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Multi-Solitons, Multi-Breathers and Multi-Rational Solutions of Integrable Extensions of the Kadomtsev–Petviashvili Equation in Three Dimensions

Athanasios S. Fokas^{1,2,3}, Yulei Cao⁴ and Jingsong He^{5,*}¹ Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK² Viterbi School of Engineering, University of Southern California, Los Angeles, CA 90089, USA³ Centre of Mathematics, Academy of Athens, 11527 Athens, Greece⁴ School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China⁵ Institute for Advanced Study, Shenzhen University, Shenzhen 518060, China

* Correspondence: hejingsong@szu.edu.cn or jshe@ustc.edu.cn

Abstract: The celebrated Korteweg–de Vries and Kadomtsev–Petviashvili (KP) equations are prototypical examples of integrable evolution equations in one and two spatial dimensions, respectively. The question of constructing integrable evolution equations in three-spatial dimensions has been one of the most important open problems in the history of integrability. Here, we study an integrable extension of the KP equation in three-spatial dimensions, which can be derived using a specific reduction of the integrable generalization of the KP equation in four-spatial and two-temporal dimensions derived in (Phys. Rev. Lett. 96, (2006) 190201). For this new integrable extension of the KP equation, we construct smooth multi-solitons, high-order breathers, and high-order rational solutions, by using Hirota’s bilinear method.



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1. Introduction

There exists a distinctive class of nonlinear evolution equations in one and two spatial dimensions, called *integrable* [1,2]. Such prototypical equations in one spatial dimension are the celebrated Korteweg–de Vries (KdV) [3,4] and the nonlinear Schrödinger (NLS) [5,6] equations. Every integrable nonlinear evolution equation in one spatial dimension has several integrable versions in two spatial dimensions. Two such physically significant generalizations of the KdV are the Kadomtsev–Petviashvili I (KPI) and II (KPII) equations. Similarly, two analogous generalizations of the NLS are the Davey–Stewartson I (DS I) and II (DSII) equations. It is important to emphasize that integrable nonlinear evolution equations arise in a variety of physical applications. For example, in the context of fluid mechanics, the KP equations arise in the weakly nonlinear, weakly dispersive, weakly two-dimensional limit of inviscid, irrotational, water waves, and in the case of KPI when the surface tension is dominant [7]. Other significant applications of the KP and DS equations are discussed, for example, in weakly dispersive media [2,8–10] in optics and hydrodynamics [2,10–13].

One of the most important open questions in the field of integrability has been the question of the existence of integrable evolution equations in three spatial dimensions. Substantial progress regarding this question was presented in [14], where 4 + 2 generalizations of the KP and of the DS equations were presented, namely equations in four spatial and two temporal dimensions. The solution of the Cauchy problem of the 4 + 2 generalization of the KP is presented in [15]. The solution of the Cauchy problem of the

$4 + 2$ generalization of the DS equation is sketched in [14] and presented in detail in [16]. Incidentally, another approach toward integrability in multidimensions is presented in [17], where it is shown that there exist integrable nonlinear evolution equations in any number of dimensions; however, these equations have the major disadvantage that they involve a nonlocal commutator.

The question of reducing integrable equations from $4 + 2$ to equations in three spatial dimensions, although discussed in many papers, including [14,15], as well as [18,19], remained open. In particular, regarding $3 + 1$ reductions of the KP equation, it should be noted that, although the real reduction discussed in [14] involves two equations, many authors have erroneously analyzed only one of these two equations, ignoring the other one. Here, we discuss a constructive approach for obtaining integrable equations in three spatial dimensions, and by implementing this approach to KP, we derive the results presented below. Our paper is a companion of [20], where a highly novel nonlinear transform pair in three spatial dimensions, capable of solving the Cauchy problem of these equations, is also derived. Specifically, regarding Equation (1) below, we note that the analysis of its Cauchy problem performed in [20] reveals that the time dependence of the associated spectral function (the nonlinear Fourier transform of the solution) is of the form $\exp(iTt - Rt)$, where T and R are real functions of the real spectral parameters k_1, k_2 , and k_3 (in the linear limit, these three variables in the Fourier space correspond to the three real variables x, y, z in the physical space). Because of the occurrence of the term Rt , it is not clear that this initial value problem is well posed. However, in the recent breakthrough presented in [20], it is explicitly shown that the Cauchy problem of Equation (1) is indeed well posed. Remarkably, the novel nonlinear Fourier transform introduced in [20], which is capable of solving (1), gives rise to a completely new transform for solving the linear limit of Equation (1), namely, the equation obtained from Equation (1) after eliminating the nonlinear term uu_x .

The following equation for the *complex-valued* function $u(x, y, z, t)$, where x, y, z, t , are real, independent variables, provides an integrable generalization of the KP equation in three-spatial dimensions,

$$u_{xt} + \alpha u_{xxxx} + \beta(uu_x)_x + \frac{\gamma}{4}u_{yy} - \frac{\gamma}{4}u_{zz} + i\frac{\gamma}{2}u_{yz} = 0, \quad u \in \mathbb{C}, \quad x, y, z, t \in \mathbb{R}, \quad (1)$$

where α, β, γ are arbitrary complex constant parameters.

As is well known, the most important property of integrable equations is that they admit a Lax pair formulation. The derivation of Equation (1), as well as its Lax pair, is presented in Section 2. Two-dimensional integrable equations, like KP and DS equations, possess a variety of particular solutions, which can be obtained using different techniques, including the inverse scattering method, the d-bar method, the Darboux transformation, the Hirota bilinear method and the KP hierarchy reduction method [21–30]. In this connection, smooth multi-soliton and high-order breather solutions, as well as high-order rational solutions of Equation (1), are constructed in Sections 3 and 4, respectively, via Hirota's bilinear method. These results are discussed further in Section 5.

2. Integrable Extensions of the KP in Three Dimensions and Their Lax Pairs

Starting with the KP equation and complexifying x, y, t , namely, introducing the complex variables

$$x = x_1 + ix_2, y = y_1 + iy_2, t = t_1 + it_2, x_1, x_2, y_1, y_2, t_1, t_2 \in \mathbb{R}, \quad (2)$$

it is shown in [14] that the equation

$$q_{\bar{t}} = \frac{1}{4}q_{\bar{x}\bar{x}\bar{x}} - \frac{3}{2}qq_{\bar{x}} + \frac{3}{4}\partial_{\bar{x}}^{-1}q_{\bar{y}\bar{y}}, \quad q \in \mathbb{C}, \quad (3)$$

is an integrable generalization of the KP equation in the 4-spatial dimensions x_1, x_2, y_1, y_2 and the two temporal dimensions t_1, t_2 . In Equation (3), ∂^{-1} means integral, bar denotes complex conjugation, and Equation (2) implies the following identities:

$$\partial_{\bar{t}} = \frac{1}{2}(\partial_{t_1} + i\partial_{t_2}), \quad \partial_{\bar{x}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad \partial_{\bar{y}} = \frac{1}{2}(\partial_{y_1} + i\partial_{y_2}), \quad (\partial_{\bar{x}}^{-1}f)(x_1, x_2) = \frac{1}{\pi} \iint_{\mathbb{R}^2} \frac{f(x'_1, x'_2)}{x - x'} dx'_1 dx'_2, \quad (4)$$

where we have assumed that f vanishes as $|x_1|, |x_2| \rightarrow \infty$. Assuming q is real and suppressing the t_2 dependence, the $(4+2)$ -dimensional KP (3) yields two $(4+1)$ -dimensional equations; see Equation (10) or equivalent Equation (11) in [14]. These equations have provoked extensive studies [18,19]. In addition, similar considerations are valid for the $(4+2)$ -dimensional Davey–Stewartson equation [16].

Replacing in Equation (3) q and y by Qq and $y \rightarrow \frac{y}{\gamma_1}$, where Q and γ_1 are complex constants, we find $\partial_{\bar{y}} \rightarrow \gamma_1 \partial_{\bar{y}}$ and

$$q_{\bar{t}} = \frac{1}{4}q_{\bar{x}\bar{x}\bar{x}} - \frac{3}{2}Qq q_{\bar{x}} + \frac{3}{4}\gamma_1^2 \partial_{\bar{x}}^{-1} q_{\bar{y}\bar{y}}. \quad (5)$$

In order to obtain three-dimensional reductions of this equation, which is the first purpose of this paper, we let $\xi = ax_1 + bx_2$, where a and b are all real constants. Hence, $\partial_{\bar{x}} = A\partial_{\bar{\xi}}$, where $A = \frac{1}{2}(a + ib)$. Replacing in Equation (5) $\partial_{\bar{x}}$ by $A\partial_{\bar{\xi}}$, we find a $(3+2)$ -dimensional generalization of the KP

$$q_{\bar{t}} = \frac{A^3}{4}\partial_{\bar{\xi}}^3 q - \frac{3A}{2}Qq q_{\bar{\xi}} + \frac{3\gamma_1^2}{4A}\partial_{\bar{\xi}}^{-1}\partial_{\bar{y}}^2 q. \quad (6)$$

Letting $\tau = \tilde{a}t_1 + \tilde{b}t_2$, where $\tilde{a}, \tilde{b}$ are real constants, and seeing $\partial_{\bar{t}} = \tilde{A}\partial_{\bar{\tau}}$, where $\tilde{A} = \frac{1}{2}(\tilde{a} + i\tilde{b})$, Equation (6) reduces to a three-dimensional KP equation for the complex-valued function q ,

$$q_{\bar{\tau}} + \alpha q_{\bar{\xi}\bar{\xi}\bar{\xi}} + \beta q q_{\bar{\xi}} + \gamma \partial_{\bar{\xi}}^{-1} \partial_{\bar{y}}^2 q = 0, \quad (7)$$

where

$$\alpha = -\frac{A^3}{4\tilde{A}}, \quad \beta = \frac{3AQ}{2\tilde{A}}, \quad \gamma = -\frac{3\gamma_1^2}{4A\tilde{A}} = \tilde{\gamma}\gamma_1^2. \quad (8)$$

Renaming in Equation (7) q, τ, ξ, y_1, y_2 by u, t, x, y, z , Equation (7) becomes Equation (1).

Remark 1. If we set $\tilde{A} = -\frac{A^3}{4}, Q = -A^2, \gamma_1 = \delta A^2$ (or equivalently $\tilde{\gamma} = \frac{3}{A^4}$) in Equation (7), then

$$q_{\bar{\tau}} + q_{\bar{\xi}\bar{\xi}\bar{\xi}} + 6q q_{\bar{\xi}} + 3\delta^2 \partial_{\bar{\xi}}^{-1} \partial_{\bar{y}}^2 q = 0, \quad (9)$$

which indeed looks like a three-dimensional extension of the usual KP equation. In order to obtain a smooth solution, which will be shown in Section 3, δ^2 is a complex-valued constant.

The second purpose of the paper is to show the integrability of the three-dimensional KP Equation (1) by providing its Lax pair and different kinds of solutions. The Lax pair of Equation (3) [14] is given by

$$D_y \mu = M \mu, \quad M = D_x^2 - q, \quad (10)$$

$$D_t \mu = N \mu, \quad N = D_x^3 - \frac{3}{2}q D_x - N_0, \quad N_0 = \frac{3}{4}(q_{\bar{x}} + (\partial_{\bar{x}}^{-1} q_{\bar{y}})), \quad (11)$$

where μ is an appropriate eigenfunction, and the operators D_x, D_y, D_t are defined by

$$D_x = \partial_{\bar{x}} + k, \quad D_y = \partial_{\bar{y}} + k^2, \quad D_t = \partial_{\bar{t}} + k^3.$$

The Lax pairs (10) and (11) can be declassified in the form

$$\partial_{\bar{y}}\mu = \tilde{M}\mu, \quad \tilde{M} = \partial_{\bar{x}}^2 + 2k\partial_{\bar{x}} - q, \quad (12)$$

$$\partial_{\bar{t}}\mu = \tilde{N}\mu, \quad \tilde{N} = \partial_{\bar{x}}^3 + 3k\partial_{\bar{x}}^2 + (3k^2 - \frac{3}{2}q)\partial_{\bar{x}} + (-\frac{3}{2}qk - N_0). \quad (13)$$

Using in these equations the scaling transformations $q \rightarrow Qq$ and $y \rightarrow \frac{y}{\gamma_1}$, we find the following Lax pair for Equation (5):

$$\partial_{\bar{y}}\mu = [\frac{1}{\gamma_1}(\partial_{\bar{x}}^2 + 2k\partial_{\bar{x}} - Qq)] \cdot \mu, \quad (14)$$

$$\partial_{\bar{t}}\mu = [\partial_{\bar{x}}^3 + 3k\partial_{\bar{x}}^2 + (3k^2 - \frac{3}{2}Qq)\partial_{\bar{x}} + (-\frac{3}{2}Qqk) - \frac{3Q}{4}(q_{\bar{x}} + \gamma_1(\partial_{\bar{x}}^{-1}q_{\bar{y}}))] \cdot \mu. \quad (15)$$

The crucial reduction $\partial_{\bar{x}} = A\partial_{\bar{\xi}}, \partial_{\bar{t}} = \tilde{A}\partial_{\bar{\tau}}$ imply the following Lax pair for (7):

$$\mu_{\bar{y}} = [\frac{1}{\gamma_1}(A^2\partial_{\bar{\xi}}^2 + 2kA\partial_{\bar{\xi}} - Qq)] \cdot \mu = M_1 \cdot \mu, \quad (16)$$

$$\mu_{\bar{t}} = \frac{1}{\tilde{A}}[A^3\partial_{\bar{\xi}}^3 + 3kA^2\partial_{\bar{\xi}}^2 + (3k^2 - \frac{3}{2}Qq)A\partial_{\bar{\xi}} + (-\frac{3}{2}Qqk) - \frac{3Q}{4}(Aq_{\bar{\xi}} + \frac{\gamma_1}{A}(\partial_{\bar{\xi}}^{-1}q_{\bar{y}}))] \cdot \mu = N_1 \cdot \mu, \quad (17)$$

where $A, \tilde{A}, Q$ are expressed by virtue of α, β, γ via equations

$$A = i(\frac{3\alpha}{\gamma})^{\frac{1}{4}}, \tilde{A} = \frac{i}{4\alpha}(\frac{3\alpha}{\gamma})^{\frac{3}{4}}, Q = \frac{\beta}{\sqrt{12\alpha\gamma}}. \quad (18)$$

3. Soliton and Breather of the Three-Dimensional KP Equation

In this section, we derive soliton and breather solutions of the three-dimensional KP equation (1) by using the Hirota bilinear method and certain perturbation technique [31]. Equation (1) can be transformed into the bilinear form

$$(D_x D_t + \alpha D_x^4 + \frac{\gamma}{4} D_y^2 - \frac{\gamma}{4} D_z^2 + \frac{\gamma}{2} i D_y D_z) f \cdot f = 0, \quad (19)$$

through the variable transformation

$$u = 12 \frac{\alpha}{\beta} (\ln f)_{xx}, \quad (20)$$

where f is a complex function, and D is the Hirota's bilinear differential operator [31] defined by

$$D_x^m D_t^n f(x, t) \cdot g(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}.$$

Based on this bilinear form, the three-dimensional KP Equation (1) admits the following N -soliton solutions:

$$u = 12 \frac{\alpha}{\beta} (\ln f)_{xx}, \quad f = \sum_{\mu=0,1} \exp \left(\sum_{j < k}^{(N)} \mu_j \mu_k A_{jk} + \sum_{j=1}^N \mu_j \eta_j \right), \quad (21)$$

with

$$\eta_j = k_j \left(x + p_j y + q_j z - \left[\frac{1}{4} (p_j + i q_j)^2 + \alpha k_j^2 \right] t \right) + \eta_j^0, \quad (22)$$

$$e^{A_{jk}} = \frac{(p_j - p_k + i q_j - i q_k)^2 \gamma - 12 \alpha (k_j - k_k)^2}{(p_j - p_k + i q_j - i q_k)^2 \gamma - 12 \alpha (k_j + k_k)^2},$$

where the above parameters must satisfy the condition

$$\gamma_I q_j^2 - 2\gamma_R p_j q_j - 4\alpha_I k_j^2 - \gamma_I p_j^2 = 0, \quad (23)$$

which assures that these solutions are smooth. Here, k_j, p_j, q_j, η_j^0 are arbitrary real parameters, and the subscript j denotes an integer. The subscripts R and I denote the real and imaginary parts of the corresponding parameters, respectively.

Taking in Equation (21) $N = 1$, we obtain the one-soliton solution of the three-dimensional KP equation:

$$u = \frac{3\beta k^2}{\alpha \cosh^2(\frac{\hat{\xi}}{2})} = \frac{3\beta k^2}{\alpha} \operatorname{sech}^2(\frac{\hat{\xi}}{2}). \quad (24)$$

The modulus of this solution is given by

$$|u| = \frac{6|\beta|k^2}{|\alpha|(\cosh \mathcal{A} + \cos \mathcal{B})}, \quad (25)$$

with

$$\begin{aligned} \hat{\xi} &= \mathcal{A} + i\mathcal{B}, \quad \alpha = \alpha_R + i\alpha_I, \quad \gamma = \gamma_R + i\gamma_I, \\ \mathcal{A} &= k \left[x + p_1 y + q_1 z - \frac{k_1}{4} (4\alpha_R k_1^2 + \gamma_R p_1^2 - \gamma_R q_1^2 - 2\gamma_I p_1 q_1) t \right], \\ \mathcal{B} &= \frac{k_1}{4} [\gamma_I q_1^2 - \gamma_I p_1^2 - 2\gamma_R p_1 q_1 - 4\alpha_I k_1^2] t. \end{aligned}$$

It can be seen from the above expression that the one-soliton is nonsingular, provided that $\cos(\mathcal{B}) \geq 0$, which means $-\frac{\pi}{2} < \mathcal{B} = \frac{k_1}{4} [\gamma_I q_1^2 - \gamma_I p_1^2 - 2\gamma_R p_1 q_1 - 4\alpha_I k_1^2] t < \frac{\pi}{2}$. For simplicity, we set $[\gamma_I q_1^2 - \gamma_I p_1^2 - 2\gamma_R p_1 q_1 - 4\alpha_I k_1^2] = 0$, as noted in Equation (23), to guarantee the smoothness of the one-soliton solution (24).

Taking in (21) $N = 2, \alpha = 1 + i$ and $\gamma = 1 - i$, imposing the constraints

$$-q_1^2 - 2p_1 q_1 - 4k_1^2 + p_1^2 = 0, \quad -q_2^2 - 2p_2 q_2 - 4k_2^2 + p_2^2 = 0,$$

and choosing $k_1 = 1, k_2 = 1, p_1 = \frac{3}{2}, p_2 = -3, q_1 = \frac{\sqrt{2}-3}{2}, q_2 = 3 + \sqrt{14}, \eta_1^0 = \eta_2^0 = 0$, we obtain a smooth two-soliton solution of the three-dimensional KP equation:

$$\begin{aligned} u &= 12 \frac{\alpha}{\beta} (\ln f)_{xx}, \quad f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \\ \eta_1 &= x + \frac{3}{2}y + \left(2\sqrt{2} - \frac{3}{2}\right)z + \frac{1-3\sqrt{2}}{4}t, \\ \eta_2 &= x - 3y + (3 + \sqrt{14})z + (7 + 3\sqrt{14})t, \\ A_{12} &= \frac{(55 + 18\sqrt{14} - 9\sqrt{2} - 2\sqrt{7}) + i(26 + 2\sqrt{7})}{(103 + 18\sqrt{14} - 9\sqrt{2} - 2\sqrt{7}) + i(14 + 2\sqrt{7})}. \end{aligned}$$

It is straightforward, using Equation (21), to derive multi-soliton solutions of the three-dimensional KP equation. Such solutions are depicted in Figure 1. The effect of changing the variables z and t is simply to move the profile of the soliton in the (x, y) -plane, without any deformations.

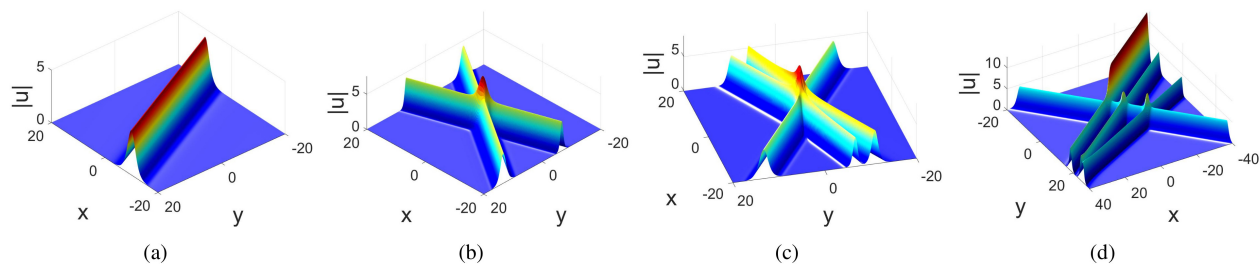


Figure 1. Plots of $|u|$ of soliton solutions for the three-dimensional KP Equation (1) generated by Equation (21) with parameters $\alpha = 1 + i, \beta = 1, \eta_j^0 (j = 1, 2, 3, \dots) = 0$ and variables $t = 0, z = 0$. (a): One-soliton solution with parameters $N = 1, \gamma = 1 + \frac{20}{3}i, k_1 = 1, p_1 = \frac{1}{2}, q_1 = 1$; (b): Two-soliton solution with parameters $N = 2, \gamma = 1 - i, k_1 = 1, k_2 = 1, p_1 = \frac{3}{2}, p_2 = -3, q_1 = \frac{\sqrt{2}-3}{2}, q_2 = 3 + \sqrt{14}$; (c): Three-soliton solution with parameters $N = 3, \gamma = 1 - i, k_1 = k_2 = k_3 = 1, p_1 = \frac{1}{2}, p_2 = -3, p_3 = -2, q_1 = \frac{\sqrt{2}-3}{2}, q_2 = 3 + \sqrt{14}, q_3 = 4$; (d): Four-soliton solution with parameters $N = 4, \gamma = 1 - i, k_1 = k_2 = k_3 = k_4 = 1, p_1 = \frac{1}{2}, p_2 = -3, p_3 = -2, p_4 = -2, q_1 = \frac{\sqrt{2}-3}{2}, q_2 = 3 + \sqrt{14}, q_3 = 4, q_4 = 0$.

In order to derive breather solutions of the three-dimensional KP equation, we use the constraints,

$$N = 2n, \quad k_j^* = k_{n+j}, \quad p_j^* = p_{n+j}, \quad q_j^* = q_{n+j}, \quad \eta_j^{0*} = \eta_{n+j}^0. \quad (26)$$

For the particular case of Equation (26),

$$N = 2, \quad k_1^* = k_2, \quad p_1^* = p_2, \quad q_1^* = q_2, \quad \eta_1^{0*} = \eta_2^0,$$

we obtain a first-order breather solution of Equation (1) $u = 12 \frac{\alpha}{\beta} (\ln f)_{xx}$ with f given below:

$$\begin{aligned} f &= 1 + [\cosh(l_1) + \sinh(l_1)] \cdot [\cosh(B_1 t) + \sinh(B_1 t)] \cdot [\cos(A_2 t) + i \sin(A_2 t)] \cdot [\cos(l_2) + i \sin(l_2)] \\ &\quad + [\cosh(l_1) + \sinh(l_1)] \cdot [\cosh(B_1 t) - \sinh(B_1 t)] \cdot [\cos(A_2 t) + i \sin(A_2 t)] \cdot [\cos(l_2) - i \sin(l_2)] \\ &\quad + \widetilde{A}_{12} [\cosh(2l_1) + \sinh(2l_1)] \cdot [\cos(2A_2 t) + i \sin(2A_2 t)], \\ \widetilde{A}_{12} &= \frac{[(p_{1I}^2 - q_{1I}^2)\gamma_R - 12\alpha_R k_{1I}^2 - 2\gamma_I p_{1I} q_{1I}] + i[(p_{1I}^2 - q_{1I}^2)\gamma_I - 12\alpha_I k_{1I}^2 + 2\gamma_R p_{1I} q_{1I}]}{[(p_{1I}^2 - q_{1I}^2)\gamma_R + 12\alpha_R k_{1I}^2 - 2\gamma_I p_{1I} q_{1I}] + i[(p_{1I}^2 - q_{1I}^2)\gamma_I + 12\alpha_I k_{1I}^2 + 2\gamma_R p_{1I} q_{1I}]}, \\ l_1 &= k_{1R} x + [k_{1R} p_{1R} - k_{1I} p_{1I}] y + [k_{1R} q_{1R} - k_{1I} q_{1I}] z + A_1 t, \\ l_2 &= k_{1I} x + [k_{1I} p_{1R} + k_{1R} p_{1I}] y + [k_{1I} q_{1R} + k_{1R} q_{1I}] z + B_2 t, \\ A_1 &= -\alpha_R k_{1R}^3 + \frac{1}{4} [-C_1 \gamma_R + 12\alpha_R k_{1I}^2 + 2C_2 \gamma_I] k_{1R} + \frac{1}{2} D_1 k_{1I}, \quad C_1 = p_{1R}^2 - p_{1I}^2 - q_{1R}^2 + q_{1I}^2, \\ B_1 &= -\alpha_I k_{1I}^3 + \frac{1}{4} [C_1 \gamma_I + 12\alpha_I k_{1R}^2 + 2C_2 \gamma_R] k_{1I} + \frac{1}{2} D_2 k_{1R}, \quad C_2 = p_{1R} q_{1R} - p_{1I} q_{1I}, \\ A_2 &= -\alpha_I k_{1R}^3 + \frac{1}{4} [-C_1 \gamma_I + 12\alpha_I k_{1I}^2 - 2C_2 \gamma_R] k_{1R} + \frac{1}{2} D_2 k_{1I}, \quad D_1 = (p_{1R} p_{1I} - q_{1R} q_{1I}) \gamma_R - (p_{1R} q_{1I} - p_{1I} q_{1R}) \gamma_I, \\ B_2 &= \alpha_R k_{1I}^3 + \frac{1}{4} [-C_1 \gamma_R - 12\alpha_R k_{1I}^2 + 2C_2 \gamma_I] k_{1I} - \frac{1}{2} D_1 k_{1R}, \quad D_2 = (p_{1R} p_{1I} - q_{1R} q_{1I}) \gamma_I - (p_{1R} q_{1I} + p_{1I} q_{1R}) \gamma_R, \end{aligned} \quad (27)$$

where the parameters satisfy

$$\begin{aligned} -2\alpha_I (p_{1R} q_{1I} + p_{1I} q_{1R}) + 2\gamma_I (q_{1R} q_{1I} - p_{1R} p_{1I}) - 8\gamma_R k_{1R} k_{1I} &= 0, \\ -C_1 - 2\alpha_I (p_{1R} q_{1R} + p_{1I} q_{1I}) + 4\gamma_R (k_{1I}^2 - k_{1R}^2) &= 0. \end{aligned} \quad (28)$$

Without loss of generality, we study the dynamics of the first-order breather in the (x, y) -plane. The velocities of the first-order breathers along the x and y directions are given by

$$V_x = \frac{k_{1R} [p_{1I} A_1 - p_{1R} B_2] + k_{1I} [p_{1R} A_1 - p_{1I} B_2]}{p_{1I} [k_{1R}^2 + k_{1I}^2]}, \quad V_y = -\frac{k_{1I} A_1 - k_{1R} B_2}{p_{1I} [k_{1R}^2 + k_{1I}^2]}. \quad (29)$$

The period of the first-order breather is given by $T = \sqrt{T_x + T_y}$, where T_x and T_y are given by

$$T_x = -\frac{2\pi[k_{1R}p_{1R} - k_{1I}p_{1I}]}{p_{1I}[k_{1R}^2 + k_{1I}^2]}, \quad T_y = \frac{2\pi k_{1R}}{p_{1I}[k_{1R}^2 + k_{1I}^2]}. \quad (30)$$

Note that changing z and t only results in moving the profile of $|u|$ in a parallel manner in the (x, y) -plane, and it shows that there exist four types of breathers:

- The line breather. If $k_{1R} = 0$, $k_{1R}p_{1R} - k_{1I}p_{1I} = 0$, $A_1 \neq 0$, $k_{1I} \neq 0$, $k_{1I}p_{1R} + k_{1R}p_{1I} \neq 0$, $B_2 \neq 0$, the first-order breather (27) is a linear breather in the (x, y) plane. The four pictures (a–d) (Figure 2) describe a series of periodic line waves that suddenly appear from the constant background plane and then annihilate in the original background.
- The oblique breather. If $k_{1R} \neq 0$, $k_{1R}p_{1R} - k_{1I}p_{1I} \neq 0$, $A_1 \neq 0$, $k_{1I} \neq 0$, $k_{1I}p_{1R} + k_{1R}p_{1I} \neq 0$, $B_2 \neq 0$, the first-order breather (27) is an oblique breather in the (x, y) plane, which has periodicity in both x and y directions; see Figure 2e.
- The x -periodic breather. If $k_{1R} = 0$, $k_{1R}p_{1R} - k_{1I}p_{1I} \neq 0$, $A_1 \neq 0$, $k_{1I} \neq 0$, $k_{1I}p_{1R} + k_{1R}p_{1I} = 0$, $B_2 \neq 0$, the first-order breather (27) has periodicity only in the x -direction in the (x, y) plane, and is localized in the y -direction; see Figure 2f.
- The y -periodic breather. If $k_{1R} \neq 0$, $k_{1R}p_{1R} - k_{1I}p_{1I} = 0$, $A_1 \neq 0$, $k_{1I} = 0$, $k_{1I}p_{1R} + k_{1R}p_{1I} \neq 0$, $B_2 = 0$, the first-order breather (27) has periodicity only in the y -direction in the (x, y) plane, and is localized in the x -direction; see Figure 2g. Since the coefficient of t in l_1 is not zero, the effect of t is to move the breather along the x -axis. Similarly, since the coefficient of z in l_2 is not zero, the effect of z is to move the breather along the y -axis.

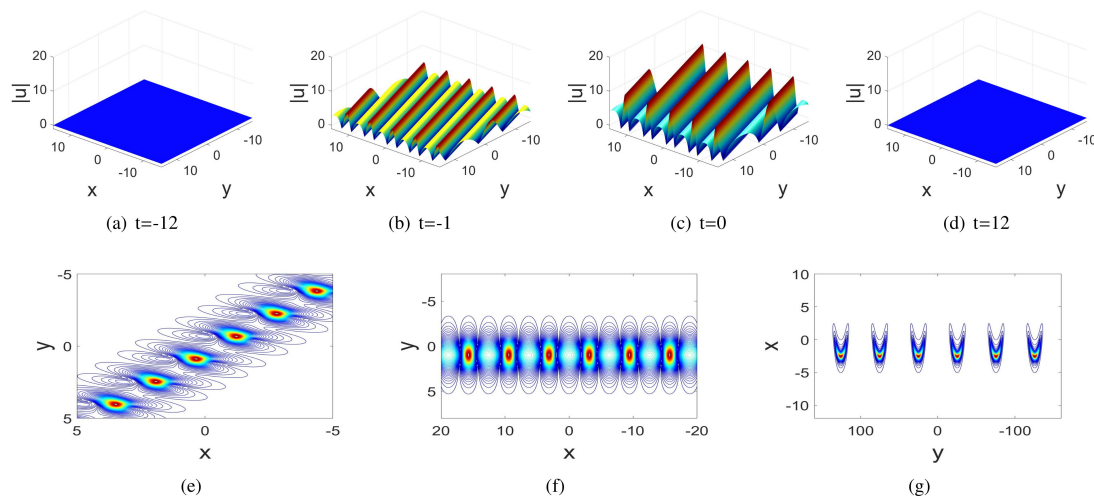


Figure 2. (a–d): Plots of $|u|$ for the first-orderline breather (27) of the three-dimensional KP Equation (1) with parameters $N = 2, \alpha = 1 + i, \beta = 1, \gamma = 1 + i, k_1 = i, p_1 = \frac{1}{3}, q_1 = \frac{1 + \sqrt{34}}{3}i, \eta_1^0 = 0, z = 0$; (e): An oblique breather with parameters $N = 2, \alpha = 1 - i, \beta = 1, \gamma = 1 + \frac{20}{3}i, k_1 = 1 + i, p_1 = 1 + 2i, q_1 = 1 + (\sqrt{6} + 2)i, \eta_1^0 = 0, z = 0, t = 0$; (f): An x -periodic breather with parameters $N = 2, \alpha = 1 - i, \beta = 1, \gamma = 1 + i, k_1 = i, p_1 = i, q_1 = \sqrt{2} + i, \eta_1^0 = \eta_2^0 = 0, z = 0, t = 0$; (g): A y -periodic breather with parameters $\alpha = 1 + 2i, \beta = 20, \gamma = 1 + i, k_1 = \frac{1}{4}, p_1 = \frac{i}{2}, q_1 = \frac{i}{2}, \eta_1^0 = 0, z = 0, t = 0$.

Taking in (26)

$$N = 4, \quad k_1^* = k_2, \quad k_3^* = k_4, \quad p_1^* = p_2, \quad p_3^* = p_4, \quad q_1^* = q_2, \quad q_3^* = q_4, \quad \eta_1^{0*} = \eta_2^0, \quad \eta_3^{0*} = \eta_4^0,$$

a second-order breather of three-dimensional KP Equation (1) can be derived. The dynamic of this solution is shown in Figure 3.

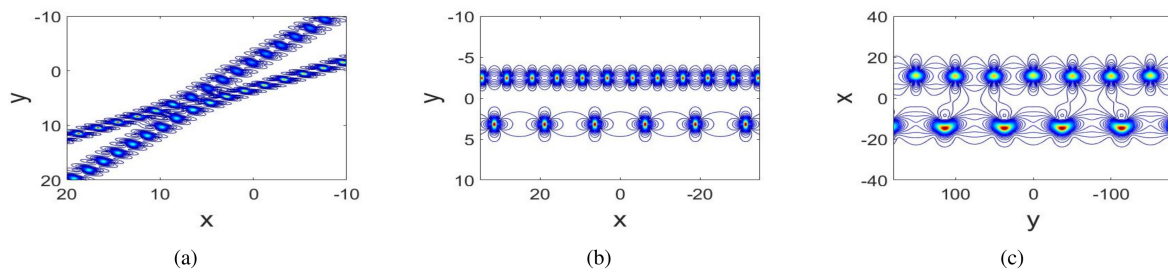


Figure 3. Second-order breathers in the (x, y) -plane of the three-dimensional KP Equation (1) with parameters $N = 4, \alpha = 1 - i, \beta = 1, \gamma = 1 + i$ and variables $z = 0, t = 0$. (a): $k_1 = 1 + i, k_3 = 1 + i, p_1 = 1 + 2i, p_3 = 1 + 3i, q_1 = 1 + (2 + \sqrt{6})i, q_3 = \frac{2 + \sqrt{8 + 2\sqrt{17}}}{\sqrt{8 + 2\sqrt{17}}} + (3 + \sqrt{8 + 2\sqrt{17}})i, \eta_1^0 = 0, \eta_3^0 = 2\pi$; (b): $k_1 = i, k_3 = \frac{i}{2}, p_1 = 2i, p_3 = 2i, q_1 = 4i, q_3 = (2 + \sqrt{7})i, \eta_1^0 = -\pi, \eta_3^0 = \pi$; (c): $k_1 = \frac{1}{4}, k_3 = \frac{1}{4}, p_1 = \frac{i}{2}, p_3 = \frac{i}{3}, q_1 = \frac{1 + \sqrt{3}}{2}i, q_3 = \frac{2 + \sqrt{17}}{6}i, \eta_1^0 = 0, \eta_3^0 = \pi$.

In addition, mixed solutions composed of solitons and breathers, are also given for $N > 2$. Taking in (21)

$$N = 3, k_1^* = k_2, p_1^* = p_2, q_1^* = q_2, \eta_1^{0*} = \eta_2^0, k_3 = 1, p_3 = -2, q_3 = 4, \eta_3^0 = 0,$$

a hybrid solution consisting of a first-order breather and one soliton is constructed, see Figure 4a. For larger N , we obtain mixed solutions consisting of several first-order breathers and solitons. For example, taking in (21),

$$N = 4, k_1^* = k_2, p_1^* = p_2, q_1^* = q_2, \eta_1^{0*} = \eta_2^0, k_3 = 1, k_4 = 1, p_3 = \frac{3}{2}, p_4 = -3, q_3 = \frac{\sqrt{2} - 3}{2}, q_4 = 3 + \sqrt{14}, \eta_3^0 = \eta_4^0 = 0,$$

we derive the hybrid solution consisting of one first-order breather and two solitons, see Figure 4b.

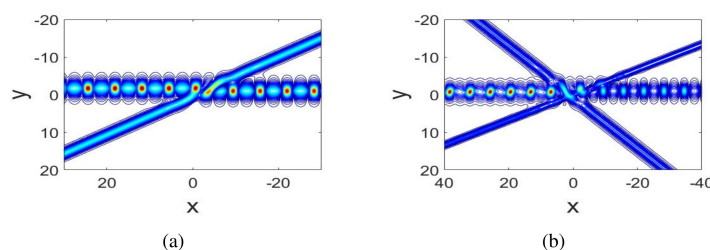


Figure 4. Hybrid solutions of the three-dimensional KP Equation (1) consisting of a breather and solitons with parameters $\alpha = 1 + i, \beta = 1, \gamma = 1 - i, \eta_j^0 (j = 1, 2, 3, 4) = 0$ and variables $z = 0, t = 0$. (a): $N = 3, k_1 = k_2^* = -i, p_1 = p_2^* = i, q_1 = q_2^* = \sqrt{2} - i, k_3 = 1, p_3 = -2, q_3 = 4$; (b): $N = 4, k_1 = k_2^* = i, p_1 = p_2^* = -i, q_1 = q_2^* = \sqrt{2} + i, k_3 = 1, k_4 = 1, p_3 = \frac{3}{2}, p_4 = -3, q_3 = \frac{\sqrt{2} - 3}{2}, q_4 = 3 + \sqrt{14}$.

4. Rational Solution of the Three-Dimensional KP Equation

In order to derive rational solutions of the three-dimensional KP Equation (1), we take in Equation (21)

$$N = 2n, \quad \eta_j^0 = i\pi \quad (1 \leq j \leq N), \quad (31)$$

and compute a suitable limit as $k_j \rightarrow 0$. In this way, the function f , defined in (21), becomes a polynomial function. This construction yields an n -th rational solution, $u = 12 \frac{\alpha}{\beta} (\ln f)_{xx}$, of Equation (1), in which f is written as follows:

$$f = f^{[n]} = \prod_{k=1}^N \theta_k + \frac{1}{2} \sum_{k,j}^{(N)} \alpha_{kj} \prod_{l \neq k,j}^N \theta_l + \dots + \frac{1}{M! 2^M} \sum_{i,j,\dots,m,n}^{(N)} \overbrace{\alpha_{kj} \alpha_{kl} \dots \alpha_{mn}}^M \prod_{p \neq k,j,\dots,m,n}^N \theta_p + \dots, \quad (32)$$

with

$$\theta_j = x + p_j y + q_j z - \frac{\gamma}{4} (p_j + i q_j)^2 t, \quad \alpha_{jk} = \frac{24\alpha}{\frac{1}{2}[(p_j - p_k)^2 - (q_j - q_k)^2] + i(p_j - p_k)(q_j - q_k)}. \quad (33)$$

To illustrate this formula of rational solutions, we provide concrete examples by taking in Equation (32)

$$N = 2, \quad p_1 = p_2^* = p_R + i p_I, \quad q_1 = q_2^* = q_R + i q_I, \quad \alpha = \alpha_R + i \alpha_I, \quad \gamma = \gamma_R + i \gamma_I, \quad (34)$$

where we impose the constraints

$$p_R = \frac{\gamma_R^2 + \gamma_I^2 - \gamma_R \sqrt{\gamma_R^2 + \gamma_I^2}}{\gamma_I \sqrt{\gamma_R^2 + \gamma_I^2}} q_R, \quad p_I = \frac{\sqrt{\gamma_R^2 + \gamma_I^2} - \gamma_R}{\gamma_I} q_I, \quad (35)$$

In this way, we obtain a first-order rational solution of the three-dimensional KP Equation (1):

$$\begin{aligned} u &= 12 \frac{\alpha}{\beta} (\ln f)_{xx}, \quad f = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \alpha_{12}, \quad \alpha_{12} = \frac{6\gamma_I^2(\alpha_R + i\alpha_I)}{(\sqrt{\gamma_R^2 + \gamma_I^2} - \gamma_R)(\gamma_R^2 + \gamma_I^2)q_I^2}, \\ \mathcal{L}_1 &= x + \frac{\gamma_R^2 + \gamma_I^2 - \gamma_R \sqrt{\gamma_R^2 + \gamma_I^2}}{\gamma_I \sqrt{\gamma_R^2 + \gamma_I^2}} q_R y + q_R z + \frac{(q_R^2 - q_I^2)(\gamma_R^2 + \gamma_I^2 - \gamma_R \sqrt{\gamma_R^2 + \gamma_I^2}) \sqrt{\gamma_R^2 + \gamma_I^2}}{2\gamma_I^2} t, \\ \mathcal{L}_2 &= \frac{\gamma_R^2 + \gamma_I^2 - \gamma_R \sqrt{\gamma_R^2 + \gamma_I^2}}{\gamma_I \sqrt{\gamma_R^2 + \gamma_I^2}} q_I y + q_I z + \frac{(\gamma_R^2 + \gamma_I^2 - \gamma_R \sqrt{\gamma_R^2 + \gamma_I^2}) \sqrt{\gamma_R^2 + \gamma_I^2}}{\gamma_I^2} q_R q_I t. \end{aligned} \quad (36)$$

If

$$\alpha_R (\sqrt{\gamma_R^2 + \gamma_I^2} - \gamma_R) > 0 \quad \text{or} \quad \alpha_I \neq 0, \quad (37)$$

this first-order rational solution is nonsingular. Without loss of generality, taking in (36) $\alpha_R = 1, \alpha_I = 1, \gamma_R = 1, \gamma_I = 1, q_R = 1, q_I = 1$, that is, $\alpha = 1 + i, \gamma = 1 + i, q = 1 + i$, we obtain the following analytical expression of first-order rational solution of the three-dimensional KP Equation (1):

$$u = 8(\sqrt{2} - 1)(1 + i) \frac{\frac{1-\sqrt{2}}{3} [x + (\sqrt{2} - 1)y + z]^2 + \frac{68+52\sqrt{2}}{3} \left(\frac{1}{2}y + \frac{\sqrt{2}+1}{2}z + t\right)^2 + 1 + i}{\frac{\sqrt{2}-1}{3} [x + (\sqrt{2} - 1)y + z]^2 + \frac{68+52\sqrt{2}}{3} \left(\frac{1}{2}y + \frac{\sqrt{2}+1}{2}z + t\right)^2 + 1 + i}. \quad (38)$$

The first-order rational solution given in (38) is a lump in the (x, y) -plane, which reaches the maximum of $8(2 - \sqrt{2})$ at $(x = (2\sqrt{2} - 2)t, y = -(\sqrt{2} + 1)z - 2t)$; see Figure 5.

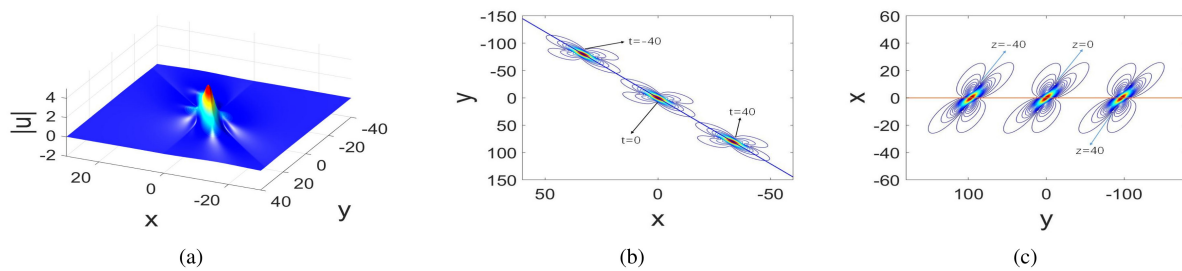


Figure 5. Plots of $|u|$ of the first-order lump solution (38) of the three-dimensional KP equation with parameters $\alpha = 1 + i, \beta = 1, \gamma = 1 + i, q_1 = q_2^* = 1 + i, p_1 = p_2^* = (\sqrt{2} - 1) + (\sqrt{2} - 1)i$; (a) a three-dimensional plot at $t = 0, z = 0$; (b) the temporal evolution in the (x, y) -plane of the contour of a first-order lump at $z = 0$, where the blue line is $x + (\sqrt{2} - 1)y + z = 0$; (c) the evolution in the (x, y) -plane of the contour of a first-order lump at $t = 0$, where the red line is $x = 0$.

Taking in Equation (32),

$$N = 2, \quad p_1 = p_2 = \frac{1}{3}, \quad q_1 = q_2^* = -i, \quad \alpha = 1, \quad \beta = -1, \quad \gamma = 2, \quad (39)$$

we obtain the rational solution

$$u = 12(\ln f)_{xx}, \quad f = \left(x - \frac{5}{9}t + \frac{1}{3}y\right)^2 + z^2 - \frac{1}{9}t^2 + 6 - \frac{2}{3}it z. \quad (40)$$

This rational solution is nonsingular when $z \neq 0$. As described in Figure 6, this solution describes the interaction of two rational parallel solitons in the (x, y) -plane. When $y = -3x$, the two rational solitons fuse together and reach the minimum value $\frac{24}{z^2 + 6}$ at $t = 0$. In general, the interaction of two solitons produces a solution with larger amplitude. Surprisingly, in this case, the amplitude of a line soliton fused by two rational solitons is significantly smaller than the solitons eventually after separation. As shown in Figure 6e–h, although the two solitons are completely separated as t tends to infinity, as time evolves, they interact, producing a downward peak, which yields a decrease of the amplitude. Finally, the two rational solitons fuse into a W-type rational soliton at $t = 0$.

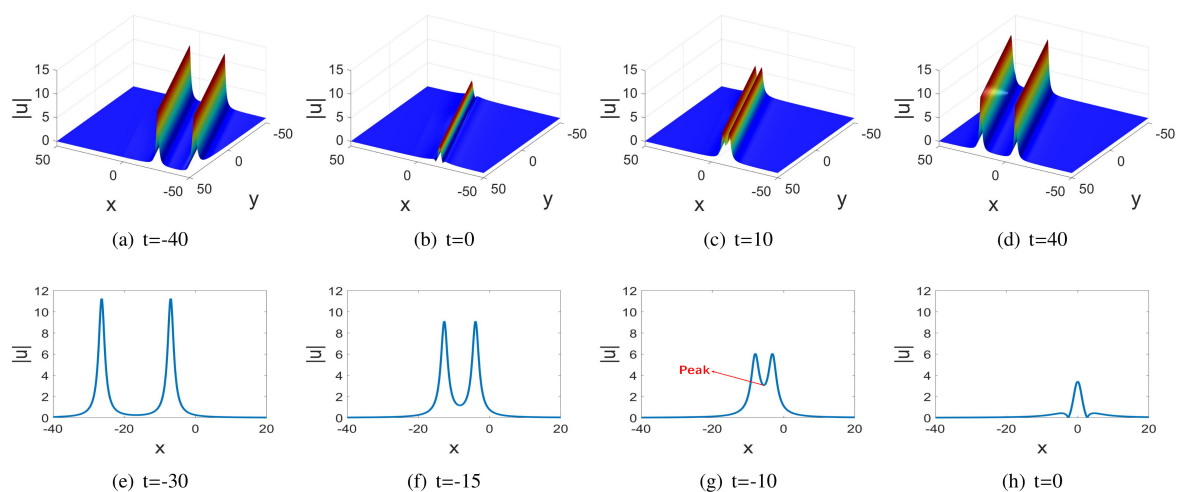


Figure 6. Plots of $|u|$ of the temporal evolution of the rational solution (40) for the three-dimensional KP equation with parameters $\alpha = 1, \beta = -1, \gamma = 2, p_1 = p_2 = \frac{1}{3}, q_1 = q_2^* = -i, z = 1$.

Following the extensive studies of the line rogue waves for $(2 + 1)$ -dimensional integrable models [32–35], it is interesting to find such solutions of the three-dimensional KP equation. The line rogue wave has infinite length and suddenly appears and disappears in the background plane. Compared with the normal rogue waves, which are localized in time and space, line rogue waves are localized in time, but are non-local in space, and also only appear in high-dimensional systems. Taking in Equation (32),

$$N = 2, \quad p_1 = p_2 = \frac{1}{3}, \quad q_1 = q_2^* = 1 + i, \quad \alpha = 1, \quad \beta = -1, \quad \gamma = 2, \quad (41)$$

we obtain

$$u = 12(\ln f)_{xx}, \quad f = \left(x - \frac{1}{18}t + \frac{1}{3}y + z\right)^2 + (z + t)^2 - \frac{2}{9}t^2 + 6 - \frac{1}{21}it(17t + 18x + 6y + 36z). \quad (42)$$

This rational solution is nonsingular when $z = 0$. This rational solution is analogous to a typical line rogue wave solution. Its visible line profile is preserved for the short period of time during the evolution process; see Figure 7.

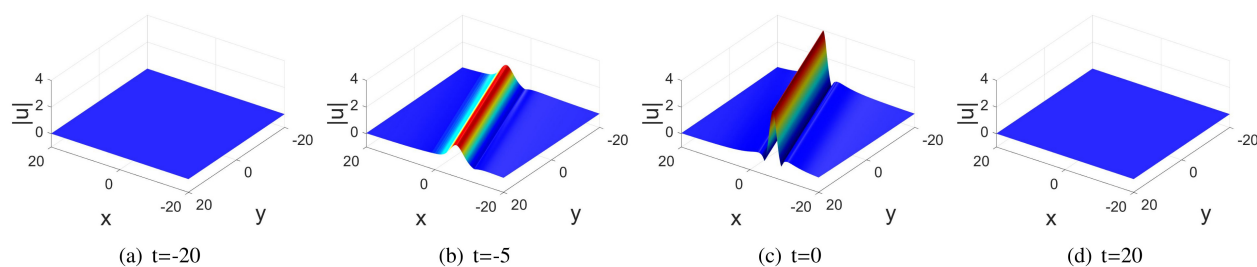


Figure 7. Plots of $|u|$ of the temporal evolution of the line rogue wave (42) for the three-dimensional KP equation with parameters $\alpha = 1, \beta = -1, \gamma = 2, p_1 = p_2 = \frac{1}{3}, q_1 = q_2^* = 1 + i, z = 0$.

5. Discussion and Conclusions

The main achievement of this work is the explicit construction of nonsingular multi-solitons, and high-order breather and high-order rational solutions of the three-dimensional KP Equation (1) using a Hirota bilinear method and the perturbation expansion technique. Moreover, different patterns of first-order breathers for the three-dimensional KP equation are discussed; see Figure 2. Furthermore, mixed solutions composed of breather and solitons are also given; see Figure 4. Additionally, high-order rational solutions of the three-dimensional KP equation are constructed by taking the long-wave limit of N-soliton solution. The first-order rational solutions are nonsingular, providing that the constraints of (35) and (37) are valid. The first-order rational solution generates a lump wave in the (x, y) -plane; its local characteristics and evolutionary process are demonstrated in Figure 5. In addition to the lump wave, the first-order rational solution (40) describes the interaction of two rational parallel solitons under the constraint (39). As shown in Figure 6, the interaction of two rational solitons surprisingly makes the amplitude of this solution significantly smaller in comparison to the amplitude of the separated line solitons. In summary, under the constraint (41), the first-order rational solution (40) in the (x, y) -plane is a line rogue wave; see Figure 7. The dynamics of the exact solutions possessed by this new three-dimensional model are rich and highly non-trivial. Hopefully, such solutions may be important in realistic applications.

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