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Article

Some New Estimates on Coordinates of Generalized Convex Interval-Valued Functions

Muhammad Bilal Khan [1], Adriana Cătaş [2,*] and Omar Mutab Alsalami [3]

1 Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan; bilal42742@gmail.com
2 Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania
3 Department of Electrical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; o.alsalami@tu.edu.sa
* Correspondence: acatas@gmail.com or acatas@uoradea.ro

Abstract: The theory of convex and nonconvex mapping has a lot of applications in the field of applied mathematics and engineering. The *Riemann integrals* are the most significant operator of interval theory, which permits the generalization of the classical theory of integrals. This study considers the well-known coordinated interval-valued Hermite–Hadamard-type and associated inequalities. To full fill this mileage, we use the introduced coordinated interval left and right preinvexity (LR-preinvexity) and *Riemann integrals* for further extension. Moreover, we have introduced some new important classes of interval-valued coordinated LR-preinvexity (preincavity), which are known as lower coordinated preinvex (preincave) and upper preinvex (preincave) interval-valued mappings, by applying some mild restrictions on coordinated preinvex (preincave) interval-valued mappings. By using these definitions, we have acquired many classical and new exceptional cases that can be viewed as applications of the main results. We also present some examples of interval-valued coordinated LR-preinvexity to demonstrate the validity of the inclusion relations proposed in this paper.

Keywords: double integral; coordinated LR-preinvex interval-valued function; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

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1. Introduction

The extended convexity of mappings is a powerful tool for dealing with a wide range of challenges in nonlinear analysis and applied analysis, including several problems in mathematical physics. Excess molar volume, heat of mixing, and excess enthalpy are common thermodynamic features of non-ideal solutions and mixes that show generalized convexity. Inequalities and other forms of extended convex mappings have also been considered important in the study of differential and integral equations such as the Kudryashov–Sinelshchikov equations and the Euler equations. Their significant effect may be seen in electrical networks, quantum relative entropy, symmetry analysis, ergodic theory, dynamic systems, equilibrium, and repulsive perturbations. As a result, generalized convexity theory and inequalities play an important role in mathematics and physics. The study of inequalities and extended convex maps is becoming increasingly popular. From the standpoint of research and analysis addressing actual application difficulties, there is a considerable literature on generalized convexity and inequality. We will discuss some of the findings of [1–7] as well as the sources listed therein.

On the other hand, a number of scholars have contributed to the development of inequalities and properties associated with generalized convexity in a variety of directions, as evidenced by the published publications [8–11] and the bibliographies cited therein. The Hermite–Hadamard inequality, which is widely used in many different branches of applied analysis, particularly in the field of optimality analysis, is one of the exceptional mathematical inequalities in the context of convex mappings. The following section explains this inequality.

Suppose that the continuous mapping $\mathcal{T} : \mathfrak{T} \to \mathbb{R}$ is a convex mapping on an interval $\mathfrak{T} = [u, \, \nu]$ of real nembers, where $u \neq \nu$. Then, we acquire the following inequality:

$$\mathcal{T}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu - u} \int_u^\nu \mathcal{T}(\varkappa) d\varkappa \leq \frac{\mathcal{T}(u) + \mathcal{T}(\nu)}{2} \tag{1}$$

If $\mathcal{T}$ is concave, then inequality (1) is reversed.

This inequality is notable because it provides error bounds for the mean value when considering a continuous convex mapping of $\mathcal{T} : \mathfrak{T} \to \mathbb{R}$, which has attracted the attention of a large number of mathematicians. On the basis of other various groups of convex mappings, such as referring to Kórus [12] with s-convex mappings, Delavar and De La Sen [13] with h-convex mappings, Abramovich and Persson [14] with N-quasiconvex mappings, and so on, there have been many studies and surveys regarding the Hermite–Hadamard-type inequalities. The reader might see [15–17] and the references therein for current developments on this important issue.

Fractional calculus has shown to be a crucial cornerstone in both mathematics and applied sciences as a very useful tool. A number of scholars have been urged by academics to consider a number of fractional calculus application difficulties. For example, mass mathematicians have studied some delicate integral inequalities involving conformable fractional integral operators [18] in the study of the Hermite–Hadamard-type integral inequalities involving conformable fractional integral operators, [19] in the Fejér-type integral inequalities via Katugampola-type fractional integral operators, [20] in the Simpson-type integral inequalities via Riemann–Liouvi-type fractional integral operators, [21] in the generalizations of the trapezoidal-type integral inequalities in the presence of k-fractional integral operators, and [22] in the generalizations of the Ostrowski-type integral inequalities in the presence of Hadamard-type fractional integral operators. We urge that interested readers study the published publications [23–28] and the bibliographies cited in them for some significant discoveries in relation to various types of fractional integrals.

As a novel non-probabilistic approach, interval analysis is a special instance of set-valued analysis. There is no doubt that interval analysis is extremely important in both pure and practical research. One of the initial aims of the interval analysis process was to examine the error estimations of finite state machines' numerical solutions. However, the interval analysis technique, which has been used in mathematical models in engineering for over fifty years as one of the ways to solve interval uncertain structural systems, is a critical cornerstone.

It is worth noting that applications in automatic error analysis [29], computer graphics [30], and neural network output optimization [31] have all been studied. Furthermore, [32–35] have a number of applications in optimization theory involving interval-valued mappings (I.V.Fs). We refer interested readers to [36,37] and the bibliographies cited in them for recent developments in the field of interval-valued mappings.

Breckner discussed the coming emerging idea of interval-valued convexity in [38].

An interval-valued mapping, $\mathcal{T} : \mathfrak{T} = [u, \, \nu] \to \mathcal{X}_C$, is called a convex interval-valued mapping if

$$\mathcal{T}(\tau\varkappa + (1-\tau)\omega) \supseteq \tau\mathcal{T}(\varkappa) + (1-\tau)\mathcal{T}(\omega), \tag{2}$$

for all $\varkappa, \, \omega \in [u, \, \nu]$, $\tau \in [0, \, 1]$, where $\mathcal{X}_C$ is the collection of all real-valued intervals. If (2) is reversed, then $\mathcal{T}$ is called concave.

Sadowska expanded the conventional Hermite–Hadamard-type inequality to the subsequent intriguing form with reference to interval integrals in a prior paper [39].

Suppose that the continuous mapping $\mathcal{T} : \mathfrak{T} \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ is an interval-valued mapping on an interval, $\mathfrak{T} = [u, \nu]$, of real numbers, where $u \neq \nu$. Then, we achieve the coming inequality:

$$\mathcal{T}\left(\frac{u+\nu}{2}\right) \supseteq \frac{1}{\nu-u} \int_u^\nu \mathcal{T}(\varkappa)d\varkappa \supseteq \frac{\mathcal{T}(u)+\mathcal{T}(\nu)}{2}. \tag{3}$$

If $\mathcal{T}$ is concave, then inequality (3) is reversed.

Ahmad et al. [39] and Khan et al. [40–43] extended these concepts and introduced different types of convexities, fuzz-numbered convexities, and constructed some new versions of inequalities in classical fractional calculus and fuzzy fractional calculus. Similarly, by considering the fractional integrals with exponential kernels for real-valued mappings proposed by Ahmad et al., in [44], Zhou et al., introduced the successive interval-valued fractional integrals with exponential kernels to obtain certain fractional integral inclusion relations for interval-valued convex mappings. We urge that interested readers study the published works [45–56] and the references therein for current developments in the field of coordinated mappings.

The current paper is motivated by the above-mentioned studies, in particular the findings developed in [50,57–65]. The interval-valued LR-preinvexity is used to create certain interval integral inclusion relations that are bound up with the extraordinary Hermite–Hadamard as well as Fejér–Hermite–Hadamard-type inequalities. We also use defined Riemann interval-valued integrals to create Pachpatte-type inclusion relations for interval-valued LR-preinvexity.

2. Preliminaries

Let $\mathcal{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathcal{Q} \in \mathcal{X}_C$ be defined by

$$\mathcal{Q} = [\mathcal{Q}_*,\ \mathcal{Q}^*] = \{\varkappa \in \mathbb{R} |\ \mathcal{Q}_* \leq \varkappa \leq \mathcal{Q}^*\}, (\mathcal{Q}_*,\ \mathcal{Q}^* \in \mathbb{R}).$$

If $\mathcal{Q}_* = \mathcal{Q}^*$, then $\mathcal{Q}$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathcal{Q}_* \geq 0$, then $[\mathcal{Q}_*,\ \mathcal{Q}^*]$ is called positive interval. The set of all positive intervals is denoted by $\mathcal{X}_C^+$ and defined as $\mathcal{X}_C^+ = \{[\mathcal{Q}_*,\ \mathcal{Q}^*] : [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathcal{X}_C \text{ and } \mathcal{Q}_* \geq 0\}$.

Let $\lambda \in \mathbb{R}$ and $\lambda \cdot \mathcal{Q}$ be defined by

$$\lambda \cdot \mathcal{Q} = \begin{cases} [\lambda\mathcal{Q}_*,\ \lambda\mathcal{Q}^*] \text{ if } \lambda > 0, \\ \{0\} \text{ if } \lambda = 0, \\ [\lambda\mathcal{Q}^*, \lambda\mathcal{Q}_*] \text{ if } \lambda < 0. \end{cases} \tag{4}$$

Then, the Minkowski difference, $\mathcal{Z} - \mathcal{Q}$; addition, $\mathcal{Q} + \mathcal{Z}$; and $\mathcal{Q} \times \mathcal{Z}$ for $\mathcal{Q}, \mathcal{Z} \in \mathcal{X}_C$ are defined by

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] + [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* + \mathcal{Q}_*,\ \mathcal{Z}^* + \mathcal{Q}^*], \tag{5}$$

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] \times [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\min\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\},\ \max\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\}] \tag{6}$$

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] - [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* - \mathcal{Q}^*,\ \mathcal{Z}^* - \mathcal{Q}_*], \tag{7}$$

For given $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$, the relation " $\supseteq$ " is defined on $\mathbb{R}_I$ by

$$[\mathcal{Q}_*,\ \mathcal{Q}^*] \supseteq [\mathcal{Z}_*,\ \mathcal{Z}^*] \text{ if and only if } \mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Z}^* \leq \mathcal{Q}^*,$$

For all $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$, it is a partial interval inclusion relation or an up-and-down (UD) relation.

On the other hand, for given $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$, the relation " $\leq_p$ " is defined on $\mathbb{R}_I$ by

$$[\mathcal{Q}_*,\ \mathcal{Q}^*] \supseteq [\mathcal{Z}_*,\ \mathcal{Z}^*] \text{ if and only if } \mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Z}^* \leq \mathcal{Q}^*,$$

For all $[\mathcal{Z}_*, \mathcal{Z}^*], [\mathcal{Q}_*, \mathcal{Q}^*] \in \mathbb{R}_I$, it is a pseudo-order relation or left-and-right (LR) relation, see [60].

Moreover, for $[\mathcal{Z}_*, \mathcal{Z}^*], [\mathcal{Q}_*, \mathcal{Q}^*] \in \mathcal{X}_C$, the Hausdorff–Pompeiu distance between intervals $[\mathcal{Z}_*, \mathcal{Z}^*]$ and $[\mathcal{Q}_*, \mathcal{Q}^*]$ is defined by

$$d_H([\mathcal{Z}_*, \mathcal{Z}^*], [\mathcal{Q}_*, \mathcal{Q}^*]) = max\{|\mathcal{Z}_* - \mathcal{Q}_*|, |\mathcal{Z}^* - \mathcal{Q}^*|\}. \tag{8}$$

It is a familiar fact that $(\mathcal{X}_C, d_H)$ is a complete metric space [65].

Theorem 1 ([46])**.** *If $\mathcal{T} : [u, \nu] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF given by $\mathcal{T}(x) = [\mathcal{T}_*(x), \mathcal{T}^*(x)]$, then $\mathcal{T}$ is Riemann-integrable over $[u, \nu]$ if and only if $\mathcal{T}_*$ and $\mathcal{T}^*$ both are Riemann-integrable over $[u, \nu]$ such that*

$$(IR)\int_u^\nu \mathcal{T}(x)dx = [(R)\int_u^\nu \mathcal{T}_*(x)dx, (R)\int_u^\nu \mathcal{T}^*(x)dx] \tag{9}$$

Khan et al., expanded the conventional Hermite–Hadamard-type inequality to the subsequent intriguing form with reference to interval integrals in a prior paper [65].

Suppose that the continuous mapping $\mathcal{T} : \mathfrak{T} \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ is an LR-preinvex interval-valued mapping on an interval $\mathfrak{T} = [u, u + \varphi(\nu, u)]$ of real nembers, where $u \neq u + \varphi(\nu, u)$. Then, we achieve the coming inequality:

$$\mathcal{T}\left(\frac{2u + \varphi(\nu, u)}{2}\right) \leq_p \frac{1}{\varphi(\nu, u)} \int_u^{u+\varphi(\nu,u)} \mathcal{T}(\varkappa)d\varkappa \leq_p \frac{\mathcal{T}(u) + \mathcal{T}(\nu)}{2}. \tag{10}$$

If $\mathcal{T}$ is LR-preincave, then inequality (3) is reversed.

Theorem 2 ([65])**.** *Let $\mathcal{T}, \varkappa : [u, u + \varphi(\nu, u)] \to \mathbb{R}_I^+$ be two LR-preinvex I.V.Fs such that $\mathcal{T}(x) = [\mathcal{T}_*(x), \mathcal{T}^*(x)]$ and $\varkappa(x) = [\varkappa_*(x), \varkappa^*(x)]$ for all $x \in [u, \nu]$. If $\mathcal{T} \times \varkappa$ is interval-Riemann-integrable, then*

$$\frac{1}{\varphi(\nu, u)} (IR) \int_u^{u+\varphi(\nu,u)} \mathcal{T}(x) \times \varkappa(x)dx \leq_p \frac{1}{3}\mathcal{M}(u, \nu) + \frac{1}{6}\mathcal{N}(u, \nu), \tag{11}$$

and

$$2\,\mathcal{T}\left(\frac{2u + \varphi(\nu, u)}{2}\right) \times \varkappa\left(\frac{2u + \varphi(\nu, u)}{2}\right) \leq_p \frac{1}{\varphi(\nu, u)} (IR) \int_u^{u+\varphi(\nu,u)} \mathcal{T}(x) \times \varkappa(x)dx + \frac{1}{6}\mathcal{M}(u, \nu) + \frac{1}{3}\mathcal{N}(u, \nu). \tag{12}$$

where $\mathcal{M}(u, \nu) = \mathcal{T}(u) \times \varkappa(u) + \mathcal{T}(\nu) \times \varkappa(\nu)$, $\mathcal{N}(u, \nu) = \mathcal{T}(u) \times \varkappa(\nu) + \mathcal{T}(\nu) \times \varkappa(u)$, $\mathcal{M}(u, \nu) = [\mathcal{M}_*(u, \nu), \mathcal{M}^*(u, \nu)]$, and $\mathcal{N}(u, \nu) = [\mathcal{N}_*(u, \nu), \mathcal{N}^*(u, \nu)]$.

Theorem 3 ([65])**.** *Let $\mathcal{T} : [u, u + \varphi(\nu, u)] \to \mathbb{R}_I^+$ be an LR-preinvex I.V.F with $u < u + \varphi(\nu, u)$ such that $\mathcal{T}(x) = [\mathcal{T}_*(x), \mathcal{T}^*(x)]$ for all $x \in [u, \nu]$. If $\mathcal{T}$ is interval-Riemann-integrable and $\Omega : [u, u + \varphi(\nu, u)] \to \mathbb{R}$, $\Omega(x) \geq 0$, symmetric with respect to $\frac{2u+\varphi(\nu,u)}{2}$, and $\int_u^{u+\varphi(\nu,u)} \Omega(x)\, dx > 0$, then*

$$\mathcal{T}\left(\frac{2u + \varphi(\nu, u)}{2}\right) \leq_p \frac{1}{\int_u^{u+\varphi(\nu,u)} \Omega(x)dx} (IR) \int_u^{u+\varphi(\nu,u)} \mathcal{T}(x)\Omega(x)dx \leq_p \frac{\mathcal{T}(u) + \mathcal{T}(\nu)}{2}. \tag{13}$$

If $\mathcal{T}$ is an LR-preincave I.V.F, then inequality (13) is reversed.

Definition 1 ([62])**.** *The IVF $\mathcal{T} : \Delta = [\rho, \varsigma] \times [u, \nu] \to \mathbb{R}^+$ is said to be a coordinated convex function on Δ if*

$$\mathcal{T}(\tau\rho + (1-\tau)\varsigma, su + (1-s)\nu) \leq \tau s\mathcal{T}(\rho, u) + \tau(1-s)\mathcal{T}(\rho, \nu) + (1-\tau)s\mathcal{T}(\varsigma, u) + (1-\tau)(1-s)\mathcal{T}(\varsigma, \nu), \tag{14}$$

for all (ρ, ς), $(u, \nu) \in \Delta$, τ and $\tau, s \in [0, 1]$. If inequality (17) is reversed, then $\mathcal{T}$ is called a coordinated concave IVF on Δ.

Definition 2 ([64]). *The I.V.F $\mathcal{T} : \Delta \to \mathbb{R}_I^+$ is said to be a coordinated LR-convex I.V.F on Δ if*

$$\mathcal{T}(\tau\rho + (1-\tau)\varsigma,\ su + (1-s)\nu) \leq_p \tau s\mathcal{T}(\rho, u) + \tau(1-s)\mathcal{T}(\rho, \nu) + (1-\tau)s\mathcal{T}(\varsigma, u) + (1-\tau)(1-s)\mathcal{T}(\varsigma, \nu), \tag{15}$$

for all $(\rho,\ \varsigma)$, $(u,\ \nu) \in \Delta$, and $\tau, s \in [0,\ 1]$. If inequality (15) is reversed, then $\mathcal{T}$ is called a coordinate LR-concave I.V.F on Δ.

Now, we recall the concept of interval-double-integrable functions.

Definition 3 ([63]). *A function $\mathcal{T} : \Delta = [\rho,\ \varsigma] \times [u,\ \nu] \to \mathbb{R}_I$ is called interval-double-integrable (ID-integrable) on Δ if there exists $B \in \mathbb{R}_I$ such that, for each ϵ, there exists $\delta > 0$ such that*

$$d(S(\mathcal{T}, P,\ \delta, \Delta),\ B) < \epsilon, \tag{16}$$

for every Riemann sum of $\mathcal{T}$ corresponding to $P \in \mathcal{P}(\delta,\ \Delta)$ and for the arbitrary choice of $(\eta_i, w_j) \in [x_{i-1},\ x_i] \times [\omega_{j-1},\ \omega_j]$ for $1 \leq i \leq k$ and $1 \leq j \leq n$. Then, we say that B is the IR-integral of $\mathcal{T}$ on Δ, and it is denoted by $B = (ID)\int_\rho^\varsigma \int_u^\nu \mathcal{T}(x, \omega)d\omega dx$ or $B = (ID)\iint_\Delta \mathcal{T}dA$.

Note that Theorem 4 is also true for interval double integrals. The collection of all double-integrable IVFs is denoted $\mathfrak{TD}_\Delta$.

Theorem 4 ([63]). *Let $\Delta = [\rho,\ \varsigma] \times [u,\ \nu]$. If $\mathcal{T} : \Delta \to \mathbb{R}_I$ is ID-integrable on Δ, then we have*

$$(ID)\int_\rho^\varsigma \int_u^\nu \mathcal{T}(x, \omega)d\omega dx = (IR)\int_\rho^\varsigma (IR)\int_u^\nu \mathcal{T}(x, \omega)d\omega dx. \tag{17}$$

Coordinated LR-Preinvex Interval-Valued Functions

Definition 4. *The I.V.F $\mathcal{T} : \Delta \to \mathbb{R}_I^+$ is said to be a coordinated LR-preinvex I.V.F on Δ if*

$$\mathcal{T}(\rho + (1-\tau)\varphi_1(\varsigma, \rho),\ u + (1-s)\varphi_2(\nu, u)\ ) \leq_p \tau s\mathcal{T}(\rho, u) + \tau(1-s)\mathcal{T}(\rho, \nu) + (1-\tau)s\mathcal{T}(\varsigma, u) + (1-\tau)(1-s)\mathcal{T}(\varsigma, \nu), \tag{18}$$

for all $(\rho,\ \varsigma)$, $(u,\ \nu) \in \Delta$, and $\tau, s \in [0,\ 1]$ where $\mathcal{T}(x) \geq_I 0$. If inequality (18) is reversed, then $\mathcal{T}$ is a called coordinate LR-preincave I.V.F on Δ.

The proof of Lemma 1 is straightforward and is omitted here.

Lemma 1. *Let $\mathcal{T} : \Delta \to \mathbb{R}_I^+$ be a coordinated I.V.F on Δ. Then, $\mathcal{T}$ is a coordinated LR-preinvex I.V.F on Δ if and only if there exist two coordinated LR-preinvex I.V.Fs, $\mathcal{T}_x : [u,\ \nu] \to \mathbb{R}_I^+$, $\mathcal{T}_x(w) = \mathcal{T}(x, w)$ and $\mathcal{T}_\omega : [\rho, \varsigma] \to \mathbb{R}_I^+$, $\mathcal{T}_\omega(y) = \mathcal{T}(y, \omega)$.*

Proof. From the definition of a coordinated LR-preinvex I.V.F, it can be easily proven. From Lemma 1, we can easily note each LR-preinvex I.V.F is a coordinated preinvex I.V.F, but the converse is not true (see Example 1). □

Theorem 5. *Let $\mathcal{T} : \Delta \to \mathbb{R}_I^+$ be an I.V.F on Δ such that*

$$\mathcal{T}(x, \omega) = [\mathcal{T}_*(x, \omega),\ \mathcal{T}^*(x, \omega)], \tag{19}$$

for all $(x, \omega) \in \Delta$. Then, $\mathcal{T}$ is a coordinated LR-preinvex I.V.F on Δ if and only if $\mathcal{T}_*(x, \omega)$ and $\mathcal{T}^*(x, \omega)$ are coordinated preinvex and preinvex functions, respectively.

Proof. Assume that $\mathcal{T}_*(x, \omega)$ and $\mathcal{T}^*(x, \omega)$ are coordinated preinvex functions on Δ. Then, from (18), for all $(\rho,\ \varsigma)$, $(u,\ \nu) \in \Delta$, τ and $s \in [0,\ 1]$ we have

$$\begin{gathered}\mathcal{T}_*(\rho + (1-\tau)\varphi_1(\varsigma, \rho),\ u + (1-s)\varphi_2(\nu, u)) \\ \leq \tau s\mathcal{T}_*(\rho, u) + t(1-s)\mathcal{T}_*(\rho, \nu) + s(1-t)\mathcal{T}_*(\rho, u) + (1-\tau)(1-s)\mathcal{T}_*(\rho, \nu),\end{gathered}$$

and

$$\begin{gathered}\mathcal{T}^*(\rho + (1-\tau)\varphi_1(\varsigma, \rho),\ u + (1-s)\varphi_2(\nu, u)) \\ \leq \tau s\mathcal{T}_*(\rho, u) + t(1-s)\mathcal{T}^*(\rho, \nu) + s(1-t)\mathcal{T}^*(\rho, u) + (1-\tau)(1-s)\mathcal{T}^*(\rho, \nu),\end{gathered}$$

Then, by (19), (4), and (5), we obtain

$$\begin{gathered}\mathcal{T}((\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)))= \\ [\mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)),\ \mathcal{T}^*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u))], \\ \leq_p \tau s[\mathcal{T}_*(\rho,u),\ \mathcal{T}^*(\rho,u)]+t(1-s)[\mathcal{T}_*(\rho,\nu),\ \mathcal{T}^*(\rho,\nu)] \\ +s(1-\tau)[\mathcal{T}_*(\rho,u),\ \mathcal{T}^*(\rho,u)]+(1-\tau)(1-s)[\mathcal{T}_*(\rho,\nu),\ \mathcal{T}^*(\rho,\nu)].\end{gathered}$$

That is

$$\begin{aligned}&\mathcal{T}(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)) \\ &\qquad \leq_p \tau s\mathcal{T}(\rho,u)+\tau(1-s)\mathcal{T}(\rho,\nu)+(1-\tau)s\mathcal{T}(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}(\varsigma,\nu),\end{aligned}$$

Hence, $\mathcal{T}$ is a coordinated LR-preinvex I.V.F on Δ. □

Conversely, let $\mathcal{T}$ be a coordinated LR-preinvex I.V.F on Δ. Then, for all $(\rho,\ \varsigma),\ (u,\ \nu)\in\Delta,\ \tau$ and $s\in[0,\ 1]$, we have

$$\begin{aligned}&\mathcal{T}(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)) \\ &\leq_p \tau s\mathcal{T}(\rho,u)+\tau(1-s)\mathcal{T}(\rho,\nu)+(1-\tau)s\mathcal{T}(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}(\varsigma,\nu).\end{aligned}$$

Therefore, again from (19), we have

$$\begin{aligned}&\mathcal{T}((\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u))) \\ &=[\mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)),\ \mathcal{T}^*(\rho+(1-\tau)\varsigma,\ u+(1-s)\varphi_2(\nu,u))].\end{aligned}$$

Again, from (4) and (5), we obtain

$$\begin{aligned}&\tau s\mathcal{T}(\rho,u)+\tau(1-s)\mathcal{T}(\rho,\nu)+(1-\tau)s\mathcal{T}(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}(\varsigma,\nu) \\ &=\tau s[\mathcal{T}_*(\rho,u),\ \mathcal{T}^*(\rho,u)]+t(1-s)[\mathcal{T}_*(\rho,\nu),\ \mathcal{T}^*(\rho,\nu)] \\ &+s(1-\tau)[\mathcal{T}_*(\rho,u),\ \mathcal{T}^*(\rho,u)]+(1-\tau)(1-s)[\mathcal{T}_*(\rho,\nu),\ \mathcal{T}^*(\rho,\nu)],\end{aligned}$$

for all $x,\omega\in\Delta$ and $\tau\in[0,\ 1]$. Then, by the coordinated LR-preinvexity of $\mathcal{T}$, we have for all $x,\omega\in\Delta$ and $\tau\in[0,\ 1]$ such that

$$\begin{aligned}&\mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)) \\ &\leq \tau s\mathcal{T}_*(\rho,u)+\tau(1-s)\mathcal{T}_*(\rho,\nu)+(1-\tau)s\mathcal{T}_*(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}_*(\varsigma,\nu),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{T}^*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u)) \\ &\leq \tau s\mathcal{T}^*(\rho,u)+\tau(1-s)\mathcal{T}^*(\rho,\nu)+(1-\tau)s\mathcal{T}^*(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}^*(\varsigma,\nu),\end{aligned}$$

Hence, the result follows.

Example 1. *We consider the I.V.Fs $\mathcal{T}:[0,\ 1]\times[0,\ 1]\to\mathbb{R}_I^+$ defined by,*

$$\mathcal{T}(x)=[x\omega,(6+e^x)(6+e^\omega)] \tag{20}$$

Since end point functions $\mathcal{T}_*(x,\omega)$ and $\mathcal{T}^*(x,\omega)$ are coordinated preinvex and preinvex functions with respect to $\varphi_1(\varsigma,\rho)=\varsigma-\rho$ and $\varphi_2(\nu,u)=\nu-u$, respectively, $\mathcal{T}(x,\omega)$ is a coordinated LR-preinvex I.V.F.

From Example 1, it can be easily seen that each coordinated LR-preinvex I.V.F is not a preinvex I.V.F.

Theorem 6. *Let Δ be a coordinated preinvex set, and let $\mathcal{T}:\Delta\to\mathbb{R}_I^+$ be an I.V.F such that*

$$\mathcal{T}(x,\omega)=[\mathcal{T}_*(x,\omega),\ \mathcal{T}^*(x,\omega)], \tag{21}$$

for all $(x,\omega)\in\Delta$. Then, $\mathcal{T}$ is a coordinated LR-preincave I.V.F on Δ if and only if $\mathcal{T}_*(x,\omega)$ and $\mathcal{T}^*(x,\omega)$ are coordinated preincave functions.

Proof. The demonstration of proof of Theorem 6 is similar to the demonstration proof of Theorem 5. □

Example 2. *We consider the I.V.Fs* $\mathcal{T} : [0,\ 1] \times [0,\ 1] \to \mathbb{R}_I^+$, *defined by*

$$\mathcal{T}(x) = [(6-e^x)(6-e^\omega), 4(6-e^x)(6-e^\omega)\ ] \tag{22}$$

Since end point functions $\mathcal{T}_*(x,\omega)$ *and* $\mathcal{T}^*(x,\omega)$ *are both coordinated preincave functions with respect to* $\varphi_1(\varsigma,\rho) = \varsigma - \rho$ *and* $\varphi_2(\nu,u) = \nu - u, \mathcal{T}(x,\omega)$ *is a coordinated LR-preincave I.V.F.*

In the next results, to avoid confusion, we do not include the symbols (R), (IR), and (ID) before the integral sign.

3. Interval Hermite–Hadamard Inequalities

In this section, we propose HH and HH-Fejér inequalities for coordinated LR-preinvex I.V.Fs and verify them with the help of some nontrivial examples.

Theorem 7. *Let* $\mathcal{T} : \Delta = [\rho,\ \rho + \varphi_1(\varsigma,\rho)] \times [u,\ u + \varphi_2(\nu,u)] \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *be a coordinate LR-preinvex IVF on* Δ *such that* $\mathcal{T}(x,\omega) = [\mathcal{T}_*(x,\omega),\ \mathcal{T}^*(x,\omega)]$ *for all* $(x,\omega) \in \Delta$, *and let condition C hold for* φ_1 *and* φ_2. *Then, the following inequality holds:*

$$\begin{aligned} &\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\ \leq_p\ &\tfrac{1}{2}\left[\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx + \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)d\omega\right] \\ &\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega)d\omega dx \\ &\leq_p \tfrac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u)dx + \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu)dx\right] \\ &+ \tfrac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega)d\omega\right] \\ &\leq_p \tfrac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{4}. \end{aligned} \tag{23}$$

If $\mathcal{T}(x)$ *is an LR-preincave I.V.F then,*

$$\begin{aligned} &\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\ \geq_p\ &\tfrac{1}{2}\left[\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx + \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)d\omega\right] \\ &\geq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega)d\omega dx \\ &\geq_p \tfrac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u)dx + \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu)dx\right] \\ &+ \tfrac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega)d\omega\right] \\ &\geq_p \tfrac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{4} \end{aligned} \tag{24}$$

Proof. Let $\mathcal{T} : \Delta \to \mathbb{R}_I^+$ be a coordinated LR-preinvex I.V.F. Then, by this hypothesis, we have

$$\begin{aligned} &4\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\ \leq_p\ &\mathcal{T}(\rho + (1-\tau)\varphi_1(\varsigma,\rho),\ u + (1-s)\varphi_2(\nu,u)) + \mathcal{T}(\varsigma + \tau\varphi_1(\varsigma,\rho),\ \nu + s\varphi_2(\nu,u)). \end{aligned}$$

By using Theorem 5, we have

$$\begin{aligned} &4\mathcal{T}_*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\ \leq\ &\mathcal{T}_*(\rho + (1-\tau)\varphi_1(\varsigma,\rho),\ u + (1-s)\varphi_2(\nu,u)) + \mathcal{T}_*(\varsigma + \tau\varphi_1(\varsigma,\rho),\ \nu + s\varphi_2(\nu,u)), \\ &4\mathcal{T}^*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\ \leq\ &\mathcal{T}^*(\rho + (1-\tau)\varphi_1(\varsigma,\rho),\ u + (1-s)\varphi_2(\nu,u)) + \mathcal{T}^*(\varsigma + \tau\varphi_1(\varsigma,\rho),\ \nu + s\varphi_2(\nu,u)). \end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned} 2\mathcal{T}_*\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) &\leq \mathcal{T}_*(x,\ u + (1-s)\varphi_2(\nu,u)) + \mathcal{T}_*(x,\ \nu + s\varphi_2(\nu,u)), \\ 2\mathcal{T}^*\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) &\leq \mathcal{T}^*(x,\ u + (1-s)\varphi_2(\nu,u)) + \mathcal{T}^*(x,\ \nu + s\varphi_2(\nu,u)), \end{aligned} \tag{25}$$

and

$$\begin{aligned} 2\mathcal{T}_*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right) &\leq \mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega)+\mathcal{T}_*((1-\tau)\rho+t\varsigma,\ \omega), \\ 2\mathcal{T}^*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right) &\leq \mathcal{T}^*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega)+\mathcal{T}^*((1-\tau)\rho+t\varsigma,\ \omega). \end{aligned} \tag{26}$$

From (25) and (26), we have

$$\begin{aligned} &2\left[\mathcal{T}_*\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\mathcal{T}^*\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right] \\ &\leq_p [\mathcal{T}_*(x,\ u+(1-s)\varphi_2(\nu,u)),\mathcal{T}^*(x,\ u+(1-s)\varphi_2(\nu,u))] \\ &+[\mathcal{T}_*(x,\ \nu+s\varphi_2(\nu,u)),\mathcal{T}^*(x,\ \nu+s\varphi_2(\nu,u))], \end{aligned}$$

and

$$\begin{aligned} &2\left[\mathcal{T}_*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right),\mathcal{T}^*\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)\right] \\ &\leq_p [\mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega),\mathcal{T}^*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega)] \\ &+[\mathcal{T}_*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega),\mathcal{T}^*(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega)], \end{aligned}$$

It follows that

$$\mathcal{T}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \mathcal{T}(x,\ u+(1-s)\varphi_2(\nu,u))+\mathcal{T}(x,\ \nu+s\varphi_2(\nu,u)) \tag{27}$$

and

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right) \leq_p \mathcal{T}(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ \omega)+\mathcal{T}(\varsigma+\tau\varphi_1(\varsigma,\rho),\ \omega) \tag{28}$$

Since $\mathcal{T}(x,.)$ and $\mathcal{T}(.,\omega)$ are both coordinated LR-preinvex-IVFs, then from inequalities (10), (27), and (28), we have

$$\mathcal{T}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega \leq_p \frac{T(x,u)+T(x,\nu)}{2}. \tag{29}$$

and

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right) \leq_p \frac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\omega)dx \leq_p \frac{\mathcal{T}(\rho,\omega)+\mathcal{T}(\varsigma,\omega)}{2}. \tag{30}$$

Dividing double inequality (29) by $\varphi_1(\varsigma,\rho)$ and integrating with respect to x over $[\rho,\ \rho+\varphi_1(\varsigma,\rho)]$, we have

$$\begin{aligned} &\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx \leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega dx \\ &\leq_p \tfrac{1}{2\varphi_1(\varsigma,\rho)}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx+\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\right]. \end{aligned} \tag{31}$$

Similarly, dividing double inequality (30) by $\varphi_2(\nu,u)$ and integrating with respect to x over $[u,\ u+\varphi_2(\nu,u)]$, we have

$$\begin{aligned} &\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega \leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega dx \\ &\leq_p \tfrac{1}{2\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega+\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\right] \end{aligned} \tag{32}$$

By adding (31) and (32), we have

$$\begin{aligned}
&\tfrac{1}{2}\left[\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx+\tfrac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\right]\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega dx\\
&\leq_p \tfrac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx+\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\right]\\
&+\tfrac{1}{4\varphi_2(\nu,u)}\left[\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega+\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\right].
\end{aligned} \tag{33}$$

Since $\mathcal{T}$ is an I.V.F, then with inequality (33), we have

$$\begin{aligned}
&\tfrac{1}{2}\left[\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx+\tfrac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\right]\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega dx\\
&\leq_p \tfrac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx+\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\right]\\
&+\tfrac{1}{4\varphi_2(\nu,u)}\left[\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega+\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\right].
\end{aligned} \tag{34}$$

From the left side of inequality (10), we have

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right)\leq_p \frac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)dx, \tag{35}$$

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right)\leq_p \frac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega. \tag{36}$$

Taking the addition of inequality (35) with inequality (36), we have

$$\begin{aligned}
&\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{2}\left[\tfrac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx+\tfrac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\right].
\end{aligned} \tag{37}$$

Now, from right side of inequality (10), we have

$$\frac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx\leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)}{2} \tag{38}$$

$$\frac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\leq_p \frac{\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{2} \tag{39}$$

$$\frac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega\leq_p \frac{\mathcal{T}(\rho,\nu)+\mathcal{T}(\rho,u)}{2}, \tag{40}$$

$$\frac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\leq_p \frac{\mathcal{T}(\varsigma,\nu)+\mathcal{T}(\varsigma,u)}{2}. \tag{41}$$

By adding inequalities (38)–(41), we have

$$\begin{aligned}
&\tfrac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx+\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\right]\\
&+\tfrac{1}{4\varphi_2(\nu,u)}\left[\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega+\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\right]\\
&\leq_p \tfrac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{4}
\end{aligned} \tag{42}$$

By combining inequalities (34), (37), and (42), we obtain the desired result. □

Example 3. *We consider the I.V.Fs $\mathcal{T}:[0,\ 1]\times[0,\ 1]\to\mathbb{R}_I^+$, defined by*

$$\mathcal{T}(x)=[2(6+e^x)(6+e^\omega),8(6+e^x)(6+e^\omega)\ ] \tag{43}$$

Since end point functions $\mathcal{T}_*(x,\omega)$, $\mathcal{T}^*(x,\omega)$ are coordinate preinvex functions, $\mathcal{T}(x,\omega)$ is a coordinate LR-preinvex I.V.F.

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right) = \left[2\left(6+e^{\frac{1}{2}}\right)^2, 8\left(6-e^{\frac{1}{2}}\right)^2\right],$$
$$\frac{1}{2}\left[\frac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)dx+\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\right]$$
$$=\left[2\left(6+e^{\frac{1}{2}}\right)(5+e), 8\left(6+e^{\frac{1}{2}}\right)(5+e)\right],$$

$$\frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)d\omega dx = \left[2(5+e)^2, 8(5+e)^2\right]$$
$$\frac{1}{4\varphi_1(\varsigma,\rho)}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)dx+\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)dx\right]+\frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)d\omega+\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)d\omega\right]$$
$$=[(5+e)(13+e), 4(5+e)(13+e)],$$

$$\frac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{4} = \left[\frac{(6+e)(20+e)+49}{2}, 2((6+e)(20+e)+49)\right],$$

that is

$$\begin{aligned}\left[2\left(6+e^{\frac{1}{2}}\right)^2, 8\left(6+e^{\frac{1}{2}}\right)^2\right] &\le_p \left[2\left(6+e^{\frac{1}{2}}\right)(5+e), 8\left(6+e^{\frac{1}{2}}\right)(5+e)\right]\\ &\le_p \left[2(5+e)^2, 8(5+e)^2\right] \le_p [(5+e)(13+e), 4(5+e)(13+e)]\\ &\le_p \left[\frac{(6+e)(20+e)+49}{2}, 2((6+e)(20+e)+49)\right].\end{aligned}$$

Hence, Theorem 7 has been verified.

Remark 1. *If $\mathcal{T}_*(x,\omega) \neq \mathcal{T}^*(x,\omega)$ with $\varphi_1(\varsigma,\rho) = \varsigma-\rho$ and $\varphi_2(\nu,u) = \nu-u$, then from (23), we acquire the following inequality (see [64]):*

$$\begin{aligned}\mathcal{T}\left(\frac{c+d}{2},\frac{\mu+x}{2}\right) &\le_p \frac{1}{2}\left[\frac{1}{d-c}\int_c^d\mathcal{T}\left(x,\frac{\mu+x}{2}\right)dx+\frac{1}{x-\mu}\int_\mu^x\mathcal{T}\left(\frac{c+d}{2},\omega\right)d\omega\right]\\ &\le_p \frac{1}{(d-c)(x-\mu)}\int_c^d\int_\mu^x\mathcal{T}(x,\omega)d\omega dx\\ &\le_p \frac{1}{4(x-\mu)}\left[\int_\mu^x\mathcal{T}(c,\omega)d\omega+\int_\mu^x\mathcal{T}(d,\omega)d\omega\right]\\ &+\frac{1}{4(d-c)}\left[\int_c^d\mathcal{T}(x,\mu)dx+\int_c^d\mathcal{T}(x,x)dx\right]\\ &\frac{\mathcal{T}(c,\mu)+\mathcal{T}(c,x)}{2}+\frac{\mathcal{T}(d,\mu)+\mathcal{T}(d,x)}{2}+\frac{\mathcal{T}(c,\mu)+\mathcal{T}(d,\mu)}{2}+\frac{\mathcal{T}(c,x)+\mathcal{T}(d,x)}{2}\end{aligned} \tag{44}$$

If $\mathcal{T}_*(x,\omega) = \mathcal{T}^*(x,\omega)$ with $\varphi_1(\varsigma,\rho) = \varsigma-\rho$ and $\varphi_2(\nu,u) = \nu-u$, then from (23) we acquire the following inequality (see [62]):

$$\begin{aligned}\mathcal{T}\left(\frac{c+d}{2},\frac{\mu+x}{2}\right) &\le \frac{1}{2}\left[\frac{1}{d-c}\int_c^d\mathcal{T}\left(x,\frac{\mu+x}{2}\right)dx+\frac{1}{x-\mu}\int_\mu^x\mathcal{T}\left(\frac{c+d}{2},\omega\right)d\omega\right]\\ &\le \frac{1}{(d-c)(x-\mu)}\int_c^d\int_\mu^x\mathcal{T}(x,\omega)d\omega dx\\ &\le \frac{1}{4(x-\mu)}\left[\int_\mu^x\mathcal{T}(c,\omega)d\omega+\int_\mu^x\mathcal{T}(d,\omega)d\omega\right]\\ &+\frac{1}{4(d-c)}\left[\int_c^d\mathcal{T}(x,\mu)dx+\int_c^d\mathcal{T}(x,x)dx\right]\\ &\le \frac{\mathcal{T}(c,\mu)+\mathcal{T}(c,x)}{2}+\frac{\mathcal{T}(d,\mu)+\mathcal{T}(d,x)}{2}+\frac{\mathcal{T}(c,\mu)+\mathcal{T}(d,\mu)}{2}+\frac{\mathcal{T}(c,x)+\mathcal{T}(d,x)}{2}.\end{aligned} \tag{45}$$

We now give anHH-Fejér inequality for coordinated LR-preinvex I.V.Fs by an inclusion relation in the following result.

Theorem 8. *Let $\mathcal{T}:\Delta = [\rho,\ \rho+\varphi_1(\varsigma,\rho)]\times[u,\ u+\varphi_2(\nu,u)] \to \mathbb{R}_I^+$ be a coordinated LR-preinvex IVF with $\rho<\varsigma$ and $u<\nu$ such that $\mathcal{T}(x,\omega) = [\mathcal{T}(x,\omega), \mathcal{T}(x,\omega)]$ for all $(x,\omega)\in\Delta$, and let condition C hold for φ_1 and φ_2. Let $\Omega:[\rho,\ \rho+\varphi_1(\varsigma,\rho)]\to\mathbb{R}$ with $(x)\ge 0$, $\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx>0$, and $\mathcal{W}:[u,\ u+\varphi_2(\nu,u)]\to\mathbb{R}$ with $\mathcal{W}(\omega)\ge 0$ and $\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)$*

$d\omega > 0$ as two symmetric functions with respect to $\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}$ and $\frac{2u+\varphi_2(\nu,u)}{2}$, respectively. Then, the following inequality holds:

$$\begin{aligned}
&\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \tfrac{1}{2}\left[\begin{array}{c} \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\Omega(x)dx \\ +\frac{1}{\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega \end{array}\right] \\
&\leq_p \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\
&\leq_p \frac{1}{4\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\Omega(x)dx + \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\Omega(x)dx\right] \\
&+\frac{1}{4\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\mathcal{W}(\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\mathcal{W}(\omega)d\omega\right] \\
&\leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{4}
\end{aligned} \tag{46}$$

Proof. Since $\mathcal{T}$ is a coordinated LR-preinvex I.V.F on Δ and it follows these functions, then by Lemma 1, there exist

$$\mathcal{T}_x : [u,\ u+\varphi_2(\nu,u)] \to \mathbb{R}_I^+,\ \mathcal{T}_x(\omega) = \mathcal{T}(x,\omega),$$
$$\mathcal{T}_\omega : [\rho,\ \rho+\varphi_1(\varsigma,\rho)] \to \mathbb{R}_I^+,\ \mathcal{T}_\omega(x) = \mathcal{T}(x,\omega).$$

Thus, from inequality (13), for each, we have

$$\mathcal{T}_x\left(\frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}_x(\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\mathcal{T}_x(u)+\mathcal{T}_x(\nu)}{2},$$

and

$$\mathcal{T}_\omega\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}\right) \leq_p \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}_\omega(x)\Omega(x)dx \leq_p \frac{\mathcal{T}_\omega(\rho)+\mathcal{T}_\omega(\varsigma)}{2}.$$

The above inequalities can be written as

$$\mathcal{T}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\mathcal{T}(x,u)+\mathcal{T}(x,\ \nu)}{2}, \tag{47}$$

and

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right) \leq_p \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\omega)\Omega(x)dx \leq_p \frac{\mathcal{T}(\rho,\omega)+\mathcal{T}(\varsigma,\omega)}{2} \tag{48}$$

Multiplying (47) by $\Omega(x)$ and then integrating the resultant with respect to x over $[\rho,\ \rho+\varphi_1(\varsigma,\rho)]$, we have

$$\begin{aligned}
&\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right)\Omega(x)dx \\
&\leq_p \frac{1}{\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\
&\leq_p \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\frac{\mathcal{T}(x,u)+\mathcal{T}(x,\ \nu)}{2}\Omega(x)dx.
\end{aligned} \tag{49}$$

Now, multiplying (48) by $\mathcal{W}(\omega)$ and then integrating the resultant with respect to ω over $[u,\ u+\varphi_2(\nu,u)]$, we have

$$\begin{aligned}
&\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega \\
&\leq_p \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)dxd\omega \\
&\leq_p \int_u^{u+\varphi_2(\nu,u)}\frac{\mathcal{T}(\rho,\omega)+\mathcal{T}(\varsigma,\omega)}{2}\mathcal{W}(\omega)d\omega.
\end{aligned} \tag{50}$$

Since $\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx > 0$ and $\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega > 0$, then by dividing (49) and (50) by $\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx > 0$ and $\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega > 0$, respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\Omega(x)dx + \frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\
&\leq_p \frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx \int_{\rho}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx, \\
&\leq_p \frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \frac{\mathcal{T}(x,u)+\mathcal{T}(x,\nu)}{4}\Omega(x)dx + \frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \frac{\mathcal{T}(\rho,\omega)+\mathcal{T}(\varsigma,\omega)}{4}\mathcal{W}(\omega)d\omega.
\end{aligned} \tag{51}$$

Now, from the left part of double inequalities (47) and (48), we obtain

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega, \tag{52}$$

and

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}\left(x, \frac{u+\nu}{2}\right)\Omega(x)dx \tag{53}$$

Summing inequalities (52) and (53), we obtain

$$\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_p \frac{1}{2}\left[\begin{array}{c} \frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\Omega(x)dx \\ + \frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega \end{array}\right]. \tag{54}$$

Similarly, from the right part of (47) and (48), we can obtain

$$\frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\rho,\nu)}{2}, \tag{55}$$

$$\frac{1}{\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega} \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega)\mathcal{W}(\omega)d\omega \leq_p \frac{\mathcal{T}(\varsigma,u)+\mathcal{T}(\varsigma,\nu)}{2}, \tag{56}$$

and

$$\frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u)\Omega(x)dx \leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)}{2} \tag{57}$$

$$\frac{1}{\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx} \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu)\Omega(x)dx \leq_p \frac{\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{2} \tag{58}$$

Adding (55)–(58) and dividing by 4, we obtain

$$\begin{aligned}
&\frac{1}{4\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{W}(\omega)d\omega}\left[\int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega)\mathcal{W}(\omega)d\omega + \int_{u}^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega)\mathcal{W}(\omega)d\omega\right] \\
&+ \frac{1}{4\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \Omega(x)dx}\left[\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u)\Omega(x)dx + \int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu)\Omega(x)dx\right] \\
&\leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,u)+\mathcal{T}(\varsigma,\nu)}{4}
\end{aligned} \tag{59}$$

Combining inequalities (51), (54), and (59), we obtain

$$
\begin{aligned}
&\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \\
&\leq_p \frac{1}{2}\left[\begin{array}{c} \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\Omega(x)dx \\ +\frac{1}{\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \omega\right)\mathcal{W}(\omega)d\omega \end{array}\right] \\
&\leq_p \frac{1}{\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\
&\leq_p \frac{1}{4\int_u^{u+\varphi_2(\nu,u)}\mathcal{W}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\mathcal{W}(\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\mathcal{W}(\omega)d\omega\right] \\
&+\frac{1}{4\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\Omega(x)dx}\left[\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\Omega(x)dx + \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\Omega(x)dx\right] \\
&\leq_p \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\rho,\nu)}{2} + \frac{\mathcal{T}(\varsigma,u)+\mathcal{T}(\varsigma,\nu)}{2} + \frac{\mathcal{T}(\rho,u)+\mathcal{T}(\varsigma,u)}{2} + \frac{\mathcal{T}(\rho,\nu)+\mathcal{T}(\varsigma,\nu)}{2}
\end{aligned}
$$

Hence, this concludes the proof. □

Remark 2. *If one takes $\mathcal{W}(\omega) = 1 = \Omega(x)$, then from (46) we achieve (23).*
If $\mathcal{T}_*(x,\omega) \neq \mathcal{T}^*(x,\omega)$ with $\varphi_1(\varsigma,\rho) = \varsigma - \rho$ and $\varphi_2(\nu,u) = \nu - u$, then from (46) we acquire the following inequality (see [64]):

$$
\begin{aligned}
\mathcal{T}\left(\frac{c+d}{2}, \frac{\mu+x}{2}\right) &\leq_p \frac{1}{2}\left[\frac{1}{\int_c^d\Omega(x)dx}\int_c^d\mathcal{T}\left(x, \frac{\mu+x}{2}\right)\Omega(x)dx + \frac{1}{\int_\mu^x\mathcal{W}(\omega)d\omega}\int_\mu^x\mathcal{T}\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\
&\leq_p \frac{1}{\int_c^d\Omega(x)dx\int_\mu^x\mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^x\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\
&\leq_p \frac{1}{4\int_\mu^x\mathcal{W}(\omega)d\omega}\left[\int_\mu^x\mathcal{T}(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^x\mathcal{T}(d,\omega)\mathcal{W}(\omega)d\omega\right] \\
&+\frac{1}{4\int_c^d\Omega(x)dx}\left[\int_c^d\mathcal{T}(x,\mu)\Omega(x)dx + \int_c^d\mathcal{T}(x,x)\Omega(x)dx\right] \\
&\leq_p \frac{\mathcal{T}(c,\mu)+\mathcal{T}(c,x)}{2} + \frac{\mathcal{T}(d,\mu)+\mathcal{T}(d,x)}{2} + \frac{\mathcal{T}(c,\mu)+\mathcal{T}(d,\mu)}{2} + \frac{\mathcal{T}(c,x)+\mathcal{T}(d,x)}{2}.
\end{aligned} \tag{60}
$$

If $\mathcal{T}_*(x,\omega) = \mathcal{T}^*(x,\omega)$, then from (46) we acquire the following inequality (see [62]):

$$
\begin{aligned}
\mathcal{T}\left(\frac{c+d}{2}, \frac{\mu+x}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\int_c^d\Omega(x)dx}\int_c^d\mathcal{T}\left(x, \frac{\mu+x}{2}\right)\Omega(x)dx + \frac{1}{\int_\mu^x\mathcal{W}(\omega)d\omega}\int_\mu^x\mathcal{T}\left(\frac{c+d}{2}, \omega\right)\mathcal{W}(\omega)d\omega\right] \\
&\leq \frac{1}{\int_c^d\Omega(x)dx\int_\mu^x\mathcal{W}(\omega)d\omega}\int_c^d\int_\mu^x\mathcal{T}(x,\omega)\Omega(x)\mathcal{W}(\omega)d\omega dx \\
&\leq \frac{1}{4\int_\mu^x\mathcal{W}(\omega)d\omega}\left[\int_\mu^x\mathcal{T}(c,\omega)\mathcal{W}(\omega)d\omega + \int_\mu^x\mathcal{T}(d,\omega)\mathcal{W}(\omega)d\omega\right] \\
&+\frac{1}{4\int_c^d\Omega(x)dx}\left[\int_c^d\mathcal{T}(x,\mu)\Omega(x)dx + \int_c^d\mathcal{T}(x,x)\Omega(x)dx\right] \\
&\leq \frac{\mathcal{T}(c,\mu)+\mathcal{T}(c,x)}{2} + \frac{\mathcal{T}(d,\mu)+\mathcal{T}(d,x)}{2} + \frac{\mathcal{T}(c,\mu)+\mathcal{T}(d,\mu)}{2} + \frac{\mathcal{T}(c,x)+\mathcal{T}(d,x)}{2}
\end{aligned} \tag{61}
$$

We now obtain some HH inequalities for the product of coordinated LR-preinvex I.V.Fs. These inequalities are refinements of some known inequalities (see [62,64]).

Theorem 9. *Let $\mathcal{T}, \varkappa : \Delta = [\rho, \rho+\varphi_1(\varsigma,\rho)] \times [u, u+\varphi_2(\nu,u)] \subset \mathbb{R}^2 \to \mathbb{R}_I^+$ be two coordinated LR-preinvex IVFs on Δ such that $\mathcal{T}(x,\omega) = [\mathcal{T}_*(x,\omega), \mathcal{T}^*(x,\omega)]$ and $\varkappa(x,\omega) = [\varkappa_*(x,\omega), \varkappa^*(x,\omega)]$ for all $(x,\omega) \in \Delta$, and let condition C hold for φ_1 and φ_2. Then, the following inequality holds:*

$$\begin{aligned}&\frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\&\leq_p \frac{1}{9}P(\rho,\varsigma,u,\ \nu)+\frac{1}{18}\mathcal{M}(\rho,\varsigma,u,\ \nu)+\frac{1}{36}\mathcal{N}(\rho,\varsigma,u,\ \nu).\end{aligned} \tag{62}$$

where

$$\begin{aligned}&P(\rho,\varsigma,u,\ \nu)=\mathcal{T}(\rho,u)\times\varkappa(\rho,u)+\mathcal{T}(\rho,\nu)\widetilde{\times}\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,u)\\&\qquad+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,\nu),\\&\mathcal{M}(\rho,\varsigma,u,\ \nu)=\mathcal{T}(\rho,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\rho,\nu)\times\varkappa(\rho,u)+\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,\nu)\\&\qquad+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,u),\\&+\mathcal{T}(\rho,u)\times\varkappa(\varsigma,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,u)\times\varkappa(\rho,u)+\mathcal{T}(\rho,\nu)\times\varkappa(\varsigma,\nu),\\&\mathcal{N}(\rho,\varsigma,u,\ \nu)=\mathcal{T}(\rho,u)\times\varkappa(\varsigma,\nu)+\mathcal{T}(\varsigma,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,u)+\\&\mathcal{T}(\varsigma,u)\times\varkappa(\rho,\nu)\ .\end{aligned}$$

and $P(\rho,\varsigma,u,\ \nu)$, $\mathcal{M}(\rho,\varsigma,u,\ \nu)$, *and* $\mathcal{N}(\rho,\varsigma,u,\ \nu)$ *are defined as follows:*

$$\begin{aligned}&P(\rho,\varsigma,u,\ \nu)=[P_*(\rho,\varsigma,u,\ \nu),\ P^*(\rho,\varsigma,u,\ \nu)],\\&\mathcal{M}(\rho,\varsigma,u,\ \nu)=[\mathcal{M}_*(\rho,\varsigma,u,\ \nu),\ \mathcal{M}^*(\rho,\varsigma,u,\ \nu)],\\&\mathcal{N}(\rho,\varsigma,u,\ \nu)=[\mathcal{N}_*(\rho,\varsigma,u,\ \nu),\ \mathcal{N}^*(\rho,\varsigma,u,\ \nu)].\end{aligned}$$

Proof. Let $\mathcal{T}$ and $\varkappa$ both be coordinated LR-preinvex I.V.Fs on $[\rho,\ \rho+\varphi_1(\varsigma,\rho)]\times[u,\ u+\varphi_2(\nu,u)]$. Then

$$\begin{aligned}&\mathcal{T}(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u))\\&\quad\leq_p \tau s\mathcal{T}(\rho,u)+\tau(1-s)\mathcal{T}(\rho,\nu)+(1-\tau)s\mathcal{T}(\varsigma,u)+(1-\tau)(1-s)\mathcal{T}(\varsigma,\nu),\end{aligned}$$

and

$$\begin{aligned}&\varkappa(\rho+(1-\tau)\varphi_1(\varsigma,\rho),\ u+(1-s)\varphi_2(\nu,u))\\&\quad\leq_p \tau s\varkappa(\rho,u)+\tau(1-s)\varkappa(\rho,\nu)+(1-\tau)s\varkappa(\varsigma,u)+(1-\tau)(1-s)\varkappa(\varsigma,\nu).\end{aligned}$$

Since $\mathcal{T}$ and $\varkappa$ are both coordinated LR-preinvex I.V.Fs, then by Lemma 1, there exist

$\mathcal{T}_x:[u,\ u+\varphi_2(\nu,u)]\to\mathbb{R}_I^+,\ \mathcal{T}_x(\omega)=\mathcal{T}(x,\omega),\ \varkappa_x:[u,\ u+\varphi_2(\nu,u)]\to\mathbb{R}_I^+,\ \varkappa_x(\omega)=\varkappa(x,\omega),$

and

$$\mathcal{T}_\omega:[\rho,\varsigma]\to\mathbb{R}_I^+,\ \mathcal{T}_\omega(x)=\mathcal{T}(x,\omega),\ \varkappa_\omega:[\rho,\varsigma]\to\mathbb{R}_I^+,\ \varkappa_\omega(x)=\varkappa(x,\omega).$$

Since $\mathcal{T}_x$, $\varkappa_x$, $\mathcal{T}_\omega$, and $\varkappa_\omega$ are I.V.Fs, then by inequality (11), we have

$$\begin{aligned}\frac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}_\omega(x)\times\varkappa_\omega(x)dx&\leq_p\frac{1}{3}[\mathcal{T}_\omega(\rho)\times\varkappa_\omega(\rho)+\mathcal{T}_\omega(\varsigma)\times\varkappa_\omega(\varsigma)]\\&+\frac{1}{6}[\mathcal{T}_\omega(\rho)\times\varkappa_\omega(\varsigma)+\mathcal{T}_\omega(\varsigma)\times\varkappa_\omega(\rho)],\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varphi_2(\nu,u)}\int_{u}^{u+\varphi_2(\nu,u)}\mathcal{T}_x(\omega)\times\varkappa_x(\omega)d\omega&\leq_p\frac{1}{3}[\mathcal{T}_x(u)\times\varkappa_x(u)+\mathcal{T}_x(\nu)\times\varkappa_x(\nu)]\\&+\frac{1}{6}[\mathcal{T}_x(u)\times\varkappa_x(\nu)+\mathcal{T}_x(u)\times\varkappa_x(\nu)].\end{aligned}$$

The above inequalities can be written as

$$\begin{aligned}&\frac{1}{\varphi_1(\varsigma,\rho)}\int_{\rho}^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)dx\\&\leq_p\frac{1}{3}[\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)]+\frac{1}{6}[\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)],\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega\\ &\leq_p \frac{1}{3}[\mathcal{T}(x,u)\times\varkappa(x,u)+\mathcal{T}(x,\nu)\times\varkappa(x,\nu)]\\ &\qquad+\frac{1}{6}[\mathcal{T}(x,u)\times\varkappa(x,u)+\mathcal{T}(x,\nu)\times\varkappa(x,\nu)].\end{aligned} \tag{64}$$

First, we solve inequality (63). By taking the integration on both sides of the inequality with respect to ω over interval $[u,\ u+\varphi_2(\nu,u)]$ and dividing both sides by $\varphi_2(\nu,u)$, we have

$$\begin{aligned}&\frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\ &\leq_p \frac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)]d\omega\\ &+\frac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)]d\omega.\end{aligned} \tag{65}$$

Now, again by inequality (11), we have

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)d\omega\\ &\leq_p \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,u)\times\varkappa(\rho,u)+\mathcal{T}(\rho,\nu)\times\varkappa(\rho,\nu)]d\omega\\ &+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\rho,u)\times\varkappa(\rho,\nu)]d\omega.\end{aligned} \tag{66}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)d\omega\\ &\leq_p \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,\nu)]d\omega\\ &+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,\nu)+\mathcal{T}(\varsigma,u)\times\varkappa(\rho,\nu)]d\omega.\end{aligned} \tag{67}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)d\omega\\ &\leq_p \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,u)\times\varkappa(\varsigma,u)+\mathcal{T}(\rho,\nu)\times\varkappa(\varsigma,\nu)]d\omega\\ &+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,u)\times\varkappa(\varsigma,\nu)+\mathcal{T}(\rho,\nu)\times\varkappa(\varsigma,u)]d\omega.\end{aligned} \tag{68}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)d\omega\\ &\leq_p \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\varsigma,u)\times\varkappa(\rho,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,\nu)]d\omega\\ &+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\varsigma,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,u)]d\omega.\end{aligned} \tag{69}$$

From (64)–(69) and inequality (65), we have

$$\begin{aligned}&\frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\ &\leq_p \frac{1}{9}P(\rho,\varsigma,u,\ \nu)+\frac{1}{18}\mathcal{M}(\rho,\varsigma,u,\ \nu)+\frac{1}{36}\mathcal{N}(\rho,\varsigma,u,\ \nu).\end{aligned}$$

Hence, this concludes the proof of the theorem. □

Theorem 10. *Let $\mathcal{T},\varkappa\ :\Delta=[\rho,\ \rho+\varphi_1(\varsigma,\rho)]\times[u,\ u+\varphi_2(\nu,u)]\subset\mathbb{R}^2\to\mathbb{R}_I^+$ be two coordinated LR-preinvex I.V.Fs such that $\mathcal{T}(x)=[\mathcal{T}_*(x,\omega),\ \mathcal{T}^*(x,\omega)]$ and $\varkappa(x)=[\varkappa_*(x,\omega),\ \varkappa^*(x,\omega)]$ for all $(x,\omega)\in\Delta$, and let condition C hold for φ_1 and φ_2. Then, the following inequality holds:*

$$\begin{aligned}&4\,\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\\ &\leq_p \frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\ &\qquad+\frac{5}{36}P(\rho,\varsigma,u,\ \nu)+\frac{7}{36}\mathcal{M}(\rho,\varsigma,u,\ \nu)+\frac{2}{9}\mathcal{N}(\rho,\varsigma,u,\ \nu).\end{aligned} \tag{70}$$

where $P(\rho,\varsigma,u,\ \nu)$, $\mathcal{M}(\rho,\varsigma,u,\ \nu)$, and $\mathcal{N}(\rho,\varsigma,u,\ \nu)$are given in Theorem 9.

Proof. Since $\mathcal{T},\varkappa\ :\Delta\to\mathbb{R}_I^+$ are two LR-preinvex I.V.Fs, then from inequality (12) we have

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&+\tfrac{1}{6}\left[\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\
&+\tfrac{1}{3}\left[\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right],
\end{aligned}
\tag{71}
$$

and

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\\
&+\tfrac{1}{6}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\right]\\
&+\tfrac{1}{3}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\right].
\end{aligned}
\tag{72}
$$

By summing inequalities (71) and (72) then multiplying the resultant inequality by 2, we obtain

$$
\begin{aligned}
&8\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{2}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&+\tfrac{2}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)d\omega\\
&+\tfrac{1}{6}\left[2\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+2\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\
&+\tfrac{1}{6}\left[2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)+2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\right]\\
&+\tfrac{1}{3}\left[2\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+2\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\
&+\tfrac{1}{3}\left[2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)+2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\right].
\end{aligned}
\tag{73}
$$

Now, with the help of integral inequality (12) for each integral on the right-hand side of (73), we have

$$
\begin{aligned}
&2\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)d\omega+\tfrac{1}{6}[\mathcal{T}(\rho,u)\times\varkappa(\rho,u)+\mathcal{T}(\rho,\nu)\times\varkappa(\rho,\nu)]\\
&\qquad+\tfrac{1}{3}[\mathcal{T}(\rho,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\rho,\nu)\times\varkappa(\rho,u)].
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)d\omega+\tfrac{1}{6}[\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,\nu)]\\
&\qquad+\tfrac{1}{3}[\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,\nu)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,u)].
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)d\omega+\tfrac{1}{6}[\mathcal{T}(\rho,u)\times\varkappa(\varsigma,u)+\mathcal{T}(\rho,\nu)\times\varkappa(\varsigma,\nu)]\\
&\qquad+\tfrac{1}{3}[\mathcal{T}(\rho,u)\times\varkappa(\varsigma,\nu)+\mathcal{T}(\rho,\nu)\times\varkappa(\varsigma,u)].
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\varsigma,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\rho,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)d\omega+\tfrac{1}{6}[\mathcal{T}(\varsigma,u)\times\varkappa(\rho,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,\nu)]\\
&\qquad+\tfrac{1}{3}[\mathcal{T}(\varsigma,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\rho,u)].
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\times\varkappa(x,u)dx+\tfrac{1}{6}[\mathcal{T}(\rho,u)\times\varkappa(\rho,u)+\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,u)]+\\
&\tfrac{1}{3}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\right].
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\times\varkappa(x,\nu)dx+\tfrac{1}{6}[\mathcal{T}(\rho,\nu)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,\nu)]+\\
&\quad\tfrac{1}{3}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\right].
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\times\varkappa(x,\nu)dx+\tfrac{1}{6}[\mathcal{T}(\rho,u)\times\varkappa(\rho,\nu)+\mathcal{T}(\varsigma,u)\times\varkappa(\varsigma,\nu)]\\
&\quad+\tfrac{1}{3}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\right].
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
&2\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\times\varkappa(x,u)dx+\tfrac{1}{6}[\mathcal{T}(\rho,\nu)\times\varkappa(\rho,u)+\mathcal{T}(\varsigma,\nu)\times\varkappa(\varsigma,u)]\\
&+\tfrac{1}{3}\left[\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)+\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\nu\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},u\right)\right]
\end{aligned}
\tag{81}
$$

From (74)–(81), we have

$$
\begin{aligned}
&8\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_p \tfrac{2}{\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&+\tfrac{2}{\varphi_2(\nu,u)}\ \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)dx\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\ \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)d\omega+\tfrac{1}{6\varphi_2(\nu,u)}\ \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)d\omega\\
&+\tfrac{1}{6\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\times\varkappa(x,u)dx+\tfrac{1}{6\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\times\varkappa(x,\nu)dx\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)d\omega+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)d\omega\\
&+\tfrac{1}{3\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,u)\times\varkappa(x,\nu)dx+\tfrac{1}{3\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}(x,\nu)\times\varkappa(x,u)dx,\\
&+\tfrac{1}{18}P(\rho,\varsigma,u,\ \nu)+\tfrac{1}{9}\mathcal{M}(\rho,\varsigma,u,\ \nu)+\tfrac{2}{9}\mathcal{N}(\rho,\varsigma,u,\ \nu).
\end{aligned}
\tag{82}
$$

Now, again with the help of integral inequality (12), for first two integrals on the right-hand side of (82) we have the following relation

$$
\begin{aligned}
&\tfrac{2}{\varphi_1(\varsigma,\rho)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\mathcal{T}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\varkappa\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}[\mathcal{T}(x,u)\times\varkappa(x,u)+\mathcal{T}(x,\nu)\times\varkappa(x,\nu)]dx\\
&+\tfrac{1}{6\varphi_1(\varsigma,\rho)}\int_\rho^{\rho+\varphi_1(\varsigma,\rho)}[\mathcal{T}(u,x)\times\varkappa(x,\nu)+\mathcal{T}(x,\nu)\times\varkappa(x,u)]dx,
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&\tfrac{2}{\varphi_2(\nu,u)}\ \int_u^{u+\varphi_2(\nu,u)}\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)\times\varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2},\omega\right)dx\\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)}\ \int_\rho^{\rho+\varphi_1(\varsigma,\rho)}\int_u^{u+\varphi_2(\nu,u)}\mathcal{T}(x,\omega)\times\varkappa(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,\omega)\times\varkappa(\rho,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\varsigma,\omega)]d\omega\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathcal{T}(\rho,\omega)\times\varkappa(\varsigma,\omega)+\mathcal{T}(\varsigma,\omega)\times\varkappa(\rho,\omega)]d\omega.
\end{aligned}
\tag{84}
$$

From (83) and (84), we have

$$
\begin{aligned}
&8\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \times \varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\
&\leq_p \tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega) \times \varkappa(x,\omega) d\omega dx \\
&+\tfrac{1}{3\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} [\mathcal{T}(x,u) \times \varkappa(x,u) + \mathcal{T}(x,\nu) \times \varkappa(x,\nu)] dx \\
&+\tfrac{1}{6\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} [\mathcal{T}(x,u) \times \varkappa(x,\nu) + \mathcal{T}(x,\nu) \times \varkappa(x,u)] dx \\
&+\tfrac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega) \times \varkappa(x,\omega) d\omega dx \\
&+\tfrac{1}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} [\mathcal{T}(\rho,\omega) \times \varkappa(\rho,\omega) + \mathcal{T}(\varsigma,\omega) \times \varkappa(\varsigma,\omega)] d\omega \\
&+\tfrac{1}{6\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} [\mathcal{T}(\rho,\omega) \times \varkappa(\varsigma,\omega) + \mathcal{T}(\varsigma,\omega) \times \varkappa(\rho,\omega)] d\omega \\
&+\tfrac{1}{6\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega) \times \varkappa(\rho,\omega) d\omega + \tfrac{1}{6\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega) \times \varkappa(\varsigma,\omega) d\omega \\
&+\tfrac{1}{6\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u) \times \varkappa(x,u) dx + \tfrac{1}{6\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu) \times \varkappa(x,\nu) dx \\
&+\tfrac{1}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega) \times \varkappa(\varsigma,\omega) d\omega + \tfrac{1}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega) \times \varkappa(\rho,\omega) d\omega \\
&+\tfrac{1}{3\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u) \times \varkappa(x,\nu) dx + \tfrac{1}{3\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu) \times \varkappa(x,u) dx \\
&+\tfrac{1}{18} P(\rho,\varsigma,u,\ \nu) + \tfrac{1}{9}\mathcal{M}(\rho,\varsigma,u,\ \nu) + \tfrac{2}{9}\mathcal{N}(\rho,\varsigma,u,\ \nu).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&8\mathcal{T}\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \times \varkappa\left(\tfrac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\
&\leq_p \tfrac{2}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega) \times \varkappa(x,\omega) d\omega dx \\
&+\tfrac{2}{3\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} [\mathcal{T}(x,u) \times \varkappa(x,u) + \mathcal{T}(x,\nu) \times \varkappa(x,\nu)] dx \\
&+\tfrac{1}{3\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} [\mathcal{T}(x,u) \times \varkappa(x,\nu) + \mathcal{T}(x,\nu) \times \varkappa(x,u)] dx \\
&+\tfrac{2}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} [\mathcal{T}(\rho,\omega) \times \varkappa(\rho,\omega) + \mathcal{T}(\varsigma,\omega) \times \varkappa(\varsigma,\omega)] d\omega \\
&+\tfrac{1}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} [\mathcal{T}(\rho,\omega) \times \varkappa(\varsigma,\omega) + \mathcal{T}(\varsigma,\omega) \times \varkappa(\rho,\omega)] d\omega \\
&+\tfrac{1}{18} P(\rho,\varsigma,u,\ \nu) + \tfrac{1}{9}\mathcal{M}(\rho,\varsigma,u,\ \nu) + \tfrac{2}{9}\mathcal{N}(\rho,\varsigma,u,\ \nu).
\end{aligned} \tag{85}
$$

Now, using integral inequality (11) for integrals on the right-hand side of (85), we have the following relation

$$
\begin{aligned}
&\tfrac{1}{\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u) \times \varkappa(x,u) dx \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,u) \times \varkappa(\rho,u) + \mathcal{T}(\varsigma,u) \times \varkappa(\varsigma,u)] + \tfrac{1}{6}[\mathcal{T}(\rho,u) \times \varkappa(\varsigma,u) + \mathcal{T}(\varsigma,u) \times \varkappa(\rho,u)]
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu) \times \varkappa(x,\nu) dx \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,\nu) \times \varkappa(\rho,\nu) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\varsigma,\nu)] + \tfrac{1}{6}[\mathcal{T}(\rho,\nu) \times \varkappa(\varsigma,\nu) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\rho,\nu)],
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,u) \times \varkappa(x,\nu) dx \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,u) \times \varkappa(\rho,\nu) + \mathcal{T}(\varsigma,u) \times \varkappa(\varsigma,\nu)] + \tfrac{1}{6}[\mathcal{T}(\rho,u) \times \varkappa(\varsigma,\nu) + \mathcal{T}(\varsigma,u) \times \varkappa(\rho,\nu)],
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_1(\varsigma,\rho)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \mathcal{T}(x,\nu) \times \varkappa(x,u) dx \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,\nu) \times \varkappa(\rho,u) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\varsigma,u)] + \tfrac{1}{6}[\mathcal{T}(\rho,\nu) \times \varkappa(\varsigma,u) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\rho,u)],
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega) \times \varkappa(\rho,\omega) d\omega \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,u) \times \varkappa(\rho,u) + \mathcal{T}(\rho,\nu) \times \varkappa(\rho,\nu)] + \tfrac{1}{6}[\mathcal{T}(\rho,u) \times \varkappa(\rho,\nu) + \mathcal{T}(\rho,\nu) \times \varkappa(\rho,u)],
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega) \times \varkappa(\varsigma,\omega) d\omega \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\varsigma,u) \times \varkappa(\varsigma,u) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\varsigma,\nu)] + \tfrac{1}{6}[\mathcal{T}(\varsigma,u) \times \varkappa(\varsigma,\nu) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\varsigma,u)],
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\rho,\omega) \times \varkappa(\varsigma,\omega) d\omega \\
&\leq_p \tfrac{1}{3}[\mathcal{T}(\rho,u) \times \varkappa(\varsigma,u) + \mathcal{T}(\rho,\nu) \times \varkappa(\varsigma,\nu)] + \tfrac{1}{6}[\mathcal{T}(\rho,u) \times \varkappa(\varsigma,\nu) + \mathcal{T}(\rho,\nu) \times \varkappa(\varsigma,u)],
\end{aligned} \tag{92}
$$

$$\frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(\varsigma,\omega) \times \varkappa(\rho,\omega)d\omega$$
$$\leq_p \frac{1}{3}[\mathcal{T}(\varsigma,u) \times \varkappa(\rho,u) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\rho,\nu)] + \frac{1}{6}[\mathcal{T}(\varsigma,u) \times \varkappa(\rho,\nu) + \mathcal{T}(\varsigma,\nu) \times \varkappa(\rho,u)]. \quad (93)$$

From (82)–(93) and inequality (85), we have

$$4\,\mathcal{T}\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \times \varkappa\left(\frac{2\rho+\varphi_1(\varsigma,\rho)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right)$$
$$\leq_p \frac{1}{\varphi_1(\varsigma,\rho)\varphi_2(\nu,u)} \int_\rho^{\rho+\varphi_1(\varsigma,\rho)} \int_u^{u+\varphi_2(\nu,u)} \mathcal{T}(x,\omega) \times \varkappa(x,\omega)d\omega dx$$
$$+\frac{5}{36}P(\rho,\varsigma,u,\ \nu) + \frac{7}{36}\mathcal{M}(\rho,\varsigma,u,\ \nu) + \frac{2}{9}\mathcal{N}(\rho,\varsigma,u,\ \nu).$$

This concludes the proof. □

4. Conclusions

As an extension of convex interval-valued functions on coordinates, we proposed the idea of interval-valued LR-preinvex functions on coordinates in this paper. For coordinated LR-preinvex interval-valued functions, Hermite–Hadamard-type inclusions have been developed. In addition, the product of two coordinated LR-preinvex interval-valued functions is examined using some novel Hermite–Hadamard-type inclusions. Other types of interval-valued LR-preinvex functions on the coordinates can be used to expand the results gained in this study. In the future, we can use fuzzy interval-valued fractional integrals on coordinates to examine Fejér–Hermite–Hadamard-type inequalities for fuzzy interval-valued coordinated LR-preinvex functions. We hope that the concepts and findings presented in this article will inspire readers to pursue additional research.

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