \delta_x^2 u^n, \delta_x^2 u^n \rangle \\
 & - 2k^{1+\alpha_2} \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, \delta_x^2 u^n \rangle - 2k \langle \delta_x^4 u^n, \delta_x^2 u^n \rangle = -2k \langle f^n, \delta_x^2 u^n \rangle.
 \end{aligned} \tag{31}$$

Since

$$\begin{aligned}
 -\langle u^n \Delta u^n + \Delta(u^n)^2, \delta_x^2 u^n \rangle &= \langle \delta_x(u^n \Delta u^n + \Delta(u^n u^n)), \delta_x u^n \rangle \\
 &= \sum_{j=1}^{J-1} h \delta_x(u_j^n \Delta u_j^n + \Delta(u_j^n u_j^n)) \delta_x u_j^n \\
 &= \sum_{j=1}^{J-1} h \delta_x(u_j^n (u_{j+1}^n - u_{j-1}^n) + (u_{j+1}^n u_{j+1}^n - u_{j-1}^n u_{j-1}^n)) \delta_x u_j^n \\
 &= \sum_{j=1}^{J-1} h \delta_x((u_j^n + u_{j+1}^n) u_{j+1}^n - (u_j^n + u_{j-1}^n) u_{j-1}^n) \delta_x u_j^n \\
 &= \sum_{j=1}^{J-1} h \delta_x(u_j^n + u_{j+1}^n) \delta_x u_{j+1}^n \delta_x u_j^n - \sum_{j=1}^{J-1} h \delta_x(u_j^n + u_{j-1}^n) \delta_x u_{j-1}^n \delta_x u_j^n \\
 &= \sum_{j=1}^{J-1} h \delta_x(u_j^n + u_{j+1}^n) \delta_x u_{j+1}^n \delta_x u_j^n - \sum_{j=0}^{J-2} h \delta_x(u_j^n + u_{j+1}^n) \delta_x u_{j+1}^n \delta_x u_j^n \\
 &= h \delta_x(u_{J-1}^n + u_J^n) \delta_x u_J^n \delta_x u_{J-1}^n - h \delta_x(u_0^n + u_1^n) \delta_x u_1^n \delta_x u_0^n \\
 &= 0.
 \end{aligned}$$

Then, we have

$$-\frac{k}{3h} \langle u^n \Delta u^n + \Delta(u^n)^2, \delta_x^2 u^n \rangle = 0.$$

For $N \geq 1$, (31) can be rearranged

$$\begin{aligned}
 &-2k \sum_{n=1}^N \langle \delta_t u^n, \delta_x^2 u^n \rangle + 2k^{1+\alpha_1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, \delta_x^2 u^n \rangle \\
 &+ 2k \sum_{n=1}^N \langle \delta_x^2 u^n, \delta_x^2 u^n \rangle - 2k^{1+\alpha_2} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, \delta_x^2 u^n \rangle \\
 &- 2k \sum_{n=1}^N \langle \delta_x^4 u^n, \delta_x^2 u^n \rangle = -2k \sum_{n=1}^N \langle f^n, \delta_x^2 u^n \rangle, \quad 1 \leq n \leq N.
 \end{aligned} \tag{32}$$

Since

$$-\langle \delta_t u^n, \delta_x^2 u^n \rangle \geq \frac{1}{2k} (|u^n|_1^2 - |u^{n-1}|_1^2),$$

then

$$-2k \sum_{n=1}^N \langle \delta_t u^n, \delta_x^2 u^n \rangle \geq |u^N|_1^2 - |u^0|_1^2. \tag{33}$$

By Lemma 8, we have

$$\begin{aligned}
 &2k \sum_{n=1}^N \langle \delta_x^2 u^n, \delta_x^2 u^n \rangle + 2k^{\alpha_1+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \langle \delta_x^2 u^p, \delta_x^2 u^n \rangle \\
 &\geq 0, \quad 1 \leq n \leq N.
 \end{aligned} \tag{34}$$

Further, by Lemmas 4, 5, 7, and 8, we have

$$\begin{aligned}
 & -2k \sum_{n=1}^N \langle \delta_x^4 u^n, \delta_x^2 u^n \rangle - 2k^{\alpha_2+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \langle \delta_x^4 u^p, \delta_x^2 u^n \rangle \\
 & = 2k \sum_{n=1}^N \sum_{j=0}^{J-1} h(\delta_x^3 u_j^n)(\delta_x^3 u_j^n) + 2k^{\alpha_2+1} \sum_{n=1}^N \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \sum_{j=0}^{J-1} h(\delta_x^3 u_j^p)(\delta_x^3 u_j^n) \\
 & = 2k \sum_{n=1}^N \sum_{j=0}^{J-1} h(\delta_x^3 u_j^n)(\delta_x^3 u_j^n) + 2k^{\alpha_2+1} h \sum_{j=0}^{J-1} \left[\sum_{n=1}^N \left(\sum_{p=1}^n \omega_{n-p}^{\alpha_2} \delta_x^3 u_j^p \right) \delta_x^3 u_j^n \right] \\
 & \geq 0, \quad 1 \leq n \leq N.
 \end{aligned} \tag{35}$$

Substituting (33)–(35) into (32), and using the Cauchy–Schwarz inequality, we have

$$|u^N|_1^2 \leq |u^0|_1^2 + 2k \sum_{n=1}^N |f^n|_1 |u^n|_1. \tag{36}$$

Similarly to Equation (30), we finish the proof of the Theorem 2. $\square$

3.2. Existence

Next, we will use the Leray–Schauder Theorem [33] to prove the existence of numerical solutions for the scheme (22).

Theorem 3. *Giving two positive integers J , N , and $u^0 \in R^{J-1}$, the Equation (22) has a solution u^n for $1 \leq n \leq N$.*

Proof. We can employ the mathematical induction to prove the Theorem 3. Since $u^0 \in R^{J-1}$, for given u^m , $1 \leq m \leq n-1$, we will prove that Equation (22) has a solution for u^n .

At the beginning, we define the mapping $\mathfrak{X} : R^{J-1} \rightarrow R^{J-1}$ by

$$\mathfrak{X}(v) := -\frac{k}{6h}(v\Delta v + \Delta(v)^2) + k\delta_x^2 v + k^{\alpha_1+1}\omega_0^{\alpha_1}\delta_x^2 v - k^{\alpha_2+1}\omega_0^{\alpha_2}\delta_x^4 v - k\delta_x^4 v.$$

Then, u^n is a solution of (22) if and only if

$$u^n = \mathfrak{X}(u^n) + \tilde{f},$$

in which

$$\tilde{f} = u^{n-1} + k^{\alpha_1+1} \sum_{p=1}^{n-1} \omega_{n-p}^{\alpha_1} \delta_x^2 u^p - k^{\alpha_2+1} \sum_{p=1}^{n-1} \omega_{n-p}^{\alpha_2} \delta_x^4 u^p + kf^n.$$

Next, we need to prove that the mapping $\mathfrak{G}(\cdot) = \mathfrak{X}(\cdot) + \tilde{f}$ has a fixed point. We consider an open ball $\mathcal{L} = A(0, r)$ in R^{J-1} endowed with the norm $\|\cdot\|$ in (23). Suppose that for $\lambda > 1$ and u^n in the boundary of $\mathcal{L}$,

$$\lambda u^n = \mathfrak{G}(u^n) = \mathfrak{X}(u^n) + \tilde{f}. \tag{37}$$

Since $\langle v\Delta v + \Delta(v)^2, v \rangle = 0$, using Lemmas 5 and 6, we obtain

$$\langle \mathfrak{X}(u^n), u^n \rangle \leq 0.$$

Taking the inner product of (37) with u^n , we have

$$\lambda \|u^n\|^2 \leq \langle \tilde{f}, u^n \rangle \leq \|\tilde{f}\| \|u^n\| \leq \frac{1}{2} (\|\tilde{f}\|^2 + \|u^n\|^2).$$

Thus,

$$\lambda \leq \frac{1}{2} \frac{\|\tilde{f}\|^2 + \|u^n\|^2}{\|u^n\|^2} = \frac{1}{2} \left(\frac{\|\tilde{f}\|^2}{\|u^n\|^2} + 1 \right) \leq \frac{\|\tilde{f}\|^2}{2r^2} + \frac{1}{2}.$$

It is noted that the above inequality contradicts with hypothesis $\lambda > 1$ for large r . Hence, (37) has no solution on $\partial\mathcal{L}$. By the Leray–Schauder Theorem [33], there is a fixed point of $\mathfrak{G}$ in the closure of $\mathcal{L}$. The proof of existence Theorem is finished. $\square$

4. Uniqueness and Convergence

4.1. Convergence

Let

$$e_j^n = U_j^n - u_j^n, 0 \leq j \leq J, 1 \leq n \leq N.$$

Subtracting (22) from (19), we obtain the following error equations

$$\begin{cases} \delta_t e_j^n + \frac{1}{6h} (e_j^n \Delta e_j^n + \Delta (e_j^n)^2) - \mathfrak{L}_1 e_j^n + \mathfrak{L}_2 e_j^n = (R_1)_j^n - (R_2)_j^n \\ \quad - (R_3)_j^n - (R_4)_j^n - (R_5)_j^n - (R_6)_j^n - \frac{1}{6h} ((R_7)_j^n + (R_8)_j^n), \\ \quad 1 \leq j \leq J-1, 1 \leq n \leq N \\ e_0^n = e_J^n = 0, \quad 1 \leq n \leq N, \\ e_j^0 = 0, \quad 0 \leq j \leq J, \end{cases} \quad (38)$$

where

$$\begin{cases} (R_7)_j^n = U_j^n \Delta e_j^n + \Delta (e_j^n U_j^n), \\ (R_8)_j^n = e_j^n \Delta U_j^n + \Delta (e_j^n U_j^n). \end{cases} \quad (39)$$

To complete the proof of convergence, we provide the following Lemmas.

Lemma 9 ([34]). (Discrete Gronwall's inequality) If A_n is a non-negative real sequence and satisfies

$$A_n \leq \tilde{c}_n + \sum_{m=0}^{n-1} \tilde{d}_m A_m, \quad n \geq 0,$$

where $\tilde{c}_n$ is non-descending and non-negative sequence, $\tilde{d}_m \geq 0$, then it holds that

$$A_n \leq \tilde{c}_n \exp\left(\sum_{m=0}^{n-1} \tilde{d}_m\right), \quad n \geq 0.$$

Lemma 10. For $\forall s, g \in V_h$, it holds that

$$\begin{aligned} (i) \quad & \langle g \Delta s, g \rangle + \langle \Delta(g)^2, s \rangle = 0; \\ (ii) \quad & \langle g \Delta g, s \rangle + \langle \Delta(gs), g \rangle = 0. \end{aligned}$$

Proof. (i) By the definition of $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned}
 & \langle g\Delta s, g \rangle + \langle \Delta(g)^2, s \rangle \\
 &= h \sum_{j=1}^{J-1} g_j (s_{j+1} - s_{j-1}) g_j + h \sum_{j=1}^{J-1} (g_{j+1} g_{j+1} - g_{j-1} g_{j-1}) s_j \\
 &= h \sum_{j=2}^J g_{j-1} s_j g_{j-1} - h \sum_{j=1}^{J-1} g_j s_{j-1} g_j + h \sum_{j=2}^J g_j s_j s_{j-1} \\
 &\quad - h \sum_{j=1}^{J-1} g_{j+1} g_{j+1} s_j \\
 &= h g_{J-1} s_J g_{J-1} - h g_0 s_1 g_0 + h g_J s_{J-1} g_J - h g_1 s_0 g_1 \\
 &= 0.
 \end{aligned}$$

The proof of (ii) are similar to (i). Thus, Lemma 10 is proved. $\square$

Lemma 11. When $U_0 = U_J = 0$ and $e_0 = e_J = 0$, then it holds that

$$\begin{aligned}
 & k \sum_{n=1}^N (\|(R_1)_j^n\| + \|(R_2)_j^n\| + \|(R_3)_j^n\| + \|(R_4)_j^n\| + \|(R_5)_j^n\| \\
 & + \|(R_6)_j^n\|) \leq C(T)(k + h^2).
 \end{aligned}$$

Proof. Utilizing the conditions $U_0 = U_J = 0$ and $e_0 = e_J = 0$.

Firstly, by (15), Lemmas 1 and 2, we have

$$\begin{aligned}
 & k \sum_{n=1}^N \|(R_1)_j^n\| + k \sum_{n=1}^N \|(R_3)_j^n\| \\
 & \leq k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h [C(n^{\alpha_1-1} k^{\alpha_1} + h^2)]^2} \\
 & \leq Ck \sum_{n=1}^N (k^{\alpha_1} n^{\alpha_1-1} + h^2) \leq Ck \left(\sum_{n=1}^N t_n^{\alpha_1-1} k \right) + C(Nk)h^2 \\
 & \leq Ck \left(\int_{t_0}^{t_N} s^{\alpha_1-1} ds \right) + C(T)h^2 \leq Ck \left(\frac{T^{\alpha_1}}{\alpha_1} \right) + C(T)h^2 \leq C(T)(k + h^2).
 \end{aligned} \tag{40}$$

Secondly,

$$\begin{aligned}
 & k \sum_{n=1}^N \|(R_5)_j^n\| = k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h ((R_5)_j^n)^2} \leq k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h (Ch^2)^2} \\
 & \leq C(T)h^2.
 \end{aligned} \tag{41}$$

Thirdly,

$$k \sum_{n=1}^N \|(R_6)_j^n\| = k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h ((R_6)_j^n)^2} \leq k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h (Ck)^2} \leq C(T)k. \tag{42}$$

Finally, by (16), Lemmas 1 and 3, we have

$$\begin{aligned}
 & k \sum_{n=1}^N \|(R_2)_j^n\| + k \sum_{n=1}^N \|(R_4)_j^n\| \\
 & \leq k \sum_{n=1}^N \sqrt{\sum_{j=1}^{J-1} h [C(n^{\alpha_2-1} k^{\alpha_2} + h^2)]^2} \\
 & \leq Ck \left(\int_{t_0}^{t_N} s^{\alpha_1-1} ds \right) + C(T)h^2 \\
 & Ck \left(\frac{T^{\alpha_2}}{\alpha_2} \right) + C(T)h^2 \leq C(T)(k + h^2).
 \end{aligned} \tag{43}$$

Combining (40)–(43), we have

$$\begin{aligned}
 & k \sum_{n=1}^N (\|(R_1)_j^n\| + \|(R_2)_j^n\| + \|(R_3)_j^n\| + \|(R_4)_j^n\| + \|(R_5)_j^n\| \\
 & + \|(R_6)_j^n\|) \leq C(T)(k + h^2).
 \end{aligned} \tag{44}$$

Therefore, we are done with this proof. $\square$

Lemma 12. Set $\hat{c}_0 := \max_{(x,t) \in [0,L] \times (0,T]} \{|u(x,t)|, |u_x(x,t)|\}$, for $U_0 = U_J = 0$, $e_0 = e_J = 0$, $1 \leq n \leq N$, we have

$$|\langle (R_8)^n, e^n \rangle| \leq 3\hat{c}_0 h \|e^n\|^2.$$

Proof. By Lemma 10, for $\forall s, g \in V_h$, it holds that

$$\langle \Delta(gs), s \rangle = \frac{1}{2} \langle s_{j+1} \Delta_+ g + s_{j-1} \Delta_- g, s \rangle,$$

then, we obtain

$$\langle \Delta(e^n U^n), e^n \rangle = \frac{1}{2} \langle e_{j+1}^n \Delta_+ U^n + e_{j-1}^n \Delta_- U^n, e^n \rangle, 1 \leq n \leq N. \tag{45}$$

Utilizing the boundary conditions $U_0 = U_J = 0$, $e_0 = e_J = 0$ and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
 |\langle e_{j+1}^n \Delta_+ U^n, e^n \rangle| & \leq \|\Delta_+ U^n\|_\infty |\langle e_{j+1}^n, e^n \rangle| \\
 & = \|\Delta_+ U^n\|_\infty \left| \sum_{j=1}^{J-1} h e_{j+1}^n e_j^n \right| \\
 & \leq \|\Delta_+ U^n\|_\infty \sum_{j=1}^{J-1} \frac{h}{2} ((e_{j+1}^n)^2 + (e_j^n)^2) \\
 & = \frac{1}{2} \|\Delta_+ U^n\|_\infty \left(\sum_{j=1}^{J-1} h (e_{j+1}^n)^2 + \|e^n\|^2 \right) \\
 & \leq \frac{1}{2} \|\Delta_+ U^n\|_\infty \left(\sum_{j=2}^J h (e_j^n)^2 + \|e^n\|^2 \right) \\
 & \leq \|\Delta_+ U^n\|_\infty \|e^n\|^2, 1 \leq n \leq N.
 \end{aligned}$$

Additionally, we can get

$$|\langle e_{j-1}^n \Delta_- U^n, e^n \rangle| \leq \|\Delta_- U^n\|_\infty \|e^n\|^2, 1 \leq n \leq N.$$

Thus, we obtain

$$\begin{aligned}
 |\langle (R_8)^n, e^n \rangle| &= |\langle e^n \Delta U^n + \Delta(e^n U^n), e^n \rangle| \\
 &= |\langle e^n \Delta U^n, e^n \rangle + \frac{1}{2} \langle e_{j+1}^n \Delta_+ U^n + e_{j-1}^n \Delta_- U^n, e^n \rangle| \\
 &\leq \|\Delta U^n\|_\infty \|e^n\|^2 + \frac{1}{2} (\|\Delta_+ U^n\|_\infty + \|\Delta_- U^n\|_\infty) \|e^n\|^2 \\
 &\leq 3\tilde{c}_0 h \|e^n\|^2, 1 \leq n \leq N.
 \end{aligned}$$

The proof is proved. $\square$

Theorem 4. (L^2 -convergence) Suppose that the problem (1)–(3) has smooth solution $u(x, t) \in C_{x,t}^{4,2}([0, L] \times (0, T])$, $\{u_j^n | 0 \leq j \leq J, 1 \leq n \leq N\}$ is the solution of difference scheme (22). We can obtain

$$\max_{1 \leq n \leq N} \|U^n - u^n\| \leq C(T)(k + h^2).$$

Proof. Taking the inner product of the (38) with e^n , summing up for n from 1 to N , utilizing (26) and Lemma 8, and noting

$$\langle e^n \Delta e^n + \Delta(e^n)^2, e^n \rangle = 0$$

we can obtain

$$\begin{aligned}
 \|e^N\|^2 &\leq \|e^0\|^2 + 2k \sum_{n=1}^N \langle R_1^n - R_2^n - R_3^n - R_4^n \\
 &\quad - R_5^n - R_6^n - \frac{1}{6h} (R_7^n + R_8^n), e^n \rangle, 1 \leq n \leq N.
 \end{aligned} \tag{46}$$

It is noted that when $e^0 = 0$, we have

$$\begin{aligned}
 \|e^N\|^2 &\leq 2k \sum_{n=1}^N (\|R_1^n\| + \|R_2^n\| + \|R_3^n\| + \|R_4^n\| + \|R_5^n\| \\
 &\quad + \|R_6^n\| \|e^n\| - 2k \sum_{n=1}^N \langle \frac{1}{6h} (R_7^n + R_8^n), e^n \rangle, 1 \leq n \leq N.
 \end{aligned} \tag{47}$$

Since

$$\begin{aligned}
 \langle R_7^n, e^n \rangle &= \langle U^n \Delta e^n + \Delta(e^n U^n), e^n \rangle \\
 &= \sum_{j=1}^{J-1} h (U_j^n \Delta e_j^n + \Delta(e_j^n U_j^n)) e_j^n \\
 &= \sum_{j=1}^{J-1} h (U_j^n (e_{j+1}^n - e_{j-1}^n) + (e_{j+1}^n U_{j+1}^n - e_{j-1}^n U_{j-1}^n)) e_j^n \\
 &= \sum_{j=1}^{J-1} h ((U_j^n + U_{j+1}^n) e_{j+1}^n - (U_j^n + U_{j-1}^n) e_{j-1}^n) e_j^n \\
 &= \sum_{j=1}^{J-1} h (U_j^n + U_{j+1}^n) e_{j+1}^n e_j^n - \sum_{j=1}^{J-1} h (U_j^n + U_{j-1}^n) e_{j-1}^n e_j^n \\
 &= \sum_{j=1}^{J-1} h (U_j^n + U_{j+1}^n) e_{j+1}^n e_j^n - \sum_{j=0}^{J-2} h (U_j^n + U_{j+1}^n) e_{j+1}^n e_j^n \\
 &= h (U_{J-1}^n + U_J^n) e_J^n e_{J-1}^n - h (U_0^n + U_1^n) e_1^n e_0^n \\
 &= 0, \quad 1 \leq n \leq N.
 \end{aligned} \tag{48}$$

Combining (48), Lemma 12 and inequality (40), (47) can be written as

$$\begin{aligned} \|e^N\|^2 &\leq 2k \sum_{n=1}^N (\|R_1^n\| + \|R_2^n\| + \|R_3^n\| \\ &\quad + \|R_4^n\| + \|R_5^n\| + \|R_6^n\|) \|e^n\| + \hat{c}_0 k \sum_{n=1}^N \|e^n\|^2, 1 \leq n \leq N. \end{aligned}$$

Taking appropriate M such that $\|e^M\| = \max_{0 \leq n \leq N} \|e^n\|$ and using Lemma 11, we obtain

$$\begin{aligned} \|e^N\| &\leq \|e^M\| \\ &\leq 2k \sum_{n=1}^N (\|R_1^n\| + \|R_2^n\| + \|R_3^n\| + \|R_4^n\| + \|R_5^n\| + \|R_6^n\|) + \hat{c}_0 k \sum_{n=1}^N \|e^n\| \\ &\leq C(T)(k + h^2) + \hat{c}_0 k \sum_{n=1}^N \|e^n\|. \end{aligned} \quad (49)$$

Further

$$(1 - \hat{c}_0 k) \|e^N\| \leq C(T)(k + h^2) + \hat{c}_0 k \sum_{n=0}^{N-1} \|e^n\|. \quad (50)$$

Using discrete Gronwall inequality, for $k < \frac{\hat{c}_0}{2}$, we obtain

$$\|e^N\| \leq 2 \exp\{2\hat{c}_0 Nk\} C(T)(k + h^2) \leq C(T)(k + h^2).$$

□

4.2. Uniqueness

Theorem 5. Under the assumptions in Theorem 4—for h is small enough and $k = o(h^{\frac{3}{4}})$ —then the difference scheme (22) has a unique solution.

Proof. Set $u^n \in R^{n-1}$ and $v^n \in R^{n-1}$, $0 \leq n \leq N$ to be the solutions of (22). Since $u^0 = v^0$, we assume $u^m = v^m$ for $0 \leq m \leq n-1$. Next, we need to prove $u^n = v^n$.

First, using (22), we have

$$\begin{aligned} &\delta_t(u_j^n - v_j^n) - \delta_x^2(u_j^n - v_j^n) + \delta_x^4(u_j^n - v_j^n) + \frac{1}{6h}(u_j^n \Delta u_j^n \\ &\quad + \Delta(u_j^n)^2 - v_j^n \Delta v_j^n - \Delta(v_j^n)^2) \\ &= k^{\alpha_1} \sum_{p=1}^n \omega_{n-p}^{\alpha_1} \delta_x^2(u_j^p - v_j^p) - k^{\alpha_2} \sum_{p=1}^n \omega_{n-p}^{\alpha_2} \delta_x^4(u_j^p - v_j^p). \end{aligned} \quad (51)$$

Second, taking the inner product of (51) with $u^n - v^n$, and using Lemmas 5, 6, and 8, we obtain

$$\begin{aligned} &\frac{1}{2k} (\|u^n - v^n\|^2 - \|u^{n-1} - v^{n-1}\|^2) \\ &\leq -\frac{1}{6h} \langle u^n \Delta u^n + \Delta(u^n)^2 - v^n \Delta v^n - \Delta(v^n)^2, u^n - v^n \rangle \\ &= -\frac{1}{6h} \langle u^n \Delta(u^n - v^n) + (u^n - v^n) \Delta v^n + \Delta(u^n - v^n)(u^n + v^n), u^n - v^n \rangle. \end{aligned}$$

Since

$$\begin{aligned} & \langle u^n \Delta(u^n - v^n) + (u^n - v^n) \Delta v^n + \Delta(u^n - v^n)(u^n + v^n), u^n - v^n \rangle \\ &= \langle (u^n - v^n) \Delta v^n + \Delta(v^n(u^n - v^n)), u^n - v^n \rangle. \end{aligned}$$

Then,

$$\begin{aligned} \|u^n - v^n\|^2 &\leq \|u^{n-1} - v^{n-1}\|^2 \\ &\quad - \frac{k}{3h} \langle (u^n - v^n) \Delta v^n + \Delta(v^n(u^n - v^n)), u^n - v^n \rangle \\ &\leq \frac{k}{3h} |\langle (u^n - v^n) \Delta v^n + \Delta(v^n(u^n - v^n)), u^n - v^n \rangle|. \end{aligned} \quad (52)$$

Further, we have

$$\begin{aligned} & |\langle (u^n - v^n) \Delta v^n + \Delta(v^n(u^n - v^n)), u^n - v^n \rangle| \\ &\leq \|\Delta v^n\|_\infty \|u^n - v^n\|^2 + \frac{1}{2} (\|\Delta_+ v^n\|_\infty + \|\Delta_- v^n\|_\infty) \|u^n - v^n\|^2. \end{aligned} \quad (53)$$

Rearranging, we have

$$\begin{aligned} \|\Delta v^n\|_\infty &= \max_{1 \leq j \leq J-1} \{ |u_{j+1}^n - v_{j-1}^n| \} \\ &\leq \max_{1 \leq j \leq J-1} \{ |v_{j+1}^n - V_{j+1}^n| + |V_{j+1}^n - V_{j-1}^n| + |V_{j-1}^n - v_{j-1}^n| \} \\ &\leq 2\|V^n - v^n\|_\infty + Ch \leq 2h^{-\frac{1}{2}} \|V^n - v^n\| + Ch \\ &\leq Ch^{-\frac{1}{2}} (k + h^2) + Ch. \end{aligned} \quad (54)$$

Additionally,

$$\begin{aligned} & |\langle (u^n - v^n) \Delta v^n + \Delta(v^n(u^n - v^n)), u^n - v^n \rangle| \\ &\leq C[h^{-\frac{1}{2}} (k + h^2) + h] \|u^n - v^n\|^2. \end{aligned} \quad (55)$$

Thus, we obtain

$$\|u^n - v^n\| \leq C(k^2 h^{-\frac{3}{2}} + kh^{\frac{1}{2}} + k) \|u^n - v^n\|. \quad (56)$$

Using inequality (56), we have $\|u^n - v^n\|^2 = 0$ for $k = o(h^{\frac{3}{4}})$ as $h \rightarrow 0$ and finish the proof. $\square$

5. Numerical Results

In this section, we solve this problem (1)–(3) with $L = T = 1$ by difference scheme (22). We provide three iterative methods [32,35,36]: the Besse relaxation algorithm (Besse), the Newton iterative method (Newton), and the linearized iterative algorithm (linearized), to solve the nonlinear system (22). Let $MaxStep = 300$ and $eps = 1.0 \times 10^{-5}$ and

$$\begin{aligned} E(h, k) &= \sqrt{h \sum_{j=1}^{J-1} (U_j^N - u_j^N)^2}, \\ rate^x &= \log_2 \left(\frac{E(2h, k)}{E(h, k)} \right), \quad rate^t = \log_2 \left(\frac{E(h, 2k)}{E(h, k)} \right). \end{aligned}$$

Example 1. In the first example, we consider the initial condition $\varphi(x) = \sin \pi x$, the source term

$$\begin{aligned} f(x, t) = & \sin 2\pi x \left(-\frac{2\alpha t^{\alpha-1}}{\Gamma(\alpha+1)} - \frac{2(\alpha+1)t^{\alpha}}{\Gamma(\alpha+2)} \right) \\ & + \pi \sin 4\pi x \left(1 - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2 \\ & + 4\pi^2 \sin 2\pi x \left(\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \frac{2t^{\alpha_1+\alpha}}{\Gamma(\alpha+\alpha_1+1)} - \frac{t^{\alpha_1+\alpha+1}}{\Gamma(\alpha+\alpha_1+2)} \right) \\ & + 16\pi^4 \sin 2\pi x \left(\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{2t^{\alpha_2+\alpha}}{\Gamma(\alpha+\alpha_2+1)} - \frac{t^{\alpha_2+\alpha+1}}{\Gamma(\alpha+\alpha_2+2)} \right) \\ & (16\pi^4 \sin 2\pi x + 4\pi^2 \sin 2\pi x) \left(1 - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right). \end{aligned}$$

and the exact solution is

$$u(x, t) = \sin 2\pi x \left(1 - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right),$$

where $\alpha, 0 < \alpha < 1$ is the regular parameter.

Table 1 lists the L^2 norm errors; the corresponding spatial convergence rate; and the total number of iterations of our scheme under different parameters α_1 and α_2 , respectively. Taking the temporal step $k = 1/1024$ and $\alpha = 0.50$, we can know from Table 1 that the spatial convergence order is about order 2. Through comparison, it can be seen that the numerical results of the three iterative methods have a small gap in the spatial direction.

Fix the spatial step $h = 1/J = 1024$ and $\alpha = 0.50$. Table 2 shows that the temporal convergence order is about order one. Through the comparison of three iteration methods, we can find that the temporal convergence order of the Basse relaxation algorithm is not very stable. In addition, the total number of iterations of the linear iterative algorithm is less than the Newton iterative method.

Taking $\alpha = 0.5$ fixed, Figure 1 shows the spatial convergence order for $N = 1024$, and Figure 2 shows the convergence order in the time direction for $J = 1024$. It can be seen that the numerical results of the convergence order are in good agreement with the theoretical analysis.

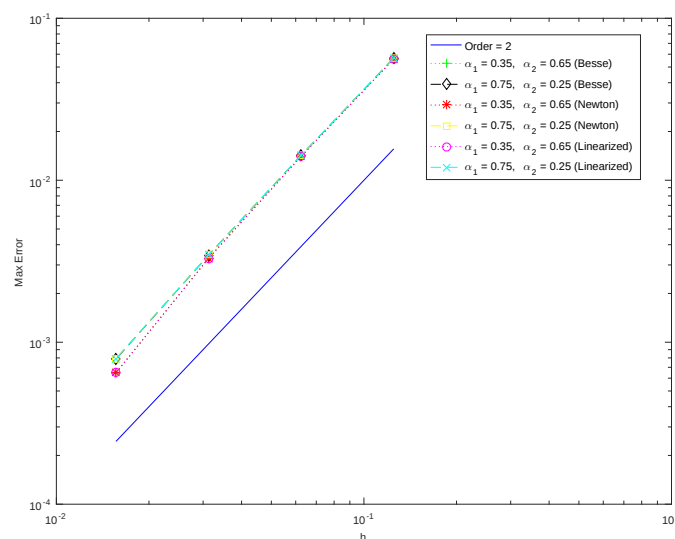


Figure 1. The error and convergence orders in space with $\alpha = 0.50$ and $k = 1/1024$, for Example 1.

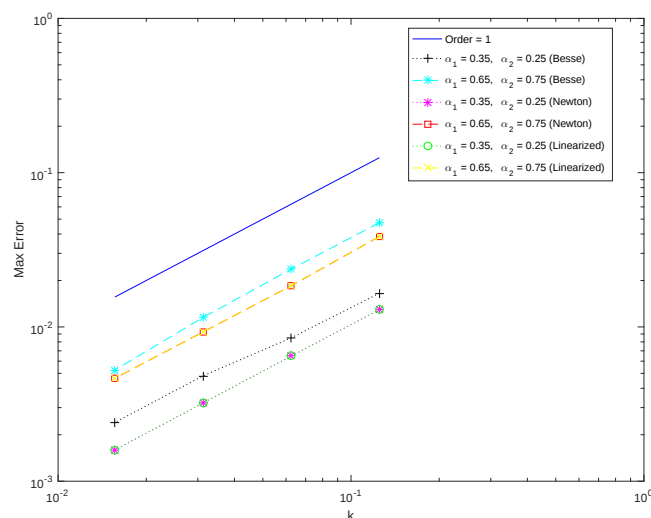


Figure 2. The error and convergence orders in time with $\alpha = 0.50$ and $h = 1/1024$, for Example 1.

Table 1. The errors and convergence rates when $k = 1/1024$ and $\alpha = 0.50$, for Example 1.

Methods	α_1, α_2	J	Error	rate ^x	Iterative
Besse	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3245×10^{-2}	—	—
		32	1.1700×10^{-2}	0.9904	—
		64	5.7776×10^{-3}	1.0179	—
		128	2.7732×10^{-3}	1.0589	—
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1483×10^{-2}	—	—
		32	1.0779×10^{-2}	0.9950	—
		64	5.3063×10^{-3}	1.0224	—
		128	2.5347×10^{-3}	1.0659	—
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8288×10^{-2}	—	—
		32	9.1218×10^{-3}	1.0035	—
		64	4.4609×10^{-3}	1.0320	—
		128	2.1079×10^{-3}	1.0816	—
Newton	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3456×10^{-2}	—	46
		32	1.1747×10^{-2}	0.9977	87
		64	5.7882×10^{-3}	1.0211	155
		128	2.7753×10^{-3}	1.0605	259
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1633×10^{-2}	—	46
		32	1.0811×10^{-2}	1.0007	87
		64	5.3130×10^{-3}	1.0249	155
		128	2.5358×10^{-3}	1.0671	259
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8353×10^{-2}	—	45
		32	9.1346×10^{-3}	1.0066	86
		64	4.4631×10^{-3}	1.0333	154
		128	2.1078×10^{-3}	1.0823	259

Table 1. Cont.

Methods	α_1, α_2	J	Error	$rate^x$	Iterative
<i>Linearized</i>	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3455×10^{-2}	—	46
		32	1.1747×10^{-2}	0.9976	87
		64	5.7879×10^{-3}	1.0212	154
		128	2.7751×10^{-3}	1.0605	259
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1632×10^{-2}	—	46
		32	1.0812×10^{-2}	1.0006	87
		64	5.3132×10^{-3}	1.0250	154
		128	2.5357×10^{-3}	1.0672	259
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8353×10^{-2}	—	45
		32	9.1347×10^{-3}	1.0066	86
		64	4.4626×10^{-3}	1.0335	153
		128	2.1077×10^{-3}	1.0822	259

Table 2. The errors and convergence rates when $h = 1/1024$ and $\alpha = 0.50$, for Example 1.

Methods	α_1, α_2	N	Error	$rate^t$	Iterative
<i>Besse</i>	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3245×10^{-2}	—	—
		32	1.1700×10^{-2}	0.9904	—
		64	5.7776×10^{-3}	1.0179	—
		128	2.7732×10^{-3}	1.0589	—
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1483×10^{-2}	—	—
		32	1.0779×10^{-2}	0.9950	—
		64	5.3063×10^{-3}	1.0224	—
		128	2.5347×10^{-3}	1.0659	—
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8288×10^{-2}	—	—
		32	9.1218×10^{-3}	1.0035	—
		64	4.4609×10^{-3}	1.0320	—
		128	2.1079×10^{-3}	1.0816	—
<i>Newton</i>	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3456×10^{-2}	—	46
		32	1.1747×10^{-2}	0.9977	87
		64	5.7882×10^{-3}	1.0211	155
		128	2.7753×10^{-3}	1.0605	259
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1633×10^{-2}	—	46
		32	1.0811×10^{-2}	1.0007	87
		64	5.3130×10^{-3}	1.0249	155
		128	2.5358×10^{-3}	1.0671	259
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8353×10^{-2}	—	45
		32	9.1346×10^{-3}	1.0066	86
		64	4.4631×10^{-3}	1.0333	154
		128	2.1078×10^{-3}	1.0823	259

Table 2. Cont.

Methods	α_1, α_2	N	Error	rate ^t	Iterative
Linearized	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	2.3455×10^{-2}	—	46
		32	1.1747×10^{-2}	0.9976	87
		64	5.7879×10^{-3}	1.0212	154
		128	2.7751×10^{-3}	1.0605	259
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	2.1632×10^{-2}	—	46
		32	1.0812×10^{-2}	1.0006	87
		64	5.3132×10^{-3}	1.0250	154
		128	2.5357×10^{-3}	1.0672	259
	$\alpha_1 = 0.75$ $\alpha_2 = 0.55$	16	1.8353×10^{-2}	—	45
		32	9.1347×10^{-3}	1.0066	86
		64	4.4626×10^{-3}	1.0335	153
		128	2.1077×10^{-3}	1.0822	259

Example 2. In the second Example, we take the exact solution

$$u(x, t) = \sin \pi x \frac{2t^\alpha}{\Gamma(\alpha + 1)}, \quad 0 < \alpha < 1.$$

Correspondingly, the initial condition is $u^0(x) = 0$ and the inhomogeneous term is

$$f(x, t) = \sin \pi x \frac{2\alpha t^{\alpha-1}}{\Gamma(\alpha + 1)} + 2\pi \sin 2\pi x \left(\frac{t^\alpha}{\Gamma(\alpha + 1)}\right)^2 + \frac{2\pi^2 \sin(\pi x) t^{\alpha+\alpha_1}}{\Gamma(\alpha + \alpha_1 + 1)} + \frac{2\pi^4 \sin(\pi x) t^{\alpha+\alpha_2}}{\Gamma(\alpha + \alpha_2 + 1)} + (\pi^4 \sin \pi x + \pi^2 \sin \pi x) \frac{2t^\alpha}{\Gamma(\alpha + 1)}.$$

It can be seen from Tables 3 and 4 that the spatial convergence order is about order two and the temporal convergence order is about order one, respectively. It can be seen that the numerical results are the same as Example 1 and the convergence order is in good agreement with the theoretical analysis.

Table 3. The errors and convergence rates when $k = 1/1024$ and $\alpha = 0.50$, for Example 2.

Methods	α_1, α_2	J	Error	rate ^x	Iterative
Besse	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	9.6163×10^{-3}	—	—
		32	2.2743×10^{-3}	2.092	—
		64	4.4453×10^{-4}	2.423	—
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	9.7560×10^{-3}	—	—
		32	2.3922×10^{-3}	2.073	—
		64	5.5698×10^{-4}	2.315	—
Newton	$\alpha_1 = 0.30$ $\alpha_2 = 0.70$	16	9.6292×10^{-3}	—	2048
		32	2.2840×10^{-3}	2.076	2048
		64	4.5341×10^{-4}	2.333	2048
	$\alpha_1 = 0.35$ $\alpha_2 = 0.65$	16	9.6486×10^{-3}	—	2048
		32	2.3005×10^{-3}	2.068	2048
		64	4.6915×10^{-4}	2.294	2048

Table 3. Cont.

Methods	α_1, α_2	J	Error	$rate^x$	Iterative
Linearized	$\alpha_1 = 0.30$	16	9.6291×10^{-3}	—	2048
	$\alpha_2 = 0.70$	32	2.2839×10^{-3}	2.076	2048
		64	4.5330×10^{-4}	2.333	2048
	$\alpha_1 = 0.35$	16	9.6484×10^{-3}	—	2048
	$\alpha_2 = 0.65$	32	2.3004×10^{-3}	2.068	2048
		64	4.6904×10^{-4}	2.294	2048

Table 4. The errors and convergence rates when $h = 1/1024$ and $\alpha = 0.50$, for Example 2.

Methods	α_1, α_2	N	Error	$rate^t$	Iterative
Besse	$\alpha_1 = 0.30$	16	1.2909×10^{-2}	—	—
	$\alpha_2 = 0.70$	32	6.4875×10^{-3}	0.993	—
		64	3.2196×10^{-3}	1.011	—
	$\alpha_1 = 0.35$	16	1.1689×10^{-2}	—	—
	$\alpha_2 = 0.65$	32	5.8697×10^{-3}	0.994	—
		64	2.9127×10^{-3}	1.011	—
Newton	$\alpha_1 = 0.30$	16	1.2818×10^{-2}	—	62
	$\alpha_2 = 0.70$	32	6.4717×10^{-3}	0.986	109
		64	3.2142×10^{-3}	1.010	192
	$\alpha_1 = 0.35$	16	1.1588×10^{-2}	—	62
	$\alpha_2 = 0.65$	32	5.8495×10^{-3}	0.986	110
		64	2.9036×10^{-3}	1.011	192
Linearized	$\alpha_1 = 0.30$	16	1.2813×10^{-2}	—	48
	$\alpha_2 = 0.70$	32	6.4754×10^{-3}	0.985	96
		64	3.2101×10^{-3}	1.012	192
	$\alpha_1 = 0.35$	16	1.1580×10^{-2}	—	48
	$\alpha_2 = 0.65$	32	5.8529×10^{-3}	0.984	96
		64	2.9054×10^{-3}	1.010	192

6. Concluding Remarks

In this paper, we propose an implicit difference scheme for a class of nonlinear fourth-order equations with the multi-term Riemann–Liouville fractional integral kernels. For the nonlinear convection term, we use the Galerkin method based on piecewise linear test functions. The Riemann–Liouville fractional integral terms are treated by convolution quadrature. The standard central difference approximation is used to discretize the spatial derivative. The stability and convergence are rigorously proved by the discrete energy method. The existence and uniqueness of the numerical solutions for nonlinear systems are proved strictly. Lastly, we introduce and compare three iterative methods for solving the nonlinear systems.

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