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Local Discontinuous Galerkin Method Coupled with Nonuniform Time Discretizations for Solving the Time-Fractional Allen-Cahn Equation

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Abstract: This paper aims to numerically study the time-fractional Allen-Cahn equation, where the time-fractional derivative is in the sense of Caputo with order $\alpha \in (0,1)$. Considering the weak singularity of the solution $u(\mathbf{x},t)$ at the starting time, i.e., its first and/or second derivatives with respect to time blowing-up as $t \to 0^+$ albeit the function itself being right continuous at $t = 0$, two well-known difference formulas, including the nonuniform L1 formula and the nonuniform L2-1_σ formula, which are used to approximate the Caputo time-fractional derivative, respectively, and the local discontinuous Galerkin (LDG) method is applied to discretize the spatial derivative. With the help of discrete fractional Gronwall-type inequalities, the stability and optimal error estimates of the fully discrete numerical schemes are demonstrated. Numerical experiments are presented to validate the theoretical results.

Keywords: time-fractional Allen-Cahn equation; nonuniform time meshes; local discontinuous galerkin method; stability and convergence

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1. Introduction

The classical Allen-Cahn equation, originally proposed by Allen and Cahn [1] to describe the motion of antiphase boundaries in crystalline solids, has subsequently been used in a wide variety of problems such as vesicle membranes, nucleation of solids, and a mixture of two incompressible fluids [2]. It has become a fundamental model equation for diffusion interface methods in materials science to study phase transitions and interface dynamics [3]. Since the Allen-Cahn equation is a nonlinear equation and it is not easy to obtain its analytical solution, various numerical methods have been proposed to solve it, for example, finite difference methods [4], finite element methods [5], local discontinuous Galerkin (LDG) methods [6], and so on. Most of these studies focused on integer-order phase-field models, implicitly assuming that the motion of the underlying particles is normal diffusion and that the spatial interactions between them are local. However, in the original formulation of the physical model [7], nonlocal interactions were part of the phase-field model, and thus in the following decades, the phase-field model was approximated by the local model by assuming slow spatial variations. Meanwhile, it has been reported that the presence of nonlocal operators in time [8] or space [9] in the phase-field model may significantly change the diffusion dynamics.

In this paper, we consider the LDG method for the following time-fractional Allen-Cahn equation

$$\begin{cases} {}_C\mathrm{D}^{\alpha}_{0,t}u - \varepsilon^2\Delta u = -F'(u) =: f(u), \ \mathbf{x} \in \Omega, \ 0 < t \le T, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \ \mathbf{x} \in \overline{\Omega} \\ u(\mathbf{x},t) = 0, \ \mathbf{x} \in \partial\Omega, \ 0 < t \le T, \end{cases} \tag{1}$$

where ε is an interface width parameter and $\Omega = (-1,1)^d$ is a bounded domain of $\mathbb{R}^d$ with $d = 1,2$. The operator ${}_C\mathrm{D}_{0,t}^{\alpha}$ denotes the Caputo-type fractional derivative of order $\alpha \in (0,1)$ in time, which is a typical example of nonlocal operators and defined as [10]

$$ {}_C\mathrm{D}_{0,t}^{\alpha} u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s} \mathrm{d}s. \tag{2} $$

The nonlinear term $F(u)$ is the interficial (or potential) energy. To facilitate the mathematical and numerical analysis of phase-field model, the following Ginzburg-Landau double-well potential has often been used [11,12]

$$ F(u) = \frac{1}{4}(1-u^2)^2. $$

This is a relatively simple phenomenological double-well potential that is commonly used in physical and geometrical applications. It was first shown in [13] that the time-fractional Allen-Cahn equation satisfies the following energy law

$$ E(u(t)) \le E(u_0), $$

where $E(u(t))$ is the total energy defined by

$$ E(u) := \int_{\Omega} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}\mathbf{x}. $$

For the time-fractional Allen-Cahn Equation (1), several numerical studies have been done. In [8], Liu et al. proposed an efficient finite-difference scheme and a Fourier spectral scheme for the time-fractional Allen-Cahn and Cahn-Hilliard phase-field equations, but there was no stability analysis or error estimate in this paper. In [13], Tang et al. proposed a class of finite difference schemes for the time-fractional phase-field equation. They also proved for the first time that the fractional phase-field model does admit an integral-type energy dissipation law. In [14], Liu et al. considered a fast algorithm based on a two-mesh finite element format for numerically solving the nonlinear spatial-fractional Allen-Cahn equation with smooth and nonsmooth solutions. In [11], Du et al. first studied the well-posedness and regularity of the time-fractional Allen-Cahn equation, and then developed several unconditionally solvable and stable numerical schemes to solve it. In [15], Huang and Stynes presented a numerical scheme to solve the time-fractional Allen-Cahn equation, which is based on the Galerkin finite element method in space and the nonuniform L1 formula in time. In [16], Hou et al. constructed a first-order scheme and a $(2-\alpha)$th-order scheme for the time-fractional Allen-Cahn equation. In [17], Jiang et al. considered the Legendre spectral method for the time-fractional Allen-Cahn equation. In a series of works [18–20], Liao et al. proposed several efficient finite difference schemes to solve the time-fractional phase-field type models.

The LDG method is a special class of discontinuous Galerkin (DG) methods, introduced first by Cockburn and Shu [21]. This type of method not only inherits the advantages of DG methods, but it can easily handle meshes with hanging nodes, cells of general shape, and different types of local spaces, so it is flexible for hp-adaptivity [22,23]. In addition, the LDG scheme is locally solvable, i.e., the auxiliary variables of the derivatives of the approximate solution can be eliminated locally. Therefore, we would like to extend the LDG method to the numerical calculation of the time-fractional Allen-Cahn Equation (1) and further enrich the numerical methods for solving such an equation. Specifically, we construct two fully discrete numerical schemes for problem (1). For the first scheme, we utilize the nonuniform L1 formula to compute the time-fractional derivative and apply the LDG method to approximate the spatial derivative. With the aid of the discrete fractional Gronwall inequality, we show that the constructed scheme is numerically stable and the optimal error estimate is proved detailedly (i.e., $(2-\alpha)$th-order accurate in time and

$(k+1)$th-order accurate in space when piecewise polynomials of up to k are used). If the solution of Equation (1) has better regularity in the time direction, we approximate the time-fractional derivative by the nonuniform L2-1_σ formula and still use the LDG method to approach the spatial derivative. The stability and convergence analysis of the scheme are also carefully investigated, and it is proved that this scheme can achieve second-order accuracy in the time direction.

The rest of the paper is organized as follows. In Section 2, we will introduce some necessary notations, projections, and corresponding interpolation properties. In Sections 3 and 4, we consider the LDG method for the time-fractional Allen-Cahn Equation (1). The stability and optimal convergence results are obtained. In Section 5, we perform some numerical experiments to verify the theoretical statements. A brief concluding remark is given in Section 6.

2. Preliminaries

Let us start by presenting some notations for the mesh, function space, and norm. We also present some projections and certain corresponding interpolation properties for the finite element spaces which will be used for the convergence analysis.

2.1. Finite Element Space and Notations

Let $\mathcal{T}_h$ be a shape-regular subdivision of Ω with elements K, Γ denotes the union of the boundary of elements $K \in \mathcal{T}_h$, i.e., $\Gamma = \cup_{K\in\mathcal{T}_h}\partial K$. Let e be a face shared by the "left" and "right" elements K_L and K_R. Define the normal vectors $\boldsymbol{\nu}_L$ and $\boldsymbol{\nu}_R$ on e pointing exterior to K_L and K_R, respectively. If φ is a function on K_L and K_R, but possibly discontinuous across e, let φ_L denote $(\varphi|_{K_L})|_e$ and φ_R denote $(\varphi|_{K_R})|_e$, the left and right trace, respectively. The associated finite element space is defined as

$$V_h = \Big\{v \in L^2(\Omega) : v|_K \in \mathcal{Q}^k(K),\ \forall K \in \mathcal{T}_h\Big\},$$
$$\Sigma_h = \Big\{\mathbf{q} = (q_1, \cdots, q_d)^T|_K \in (L^2(\Omega))^d : q_l|_K \in \mathcal{Q}^k(K),\ l = 1, \cdots, d,\ \forall K \in \mathcal{T}_h\Big\},$$

where $\mathcal{Q}^k(K)$ denotes the space of polynomials of degrees at most $k \geq 0$ defined on K. In particular, for one-dimensional case, we have $\mathcal{Q}^k(K) = \mathcal{P}^k(K)$.

We define the inner product over the element K by

$$(u, v)_K = \int_K uv\mathrm{d}K,\ \langle u, v\rangle_{\partial K} = \int_{\partial K} uv\mathrm{d}s,$$
$$(\mathbf{p}, \mathbf{q})_K = \int_K \mathbf{p}\cdot\mathbf{q}\mathrm{d}K,\ \langle \mathbf{p}, \mathbf{q}\rangle_{\partial K} = \int_{\partial K} \mathbf{p}\cdot\mathbf{q}\mathrm{d}s,$$

for scalar variables u, v and vector variables $\mathbf{p}$, $\mathbf{q}$ respectively. The inner products on Ω are defined as

$$(u, v)_\Omega = \sum_K (u, v)_K,\ (\mathbf{p}, \mathbf{q})_\Omega = \sum_K (\mathbf{p}, \mathbf{q})_K.$$

Furthermore, the L^2 norm on the domain Ω and the boundary Γ are given by

$$\|u\|^2_\Omega = (u, u)_\Omega,\ \|u\|^2_\Gamma = \langle u, u\rangle_\Gamma,$$
$$\|\mathbf{p}\|^2_\Omega = (\mathbf{p}, \mathbf{p})_\Omega,\ \|\mathbf{p}\|^2_\Gamma = \langle \mathbf{p}, \mathbf{p}\rangle_\Gamma.$$

For any nonnegative integer m, $H^m(\Omega)$ denotes the standard Sobolev space with its associated norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$.

2.2. Projections and Interpolation Properties

In this subsection, we follow [24] to define the projections in one- and two-dimensional space, respectively.

One-dimensional case. Assume that the mesh consisting of cells $K_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, for $1 \le j \le N$, where $-1 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1$, covers $\overline{\Omega}$ =$[-1,1]$. Denote $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and $h = \max_{1 \le j \le N} h_j$. We assume $\mathcal{T}_h$ is quasi-uniform mesh in this case; namely, there exists a fixed positive constant ν independent of h such that $\nu h \le h_j \le h$ for $j = 1, \ldots, N$, as h goes to zero. We introduce the standard L^2 projection of a function $u \in L^2(\Omega)$ into the finite element space V_h, denoted by $\mathscr{P}_h u$, which is a unique function in V_h satisfying

$$\int_{K_j} (\mathscr{P}_h u - u) v_h \mathrm{d}x = 0, \ \forall v_h \in \mathcal{P}^k(K_j), \ j = 1, \ldots, N. \tag{3}$$

For any given function $u \in H^1(\Omega)$ and an arbitrary element K_j, the special Gauss-Radau projection of u, denoted by $\mathscr{P}_h^{\pm} u$, is the unique function in V_h satisfying, for each j,

$$\int_{K_j} (\mathscr{P}_h^{+} u - u) v_h \mathrm{d}x = 0, \ \forall v_h \in \mathcal{P}^{k-1}(K_j), \ (\mathscr{P}_h^{+} u)^{+}_{j-\frac{1}{2}} = u(x^{+}_{j-\frac{1}{2}}), \tag{4}$$

$$\int_{K_j} (\mathscr{P}_h^{-} u - u) v_h \mathrm{d}x = 0, \ \forall v_h \in \mathcal{P}^{k-1}(K_j), \ (\mathscr{P}_h^{-} u)^{-}_{j+\frac{1}{2}} = u(x^{-}_{j+\frac{1}{2}}). \tag{5}$$

Two-dimensional case. Let $\mathcal{T}_h = \{K_{ij}\}_{i=1,\ldots,N_x}^{j=1,\ldots,N_y}$ denote a subdivision of $\Omega = (-1,1)^2$ with rectangular element $K_{ij} = I_i \times J_j$, where $I_i = (x_{i-1/2} - x_{i+1/2})$ and $J_j = (y_{j-1/2}, y_{j+1/2})$, with the length $h_i^x = x_{i+1/2} - x_{i-1/2}$ and width $h_j^y = y_{j+1/2} - y_{j-1/2}$. Let $h_{ij} = \max\{h_i^x, h_j^y\}$ and denote $h = \max_{K_{ij} \in \mathcal{T}_h} h_{ij}$. We also assume $\mathcal{T}_h$ is quasi-uniform in this case; namely, there exists a fixed positive constant ν independent of h such that $\nu h \le \min\{h_i^x, h_j^y\} \le h$ for $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. Similar to the one-dimensional case, we need to introduce a suitable projection $\mathscr{P}_h^{\pm}$. The projection for the scalar function is defined as

$$\mathscr{P}_h^{-} = \mathscr{P}_{h,x}^{-} \times \mathscr{P}_{h,y}^{-}, \tag{6}$$

where the subscripts x and y indicate that the one-dimensional projection $\mathscr{P}_h^{-}$ defined by (5) is applied with respect to the corresponding variable.

Let $\mathscr{P}_{h,x}$ and $\mathscr{P}_{h,y}$ be the standard L^2 projections in the x and y directions, respectively. The projection Π_h^{+} for vector-valued function $\mathbf{q} = (q_1(x,y), q_2(x,y)) \in [H^1(\Omega)]^2$ is defined by

$$\Pi_h^{+} \mathbf{q} = \left(\mathscr{P}_{h,x}^{+} \times \mathscr{P}_{h,y} \right) : \ [H^1(\Omega)]^2 \to [\mathcal{Q}^k(I_i \times J_j)]^2,$$

which satisfies

$$\begin{aligned}
&\int_{I_i} \int_{J_j} (\Pi_h^{+} \mathbf{q} - \mathbf{q}) \cdot \nabla w \mathrm{d}x \mathrm{d}y, \ \forall w \in \mathcal{Q}^k(I_i \times J_j), \\
&\int_{J_j} (\Pi_h^{+} \mathbf{q}(x_{i-1/2}, y) - \mathbf{q}(x_{i-1/2}, y)) \cdot \mathbf{n} w(x^{+}_{i-1/2}, y) \mathrm{d}y = 0, \ \forall w \in \mathcal{Q}^k(I_i \times J_j), \\
&\int_{I_i} \left(\Pi_h^{+} \mathbf{q}(x, y_{j-1/2}) - \mathbf{q}(x, y_{j-1/2}) \right) \cdot \mathbf{n} w(x, y^{+}_{j-1/2}) \mathrm{d}x = 0, \ \forall w \in \mathcal{Q}^k(I_i \times J_j),
\end{aligned} \tag{7}$$

where $\mathbf{n}$ is the outward unit normal vector of the domain integrated.

Interpolation properties. The projections defined above have the following approximation properties. If $u \in H^{k+1}(\Omega)$, we have (see Lemma 2.4 in [25])

$$\| \mathscr{P}_h^{\pm} u - u \|_{\Omega} \le C h^{k+1} \| u \|_{H^{k+1}(\Omega)}, \tag{8}$$

$$\| \Pi_h^{+} \mathbf{q} - \mathbf{q} \|_{\Omega} \le C h^{k+1} \| \mathbf{q} \|_{H^{k+1}(\Omega)}. \tag{9}$$

The projection $\mathscr{P}_h^-$ on the Cartesian meshes has the following superconvergence property (see Lemma 3.7 in [25]).

Lemma 1. *Assume $u \in H^{k+2}(\Omega)$, $\mathbf{q} \in \Sigma_h$, then the projection defined by (6) satisfies*

$$\left|(u - \mathscr{P}_h^- u, \nabla \cdot \mathbf{q})_\Omega - (u - \widehat{\mathscr{P}_h^- u}, \mathbf{q} \cdot \mathbf{n})_\Gamma\right| \le Ch^{k+1}\|u\|_{H^{k+2}(\Omega)}\|\mathbf{q}\|_\Omega,$$

where the "hat" term is the numerical flux.

3. Nonuniform L1–LDG Scheme

In this section, Equation (1) is first transformed into a first-order system of differential equations. Then the L1 method on nonuniform meshes is applied to the time-fractional derivative and the spatial derivative is approximated by the LDG method, and a fully discrete numerical scheme is obtained. The stability analysis and error estimate of the scheme is given by choosing suitable numerical fluxes.

3.1. The Fully Discrete Numerical Scheme and Its Stability Analysis

The usual notations of the nonuniform L1 formula are introduced here. Let M be a positive integer. Set $t_n = T(n/M)^r$ for $n = 0, 1, \ldots, M$, where the temporal mesh grading parameter $r \ge 1$ is chosen by the user. Denote $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots, M$ be the time mesh sizes. It is easy to see that when $r = 1$, the mesh is uniform.

For $n \ge 1$, we approximate the Caputo fractional derivative ${}_C\mathrm{D}_{0,t}^\alpha u(\mathbf{x}, t_n)$ by the well-known L1 formula [26]

$$\begin{aligned} {}_C\mathrm{D}_{0,t}^\alpha u(\mathbf{x}, t_n) &\approx \Upsilon_t^\alpha u(\mathbf{x}, t_n) \\ &:= \frac{d_{n,1}}{\Gamma(2-\alpha)}u^n - \frac{d_{n,n}}{\Gamma(2-\alpha)}u^0 + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{n-1} u^{n-i}(d_{n,i+1} - d_{n,i}), \end{aligned} \tag{10}$$

where $d_{n,i} = [(t_n - t_{n-i})^{1-\alpha} - (t_n - t_{n-i+1})^{1-\alpha}]/\tau_{n-i+1}$ for $i = 1, \ldots, n$. For simplicity, if there is no confusion, we denote $u^n = u(\mathbf{x}, t_n)$.

Set $a_{n-k}^{(n)} = d_{n,n-k+1}/\Gamma(2-\alpha)$ for $k = 1, \ldots, n$ and

$$P_{n-k}^{(n)} = \frac{1}{a_0^{(k)}}\begin{cases} 1, & k = n, \\ \sum\limits_{j=k+1}^{n} (a_{j-k-1}^{(j)} - a_{j-k}^{(j)})P_{n-j}^{(n)}, & 1 \le k \le n-1. \end{cases}$$

Therefore, the approximate scheme (10) can be written as $\Upsilon_t^\alpha u^n = \sum_{i=1}^{n} a_{n-i}^{(n)}(u^i - u^{i-1})$ for $n = 1, \ldots, M$. It follows from Lemma 2.1 in the literature [27] that the coefficient coefficients $\{P_{n-k}^{(n)}\}$ satisfies

$$\sum_{k=1}^{n} P_{n-k}^{(n)} \le (t_n)^\alpha/\Gamma(1+\alpha). \tag{11}$$

Denote the truncation error R_1^n as

$$R_1^n = {}_C\mathrm{D}_{0,t}^\alpha u(\mathbf{x}, t_n) - \Upsilon_t^\alpha u(\mathbf{x}, t_n).$$

Lemma 2 ([26])**.** *Assume that $\|\partial^l u(\mathbf{x}, t)/\partial t^l\|_\Omega \le Ct^{l-\alpha}$ for $l = 0, 1, 2$. Then the following identity holds*

$$\|R_1^n\|_\Omega \le Cn^{-\min\{2-\alpha, r\alpha\}}.$$

Lemma 3 ([27])**.** *Assume that* $u(\mathbf{x},\cdot) \in C^2((0,T])$ *and* $\|\partial^l u(\mathbf{x},t)/\partial t^l\|_\Omega \le Ct^{l-\alpha}$ *for* $l = 0,1,2$. *Then the following identity holds*

$$\sum_{j=1}^{n} P_{n-j}^{(n)}|R_1^j| \le C\left(\alpha^{-1}T^\alpha M^{-r\alpha} + \frac{r^2}{1-\alpha}4^{r-1}T^\alpha M^{-\min\{r\alpha,2-\alpha\}}\right),\ n \ge 1. \tag{12}$$

As the usual treatment, we would like to introduce the auxiliary variable $\mathbf{p} = \nabla u$ and consider the equivalent first-order system

$${}_C\mathrm{D}_{0,t}^\alpha u - \epsilon^2 \nabla\cdot\mathbf{p} - f(u) = 0, \tag{13a}$$

$$\mathbf{p} - \nabla u = 0. \tag{13b}$$

Then the weak formulation of (13) at t_n can be written as

$$({}_C\mathrm{D}_{0,t}^\alpha u^n, v)_K + \epsilon^2(\mathbf{p}^n, \nabla v)_K - \epsilon^2\langle \mathbf{p}^n\cdot\mathbf{n}, v\rangle_{\partial K} - (f(u^n), v)_K = 0, \tag{14a}$$

$$(\mathbf{p}^n, \mathbf{w})_K + (u^n, \nabla\cdot\mathbf{w})_K - \langle u^n, \mathbf{w}\cdot\mathbf{n}\rangle_{\partial K} = 0, \tag{14b}$$

where v, $\mathbf{w}$ are test functions.

Let $(U_h^n, \mathbf{P}_h^n) \in (V_h, \Sigma_h)$ be the approximation of u^n and $\mathbf{p}^n$, respectively. Based on (14), a fully discrete nonuniform L1–LDG method is: find $(U_h^n, \mathbf{P}_h^n) \in (V_h, \Sigma_h)$ such that for all test functions $(v_h, \mathbf{w}_h) \in (V_h, \Sigma_h)$,

$$(\Upsilon_t^\alpha U_h^n, v_h)_K + \epsilon^2(\mathbf{P}_h^n, \nabla v_h)_K - \epsilon^2\langle \widehat{\mathbf{P}_h^n}\cdot\mathbf{n}, v_h\rangle_{\partial K} - (f(U_h^n), v_h)_K = 0, \tag{15a}$$

$$(\mathbf{P}_h^n, \mathbf{w}_h)_K + (U_h^n, \nabla\cdot\mathbf{w}_h)_K - \langle \widehat{U_h^n}, \mathbf{w}_h\cdot\mathbf{n}\rangle_{\partial K} = 0. \tag{15b}$$

All the "hat" terms are numerical fluxes which are yet to be determined. The freedom in choosing numerical fluxes can be utilized for designing a scheme that enjoys a certain stability property. Here alternative flux is chosen

$$\widehat{U_h^n}|_e = U_{h,L}^n,\ \widehat{\mathbf{P}_h^n}|_e = \mathbf{P}_{h,R}^n, \tag{16}$$

or

$$\widehat{U_h^n}|_e = U_{h,R}^n,\ \widehat{\mathbf{P}_h^n}|_e = \mathbf{P}_{h,L}^n. \tag{17}$$

Summing Equation (15) over all elements yields

$$(\Upsilon_t^\alpha U_h^n, v_h)_\Omega + \epsilon^2(\mathbf{P}_h^n, \nabla v_h)_\Omega - \epsilon^2\langle \widehat{\mathbf{P}_h^n}\cdot\mathbf{n}, v_h\rangle_\Gamma - (f(U_h^n), v_h)_\Omega = 0, \tag{18a}$$

$$(\mathbf{P}_h^n, \mathbf{w}_h)_\Omega + (U_h^n, \nabla\cdot\mathbf{w}_h)_\Omega - \langle \widehat{U_h^n}, \mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma = 0. \tag{18b}$$

Next, we study the stability of scheme (18) using the numerical flux (16). The case of choosing numerical flux (17) is almost the same, so is omitted here. Firstly, we state a discrete fractional Gronwall inequality and a property of the nonuniform L1 scheme.

Lemma 4 ([28])**.** *For any finite time* $t_M = T > 0$ *and a given nonnegative sequence* $(\lambda_l)_{l=0}^{M-1}$, *assume that there exists a constant* λ, *independent of time-steps, such that* $\lambda \ge \sum_{l=0}^{M-1}\lambda_l$. *Suppose that the grid function* $\{u^n | n \ge 0\}$ *satisfies*

$$\Upsilon_t^\alpha (u^n)^2 \le \sum_{l=1}^{n}\lambda_{n-l}(u^l)^2 + \phi^n u^n + (\psi^n)^2,\ 1 \le n \le M, \tag{19}$$

where $\{\phi^n, \psi^n | 1 \leq n \leq M\}$ *are nonnegative sequences. If the maximum time-step* $\tau_M \leq (2\Gamma(2-\alpha)\lambda)^{-\frac{1}{\alpha}}$*, it holds that, for* $1 \leq n \leq M$*,*

$$u^n \leq 2E_{\alpha,1}(2\lambda t_n^\alpha)\left(u^0 + \max_{1\leq k\leq n}\sum_{j=1}^{k} P_{k-j}^{(k)}\phi^j + \sqrt{\Gamma(1-\alpha)}\max_{1\leq k\leq n}\{t_k^{\alpha/2}\psi^k\}\right). \tag{20}$$

Lemma 5 ([29])**.** *Let the functions* $u^n = u(\mathbf{x}, t_n)$ *be in* $L^2(\Omega)$ *for* $n = 0, 1, \ldots, M$*. Then, one has the following inequality*

$$(\Upsilon_t^\alpha u^n, u^n)_\Omega \geq \frac{1}{2}\Upsilon_t^\alpha \|u^n\|_\Omega^2.$$

Theorem 1. *The solution* U_h^n *of the fully discrete nonuniform L1–LDG scheme* (18) *satisfies*

$$\|U_h^n\|_\Omega \leq 2E_{\alpha,1}(4t_n^\alpha)\|U_h^0\|_\Omega,\; n = 1, \ldots, M.$$

Proof. Taking the test functions in scheme (18) as $v_h = U_h^n$ and $\mathbf{w}_h = \epsilon^2\mathbf{P}_h^n$, we obtain

$$(\Upsilon_t^\alpha U_h^n, U_h^n)_\Omega + \epsilon^2(\mathbf{P}_h^n, \nabla U_h^n)_\Omega - \epsilon^2\langle\widehat{\mathbf{P}_h^n}\cdot\mathbf{n}, U_h^n\rangle_\Gamma + \left((U_h^n)^3 - U_h^n, U_h^n\right)_\Omega = 0, \tag{21a}$$

$$\epsilon^2(\mathbf{P}_h^n, \mathbf{P}_h^n)_\Omega + \epsilon^2(U_h^n, \nabla\cdot\mathbf{P}_h^n)_\Omega - \epsilon^2\langle\widehat{U_h^n}, \mathbf{P}_h^n\cdot\mathbf{n}\rangle_\Gamma = 0. \tag{21b}$$

Adding the two equations in (21) and using (16), we have that

$$(\Upsilon_t^\alpha U_h^n, U_h^n)_\Omega + \epsilon^2\|\mathbf{P}_h^n\|_\Omega^2 + \|(U_h^n)^2\|_\Omega^2 = \|U_h^n\|_\Omega^2, \tag{22}$$

which indicates that

$$(\Upsilon_t^\alpha U_h^n, U_h^n)_\Omega \leq \|U_h^n\|_\Omega^2. \tag{23}$$

Invoking Lemma 5, we derive that

$$\Upsilon_t^\alpha\|U_h^n\|_\Omega^2 \leq 2\|U_h^n\|_\Omega^2. \tag{24}$$

Therefore, applying Lemma 4 with $u^n = \|U_h^n\|_\Omega$, $\phi^n = \psi^n = 0$, $\lambda_0 = 2$, and $\lambda_j = 0$ for $1 \leq j \leq M-1$, we have

$$\|U_h^n\|_\Omega \leq 2E_{\alpha,1}(4t_n^\alpha)\|U_h^0\|_\Omega.$$

It completes the proof. □

Remark 1.

(i) *We point out that the stability analysis in Theorem 1 can be further improved by mathematical induction. Following the discussions given in (Theorem 4.4 in [30]), we deduce that*

$$\|U_h^n\|_\Omega \leq \|U_h^0\|_\Omega.$$

(ii) *It could be interesting to check the energy stability (i.e.,* $E(U_h^n) \leq E(U_h^0)$ *for all* $n \geq 1$*, where* $E(U_h^n) = \int_\Omega \frac{\epsilon^2}{2}|\mathbf{P}_h^n|^2 + F(U_h^n)\mathrm{d}\mathbf{x}$*) of the fully discrete numerical scheme* (18)*, although we cannot give the theoretical analysis at present. As seen in [19], the main difficulty is to prove the positive semi-definite of the quadratic form* $(\Upsilon_t^\alpha U_h^n, U_h^n - U_h^{n-1})_\Omega$*. In fact, with the help of Lemma 3.1 in [13], we can show the energy stability for the uniform case (i.e., the L1 formula on uniform meshes for the time-fractional derivative and the LDG method for the space approximation).*

(iii) *The stability mentioned in Theorem 1 is about the initial value, so we can regard this stability as a priori stability.*

3.2. Optimal Error Estimate

Suppose the exact solution $u(\mathbf{x}, t)$ of Equation (1) has the following smoothness properties:

$$u \in L^\infty\Big((0,T];H^{k+2}(\Omega)\Big),\ \Big|\partial^l u(\mathbf{x},t)/\partial t^l\Big| \le C(1+t^{\alpha-l}) \text{ for } 0<t\le T \text{ and } l=0,1,2. \tag{25}$$

Such a regularity assumption with respect to time t is often used, see for instance [15,19,30–39]. It implies that the solution $u(\mathbf{x},t)$ likely behaves a weak singularity at the starting time $t=0$, i.e., $|\partial u(\mathbf{x},t)/\partial t|$ and /or $|\partial^2 u(\mathbf{x},t)/\partial t^2|$ blow up as $t\to 0^+$ albeit $u(\mathbf{x},t)$ is continuous on $[0,T]$. Since it has been shown in [13] that the time-fractional Allen-Cahn Equation (1) satisfies the maximum principle, namely,

$$|u(\mathbf{x},t)|\le 1 \quad \text{for} \quad t>0 \quad \text{if} \quad |u(\mathbf{x},0)|\le 1,$$

we assume that the nonlinear term $f(u)$ satisfies

$$\max|f'(u)|\le L, \tag{26}$$

where L is a positive constant. For simplicity, we denote

$$e_u^n = u^n - U_h^n = u^n - Pu^n + Pu^n - U_h^n = u^n - Pu^n + Pe_u^n, \tag{27a}$$

$$e_{\mathbf{p}}^n = \mathbf{p}^n - \mathbf{P}_h^n = \mathbf{p}^n - \Pi\mathbf{p}^n + \Pi\mathbf{p}^n - \mathbf{P}_h^n = \mathbf{p}^n - \Pi\mathbf{p}^n + \Pi e_{\mathbf{p}}^n. \tag{27b}$$

We choose the projection as follows

$$\begin{aligned} &(P,\Pi) = (\mathscr{P}_h^-,\mathscr{P}_h^+) \ \text{ in one dimension,}\\ &(P,\Pi) = (\mathscr{P}_h^-,\Pi_h^+) \ \text{ in two-dimensions,} \end{aligned} \tag{28}$$

which are defined in Section 2.2.

Subtracting (18) from (14), we have the error equation

$$\begin{aligned} &({}_C\mathrm{D}_{0,t}^\alpha u^n - \Upsilon_t^\alpha U_h^n, v_h)_\Omega + \epsilon^2(\mathbf{p}^n - \mathbf{P}_h^n, \nabla v_h)_\Omega - \epsilon^2\langle(\mathbf{p}^n - \widehat{\mathbf{P}_h^n})\cdot\mathbf{n}, v_h\rangle_\Gamma\\ &\quad - (f(u^n) - f(U_h^n), v_h)_\Omega = 0, \end{aligned} \tag{29a}$$

$$(\mathbf{p}^n - \mathbf{P}_h^n, \mathbf{w}_h)_\Omega + (u^n - U_h^n, \nabla\cdot\mathbf{w}_h)_\Omega - \langle(u^n - \widehat{U_h^n}), \mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma = 0. \tag{29b}$$

Now we show the error estimate for Equation (29).

Theorem 2. *Let u^n be the exact solution of Equation (1) which satisfies the smoothness assumption (25), and U_h^n be the numerical solution of the nonuniform L1–LDG scheme (18). If $f(u)$ satisfies the condition (26), then for $n=1,2,\dots,M$, the following estimate holds*

$$\|u^n - U_h^n\|_\Omega \le C\Big(M^{-\min\{2-\alpha, r\alpha\}} + h^{k+1}\Big), \tag{30}$$

where C is a positive constant independent of M and h.

Proof. By taking the test functions $v_h = Pe_u^n$ and $\mathbf{w}_h = \epsilon^2\Pi e_{\mathbf{p}}^n$ in (29) and applying (27), we arrive at

$$(\Upsilon_t^\alpha Pe_u^n, Pe_u^n)_\Omega + \epsilon^2(\Pi e_{\mathbf{p}}^n, \Pi e_{\mathbf{p}}^n)_\Omega - (f(u^n) - f(U_h^n), Pe_u^n)_\Omega = RHS, \tag{31}$$

where $R_1^n = {}_C\mathrm{D}_{0,t}^\alpha u(\mathbf{x},t_n) - \Upsilon_t^\alpha u(\mathbf{x},t_n)$ and

$$\begin{aligned} RHS =& -(\Upsilon_t^\alpha(u^n - Pu^n), Pe_u^n)_\Omega - (R_1^n, Pe_u^n)_\Omega - \epsilon^2(\mathbf{p}^n - \Pi\mathbf{p}^n, \nabla Pe_u^n)_\Omega\\ &+ \epsilon^2\langle(\mathbf{p}^n - \widehat{\Pi\mathbf{p}^n})\cdot\mathbf{n}, Pe_u^n\rangle_\Gamma - \epsilon^2(\mathbf{p}^n - \Pi\mathbf{p}^n, \Pi e_{\mathbf{p}}^n)_\Omega - \epsilon^2(u^n - Pu^n, \nabla\cdot\Pi e_{\mathbf{p}}^n)_\Omega\\ &+ \epsilon^2\langle(u^n - \widehat{Pu^n}), \Pi e_{\mathbf{p}}^n\cdot\mathbf{n}\rangle_\Gamma - \epsilon^2(\Pi e_{\mathbf{p}}^n, \nabla Pe_u^n)_\Omega + \epsilon^2\langle\widehat{\Pi e_{\mathbf{p}}^n}\cdot\mathbf{n}, Pe_u^n\rangle_\Gamma\\ &- \epsilon^2(Pe_u^n, \nabla\cdot\Pi e_{\mathbf{p}}^n)_\Omega + \epsilon^2\langle\widehat{Pe_u^n}, \Pi e_{\mathbf{p}}^n\cdot\mathbf{n}\rangle_\Gamma. \end{aligned} \tag{32}$$

Making use of flux (16) and the property of projections, it is obvious to see that

$$\begin{aligned} RHS = &-(\Upsilon_t^\alpha(u^n - Pu^n), Pe_u^n)_\Omega - (R_1^n, Pe_u^n)_\Omega - \epsilon^2(\mathbf{p}^n - \Pi\mathbf{p}^n, \Pi e_{\mathbf{p}}^n)_\Omega \\ &- \epsilon^2(u^n - Pu^n, \nabla\cdot\Pi e_{\mathbf{p}}^n)_\Omega + \epsilon^2\langle(u^n - \widehat{Pu^n}), \Pi e_{\mathbf{p}}^n\cdot\mathbf{n}\rangle_\Gamma. \end{aligned} \tag{33}$$

By using Cauchy-Schwarz inequality and Lemma 1, RHS can be estimated as follows

$$\begin{aligned} |RHS| = &\|\Upsilon_t^\alpha(u^n - Pu^n)\|_\Omega\|Pe_u^n\|_\Omega + \|R_1^n\|_\Omega\|Pe_u^n\|_\Omega + \epsilon^2\|\mathbf{p}^n - \Pi\mathbf{p}^n\|_\Omega\|\Pi e_{\mathbf{p}}^n\|_\Omega \\ &+ Ch^{k+1}\|\Pi e_{\mathbf{p}}^n\|_\Omega \\ \le\ & Ch^{k+1}\left(\|Pe_u^n\|_\Omega + \|\Pi e_{\mathbf{p}}^n\|_\Omega\right) + \|R_1^n\|_\Omega\|Pe_u^n\|_\Omega, \end{aligned} \tag{34}$$

where C is a positive constant dependent on $\|u\|_{L^\infty((0,T];H^{k+2}(\Omega))}$.

Now we estimate the nonlinear term in (31). It is obvious to see that

$$\begin{aligned} &(f(U_h^n) - f(u^n), Pe_u^n)_\Omega \\ &= (f(Pu^n) - f(u^n), Pe_u^n)_\Omega - (f(Pu^n) - f(U_h^n), Pe_u^n)_\Omega \\ &= \left(f'(\xi)(Pu^n - u^n), Pe_u^n\right)_\Omega - (f(Pu^n) - f(U_h^n), Pe_u^n)_\Omega \\ &= I + II, \end{aligned} \tag{35}$$

where $\xi = \theta u^n + (1-\theta)Pu^n$ with $0 \le \theta \le 1$. Then, using the Cauchy-Schwarz inequality, Young's inequality, interpolation property (8), and (26), we can derive

$$\begin{aligned} |I| &\le \|f'\|_{L^\infty(\Omega)}|(Pu^n - u^n, Pe_u^n)_\Omega| \\ &\le C\|Pe_u^n\|_\Omega^2 + Ch^{2k+2}. \end{aligned} \tag{36}$$

It follows from the definition of $f(u)$ (i.e., $f(u) = u - u^3$) that

$$f(u) - f(v) = f'(u)(u - v) - (u - v)^3 + 3u(u - v)^2. \tag{37}$$

Therefore, II can be rewritten as

$$\begin{aligned} II &= -(f(Pu^n) - f(U_h^n), Pe_u^n)_\Omega \\ &= -\left(f'(Pu^n)(Pu^n - U_h^n) - (Pu^n - U_h^n)^3 + 3Pu^n(Pu^n - U_h^n)^2, Pe_u^n\right)_\Omega \\ &= -\left(f'(Pu^n)Pe_u^n - (Pe_u^n)^3 + 3Pu^n(Pe_u^n)^2, Pe_u^n\right)_\Omega \\ &= \left((Pe_u^n)^3, Pe_u^n\right)_\Omega - \left(f'(Pu^n)Pe_u^n + 3Pu^n(Pe_u^n)^2, Pe_u^n\right)_\Omega. \end{aligned} \tag{38}$$

From (26) and the Cauchy-Schwarz inequality, it is obvious to see that

$$\left|(f'(Pu^n)Pe_u^n + 3Pu^n(Pe_u^n)^2, Pe_u^n)_\Omega\right| \le C\|Pe_u^n\|_\Omega^2 + \|(Pe_u^n)^2\|_\Omega^2. \tag{39}$$

Combining Equations (31), (34), (36), (38) and (39), we have

$$\begin{aligned} &(\Upsilon_t^\alpha Pe_u^n, Pe_u^n)_\Omega + \|\Pi e_{\mathbf{p}}^n\|_\Omega^2 + \|(Pe_u^n)^2\|_\Omega^2 \\ &\le Ch^{k+1}\left(\|Pe_u^n\|_\Omega + \|\Pi e_{\mathbf{p}}^n\|_\Omega\right) + \|R_1^n\|_\Omega\|Pe_u^n\|_\Omega + Ch^{k+1}\|\Pi e_{\mathbf{p}}^n\|_\Omega \\ &\quad + C\|Pe_u^n\|_\Omega^2 + \|(Pe_u^n)^2\|_\Omega^2 + Ch^{2k+2} \\ &\le C\|Pe_u^n\|_\Omega^2 + \|\Pi e_{\mathbf{p}}^n\|_\Omega^2 + \|(Pe_u^n)^2\|_\Omega^2 + Ch^{2k+2} + \|R_1^n\|_\Omega\|Pe_u^n\|_\Omega. \end{aligned} \tag{40}$$

Invoking Lemma 5, one has

$$\Upsilon_t^\alpha\|Pe_u^n\|_\Omega^2 \le 2C\|Pe_u^n\|_\Omega^2 + 2Ch^{2k+2} + 2\|R_1^n\|_\Omega\|Pe_u^n\|_\Omega. \tag{41}$$

Letting $\lambda_0 = 2C$, $\lambda_j = 0$ for $1 \le j \le M-1$, $u^n = \|Pe_u^n\|_\Omega, \phi^n = 2\|R_1^n\|_\Omega$, and $\psi^n = \sqrt{2C}h^{k+1}$ in Lemma 4, we can obtain from (41) that

$$\|Pe_u^n\|_\Omega \le 2E_{\alpha,1}(4Ct_n^\alpha)\left(2\max_{1\le k\le n}\sum_{j=1}^{k}P_{k-j}^{(k)}\|R_1^j\|_\Omega + \sqrt{2C\Gamma(1-\alpha)}\max_{1\le k\le n}\{t_k^{\alpha/2}h^{k+1}\}\right), \tag{42}$$

provided that the maximum time-step $\tau_M \le (4C\Gamma(2-\alpha))^{-1/\alpha}$. With the help of Lemma 3 and inequality (11), we have

$$\|Pe_u^n\|_\Omega \le C\left(M^{-\min\{r\alpha,2-\alpha\}} + h^{k+1}\right). \tag{43}$$

By using the interpolation property (8) and the triangle inequality, the desired estimate follows immediately. $\square$

As a conclusion of this section, we present the Algorithm 1 based on the nonuniform L1–LDG scheme (15).

Algorithm 1 The nonuniform L1–LDG scheme for solving the time-fractional Allen-Cahn equation.

Input: the order of time-fractional derivative α, interface width parameter ϵ, temporal mesh grading parameter r.
Output: nodal values of numerical solution U_h^n at t_n.

Construct a shape-regular subdivision $\mathcal{T}_h$ of Ω with $N_x \times N_y$ elements and define basis functions $\{\varphi_K^i\}_{i=1}^l$.
Give the global number and coordinates of nodes.
for $K \in \mathcal{T}_h$ **do**
  Compute the $l \times l$ mass and convection matrices $A_1^{(K)}$, $A_2^{(K)}$, and $A_3^{(K)}$ on K with components

$$(A_1^{(K)})_{ij} = (\varphi_K^j, \varphi_K^i),\ (A_2^{(K)})_{ij} = (\varphi_K^j, (\varphi_K^i)_x),\ (A_3^{(K)})_{ij} = (\varphi_K^j, (\varphi_K^i)_y).$$

  Combine the boundary conditions to calculate the $l \times l$ stiffness matrices generated by interface ∂K with entries

$$(A_4^{(K)})_{ij} = \langle \varphi_{K,R}^j n_1, \varphi_K^i\rangle,\ (A_5^{(K)})_{ij} = \langle \varphi_{K,R}^j n_2, \varphi_K^i\rangle,$$

$$(A_6^{(K)})_{ij} = \langle \varphi_{K,L}^j, \varphi_K^i n_1\rangle,\ (A_7^{(K)})_{ij} = \langle \varphi_{K,L}^j, \varphi_K^i n_2\rangle.$$

  Assemble matrices $A_1^{(K)}$-$A_7^{(K)}$ to A_1-A_7 by the global number.
end for
Construct nonuniform time meshes $t_n = T(n/M)^r$, $n = 0, 1, \ldots, M$ with time mesh sizes $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, M$.
Introduce a vector $W^n = [U^n, P_1^n, P_2^n]^T$ with $3l\,(N_xN_y)$ unknown coefficients (nodal values of U_h^n, $P_{1,h}^n$ and $P_{2,h}^n$) as components, where U^n, P_1^n and P_2^n are vectors consisting of $\{u_K^{n,i}\}_{i=1}^l$, $\{p_{1,K}^{n,i}\}_{i=1}^l$ and $\{p_{2,K}^{n,i}\}_{i=1}^l$, respectively.
Choose initial value W^0.
for $n = 1$ **do**
  Set $\beta = \frac{\Gamma(2-\alpha)}{d_{n,1}}$.
  for $K \in \mathcal{T}_h$ **do**

Calculate of the $l \times l$ matrix $A_8^{(K)}$ corresponding to nonlinear term on K at the time level t_{n-1} with components

$$(A_8^{(K)})_{ij} = \left(\left(\sum_{i=1}^{l} u_K^{n-1,i} \varphi_K^i \right)^2 \varphi_K^j, \varphi_K^i \right),$$

then assemble A_8 according to global number.

end for

Define a zero matrix $(O)_{ij} = 0$ of size $l\,(N_x N_y) \times l\,(N_x N_y)$. Then the global stiffness matrix and the global load vector are

$$A = \begin{bmatrix} (1-\beta)A_1 & \epsilon^2\beta(A_2 - A_4) & \epsilon^2\beta(A_3 - A_5) \\ A_2 - A_6 & A_1 & O \\ A_2 - A_7 & O & A_1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \beta \frac{d_{n,n}}{\Gamma(2-\alpha)} A_1 \\ O \\ O \end{bmatrix} W^0 - \begin{bmatrix} \beta A_8 \\ O \\ O \end{bmatrix} W^0.$$

Solve

$$AW^n = B.$$

end for

for $n = 2, \ldots, M$ **do**

Set $\beta = \frac{\Gamma(2-\alpha)}{d_{n,1}}$.

for $K \in \mathcal{T}_h$ **do**

Assemble the matrices A_8 and A_9 associated with the nonlinear term at moments t_{n-1} and t_{n-2}, respectively. Their components on K are

$$(A_8^{(K)})_{ij} = \left(\left(\sum_{i=1}^{l} u_K^{n-1,i} \varphi_K^i \right)^2 \varphi_K^j, \varphi_K^i \right), \quad i, j = 1 \ldots, l,$$

and

$$(A_9^{(K)})_{ij} = \left(\left(\sum_{i=1}^{l} u_K^{n-2,i} \varphi_K^i \right)^2 \varphi_K^j, \varphi_K^i \right), \quad i, j = 1 \ldots, l.$$

end for

Assemble the global stiffness matrix and the global load vector

$$A = \begin{bmatrix} (1-\beta)A_1 & \epsilon^2\beta(A_2 - A_4) & \epsilon^2\beta(A_3 - A_5) \\ A_2 - A_6 & A_1 & O \\ A_2 - A_7 & O & A_1 \end{bmatrix}$$

and

$$B = \sum_{s=1}^{n-1} \begin{bmatrix} \beta \frac{d_{n,s} - d_{n,s+1}}{\Gamma(2-\alpha)} A_1 \\ O \\ O \end{bmatrix} W^{n-s} + \begin{bmatrix} \beta \frac{d_{n,n}}{\Gamma(2-\alpha)} A_1 \\ O \\ O \end{bmatrix} W^0 - \begin{bmatrix} 2\beta A_8 \\ O \\ O \end{bmatrix} W^{n-1} + \begin{bmatrix} \beta A_9 \\ O \\ O \end{bmatrix} W^{n-2}.$$

Solve

$$AW^n = B.$$

end for

From Theorem 2, it can be seen that the scheme (18) can reach the optimal convergence order $\mathcal{O}(M^{-(2-\alpha)})$ in the time direction when the grid parameter $r \geq (2-\alpha)/\alpha$. However, the numerical solution generated by (18) will be limited to $(2-\alpha)$th-order accurate in time, even if the solution is sufficiently smooth. Therefore, in the next section, we will study a higher-order numerical algorithm for the time-fractional Allen-Cahn Equation (1).

4. Nonuniform L2-1_σ–LDG Scheme

In the section, we propose a fully discrete nonuniform L2-1_σ–LDG scheme for solving the time-fractional Allen-Cahn Equation (1), which is based on the L2-1_σ approximation in the temporal direction and the LDG method in the spatial direction. The stability and the convergence of the scheme are proved rigorously.

4.1. The Fully Discrete Numerical Scheme and Its Stability Analysis

The usual notations of the nonuniform L2-1_σ formula are introduced here. Let M be a positive integer. Set $t_n = T(n/M)^r$ for $n = 0, 1, \ldots, M$, where the temporal mesh grading parameter $r \geq 1$ is chosen by the user. Denote $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots, M$ be the time mesh sizes. Set $t_{n+\sigma} = t_n + \sigma\tau_{n+1}$, $u^{n+\sigma} = u(\mathbf{x}, t_{n+\sigma})$, and $u^{n,\sigma} = \sigma u^{n+1} + (1-\sigma)u^n$ for $\sigma \in [0, 1]$, $n = 0, 1, \ldots, M-1$.

The Caputo fractional derivative ${}_C\mathrm{D}^{\alpha}_{0,t}u$ can be approximated at the point $t_{n+\sigma}$ $(n = 0, 1, \ldots, M-1)$ by the L2-1_σ formula [35]

$$
\begin{aligned}
{}_C\mathrm{D}^{\alpha}_{0,t}u(\mathbf{x}, t_{n+\sigma}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+\sigma}} \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{\mathrm{d}s}{(t_{n+\sigma} - s)^{\alpha}} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{\mathrm{d}s}{(t_{n+\sigma} - s)^{\alpha}} \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_{t_n}^{t_{n+\sigma}} \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{\mathrm{d}s}{(t_{n+\sigma} - s)^{\alpha}} \\
&\approx g_{n,n}u^{n+1} - \sum_{j=0}^{n} (g_{n,j} - g_{n,j-1})u^j \\
&:= \mathfrak{R}^{\alpha}_t u^{n+\sigma}.
\end{aligned}
\tag{44}
$$

Here $g_{0,0} = \tau_1^{-1}a_{0,0}$, $g_{n,-1} = 0$, and for $n \geq 1$, it holds that

$$
g_{n,j} = \begin{cases}
\tau_{j+1}^{-1}(a_{n,0} - b_{n,0}), & j = 0, \\
\tau_{j+1}^{-1}(a_{n,j} + b_{n,j-1} - b_{n,j}), & 1 \leq j \leq n-1, \\
\tau_{j+1}^{-1}(a_{n,n} + b_{n,n-1}), & j = n.
\end{cases}
$$

$$
\begin{aligned}
a_{n,n} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_n}^{t_{n+\sigma}} (t_{n+\sigma} - s)^{-\alpha} \mathrm{d}s = \frac{\sigma^{1-\alpha}}{\Gamma(2-\alpha)} \tau_{n+1}^{1-\alpha}, \ n \geq 0, \\
a_{n,j} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j=1}} (t_{n+\sigma} - s)^{-\alpha} \mathrm{d}s, \ n \geq 1, \ 0 \leq j \leq n-1, \\
b_{n,j} &= \frac{2}{\Gamma(1-\alpha)(t_{j+2} - t_j)} \int_{t_j}^{t_{j+1}} (t_{n+\sigma} - s)^{-\alpha}(s - t_{j+1/2}) \mathrm{d}s, \ n \geq 1, \ 0 \leq j \leq n-1.
\end{aligned}
$$

Define the discrete convolution kernel $A^{n+1,\sigma}_{n+1-j} = g_{n,j}$, $\nabla_t u^{j+1} = u^{j+1} - u^j$ for $0 \leq j \leq n$ and $0 \leq n \leq M-1$. Then, the L2-1_σ discretization can be rewritten as

$$
\mathfrak{R}^{\alpha}_t u^{n+\sigma} = \sum_{j=0}^{n} A^{n+1,\sigma}_{n+1-j} \nabla_t u^{j+1}, \ n = 0, 1, \ldots, M-1.
$$

By referring to [40], the discrete convolution kernel $P_{n+1-j}^{n+1,\sigma}$ are defined as

$$P_1^{n+1,\sigma} = \frac{1}{A_1^{n+1,\sigma}},\ P_{n+1-j}^{n+1,\sigma} = \frac{1}{A_1^{j+1,\sigma}} \sum_{i=j+1}^{n} \left(A_{i-j}^{i+1,\sigma} - A_{i-j+1}^{i+1,\sigma} \right) P_{n+1-i}^{n+1,\sigma}.$$

The discrete convolution kernels satisfy the following properties

$$\sum_{j=i}^{n} P_{n+1-j}^{n+1,\sigma} A_{j-i+1}^{j+1,\sigma} = 1,\ \text{for } 0 \le i \le n \le M-1, \tag{45}$$

and

$$\sum_{j=0}^{n} P_{n+1-j}^{n+1,\sigma} \omega_{1+m\alpha-\alpha}(t_{j+1}) \le \pi_A \omega_{1+m\alpha}(t_{n+1}),\ \text{for } 0 \le n \le M-1 \text{ and } m = 0,1, \tag{46}$$

where $\omega_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ and π_A is a positive constant.

Let $\mathbf{p} = \nabla u$, then the weak form of the time-fractional Allen-Cahn Equation (1) at $t_{n+\sigma}$ is formulated as

$$\begin{aligned} &\left(({}_C D_{0,t}^\alpha u)^{n+\sigma}, v \right)_\Omega - \epsilon^2 (\nabla \cdot \mathbf{p}^{n+\sigma}, v) + \epsilon^2 (\nabla \cdot \mathbf{p}^{n,\sigma}, v) + \epsilon^2 (\mathbf{p}^{n,\sigma}, \nabla v)_\Omega \\ &\quad - \epsilon^2 \langle \mathbf{p}^{n,\sigma} \cdot \mathbf{n}, v \rangle_\Gamma + \left(f(u^{n,\sigma}) - f(u^{n+\sigma}), v \right)_\Omega - (f(u^{n,\sigma}), v)_\Omega = 0, \end{aligned} \tag{47a}$$

$$(\mathbf{p}^{n,\sigma}, \mathbf{w})_\Omega + (u^{n,\sigma}, \nabla \cdot \mathbf{w})_\Omega - \langle u^{n,\sigma}, \mathbf{w} \cdot \mathbf{n} \rangle_\Gamma = 0, \tag{47b}$$

where v, $\mathbf{w}$ are test functions.

By using the LDG method presented in Section 3 for the spatial discretization and the nonuniform L2-1_σ formula to time. Then we can define the fully discrete nonuniform L2-1_σ–LDG scheme as follows: find $(U_h^{n,\sigma}, \mathbf{P}_h^{n,\sigma}) \in (V_h, \Sigma_h)$ such that for all test functions $v_h \in V_h$ and $\mathbf{w}_h \in \Sigma_h$

$$\left(\Re_t^\alpha U_h^{n+\sigma}, v_h \right)_\Omega + \epsilon^2 (\mathbf{P}_h^{n,\sigma}, \nabla v_h)_\Omega - \epsilon^2 \langle \widehat{\mathbf{P}_h^{n,\sigma}} \cdot \mathbf{n}, v_h \rangle_\Gamma - \left(f(U_h^{n,\sigma}), v \right)_\Omega = 0, \tag{48a}$$

$$(\mathbf{P}_h^{n,\sigma}, \mathbf{w}_h)_\Omega + (U_h^{n,\sigma}, \nabla \cdot \mathbf{w}_h)_\Omega - \langle \widehat{U_h^{n,\sigma}}, \mathbf{w}_h \cdot \mathbf{n} \rangle_\Gamma = 0. \tag{48b}$$

Here the "numerical fluxes" are chosen as (16).

To show the stability of the proposed nonuniform L2-1_σ–LDG scheme, we need some important lemmas.

Lemma 6 ([28])**.** *For any finite time $t_M = T > 0$ and a given nonnegative sequence $(\lambda_l)_{l=0}^{M-1}$, assume that there exists a constant Λ, independent of time-steps, such that $\sum_{l=0}^{M-1} \lambda_l \le \Lambda$. Let $\sigma = 1 - \alpha/2$ and suppose that the grid function $\{u^{n+1} | n \ge 0\}$ satisfies*

$$\sum_{i=0}^{n} A_{n+1-i}^{n+1,\sigma} \nabla_t (u^{i+1})^2 \le \sum_{i=0}^{n} \lambda_{n-i} (u^{i,\sigma})^2 + \phi^{n+1} u^{n,\sigma} + (\psi^{n+1})^2,\ 0 \le n \le M-1,$$

where $\{\phi^{n+1}, \psi^{n+1} | 0 \le n \le M-1\}$ are nonnegative sequences. If the maximum time-step $\tau_M \le (2\pi_A \Gamma(2-\alpha)\Lambda)^{-1/\alpha}$, it holds that, for $0 \le n \le M-1$,

$$u^{n+1} \le 2E_{\alpha,1}(2\pi_A \Lambda t_{n+1}^\alpha) \left(u^0 + \max_{0 \le i \le n} \sum_{j=0}^{i} P_{i-j+1}^{i+1,\sigma} \phi^j + \sqrt{\pi_A \Gamma(1-\alpha)} \max_{0 \le j \le n} \{ t_{j+1}^{\alpha/2} \psi^{j+1} \} \right).$$

Here $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ is the Mittag-Leffler function.

Lemma 7 ([29]). *Suppose $\sigma = 1 - \alpha/2$. For any function $u^{n+1} (0 \le n \le M-1)$, we have the following inequality*

$$(\Re_t^\alpha u^{n+\sigma}, u^{n,\sigma})_\Omega \ge \frac{1}{2} \Re_t^\alpha (\|u\|_\Omega^2)^{n+\sigma}.$$

Theorem 3. *If the graded mesh satisfies the maximum time-step condition $\tau_M \le (4\pi_A \Gamma(2-\alpha))^{-1/\alpha}$, then the solution U_h^{n+1} of the fully discrete nonuniform L2-1_σ–LDG scheme (48) satisfies*

$$\|U_h^{n+1}\|_\Omega \le 2E_{\alpha,1}(4\pi_A t_{n+1}^\alpha)\|U_h^0\|_\Omega, \; n = 0, 1, \ldots, M-1.$$

Proof. Taking the test functions $(v_h, \mathbf{w}_h) = (U_h^{n,\sigma}, \epsilon^2 \mathbf{P}_h^{n,\sigma})$ in (48) and integrating by parts, we get

$$(\Re_t^\alpha U_h^{n+\sigma}, U_h^{n,\sigma})_\Omega + \epsilon^2 \|\mathbf{P}_h^{n,\sigma}\|_\Omega^2 + + \Big((U_h^{n,\sigma})^3 - U_h^{n,\sigma}, U_h^{n,\sigma}\Big)_\Omega = 0.$$

By virtue of Lemma 7 and Cauchy-Schwarz inequality, we obtain

$$\Big(\Re^\alpha \|U_h\|_\Omega^2\Big)^{n+\sigma} \le 2\|U_h^{n,\sigma}\|_\Omega^2. \tag{49}$$

Using Lemma 6, it follows from (49) that

$$\|U_h^{n+1}\|_\Omega \le 2E_{\alpha,1}(4\pi_A t_{n+1}^\alpha)\|U_h^0\|_\Omega, \; n = 0, 1, \ldots, M-1.$$

The proof is completed. □

4.2. Optimal Error Estimate

In this subsection, we give the optimal error estimate for the fully discrete nonuniform L2-1_σ–LDG scheme (48) of Equation (1). Suppose the exact solution $u(\mathbf{x}, t)$ of (1) has the following smoothness properties

$$u \in L^\infty\Big((0, T]; H^{k+2}(\Omega)\Big), \; \Big|\partial^l u(\mathbf{x}, t)/\partial t^l\Big| \le C(1 + t^{\alpha - l}) \text{ for } 0 < t \le T \text{ and } l = 0, 1, 2, 3. \tag{50}$$

The same as the nonuniform L1–LDG scheme, we assume that the nonlinear term $f(u)$ satisfies the condition (26).

Lemma 8 ([33]). *Suppose $\sigma = 1 - \alpha/2$. Then for any function $u(t) \in C^3(0, T]$, one has*

$$\big|({}_C D_{0,t}^\alpha u)^{n+\sigma} - \Upsilon_t^\alpha u^{n+\sigma}\big| \le C t_{n+\sigma}^{-\alpha} \Big(\psi_u^{n+\sigma} + \max_{1 \le s \le n} \{\psi_u^{n,s}\}\Big) \text{ for } n = 0, 1, \ldots, M-1,$$

where

$$\begin{aligned}
\psi_u^{n+\sigma} &= \tau_{n+1}^{3-\alpha} t_{n+\sigma}^\alpha \sup_{s \in (t_n, t_{n+1})} |u'''(s)| \text{ for } n = 1, 2, \ldots, M-1, \\
\psi_u^{n,1} &= \tau_1^\alpha \sup_{s \in (0, t_1)} \Big(s^{1-\alpha} |(I_{2,1} u(s))' - u'(s)|\Big) \text{ for } n = 1, 2, \ldots, M-1, \\
\psi_u^{n,s} &= \tau_{n+1}^{-\alpha} \tau_i^2 (\tau_i + \tau_{i+1}) t_i^\alpha \sup_{s \in (t_{i-1}, t_{i+1})} |u'''(s)| \text{ for } 2 \le i \le n \le M-1,
\end{aligned}$$

and $I_{2,1} u(s)$ is the quadratic polynomial that interpolates to $u(s)$ at the points t_{s-1}, t_s and t_{s+1}.

Lemma 9 ([33]). *Suppose that $u \in C[0, T] \cap C^3(0, T]$ satisfies the condition (50). Then we have*

$$\begin{aligned}
\psi_u^{n+\sigma} &\le CM^{-\min\{r\alpha, 3-\alpha\}} \text{ for } n = 0, 1, \ldots, M-1, \\
\psi_u^{n,s} &\le CM^{-\min\{r\alpha, 3-\alpha\}} \text{ for } s = 1, \ldots, M-1, \; n \ge 1.
\end{aligned}$$

In Section 3.2, we give the convergence analysis for the nonuniform L1–LDG scheme. The same proof idea can be extended to the nonuniform L2-1_σ–LDG scheme. However, the proof would be somewhat more complicated. Following the similar line as before, we obtain the following error equation

$$\begin{aligned}&\big(({}_C\mathrm{D}^{\alpha}_{0,t}u)^{n+\sigma}-\Re^{\alpha}_t U_h^{n+\sigma},v_h\big)_\Omega+\epsilon^2(e_{\mathbf{p}}^{n,\sigma},\nabla v_h)_\Omega-\epsilon^2\langle\widehat{e_{\mathbf{p}}^{n,\sigma}}\cdot\mathbf{n},v_h\rangle_\Gamma\\&\quad=\big(f(u^{n,\sigma})-f(U_h^{n,\sigma}),v_h\big)_\Omega+(R_2^{n,\sigma},v_h)_\Omega,\end{aligned}\tag{51a}$$

$$(e_{\mathbf{p}}^{n,\sigma},\mathbf{w}_h)_\Omega+(e_u^{n,\sigma},\nabla\cdot\mathbf{w}_h)_\Omega-\langle\widehat{e_u^{n,\sigma}},\mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma=0,\tag{51b}$$

where $(v_h,\mathbf{w}_h)\in V_h\times\Sigma_h$ are test functions, $R_2^{n+\sigma}=\epsilon^2(\nabla\cdot\mathbf{p}^{n+\sigma}-\nabla\cdot\mathbf{p}^{n,\sigma})+f(u^{n+\sigma})-f(u^{n,\sigma})$, $e_u^{n,\sigma}$ and $e_{\mathbf{p}}^{n,\sigma}$ are the errors with the decompositions

$$e_u^{n+1}=u^{n+1}-U_h^{n+1}=u^{n+1}-Pu^{n+1}+Pu^{n+1}-U_h^{n+1}=u^{n+1}-Pu^{n+1}+Pe_u^{n+1},\tag{52a}$$

$$e_{\mathbf{p}}^{n+1}=\mathbf{p}^{n+1}-\mathbf{P}_h^{n+1}=\mathbf{p}^{n+1}-\Pi\mathbf{p}^{n+1}+\Pi\mathbf{p}^{n+1}-\mathbf{P}_h^{n+1}=\mathbf{p}^{n+1}-\Pi\mathbf{p}^{n+1}+\Pi e_{\mathbf{p}}^{n+1}.\tag{52b}$$

Here P and Π are the projections defined in (28).

Theorem 4. *Assume that the solution u of the problem (1) satisfies the condition (50) and ${}_C\mathrm{D}^{\alpha}_{0,t}u\in L^\infty((0,T];H^{k+1}(\Omega))$. Let U_h^n be the numerical solution of the fully discrete LDG scheme (48). Suppose $\sigma=1-\alpha/2$, $f(u)$ satisfies the condition (26), and the nonuniform mesh satisfies the maximum time-step condition $\tau_M\le(4\pi_A\Gamma(2-\alpha))^{-1/\alpha}$, then for $n=1,2,\dots,M$, the following estimate holds*

$$\|u^n-U_h^n\|\le C\Big(M^{-\min\{r\alpha,2\}}+h^{k+1}\Big),$$

where C is a positive constant independent of M and h.

Proof. Substituting (52) into (51), we deduce that

$$\begin{aligned}&\big(\Re^{\alpha}_t(Pe_u)^{n+\sigma},v_h\big)_\Omega+\epsilon^2(\Pi e_{\mathbf{p}}^{n,\sigma},\nabla v_h)_\Omega-\epsilon^2\langle\widehat{\Pi e_{\mathbf{p}}^{n,\sigma}}\cdot\mathbf{n},v_h\rangle_\Gamma-\big(f(u^{n,\sigma})-f(U_h^{n,\sigma}),v_h\big)_\Omega\\&\quad=-\big(\Re^{\alpha}_t(u-Pu)^{n+\sigma},v_h\big)_\Omega-\epsilon^2(\mathbf{p}^{n,\sigma}-\Pi\mathbf{p}^{n,\sigma},\nabla v_h)_\Omega\\&\qquad+\epsilon^2\langle(\mathbf{p}^{n,\sigma}-\widehat{\Pi\mathbf{p}^{n,\sigma}})\cdot\mathbf{n},v_h\rangle_\Gamma-(\zeta^{n+\sigma},v_h)_\Omega+(R_2^{n+\sigma},v_h)_\Omega,\end{aligned}\tag{53a}$$

$$\begin{aligned}&(\Pi e_{\mathbf{p}}^{n,\sigma},\mathbf{w}_h)_\Omega+(Pe_u^{n,\sigma},\nabla\cdot\mathbf{w}_h)_\Omega-\langle\widehat{Pe_u^{n,\sigma}},\mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma\\&\quad=-(\mathbf{p}^{n,\sigma}-\Pi\mathbf{p}^{n,\sigma},\mathbf{w}_h)_\Omega-(u^{n,\sigma}-Pu^{n,\sigma},\nabla\cdot\mathbf{w}_h)_\Omega+\langle u^{n,\sigma}-\widehat{Pu^{n,\sigma}},\mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma,\end{aligned}\tag{53b}$$

where $\zeta^{n+\sigma}=({}_C\mathrm{D}^{\alpha}_{0,t}u)^{n+\sigma}-\Re^{\alpha}_t u^{n+\sigma}$ represents truncation error. Making use of the interpolation properties in Section 2.2, we obtain

$$\begin{aligned}&\big(\Re^{\alpha}_t(Pe_u)^{n+\sigma},v_h\big)_\Omega+\epsilon^2(\Pi e_{\mathbf{p}}^{n,\sigma},\nabla v_h)_\Omega-\epsilon^2\langle\widehat{\Pi e_{\mathbf{p}}^{n,\sigma}}\cdot\mathbf{n},v_h\rangle_\Gamma-\big(f(u^{n,\sigma})-f(U_h^{n,\sigma}),v_h\big)_\Omega\\&\quad=-\big(\Re^{\alpha}_t(u-Pu)^{n+\sigma},v_h\big)_\Omega-(\zeta^{n+\sigma},v_h)_\Omega+(R_2^{n+\sigma},v_h)_\Omega,\end{aligned}\tag{54a}$$

$$\begin{aligned}&(\Pi e_{\mathbf{p}}^{n,\sigma},\mathbf{w}_h)_\Omega+(Pe_u^{n,\sigma},\nabla\cdot\mathbf{w}_h)_\Omega-\langle\widehat{Pe_u^{n,\sigma}},\mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma\\&\quad=-(\mathbf{p}^{n,\sigma}-\Pi\mathbf{p}^{n,\sigma},\mathbf{w}_h)_\Omega-(u^{n,\sigma}-Pu^{n,\sigma},\nabla\cdot\mathbf{w}_h)_\Omega+\langle u^{n,\sigma}-\widehat{Pu^{n,\sigma}},\mathbf{w}_h\cdot\mathbf{n}\rangle_\Gamma.\end{aligned}\tag{54b}$$

Setting $(v_h, \mathbf{w}_h) = (Pe_u^{n,\sigma}, \epsilon^2 \Pi e_{\mathbf{p}}^{n,\sigma})$ in (54) and integrating by parts, we arrive at

$$\begin{aligned}
&\big(\Re_t^\alpha (Pe_u)^{n+\sigma}, Pe_u^{n,\sigma}\big)_\Omega + \epsilon^2 \|\Pi e_{\mathbf{p}}^{n,\sigma}\|_\Omega^2 - \big(f(u^{n,\sigma}) - f(U_h^{n,\sigma}), Pe_u^{n,\sigma}\big)_\Omega \\
&\quad = -\big(\Re_t^\alpha (u - Pu)^{n+\sigma}, Pe_u^{n,\sigma}\big)_\Omega - (\zeta^{n+\sigma}, Pe_u^{n,\sigma})_\Omega + (R_2^{n+\sigma}, Pe_u^{n,\sigma})_\Omega, \\
&\quad\quad - \epsilon^2 (\mathbf{p}^{n,\sigma} - \Pi \mathbf{p}^{n,\sigma}, \Pi e_{\mathbf{p}}^{n,\sigma})_\Omega - \epsilon^2 (u^{n,\sigma} - Pu^{n,\sigma}, \nabla \cdot \Pi e_{\mathbf{p}}^{n,\sigma})_\Omega \\
&\quad\quad + \epsilon^2 \langle u^{n,\sigma} - \widehat{Pu^{n,\sigma}}, \Pi e_{\mathbf{p}}^{n,\sigma} \cdot \mathbf{n} \rangle_\Gamma.
\end{aligned} \tag{55}$$

Applying the Cauchy-Schwarz inequality, interpolation property (9), and Lemma 1, we can bound the right hand side of (55) by

$$\begin{aligned}
&\big(\Re_t^\alpha (Pe_u)^{n+\sigma}, Pe_u^{n,\sigma}\big)_\Omega + \epsilon^2 \|\Pi e_{\mathbf{p}}^{n,\sigma}\|_\Omega^2 - \big(f(u^{n,\sigma}) - f(U_h^{n,\sigma}), Pe_u^{n,\sigma}\big)_\Omega \\
&\quad \le \big(\|\Re_t^\alpha (u - Pu)^{n+\sigma}\|_\Omega + \|\zeta^{n+\sigma}\|_\Omega + \|R_2^{n+\sigma}\|_\Omega\big) \|Pe_u^{n,\sigma}\|_\Omega \\
&\quad\quad + \epsilon^2 \|\mathbf{p}^{n,\sigma} - \Pi \mathbf{p}^{n,\sigma}\|_\Omega \|\Pi e_{\mathbf{p}}^{n,\sigma}\|_\Omega + Ch^{k+1} \|\Pi e_{\mathbf{p}}^{n,\sigma}\|_\Omega \\
&\quad \le \big(\|\Re_t^\alpha (u - Pu)^{n+\sigma}\|_\Omega + \|\zeta^{n+\sigma}\|_\Omega + \|R_2^{n+\sigma}\|_\Omega\big) \|Pe_u^{n,\sigma}\|_\Omega + Ch^{k+1} \|\Pi e_{\mathbf{p}}^{n,\sigma}\|_\Omega.
\end{aligned} \tag{56}$$

By using an analysis similar to that in (35), we can obtain the following estimate

$$\begin{aligned}
\big(\Re_t^\alpha (Pe_u)^{n+\sigma}, Pe_u^{n,\sigma}\big)_\Omega \le & \big(\|\Re_t^\alpha (u - Pu)^{n+\sigma}\|_\Omega + \|\zeta^{n+\sigma}\|_\Omega + \|R_2^{n+\sigma}\|_\Omega\big) \|Pe_u^{n,\sigma}\|_\Omega \\
& + C\|Pe_u^{n,\sigma}\|_\Omega^2 + Ch^{2k+2}.
\end{aligned} \tag{57}$$

According to interpolation property (8), we can get

$$\begin{aligned}
&\|\Re_t^\alpha (u - Pu)^{n+\sigma}\|_\Omega \\
&\quad = \big\|\Re_t^\alpha (u - Pu)^{n+\sigma} - ({}_C\mathrm{D}_{0,t}^\alpha (u - Pu))^{n+\sigma} + ({}_C\mathrm{D}_{0,t}^\alpha (u - Pu))^{n+\sigma}\big\|_\Omega \\
&\quad \le \big\|-({}_C\mathrm{D}_{0,t}^\alpha u)^{n+\sigma} + \Re_t^\alpha u^{n+\sigma} + P\big(({}_C\mathrm{D}_{0,t}^\alpha u)^{n+\sigma} - \Re_t^\alpha u^{n+\sigma}\big)\big\|_\Omega \\
&\quad\quad + \|({}_C\mathrm{D}_{0,t}^\alpha (u - Pu))^{n+\sigma}\|_\Omega \\
&\quad \le C\|\zeta^{n+\sigma}\|_{H^1(\Omega)} + Ch^{k+1} \|({}_C\mathrm{D}_{0,t}^\alpha u)^{n+\sigma}\|_{H^{k+1}(\Omega)}.
\end{aligned} \tag{58}$$

Next, we estimate $\max\limits_{0 \le n \le M-1} \{t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega\}$. When $n = 0$, it follows from the assumption of u that there exists a constant C such that

$$t_\sigma^\alpha \|R_2^{n+\sigma}\|_\Omega \le Ct_1^\alpha \le CM^{-r\alpha}.$$

When $n \ge 1$, applying (50) and Lemma 9 in the literature [33], we obtain

$$\begin{aligned}
t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega \le Ct_{n+\sigma}^\alpha \tau_{n+1}^2 t_n^{\alpha-2} &\le C(n+1)^{r\alpha} M^{-r\alpha} M^{-2r} n^{r\alpha-2} M^{-r\alpha+2r} \\
&\le C(n/M)^{2r\alpha-2} M^{-2},
\end{aligned}$$

where we have used $\tau_{n+1} \le CTM^{-r} n^{r-1}$ $(n = 0, 1, \ldots, M-1)$ in the second inequality. As a consequence,

$$t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega \le \begin{cases} CM^{-2}, & n = 1, 2, \ldots, M-1, \ r \ge 1/\alpha, \\ CM^{-2\alpha}, & n = 1, 2, \ldots, M-1, \ 1 \le r < 1/\alpha. \end{cases}$$

Combining the above two cases, we have

$$\max_{0 \le n \le M-1} \{t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega\} \le CM^{-\min\{r\alpha, 2\}}. \tag{59}$$

By using (58), (59), and Lemmas 8 and 9, we arrive at

$$
\begin{aligned}
&\|\Re_t^\alpha (u - Pu)^{n+\sigma}\|_\Omega + \|\zeta^{n+\sigma}\|_\Omega + \|R_2^{n+\sigma}\|_\Omega \\
&\quad \le C\|\zeta^{n+\sigma}\|_{H^1(\Omega)} + Ch^{k+1}\|({}_C\mathrm{D}_{0,t}^\alpha u)^{n+\sigma}\|_{H^{k+1}(\Omega)} + t_{n+\sigma}^{-\alpha} t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega \\
&\quad \le Ct_{n+\sigma}^{-\alpha} \max_{1\le n\le M-1} \left(t_{n+\sigma}^\alpha \|\zeta^{n+\sigma}\|_{H^1(\Omega)} + t_{n+\sigma}^\alpha \|R_2^{n+\sigma}\|_\Omega \right) + Ch^{k+1} \\
&\quad \le Ct_{n+\sigma}^{-\alpha} \left(C \max_{0\le n\le M-1} \left\{ \|\psi_u^{n+\sigma}\|_{H^1(\Omega)} + \left\{ \max_{1\le s\le n} \|\psi_u^{n,s}\|_{H^1(\Omega)} \right\} \right\} + M^{-\min\{r\alpha,2\}} \right) \\
&\qquad + Ch^{k+1} \\
&\quad \le Ct_{n+\sigma}^{-\alpha} \left(M^{-\min\{r\alpha,3-\alpha\}} + M^{-\min\{r\alpha,2\}} \right) + Ch^{k+1} \\
&\quad \le Ct_{n+\sigma}^{-\alpha} M^{-\min\{r\alpha,3-\alpha\}} + Ch^{k+1}.
\end{aligned} \tag{60}
$$

Substituting (60) into (57) and applying Lemma 7, we thus get

$$
\Re_t^\alpha (\|Pe_u\|_\Omega^2)^{n+\sigma} \le \left(Ct_{n+\sigma}^{-\alpha} M^{-\min\{r\alpha,3-\alpha\}} + Ch^{k+1} \right) \|Pe_u^{n,\sigma}\|_\Omega + C\|Pe_u^{n,\sigma}\|_\Omega^2 + Ch^{2k+2}. \tag{61}
$$

Then, invoking Lemmas 6 and (46), one has

$$
\begin{aligned}
\|Pe_u^{n+1}\|_\Omega &\le 2E_{\alpha,1}(2C\pi_A t_{n+1}^\alpha) \left(\max_{0\le i\le n} \sum_{j=0}^{i} P_{i-j+1}^{i+1,\sigma} 2(Ct_{j+\sigma}^{-\alpha} M^{-\min\{r\alpha,3-\alpha\}} + Ch^{k+1}) \right. \\
&\qquad \left. + \sqrt{\pi_A \Gamma(1-\alpha)} \max_{0\le j\le n} \left\{ \sqrt{C} t_{j+1}^{\alpha/2} h^{k+1} \right\} \right) \\
&\le C \max_{0\le i\le n} \sum_{j=0}^{i} P_{i-j+1}^{i+1,\sigma} \left(\omega_{1-\alpha}(t_{j+1}) M^{-\min\{r\alpha,2\}} + h^{k+1} \right) + Ch^{k+1} \\
&\le CM^{-\min\{r\alpha,2\}} + Ch^{k+1},
\end{aligned} \tag{62}
$$

provided that the maximum time-step $\tau_M \le (4\pi_A \Gamma(2-\alpha))^{-1/\alpha}$. By use of the triangle inequality, the interpolation properties (8) and (9), and utilizing (62) yields the desired result. This completes the proof. □

5. Numerical Examples

The purpose of this section is to numerically validate the accuracy and efficiency of proposed Schemes (18) and (48) for solving the time-fractional Allen-Cahn Equation (1) with initial singularity. All the algorithms are implemented using MATLAB R2016a, which were run in a 3.10 GHz PC having 16GB RAM and Windows 10 operating system.

Example 1. *Consider the following two-dimensional time-fractional Allen-Cahn equation with a source term $f(x,y,t)$*

$$
\begin{cases}
{}_C\mathrm{D}_{0,t}^\alpha u(x,y,t) - \Delta u(x,y,t) = u(x,y,t) - u^3(x,y,t) + f(x,y,t), \\
\qquad\qquad\qquad\qquad\qquad\qquad (x,y) \in \Omega,\ t \in (0, \tfrac{1}{4}], \\
u(x,y,0) = 0,\ (x,y) \in \Omega, \\
u(x,y,t) = 0,\ (x,y) \in \partial\Omega,\ t \in (0, \tfrac{1}{4}],
\end{cases}
$$

where $0 < \alpha < 1$, $\Omega = (-1,1) \times (-1,1)$, *and the source term is given by*

$$\begin{aligned} f(x,y,t) = & \left(\Gamma(\alpha+1) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \right) (x+1)^2(x-1)^2(y+1)^2(y-1)^2 \\ & - 4(t^\alpha + t^2)(3x^2-1)(y+1)^2(y-1)^2 \\ & - 4(t^\alpha + t^2)(3y^2-1)(x+1)^2(x-1)^2 \\ & - (t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2 \\ & + \left[(t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2 \right]^3. \end{aligned}$$

The analytical solution is given by $u(x,y,t) = (t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2$.

The purpose of Example 1 is to demonstrate the effectiveness of the nonuniform L1–LDG scheme (18) with the numerical flux (16) for the time-fractional Allen-Cahn equation with weak singularity solution. The L^2-norm errors and convergence orders of the numerical solution U_h^n at $t = \frac{1}{4}$ are shown in Tables 1–4. From Tables 1 and 2, one can see that the convergence orders of scheme (18) in the temporal direction are close to $\min\{2-\alpha, r\alpha\}$. In Tables 3 and 4, we take $r = (2-\alpha)/\alpha$ and $\alpha = 0.4$, 0.6, 0.8, and the orders of convergence for U_h^n are closed to $(k+1)$ in space. These numerical results coincide with Theorem 2.

Table 1. The L^2-norm errors and temporal convergence orders for Example 1 using scheme (18), $M = N_x = N_y, k = 1, T = 1/4, r = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20	1.7270×10^{-2}	–	9.4316×10^{-3}	–	3.0600×10^{-3}	–
40	1.4687×10^{-2}	0.2337	6.7438×10^{-3}	0.4840	1.9214×10^{-3}	0.6736
60	1.3176×10^{-2}	0.2677	5.4723×10^{-3}	0.5153	1.4451×10^{-3}	0.7026
80	1.2143×10^{-2}	0.2840	4.6982×10^{-3}	0.5301	1.1723×10^{-3}	0.7272
100	1.1372×10^{-2}	0.2940	4.1657×10^{-3}	0.5392	9.9365×10^{-4}	0.7409

Table 2. The L^2-norm errors and temporal convergence orders for Example 1 using scheme (18), $M = N_x = N_y, k = 1, T = 1/4, r = (2-\alpha)/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20	4.7878×10^{-3}	–	3.2260×10^{-3}	–	2.3836×10^{-3}	–
40	1.5255×10^{-3}	1.6501	1.0086×10^{-3}	1.6773	7.4703×10^{-4}	1.6739
60	7.2958×10^{-4}	1.8191	4.7799×10^{-4}	1.8418	4.9633×10^{-4}	1.0084
80	4.2488×10^{-4}	1.8794	3.0615×10^{-4}	1.5486	3.6894×10^{-4}	1.0310
100	2.7737×10^{-4}	1.9111	2.2904×10^{-4}	1.3004	2.9212×10^{-4}	1.0463

Table 3. The L^2-norm errors and spatial convergence orders for Example 1 using scheme (18), $M = 500, T = 1/4, r = (2-\alpha)/\alpha, k = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
$N_x \times N_y$	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20×20	4.2289×10^{-3}	–	3.1975×10^{-3}	–	2.3945×10^{-3}	–
40×40	1.3486×10^{-3}	1.6488	1.0092×10^{-3}	1.6637	7.4296×10^{-4}	1.6884
60×60	6.4661×10^{-4}	1.8130	4.8193×10^{-4}	1.8229	3.5275×10^{-4}	1.8371
80×80	3.7761×10^{-4}	1.8697	2.8048×10^{-4}	1.8816	2.0445×10^{-4}	1.8960
100×100	2.4726×10^{-4}	1.8976	1.8299×10^{-4}	1.9139	1.3286×10^{-4}	1.9316

Table 4. The L^2-norm errors and spatial convergence orders for Example 1 using scheme (18), $M = 1000$, $T = 1/4$, $r = (2-\alpha)/\alpha$, $k = 2$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
$N_x \times N_y$	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
10×10	1.8150×10^{-2}	–	1.3377×10^{-2}	–	9.5653×10^{-3}	–
20×20	2.4770×10^{-3}	2.8733	1.8303×10^{-3}	2.8696	1.3144×10^{-3}	2.8634
30×30	7.5324×10^{-4}	2.9360	5.5713×10^{-4}	2.9335	4.0074×10^{-4}	2.9295
40×40	3.2169×10^{-4}	2.9574	2.3829×10^{-4}	2.9523	1.7180×10^{-4}	2.9441

Example 2. *Consider the following two-dimensional time-fractional Allen-Cahn equation with a source term $f(x,y,t)$*

$$\begin{cases} {}_C\mathrm{D}^{\alpha}_{0,t}u(x,y,t) - 0.1\Delta u(x,y,t) = u(x,y,t) - u^3(x,y,t) + f(x,y,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,y) \in \Omega,\ t \in (0, \frac{1}{4}], \\ u(x,y,0) = 0,\ (x,y) \in \Omega, \\ u(x,y,t) = 0,\ (x,y) \in \partial\Omega,\ t \in (0, \frac{1}{4}], \end{cases}$$

where $0 < \alpha < 1$, $\Omega = (-1,1) \times (-1,1)$, and the source term is given by

$$\begin{aligned} f(x,y,t) = &\left(\Gamma(\alpha+1) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)(x+1)^2(x-1)^2(y+1)^2(y-1)^2 \\ &- 0.4(t^\alpha + t^2)(3x^2-1)(y+1)^2(y-1)^2 \\ &- 0.4(t^\alpha + t^2)(3y^2-1)(x+1)^2(x-1)^2 \\ &- (t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2 \\ &+ \left[(t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2\right]^3. \end{aligned}$$

The solution $u(x,y,t) = (t^\alpha + t^2)(x+1)^2(x-1)^2(y+1)^2(y-1)^2$ solves this equation.

It is clear that the exact solution u of Example 2 satisfies the regularity assumption (50), so we use the proposed nonuniform L2-1_σ–LDG scheme (48) to solve this problem. Tables 5 and 6 report the numerical errors and convergence orders in the temporal direction. The data in these tables demonstrate that the temporal convergence order of the numerical solution U_h^n is $\min\{2, r\alpha\}$. In order to test the convergence order of the scheme in spatial direction, we fix sufficiently small temporal step ($M = 500$ for $k = 1$ and $M = 3000$ for $k = 2$) and vary the spatial step sizes. Tables 7 and 8 list the numerical results for different values of α, where the $(k+1)$-th order convergence of scheme (48) in spatial direction can be achieved.

Table 5. The L^2-norm errors and temporal convergence orders for Example 2 using scheme (48), $M = N_x = N_y$, $k = 1$, $T = 1/4$, $r = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20	1.5147×10^{-2}	–	5.3376×10^{-3}	–	2.0856×10^{-3}	–
40	1.1155×10^{-2}	0.4413	3.4459×10^{-3}	0.6313	7.5052×10^{-4}	1.4745
60	9.3446×10^{-3}	0.4368	2.6785×10^{-3}	0.6214	5.4056×10^{-4}	0.8094
80	8.2484×10^{-3}	0.4338	2.2427×10^{-3}	0.6172	4.2859×10^{-4}	0.8068
100	7.4915×10^{-3}	0.4313	1.9552×10^{-3}	0.6148	3.5808×10^{-4}	0.8055

Table 6. The L^2-norm errors and temporal convergence orders for Example 2 using scheme (48), $M = N_x = N_y, k = 1, T = 1/4, r = (3 - \alpha)/\alpha$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20	4.8695×10^{-3}	–	2.5966×10^{-3}	–	1.9295×10^{-3}	–
40	1.4639×10^{-3}	1.7340	7.5597×10^{-4}	1.7802	5.4683×10^{-4}	1.8190
60	6.9287×10^{-4}	1.8448	3.5345×10^{-4}	1.8751	2.5311×10^{-4}	1.8998
80	4.0253×10^{-4}	1.8878	2.0395×10^{-4}	1.9114	1.4529×10^{-4}	1.9296
100	2.6277×10^{-4}	1.9113	1.3256×10^{-4}	1.9309	9.4126×10^{-5}	1.9454

Table 7. The L^2-norm errors and spatial convergence orders for Example 2 using scheme (48), $M = 500, T = 1/4, r = (3 - \alpha)/\alpha, k = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
$N_x \times N_y$	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
20×20	3.4951×10^{-3}	–	2.5737×10^{-3}	–	1.9337e-03	–
40×40	1.0436×10^{-3}	1.7438	7.4714×10^{-4}	1.7844	5.4817×10^{-4}	1.8187
60×60	4.9189×10^{-4}	1.8551	3.4871×10^{-4}	1.8794	2.5370×10^{-4}	1.9001
80×80	2.8521×10^{-4}	1.8946	2.0099×10^{-4}	1.9153	1.4561×10^{-4}	1.9301
100×100	1.8614×10^{-4}	1.9123	1.3053×10^{-4}	1.9343	9.4320×10^{-5}	1.9460

Table 8. The L^2-norm errors and spatial convergence orders for Example 2 using scheme (48), $M = 3000, T = 1/4, r = (3 - \alpha)/\alpha, k = 2$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
$N_x \times N_y$	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
10×10	1.2098×10^{-2}	–	7.7466×10^{-3}	–	4.8298×10^{-3}	–
20×20	1.6927×10^{-3}	2.8374	1.1255×10^{-3}	2.7830	7.5531×10^{-4}	2.6768
30×30	5.1633×10^{-4}	2.9283	3.4491×10^{-4}	2.9169	2.3353×10^{-4}	2.8950
40×40	2.2074×10^{-4}	2.9539	1.4765×10^{-4}	2.9492	1.0020×10^{-4}	2.9413

6. Concluding Remarks

This paper focuses on the numerical algorithms for the time-fractional Allen-Cahn equation with a weak singularity solution. In the time direction, it is discretized by the nonuniform L1 scheme and the nonuniform L2-1_σ scheme, respectively. In the spatial direction, the LDG method is utilized. By the discrete fractional Gronwall-type inequalities, the L^2 stability and optimal error estimates of these two schemes are proved in detail. Finally, the efficiency and accuracy of proposed fully discrete schemes are verified by some numerical examples. In future work, we extend the technique of coupling the LDG method with the nonuniform time discretization to solve the space-time fractional phase-field model.

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