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Solution of the Ill-Posed Cauchy Problem for Systems of Elliptic Type of the First Order

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Abstract: We study, in this paper, the Cauchy problem for matrix factorizations of the Helmholtz equation in the space $\mathbb{R}^m$. Based on the constructed Carleman matrix, we find an explicit form of the approximate solution of this problem and prove the stability of the solutions.

Keywords: Carleman matrix; stability of the Cauchy problem; regularized solution; approximate solution

MSC: 35J46; 35J56

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1. Introduction

The Cauchy problem is one of the important problems in the theory of differential equations and partial differential equations. There are some questions about Cauchy's problems:

(1) Is there a solution (though only locally)?
(2) If so, what is its domain of definition?
(3) If the solution is unique, is the problem well-posed? That is, does the solution continuously depend on the initial data?

Recently, many authors have become interested in Cauchy problems that are ill-posed because certain physical situations correspond to these problems. Ill-posed problems occur also for example in astronomical observations, computer technology, synthesis of automatic systems, management and planning, optimization of control, geophysics, etc. Hadamard, in 1902, proposed the concept of a well-posed problem. Hadamard expressed the opinion that boundary value problems whose solutions do not satisfy certain continuity conditions are not physically meaningful, and he presented examples of such problems. It was subsequently found that Hadamard's opinion was erroneous. It turned out that many problems of mathematical physics which are ill-posed in the sense of Hadamard and, in particular, problems noted by Hadamard himself have real physical content. It also turned out that ill-posed problems arise in many other areas of mathematics which are connected with applications. The concept of conditional correctness first appeared in the work of Tikhonov [1,2]. The study of uniqueness issues in a conditionally well-posed formulation does not essentially differ from the study in a classically well-posed formulation, and the stability of the solution from the data of the problem is required only from those variations of the data that do not deduce solutions from the well-posedness set. After establishing the uniqueness and stability theorems in the study of the conditional correctness of ill-posed problems, the question arises of constructing effective solution methods, i.e., construction of regularizing operators. Such a mathematical analysis problem as the problem of differentiation is ill-posed if it is connected with processing experimental data (see, for instance [3]).

The most actively developing modern area of scientific knowledge is the theory of correctly and incorrectly posed problems, most of which have practical value and require decision making in uncertain or contradictory conditions. The development and justification of methods for solving such a complex type of problems as ill-posed ones is an important current concern. The ill-posed problems theory is an apparatus of scientific research for many scientific areas, such as differentiation of approximately given functions, solving inverse boundary value problems, solving problems of linear programming and control systems, solving systems of linear equations ill-conditioned or degenerate, etc. Carleman-type formulas allow us to find a solution to an elliptic equation if the Cauchy data are known only on a part of the domain boundary. In [4], Carleman obtained a formula for Cauchy—Riemann equations in some domain. Using his idea, Goluzin and Krylov [5] established a formula for determining the values of analytic functions from data known only on a portion of the boundary, already for arbitrary domains. A formula of the Carleman type, in which the Carleman function is used, has been established by Lavrent'ev [6–8].

We are well aware that the Cauchy problem for elliptic equations, as well as systems of elliptic equations, is ill-posed. You can look extensively in the literature (see, for example, [1–5,9–11]). The Carleman matrix or Carleman function for some elliptic equations and systems was considered in the following studies [12–29].

Based on [6,7,12–14] we have constructed the Carleman matrix and based on it the approximate solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. Boundary value problems, as well as numerical solutions of some problems, are considered in [30–39]. When solving correct problems, sometimes, it is not possible to find the value of the vector function on the entire boundary. Finding it on the whole border for elliptical systems is an important issue (see, for example, [40]).

In the following, we present some notations used in the paper.

We consider $k \in \mathbb{N},\ k \geq 1,\ m = 2k$, and the Euclidean space $\mathbb{R}^m$. Let

$$x = (x_1, \dots, x_m) \in \mathbb{R}^m, \quad y = (y_1, \dots, y_m) \in \mathbb{R}^m,$$

and

$$x' = (x_1, \dots, x_{m-1}) \in \mathbb{R}^{m-1}, \quad y' = (y_1, \dots, y_{m-1}) \in \mathbb{R}^{m-1}.$$

We denote by:

$$r = |y - x|, \quad \alpha = |y' - x'|, \quad z = i\sqrt{a^2 + \alpha^2} + y_m, \quad a \geq 0,$$

$$\partial_x = (\partial_{x_1}, \dots, \partial_{x_m})^T, \quad \partial_x = \zeta^T, \quad \zeta^T = \begin{pmatrix} \zeta_1 \\ \dots \\ \zeta_m \end{pmatrix} \text{-transposed vector} \quad \zeta,$$

$$V(x) = (V_1(x), \dots, V_n(x))^T, \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 2,$$

$$E(u) = \begin{Vmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & u_n \end{Vmatrix} \text{-diagonal matrix}, u = (u_1, \dots, u_n) \in \mathbb{R}^n.$$

We also consider a bounded simply-connected domain $\Omega \subset \mathbb{R}^m$, having a piecewise smooth boundary $\partial\Omega = \Sigma \bigcup T$, where Σ is a smooth surface lying in the half-space $y_m > 0$ and T is the plane $y_m = 0$.

$D(\zeta^T)$ is an $(n \times n)$-dimensional matrix satisfying:

$$D^*(\zeta^T)D(\zeta^T) = E((|\zeta|^2 + \lambda^2)v^0),$$

where $D^*(\zeta^T)$ is the Hermitian conjugate matrix of $D(\zeta^T)$, $\lambda \in \mathbb{R}$, the elements of the matrix $D(\zeta^T)$ consist of a set of linear functions with constant coefficients from the complex plane $\mathbb{C}$.

Let us consider the following first-order systems of linear partial differential equations with constant coefficients

$$D(\partial_x)V(x) = 0, \tag{1}$$

in the domain Ω, where $D(\partial_x)$ is the matrix differential operator of the first order.

In addition, consider the set

$$S(\Omega) = \{V : \overline{\Omega} \longrightarrow \mathbb{R}^n \mid V \text{ is continuous on } \overline{\Omega} = \Omega \cup \partial\Omega \text{ and } V \text{ satisfies the system (1)}\}.$$

2. Statement of the Cauchy Problem

The Cauchy problem for system (1) is formulated as follows:

Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a continuous given function on Σ.

Problem 1. *Suppose $V(y) \in S(\Omega)$ and*

$$V(y)|_\Sigma = f(y), \quad y \in \Sigma. \tag{2}$$

Our purpose is to determine the function $V(y)$ in the domain Ω when its values are known on Σ.

If $V(y) \in S(\Omega)$, then

$$V(x) = \int\limits_{\partial\Omega} L(y, x; \lambda)V(y)ds_y, \quad x \in \Omega, \tag{3}$$

where

$$L(y, x; \lambda) = \Big(E\Big(\varphi_m(\lambda r)v^0\Big)D^*(\partial_x)\Big)D(t^T),$$

$t = (t_1, \ldots, t_m)$—is the unit exterior normal, at $y \in \partial\Omega$, $\varphi_m(\lambda r)$—is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$, that is (see [41]):

$$\varphi_m(\lambda r) = P_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$P_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k, \quad k \geq 1. \tag{4}$$

Let $K(z)$ be an entire function taking real values for real z, $(z = a + ib,\ a, b \in \mathbb{R})$ such that

$$K(a) \neq 0,\ \sup_{b \geq 1}\left|b^p K^{(p)}(z)\right| = B(a, p) < \infty,$$
$$-\infty < a < \infty, \quad p = \overline{0, m}. \tag{5}$$

For $y \neq x$, we define the function

$$\Psi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \mathrm{Im}\left[\frac{K(z)}{z - x_m}\right] \frac{a\, I_0(\lambda a)}{\sqrt{a^2 + \alpha^2}} da, \quad m = 2k, \quad k \geq 1, \tag{6}$$

where $c_m = (-1)^{k-1}(k-1)!(m-2)\omega_m$; $I_0(\lambda a) = J_0(i\lambda a)$ is the Bessel function of the first kind of zero order [9], ω_m is the area of a unit sphere in $\mathbb{R}^m$.

Choosing
$$K(z) = \exp(\sigma z), \quad K(x_m) = \exp(\sigma x_m), \quad \sigma > 0, \tag{7}$$
in (6), we get
$$\Psi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma z)}{z - x_m}\right] \frac{a\, I_0(\lambda a)}{\sqrt{a^2 + \alpha^2}} da. \tag{8}$$

The integral representation (3) holds if we put
$$\Psi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda), \tag{9}$$
instead of $\varphi_m(\lambda r)$, where $g_\sigma(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Hence, (3) can be written as:
$$V(x) = \int_{\partial\Omega} L_\sigma(y, x; \lambda) V(y) ds_y, \quad x \in \Omega, \tag{10}$$
$$L_\sigma(y, x; \lambda) = \Big(E\Big(\Psi_\sigma(y, x; \lambda) v^0\Big) D^*(\partial_x)\Big) D(t^T).$$

3. Regularized Solution of Problem (1) and (2)

Let $K(\lambda, x)$ be a bouded function on compact subsets of Ω.

Theorem 1. *Suppose $V(y) \in S(\Omega)$ satisfies the boundary condition*
$$|V(y)| \le M, \quad y \in T. \tag{11}$$

If
$$V_\sigma(x) = \int_\Sigma L_\sigma(y, x; \lambda) V(y) ds_y, \quad x \in \Omega, \tag{12}$$
then
$$|V(x) - V_\sigma(x)| \le M K(\lambda, x) \sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in \Omega. \tag{13}$$

Proof. From (10) and (12), we have
$$V(x) = \int_\Sigma L_\sigma(y, x; \lambda) V(y) ds_y + \int_T L_\sigma(y, x; \lambda) V(y) ds_y$$
$$= V_\sigma(x) + \int_T L_\sigma(y, x; \lambda) V(y) ds_y, \quad x \in \Omega.$$

Using (11), we obtain
$$\begin{aligned} |V(x) - V_\sigma(x)| &\le \left| \int_T L_\sigma(y, x; \lambda) V(y) ds_y \right| \\ &\le \int_T |L_\sigma(y, x; \lambda)||V(y)| ds_y \le \int_T |L_\sigma(y, x; \lambda)| ds_y, \quad x \in \Omega. \end{aligned} \tag{14}$$

Next, the following integrals are estimated: $\int\limits_T |\Psi_\sigma(y,x;\lambda)|ds_y$, $\int\limits_T \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j}\right|ds_y$, $j=\overline{1,m-1}$ and $\int\limits_T \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y$.

We separate the imaginary part of (8), and we obtain

$$\Psi_\sigma(y,x;\lambda) = \frac{e^{\sigma(y_m-x_m)}}{c_m}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty \frac{\cos\sigma\sqrt{a^2+\alpha^2}}{a^2+r^2}\,aI_0(\lambda a)\,da\right.$$
$$\left.-\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty \frac{(y_m-x_m)\sin\sigma\sqrt{a^2+\alpha^2}}{a^2+r^2}\,\frac{aI_0(\lambda a)}{\sqrt{a^2+\alpha^2}}\,da\right],\quad x_m>0. \tag{15}$$

Using equality (15), as well as inequality

$$I_0(\lambda a)\le\sqrt{\frac{2}{\lambda\pi a}}, \tag{16}$$

we obtain the following estimate:

$$\int\limits_T |\Psi_\sigma(y,x;\lambda)|ds_y \le K(\lambda,x)\sigma^k e^{-\sigma x_m},\quad \sigma>1,\quad x\in\Omega, \tag{17}$$

next, we use the following equality to estimate the second integral

$$\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j} = \frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j-x_j)\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2,\quad j=\overline{1,m-1}. \tag{18}$$

Using this equality, as well as equality (15) and inequality (16), we have:

$$\int\limits_T \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j}\right|ds_y \le K(\lambda,x)\sigma^k e^{-\sigma x_m},\quad \sigma>1,\quad x\in\Omega. \tag{19}$$

Now, we estimate the integral $\int\limits_T \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y$.

Similarly, using equality (15) and inequality (16), we obtain the following estimate:

$$\int\limits_T \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y \le K(\lambda,x)\sigma^k e^{-\sigma x_m},\quad \sigma>1,\quad x\in\Omega, \tag{20}$$

Using the obtained estimates (17), (19) and (20), the inequality (13) holds. □

Corollary 1.
$$\lim_{\sigma\to\infty} V_\sigma(x) = V(x),$$
uniformly, on each compact set from the domain Ω.

Theorem 2. *Suppose* $V(y)\in A(\Omega)$ *satisfies* (11) *and*

$$|V(y)|\le\delta,\quad 0<\delta<e^{-\sigma\bar{y}_m}, \tag{21}$$

on a smooth surface Σ, *where* $\bar{y}_m = \max\limits_{y\in\Sigma} y_m$.

Then

$$|V(x)| \leq MK(\lambda, x)\sigma^k \delta^{\frac{x_m}{y_m}}, \quad \sigma > 1, \quad x \in \Omega. \tag{22}$$

Proof. We will write the integral formula (10) in the following form:

$$V(x) = \int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y + \int_{T} L_\sigma(y, x; \lambda)V(y)ds_y, \quad x \in \Omega.$$

Hence

$$|V(x)| \leq \left|\int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y\right| + \left|\int_{T} L_\sigma(y, x; \lambda)V(y)ds_y\right|, \quad x \in \Omega. \tag{23}$$

Using inequality (21), we estimate the first term of inequality (23).

$$\begin{aligned} \left|\int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y\right| &\leq \int_{\Sigma} |L_\sigma(y, x; \lambda)||V(y)|ds_y \\ &\leq \delta \int_{\Sigma} |L_\sigma(y, x; \lambda)|ds_y, \quad x \in \Omega. \end{aligned} \tag{24}$$

Next, the following integrals $\int_{\Sigma} |\Psi_\sigma(y, x; \lambda)|ds_y$, $\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_j}\right|ds_y$, $j = \overline{1, m-1}$ and $\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_m}\right|ds_y$ will be estimated.

Using (15) and (16), we obtain:

$$\int_{\Sigma} |\Psi_\sigma(y, x; \lambda)|ds_y \leq K(\lambda, x)\sigma^k e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{25}$$

Further, using (15) and (18), as well as (16), we have:

$$\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_j}\right|ds_y \leq K(\lambda, x)\sigma^k e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in \Omega, \tag{26}$$

Finally, to estimate the integral $\int_{T} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_m}\right|ds_y$, we similarly use (15) and (16), and we obtain:

$$\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_m}\right|ds_y \leq K(\lambda, x)\sigma^k e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{27}$$

From (25)–(27) follows

$$\left|\int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y\right| \leq K(\lambda, x)\sigma^k \delta\, e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{28}$$

For the second integral of the inequality (23), we know:

$$\left|\int_{T} L_\sigma(y, x; \lambda)V(y)ds_y\right| \leq MK(\lambda, x)\sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in \Omega. \tag{29}$$

From (23), (28), and (29), we finally obtain:

$$|V(x)| \leq \frac{K(\lambda,x)\sigma^k}{2}(\delta e^{\sigma\bar{y}_m} + M)e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in \Omega. \tag{30}$$

Considering

$$\sigma = \frac{1}{\bar{y}_m} \ln \frac{M}{\delta}, \tag{31}$$

(22) holds. □

Suppose $V(y) \in S(\Omega)$ is defined on Σ and $f_\delta(y)$ is its approximation with an error $0 < \delta < e^{-\sigma\bar{y}_m}$. Then

$$\max_{\Sigma}|V(y) - f_\delta(y)| \leq \delta. \tag{32}$$

We put

$$V_{\sigma(\delta)}(x) = \int_{\Sigma} L_\sigma(y, x; \lambda) f_\delta(y) ds_y, \quad x \in \Omega. \tag{33}$$

We now establish the following theorem.

Theorem 3. *If $V(y) \in S(\Omega)$ satisfy (11) on the plane $y_m = 0$, then*

$$\left|V(x) - V_{\sigma(\delta)}(x)\right| \leq MK(\lambda, x)\sigma^k \delta^{\frac{x_m}{\bar{y}_m}}, \quad \sigma > 1, \quad x \in \Omega. \tag{34}$$

Proof. Using integral representations (10) and (33), we obtain the following equality

$$V(x) - V_{\sigma(\delta)}(x) = \int_{\partial\Omega} L_\sigma(y, x; \lambda) V(y) ds_y - \int_{\Sigma} L_\sigma(y, x; \lambda) f_\delta(y) ds_y$$

$$= \int_{\Sigma} L_\sigma(y, x; \lambda) V(y) ds_y + \int_{T} L_\sigma(y, x; \lambda) V(y) ds_y - \int_{\Sigma} L_\sigma(y, x; \lambda) f_\delta(y) ds_y$$

$$= \int_{\Sigma} L_\sigma(y, x; \lambda)\{V(y) - f_\delta(y)\} ds_y + \int_{T} L_\sigma(y, x; \lambda) V(y) ds_y.$$

Using (11) and (32), we get

$$\left|V(x) - V_{\sigma(\delta)}(x)\right| = \left|\int_{\Sigma} L_\sigma(y, x; \lambda)\{V(y) - f_\delta(y)\} ds_y\right| +$$

$$\left|\int_{T} L_\sigma(y, x; \lambda) V(y) ds_y\right| \leq \int_{\Sigma} |L_\sigma(y, x; \lambda)||\{V(y) - f_\delta(y)\}| ds_y +$$

$$\int_{T} |L_\sigma(y, x; \lambda)||V(y)| ds_y \leq \delta \int_{\Sigma} |L_\sigma(y, x; \lambda)| ds_y + \int_{T} |L_\sigma(y, x; \lambda)| ds_y.$$

Here, we similarly repeat the proof of Theorems 1 and 2, and we obtain the following estimate:

$$\left|V(x) - V_{\sigma(\delta)}(x)\right| \leq \frac{K(\lambda, x)\sigma^k}{2}(\delta e^{\sigma\bar{y}_m} + 1)e^{-\sigma x_m}.$$

Similarly, choosing σ in the form (31), we prove the validity of (34). □

Corollary 2.
$$\lim_{\delta\to 0} V_{\sigma(\delta)}(x) = V(x),$$
uniformly, on every compact set from the domain Ω.

The following example illustrates the possibility of incorrect formulation of the classical Cauchy problem for system (1).

Example 1. *Prove that the Cauchy problem for the following systems of linear partial differential equations is ill-posed.*
$$\begin{cases} \partial_{x_1}u_1 - \partial_{x_2}u_2 = 0, \\ \partial_{x_2}u_1 + \partial_{x_1}u_2 = 0, \\ -\partial_{x_1}u_3 + \partial_{x_2}u_4 = 0, \\ \partial_{x_2}u_3 + \partial_{x_1}u_4 = 0. \end{cases}$$
Solutions to this system will be sought in the form
$$u_1 = V_1 e^{i(\lambda x_1 + \mu x_2)}, \quad u_2 = V_2 e^{i(\lambda x_1 + \mu x_2)},$$
$$u_3 = V_3 e^{i(\lambda x_1 + \mu x_2)}, \quad u_4 = V_4 e^{i(\lambda x_1 + \mu x_2)}.$$
Substituting these notations into the system, we obtain
$$\lambda^2 + \mu^2 = 0, \quad V_1 = \frac{\lambda}{\mu} V_2,$$
$$\lambda^2 + \mu^2 = 0, \quad V_3 = \frac{\lambda}{\mu} V_4.$$
We choose the following $\mu = n,\ \lambda = -in$. *Then*
$$u_{1n} = V_{1n} e^{nx_1 - inx_2}, \quad u_{2n} = -iV_{1n} e^{nx_1 - inx_2},$$
$$u_{3n} = V_{3n} e^{i(\lambda x_1 + \mu x_2)}, \quad u_{4n} = -iV_{3n} e^{nx_1 - inx_2}.$$
Separating the real part, we find solutions
$$u_{1n} = V_{1n} e^{nx_1} \cos nx_2, \quad u_{2n} = V_{1n} e^{nx_1} \sin nx_2,$$
$$u_{3n} = V_{3n} e^{nx_1} \cos nx_2, \quad u_{4n} = V_{3n} e^{nx_1} \sin nx_2.$$
The constants V_{1n} *and* V_{3n} *are given by the formula* $V_{1n} = V_{3n} = e^{-\sqrt{n}}$.
Solution sequence example
$$u_{1n} = e^{-\sqrt{n}} e^{nx_1} \cos nx_2, \quad u_{2n} = e^{-\sqrt{n}} e^{nx_1} \sin nx_2,$$
$$u_{3n} = e^{-\sqrt{n}} e^{nx_1} \cos nx_2, \quad u_{4n} = e^{-\sqrt{n}} e^{nx_1} \sin nx_2.$$
The point is that the solution (u_{1n}, u_{2n}), (u_{3n}, u_{4n}) *satisfies at* $x_1 = 0$ *the following initial data:*
$$u_{1n}(0, x_2) = \varphi_{1n}(x) = e^{-\sqrt{n}} \cos nx_2, \quad u_{2n}(0, x_2) = \varphi_{2n}(x) = e^{-\sqrt{n}} \sin nx_2,$$
$$u_{3n}(0, x_2) = \varphi_{3n}(x) = e^{-\sqrt{n}} \cos nx_2, \quad u_{4n}(0, x_2) = \varphi_{4n}(x) = e^{-\sqrt{n}} \sin nx_2.$$

At $n \to \infty$, these initial data tend to zero. Moreover, their derivatives $\varphi_{1n}^{(k)}(x)$, $\varphi_{2n}^{(k)}(x)$, $\varphi_{3n}^{(k)}(x)$, $\varphi_{4n}^{(k)}(x)$ of orders of $k = 1, 2, ..., p$ tend to zero as $n \to \infty$. (Here, $p-$ is an arbitrary fixed natural number.) Indeed,

$$\left.\begin{array}{l} \varphi_{1n}(x) = \pm n^k e^{-\sqrt{n}} \cos nx_2 \\ \varphi_{2n}(x) = \pm n^k e^{-\sqrt{n}} \sin nx_2 \end{array}\right\}, \text{ if } k- \text{ is even,}$$

$$\left.\begin{array}{l} \varphi_{1n}(x) = \pm n^k e^{-\sqrt{n}} \sin nx_2 \\ \varphi_{2n}(x) = \pm n^k e^{-\sqrt{n}} \cos nx_2 \end{array}\right\}, \text{ if } k- \text{ is odd,}$$

$$\left.\begin{array}{l} \varphi_{3n}(x) = \pm n^k e^{-\sqrt{n}} \cos nx_2 \\ \varphi_{4n}(x) = \pm n^k e^{-\sqrt{n}} \sin nx_2 \end{array}\right\}, \text{ if } k- \text{ is even,}$$

$$\left.\begin{array}{l} \varphi_{3n}(x) = \pm n^k e^{-\sqrt{n}} \sin nx_2 \\ \varphi_{4n}(x) = \pm n^k e^{-\sqrt{n}} \cos nx_2 \end{array}\right\}, \text{ if } k- \text{ is odd.}$$

On the other hand, $u_{1n}(x_1, x_2)$, $u_{2n}(x_1, x_2)$, $u_{3n}(x_1, x_2)$, $u_{4n}(x_1, x_2)$ is unbounded for any x_1.

We see that no matter what norm we choose to estimate the value of the initial data, we will not be able to assert that the smallness of this norm implies the smallness of the solution (the solution is estimated here by the maximum of its modulus). As admissible norms for the initial data, we here admit the following norms:

$$\|\varphi_1(x)\|_p = \max_{0 \le k \le p} \sup_{x_2} \left|\varphi_1^{(k)}(x)\right|,$$

$$\|\varphi_2(x)\|_p = \max_{0 \le k \le p} \sup_{x_2} \left|\varphi_2^{(k)}(x)\right|,$$

$$\|\varphi_3(x)\|_p = \max_{0 \le k \le p} \sup_{x_2} \left|\varphi_3^{(k)}(x)\right|,$$

$$\|\varphi_4(x)\|_p = \max_{0 \le k \le p} \sup_{x_2} \left|\varphi_4^{(k)}(x)\right|.$$

That is, there is no continuous dependence on the initial data and, therefore, the problem is set incorrectly. Thus, this problem does not have stability properties and, therefore, is ill-posed. We have seen that the solution of the Cauchy problem for this system is unstable. If we narrow the class of solutions under consideration to a compact set, then the problem becomes conditionally well-posed. To estimate the conditional stability, we can apply the results of the above theorems.

4. Conclusions

We have determined, in this paper, a regularized solution to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in $\mathbb{R}^m$. We supposed that there exists a continuously differentiable solution in a closed domain. We have obtained a formula for the continuation of the solution, and we have determined a regularization formula for the case when, instead of the Cauchy data, their continuous approximations are given, with a given error.

Based on the constructed Carleman matrix, we find an explicit form of approximate solution of this problem and prove the stability of the solutions. To obtain the approximate solution, it is necessary to build a family of fundamental solutions of the Helmholtz operator, which are parameterized by some entire function $K(z)$, depending on the dimension of the space. It is necessary to choose the function $K(z)$ in such a way as to ensure convergence. Based on the results from [24,26–29,40], we obtain better results due to the special choice of the function $K(z)$. Hence, we have established a family $V_{\sigma(\delta)}(x) = V(x, f_\delta)$ of vector functions, named a regularized solution, such that for certain $\sigma = \sigma(\delta)$, $\delta \to 0$, conveniently chosen, it converges to a solution $V(x)$ at $x \in \Omega$.

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