fractal and fractional

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Local and Global Existence and Uniqueness of Solution for Time-Fractional Fuzzy Navier–Stokes Equations

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Abstract: Navier–Stokes (NS) equation, in fluid mechanics, is a partial differential equation that describes the flow of incompressible fluids. We study the fractional derivative by using fractional differential equation by using a mild solution. In this work, anomaly diffusion in fractal media is simulated using the Navier–Stokes equations (NSEs) with time-fractional derivatives of order $\beta \in (0,1)$. In $\mathbf{H}^{\gamma,\wp}$, we prove the existence and uniqueness of local and global mild solutions by using fuzzy techniques. Meanwhile, we provide a local moderate solution in Banach space. We further show that classical solutions to such equations exist and are regular in Banach space.

Keywords: Navier–Stokes equation; Caputo fractional derivative; Mittag–Leffler function; mild solution; fuzzy fractional differential equation; regularity

MSC: 34K37; 34B15

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1. Introduction

The NSEs express the conservation of mass and momentum in incompressible Newtonian fluid dynamics ranging from large-scale atmospheric motions to ball-bearing lubrication. The equation is a generalization of the equation proposed by Swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids. Later on in 1821 French engineer Claude-Louis Navier work on it. In the middle of the 19th century, British physicist and mathematician Sir George Gabriel Stokes improved on this work. They are sometimes accompanied by an equation of state relating to pressure, temperature and density. They arise from applying Isaac Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term—hence describing viscous flow. The difference between them and the closely related Euler equations is that NSEs take viscosity into account while the Euler equations model only inviscid flow. As a result, the Navier–Stokes are a parabolic equation and therefore have better analytic properties, at the expense of having less mathematical structure (e.g., they are never completely integrable). The NS equations are useful because they describe the physics of many phenomena of scientific and engineering interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The NSEs in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with Maxwell's equations, they can be used to model and study magnetohydrodynamics. The NSEs are also of great interest in a purely mathematical sense. Despite their wide range of practical uses, it has not yet been proven whether smooth solutions always exist in three dimensions—i.e., they are infinitely differentiable at all points in the domain. This is called the NS existence and smoothness problem. More information can be found in Cannone's [1] and Varnhorn's [2] monographs (see, for example, Lemarie-Rieusset [3] and Von Wahl [4]); there are so many phenomena for a system that explaining their existence, regularity, and boundary conditions requires the complete strength of the mathematical theory.

It is worth noting that Leray first discovered that the boundary-value problem for time-dependent NSEs has a unique smooth solution for specific time intervals if the data are sufficiently smooth. Many authors have investigated the existence of mild, weak, and strong solutions for NSEs since then; for example, Heck et al. [5], Chemin and Gallagher [6], Choe [7], Giga [8], Raugel [9], Almeida and Ferreira [10], wabuchi and Takada [11], Koch et al. [12], Masmoudi and Wong [13], Amrouche and Rejaiba [14], Chemin et al. [15], Danchin [16] and Kozono [17].

Fractional calculus has grown in popularity in recent decades, owing to its demonstrated applications in a variety of seemingly diverse and large-ranging fields of science and engineering, such as fluid flow, rheology, dynamical processes, porous structures, diffusive transport akin to diffusion, control theory of dynamical systems and viscoelasticity, etc., for example [18–22]. The models given by partial differential equations with fractional derivatives are the most important. Not only physicists, but even pure mathematicians, are interested in such models.

According to recent theoretical and experimental findings, the classical diffusion equation fails to characterize diffusion phenomena in heterogeneous porous media with fractal properties. What changes are made to the classical diffusion equation to make it suitable for describing anomalous diffusion phenomena? For researchers, this is an interesting challenge. Since it has been acknowledged as one of the greatest methods for characterizing long memory processes, fractional calculus helps model anomalous diffusion processes. As a result, presenting the generalized NSEs with a Caputo fractional derivative operator, which can be used to model anomalous diffusion in fractal media, is logical and significant. Its evolutions act in a far more complex manner than standard inter-order evolutions, making study more difficult.

The most effort has been paid to attempts to acquire numerical and analytical solutions to time-fractional NSEs [23–25]. We are only aware of a few conclusions about mild solutions of existence and regularity for time-fractional NSEs. Carvalho-Neto [26] recently discussed the existence-uniqueness of global and local mild solutions for time-fractional NSEs. Niazi et al. [27], Iqbal et al. [28] and Shafqat et al. [29] investigated the existence-uniqueness of the fuzzy fractional evolution equation.

Zhou and Peng [30] worked on time-fractional NSEs in an open set:

$$\begin{aligned}
&\partial_\omega^\alpha U - \nu\Delta U + (U.\triangledown)U = -\nabla p + f,\ \omega > 0,\\
&\nabla.U = 0,\\
&U|_{\partial\Omega} = 0,\\
&U(0,\mathfrak{X}) = a,
\end{aligned} \tag{1}$$

where ∂_ω^α is Caputo fractional derivative of order $\alpha \in (0,1), U = (U_1(\omega,\mathfrak{X}), U_2(\omega,\mathfrak{X}), \ldots, U_n(\omega,\mathfrak{X}))$ represents velocity field at point $\mathfrak{X} \in \Omega$ and time $\omega > 0, p = p(\omega,\mathfrak{X})$ is pressure, v is viscosity, $f = f(\omega,\mathfrak{X})$ is external force and $a = a(\mathfrak{X})$ is initial velocity.

In this paper, we investigate the below time-fractional NSEs in an open set $\Omega \subset \mathbf{R^n}(n \geq 3)$, which is motivated by the above discussion:

$$\begin{aligned}
&\partial_\omega^\beta U - v\Delta U + (U.\nabla)U = -\nabla p + g,\ \omega > 0,\\
&\nabla.U = 0,\\
&U|_{\partial\Omega} = 0,\\
&U(0,y) = by\sin\gamma,
\end{aligned} \tag{2}$$

where ∂_ω^β is Caputo fractional derivative of order $\beta \in (0,1), U = (U_1(\omega,\mathfrak{X}), U_2(\omega,\mathfrak{x}), \ldots, U_n(\omega,\mathfrak{X}))$ represents velocity field at a point and time $\omega > 0, p = p(\omega,\mathfrak{X})$ is pressure, v is viscosity, $g = g(\omega,\mathfrak{X})$ is gravitational force and $by\sin\gamma = by(\mathfrak{X})\sin\gamma$ is initial velocity. We will suppose that the boundary of Ω is smooth.

To begin, the pressure term is removed by using the Helmholtz projector P to Equation (2), which transforms Equation (2) into Equation (3) as:

$$\begin{aligned}
&\partial_\omega^\beta U - vP\Delta U + P(U.\triangledown)U = Pg,\ \omega > 0,\\
&\triangledown.U = 0,\\
&U|_{\partial\Omega} = 0,\\
&U(0,y) = by\sin\gamma.
\end{aligned}$$

The operator $-vP\Delta$ with Dirichlet boundary conditions is effectively the Stokes operator $\mathcal{A}$ in the divergence-free function space under consideration. Then, in the abstract form illustrated below, we rewrite (2).

$$\begin{aligned}
&{}^C\mathcal{D}_\omega^\beta U = -\mathcal{A}U + \mathfrak{G}(U,U) + P\mathfrak{g},\ \omega > 0,\\
&U(0) = by\sin\gamma,
\end{aligned} \tag{3}$$

where $\mathfrak{G}(U,v) = -P(U.\triangledown)v$. The solution to Equation (2) is also the solution to Equation (3) if the Helmholtz projection P and the Stokes operator $\mathcal{A}$ make sense.

The purpose of this research is to demonstrate that global and local mild solutions to Equation (2) in $\mathbf{H}^{\beta,\wp}$ exist and are unique. We further show that if $P\mathfrak{g}$ is Hölder continuous, there exists a single classical solution $U(\omega)$ such that AU and ${}^CD_\omega^\gamma U(\omega)$ are Hölder continuous in $\mathbf{S}_\wp$. The following is a breakdown of the structure of the paper. In Section 2, we go through numerous notations, definitions, and background information. Before moving on to the local mild solution in $\mathbf{H}^{\beta,\wp}$, Section 3 looks at existence and uniqueness of global mild solution in $\mathbf{H}^{\beta,\wp}$ of issue (3). In Section 4, we use the iteration method to determine the existence and regularity of a classical solution to the issue (2) in $\mathbf{S}_\wp$. Finally, in Section 4, a conclusion is provided.

2. Preliminaries

We establish notations, definitions and introductory facts in this section, which will be used throughout the work.

Assume $\Omega = \{(\mathfrak{X}_1,\ldots,\mathfrak{X}_n) : \mathfrak{X}_n > 0\}$ to be open subset of $\mathbf{R^n}$, where $n \geq 3$. Assume $1 < \wp < \infty$. Then there is bounded projection P called the Hodge projection on $(\mathcal{L}^\wp(\Omega))^n$, whose range is the closure of:

$$\mathcal{C}_\sigma^\infty(\Omega) := \{u \in (\mathcal{C}^\infty(\Omega))^n : \nabla.u = 0,\ u \text{ has compact support in } \Omega\},$$

and whose null space is the closure of:

$$\{u \in (\mathcal{C}^\infty(\Omega))^n : u = \nabla\varpi, \varpi \in \mathcal{C}^\infty(\Omega)\}.$$

For notational convenience, let $\mathbf{S}_\wp := \overline{\mathcal{C}_\sigma^\infty(\Omega)}^{|.|_\wp}$, which is a closed subspace of $(\mathcal{L}^\wp(\Omega))^n$. $(\mathcal{W}^{m,\wp}(\Omega))^n$ be a Sobolev space with norm $|.|_{m,\wp}$.

$\mathcal{A} = -vP\Delta$ represents a Stokes operator in $\mathbf{S}_\wp$ whose domain is $\mathcal{D}_\wp(\mathcal{A}) = \mathcal{D}_\wp(\Delta)\bigcap\mathbf{S}_\wp$; here,

$$\mathcal{D}_\wp(\Delta) = \{U \in (\mathcal{W}^2,\wp(\Omega))^n : U|_{\partial\Omega} = 0\}.$$

It is well known that the closed linear operator $-\mathcal{A}$ forms the bounded analytic semigroup $\{e^{-\omega\mathcal{A}}\}$ on $\mathbf{S}_\wp$.

So as to state our results, we need to introduce the definitions of the fractional spaces associated with $-\mathcal{A}$. For $\gamma > 0$ and $U \in \mathbf{S}_\wp$, define:

$$A^{-\gamma}U = \frac{1}{\Gamma(\gamma)}\int_0^\infty \omega^{\gamma-1}e^{-\omega A}Ud\omega.$$

Then $\mathcal{A}^{-\gamma}$ is a bounded, one-to-one operator on $\mathbf{S}_{\wp}$. Let $\mathcal{A}^{\gamma}$ be the inverse of $\mathcal{A}^{-\beta}$. For $\gamma > 0$, we denote the space $\mathbf{H}^{\gamma,\wp}$ by the range of $\mathcal{A}^{-\gamma}$ with the norm:

$$|U|_{H^{\gamma,\wp}} = |\mathcal{A}^{\gamma} u|_{\wp}.$$

It is easy to check that $e^{-\omega\mathcal{A}}$ extends (or restricts) to a bounded analytic semigroup on $\mathbf{H}^{\gamma,\wp}$. For more details, we refer to Van Wahl [4].

Define $\mathcal{D} : \mathbf{E^1} \times \mathbf{E^1} \to \mathbf{R}_+$ by equation

$$D(\mathfrak{X}, y) = \sup_{0 \leqslant \gamma \leqslant 1} \max\{[U]^{\gamma}, [v]^{\gamma}\};$$

d is the Hausdorff metric for non-empty compact sets in $\mathbf{R^n}$.

$\mathcal{D}$ is a metric in $\mathbf{E^1}$. By using the following results:

(i) $(\mathbf{E^1}, \mathcal{D})$ is complete metric space;
(ii) $\mathcal{D}(\mathfrak{X} \oplus z, y \oplus z) = \mathcal{D}(\mathfrak{X}, y)\ \forall\ \mathfrak{X}, y, z \in \mathbf{E^1}$;
(iii) $\mathcal{D}(k\mathfrak{X}, ky) = |k|\mathcal{D}(\mathfrak{X}, y)\ \forall\ \mathfrak{X}, y \in \mathbf{E^1}$ and $k \in \mathbf{R^n}$;
(iv) $\mathcal{D}(\mathfrak{X} \oplus y, z \oplus e) \leqslant \mathcal{D}(\mathfrak{X}, z) \oplus \mathcal{D}(y, e)\ \forall\ \mathfrak{X}, y, z, \in 2\mathbf{E^1}$.

Let χ be a Banach space and $\mathbf{S}$ be an interval of $\mathbf{R^n}$. $\mathcal{C}(\mathbf{S}, \chi)$ denotes the set of all continuous χ-valued functions. For $0 < \vartheta < 1, \mathcal{C}^{\vartheta}(\mathbf{S}, \chi)$ stands for the set of all functions which are Holder continuous with the exponent ϑ.

Assume $\beta \in (0, 1]$ and $v : [0, \infty) \to \chi$. The fractional integral of order β with the lower limit zero for the function v is defined as:

$${}_{a}\mathcal{D}^{\wp}_{\omega}\mathfrak{g}(\omega) = \left(\frac{d}{d\omega}\right)^{n+1} \int_{a}^{\omega} (\omega - \tau)^{n-\wp}\mathfrak{g}(\tau)d\tau, \quad (n \leq \wp \leq n+1),$$

provided the right hand-side is pointwise defined on $[0, \infty)$, where $\mathfrak{g}_{\beta}$ denotes the RL kernel:

$$\mathfrak{g}_{\beta}(\omega) = \frac{\omega^{\beta-1}}{\Gamma(\beta)},\ \omega > 0,$$

and Γ is the usual γ function. In case $\beta = 0$, we denote $\mathfrak{g}_0(\omega) = \delta(\omega)$; the Dirac measure is concentrated at the origin.

Further, for a function $w : [0, \infty) \to \mathbb{R}$, the Caputo derivative of order $\gamma \in \mathbb{R}_+$ is defined by:

$${}^{c}_{0}D^{\gamma}_{t} w(t) = {}^{L}_{0}D^{\gamma}_{t}\Big(w(t) - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{k!} t^{k}\Big), \quad t \geq 0,\ n-1 < \gamma < n.$$

We refer the reader to Kilbas et al. [31] for further information. Let us look at the Mittag–Leffler special functions in general:

$$\begin{aligned} E_{\beta}(-\omega^{\beta}\mathcal{A}) &= \int_{0}^{\infty} \mathcal{M}_{\beta}(s) e^{-s\omega^{\beta}\mathcal{A}} ds, \\ E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) &= \int_{0}^{\infty} \beta s \mathcal{M}_{\beta}(s) e^{-s\omega^{\beta}\mathcal{A}} ds. \end{aligned}$$

Definition 1 ([32])**.** *The Wright function ψ_{β} is defined by:*

$$\begin{aligned} \psi_{\alpha}(\theta) &= \sum_{n=0}^{\infty} \frac{(-\theta)^{n}}{n!\Gamma(-\beta n + 1 - \beta)} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-\theta)^{n}}{(n-1)!} \Gamma(n\beta) \sin(n\pi\beta), \end{aligned}$$

where $\theta \in \mathcal{C}$ *with* $0 < \beta < 1$.

Proposition 1.

$$\begin{aligned}(i)\ E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} E_{\beta,\beta}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu;\\ (ii)\ \mathcal{A}^{\gamma}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} \mu^{\gamma}E_{\beta,\beta}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu.\end{aligned}$$

Proof. (i) In view of $\int_0^\infty \beta s M_\beta(s)e^{-s\omega}ds = E_{\beta,\beta}(-\omega)$ and Fubini theorem, we get:

$$\begin{aligned}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) &= \int_0^\infty \beta s M_\beta(s)e^{-s\omega^{\beta}\mathcal{A}}ds\\ &= \frac{1}{2\pi i}\int_0^\infty \beta s M_\beta(s)\int_{\Gamma_\theta} e^{-\mu s\omega^{\beta}}(\mu I+\mathcal{A})^{-1}d\mu ds\\ &= \frac{1}{2\pi i}\int_{\Gamma_\theta} E_{\beta,\beta}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu,\end{aligned}$$

where Γ_θ is a suitable integral path;

(ii) A similar argument shows that:

$$\begin{aligned}\mathcal{A}^{\alpha}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) &= \int_0^\infty \beta s M_\beta(s)A^{\beta}e^{-s\omega^{\beta}\mathcal{A}}ds\\ &= \frac{1}{2\pi i}\int_0^\infty \beta s M_\beta(s)\int_{\Gamma_\theta} \mu^{\alpha}e^{-\mu s\omega^{\beta}}(\mu I+\mathcal{A})^{-1}d\mu ds\\ &= \frac{1}{2\pi i}\int_{\Gamma_\theta} \mu^{\alpha}E_{\beta,\beta}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu.\end{aligned}$$

□

We also have the results below.

Lemma 1 ([33])**.** *For* $\omega > 0, E_\beta(-\omega^\beta\mathcal{A})$ *and* $E_{\beta,\beta}(-\omega^\beta\mathcal{A})$ *are continuous in the uniform operator topology. Moreover, for every* $r > 0$*, the continuity is uniform on* $[r,\infty)$.

Lemma 2 ([33])**.** *Let* $0 < \beta < 1$*. Then,*

(i) *for all* $U \in \chi$, $\lim_{t\to 0^+} E_\beta(-\omega^\beta\mathcal{A})U = U$;

(ii) *for all* $U \in \mathcal{D}(\mathcal{A})$ *and* $\omega > 0$, ${}^{\mathcal{C}}\mathcal{D}^{\beta}_{\omega}E_\beta(-\omega^\beta\mathcal{A})U = -\mathcal{A}E_\beta(-\omega^\beta A)U$;

(iii) *for all* $U \in \chi$, $E'_\beta(-\omega^\beta\mathcal{A})U = -\omega^{\beta-1}\mathcal{A}E_{\beta,\beta}(-\omega^\beta\mathcal{A})U$;

(iv) *for* $\omega > 0, E_\beta(-\omega^\beta\mathcal{A})U = I^{1-\beta}_{\omega}\Big(\omega^{\beta-1}E_{\beta,\beta}(-\omega^\beta A)u\Big)$.

Before presenting the definition of a mild solution of the problem (3), we give the following lemma for a given function $h : [0,\infty) \to \chi$. For more details we refer to Zhou [32,34].

Lemma 3. *If* $\chi(t)$ *is solution of Equation (3) for* $U(0) = by \sin\gamma$*, then* $U(t)$ *is given.*

$$U(t) = t^{\gamma-1}b\sin\gamma + \frac{1}{\sqrt{\gamma}}\int_0^t (t-s)^{\gamma-1}[-\mathcal{A}U + \mathfrak{G}(U,U) + P\mathfrak{g}]ds \tag{4}$$

holds, then:

$$U(t) = t^{\gamma-1}P_\gamma(t)b\sin\gamma + \int_0^t (t-s)^{\gamma-1}P_\gamma(t-s)[\mathfrak{G}(U,U) + P\mathfrak{g}]ds,$$

where

$$P_\gamma(t) = \int_0^\infty q(\theta)M_\gamma(\theta)Q(t^\gamma\theta)d\theta.$$

We adopt the following definitions of the mild solution to the problem (3), which were inspired by the previous section.

Definition 2. *A function $U : [0,\infty) \to \mathbf{H}^{\gamma,\wp}$ is called a global mild solution of problem (3) in $\mathbf{H}^{\gamma,\wp}$, if $U \in \mathcal{C}([0,\infty), \mathbf{H}^{\gamma,\wp})$ and for $\omega \in [0,\infty)$,*

$$\begin{aligned} U(\omega) =& E_\beta(-\omega^\beta \mathcal{A})b\sin\gamma + \int_0^\omega (\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(U(s),U(s))ds \\ &+ \int_0^\omega (\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\omega\mathcal{A})P\mathfrak{g}(s))ds. \end{aligned} \tag{5}$$

Definition 3. *Assume $0 < \Im < \infty$. A function $U : [0,\Im] \to \mathbf{H}^{\gamma,\wp}$ (or $\mathbf{S}_\wp$) is called a local mild solution of problem (3) in $\mathbf{H}^{\gamma,\wp}$ (or $\mathbf{S}_\wp$), if $U \in \mathcal{C}([0,\Im], \mathbf{H}^{\gamma,\wp})$ (or $\mathcal{C}([0,\Im],\mathbf{S}_\wp)$) and u satisfies (5) for $\omega \in [0,\Im]$.*

Two operators, ϕ and $\mathfrak{g}$, are defined as follows for convenience:

$$\begin{aligned} \varpi(\omega) &= \int_0^\omega (\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s))ds \\ \mathfrak{g}(U,v)(\omega) &= \int_0^\omega (\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(u(s),u(s))ds. \end{aligned}$$

The following fixed point result is used in further proofs.

Lemma 4 ([1])**.** *Assume that $(\chi, ||.||\chi)$ is a Banach space, that $\mathfrak{G} : \chi\times\chi \to \chi$ is a bilinear operator, and that $\mathcal{L}$ is a positive real number:*

$$\mathcal{D}_\chi(\mathfrak{G}(U,v)) \le \mathcal{L}\mathcal{D}_\chi(U,v), \forall U, v \in \chi.$$

Then for any $U_0 \in \chi$ with $||U_0||_\chi < \frac{1}{4\mathcal{L}}$, the equation $U = U_0 + \mathfrak{G}(U,U)$ has unique solution $U \in \chi$.

3. Global and Local Existence in $H^{\gamma,\wp}$

The major goal of this part is to demonstrate adequate requirements for the existence-uniqueness of a mild solution to the problem (3) in $\mathbf{H}^{\gamma,\wp}$. To this purpose, we make the following assumptions:

$P\mathfrak{g}$ is continuous for $\omega > 0$ and $\mathcal{D}_\wp(P\mathfrak{g}(\omega)) = o(\omega^{-\beta(1-\gamma)})$ as $\omega \to 0$ for $0 < \gamma < 1$.

Lemma 5 ([35] (see also [36]))**.** *Let $1 < \wp < \infty$ and $\gamma_1 \le \gamma_2$. Then there exist $\mathcal{C} = \mathcal{C}(\gamma_1,\gamma_2)$ such that*

$$\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(e^{-\omega\mathcal{A}}v) \le \mathcal{C}\omega^{-(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_1,\wp}}(v), \omega > 0$$

for $v \in \mathbf{H}^{\gamma_1,\wp}$. Furthermore, $\lim\limits_{\omega\to 0}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(e^{-\omega\mathcal{A}}v) = 0$.

Now, a necessary technical lemma that will help us prove the section's final main theorems.

Lemma 6. *Assume $1 < \wp < \infty$ and $\gamma_1 \le \gamma_2$. Then for any $\Im > 0$, there exist a constant $\mathcal{C}_1 = \mathcal{C}_1(\gamma_1,\gamma_2) > 0$, such that the following holds:*

$$\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(E_{\beta,1}(-\omega^\beta A)v) \le \mathcal{C}_1\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v) \text{ and } D_{\mathbf{H}^{\gamma_2,\wp}}(E_{\beta,\beta}(-\omega^\beta A)v) \le \mathcal{C}_1\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v)$$

for all $v \in \mathbf{H}^{\gamma_1,\wp}$ and $\omega \in (0,\Im]$. Moreover,

$$\lim_{\omega\to 0}\omega^{\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(E_\beta(-\omega^\beta Av)) = 0.$$

Proof. Suppose $v \in \mathbf{H}^{\gamma_1,\wp}$. By using Lemma 5,

$$\begin{aligned}
\mathcal{D}_{\mathbf{H}^{\gamma_2,q}}(E_{\beta,1}(-\omega^{\beta}\mathcal{A})v) &\leq \int_0^{\infty} \mathcal{M}_{\beta}(s)e^{-s\omega^{\beta}\mathcal{A}}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v) \\
&\leq \left(\mathcal{C}\int_0^{\infty} \mathcal{M}_{\beta}(s)s^{-(\gamma_2-\gamma_1)}ds\right)\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v) \\
&\leq \mathcal{C}_1\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v).
\end{aligned}$$

Lebesgue's dominated convergence theorem demonstrates that:

$$\lim_{\omega\to 0}\omega^{\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(E_{\beta}(-\omega^{\beta}\mathcal{A})v) \leq \int_0^{\infty}\mathcal{M}_{\beta}(s)\lim_{\omega\to 0}\omega^{\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(e^{-s\omega^{\beta}\mathcal{A}}v)ds = 0.$$

Similarly,

$$\begin{aligned}
\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(E_{\beta,\beta}(-\omega^{\beta}\mathcal{A})v) &\leq \int_0^{\infty} \beta s\mathcal{M}_{\beta}(s)e^{-s\omega^{\beta}\mathcal{A}}\mathcal{D}_{\mathbf{H}^{\gamma_2,\wp}}(v)ds \\
&\leq \left(\beta\mathcal{C}\int_0^{\infty} \beta\mathcal{M}_{\beta}(s)s^{1-(\gamma_2-\gamma_1)}ds\right)\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_1,\wp}}(v) \\
&\leq \mathcal{C}_1\omega^{-\beta(\gamma_2-\gamma_1)}\mathcal{D}_{\mathbf{H}^{\gamma_1,\wp}}(v).
\end{aligned}$$

If $\mathcal{C}_1 = \mathcal{C}_1(\beta,\gamma_1,\gamma_2)$ is a constant which is satisfying the following:

$$\mathcal{C}_1 \geq \mathcal{C}\max\left\{\frac{\Gamma(1-\gamma_2+\gamma_1)}{\Gamma(1+\beta(\gamma_1-\gamma_2))}, \frac{\beta\Gamma(2-\gamma_2+\gamma_1)}{\Gamma(1+\beta(1+\gamma_1-\gamma_2))}\right\}.$$

□

4. Global Existence in $\mathbf{H}^{\gamma,\wp}$

In this section, our aim is to find the global mild solution of the problem (3) in $\mathbf{H}^{\gamma,\wp}$. For convenience, we denote:

$$\begin{aligned}
\mathcal{M}(\omega) &= \sup_{s\in(0,\omega]}\{s^{\beta(1-\gamma)}\mathcal{D}_{\wp}(P\mathfrak{g}(s))\}, \\
\mathcal{B}_1 &= \mathcal{C}_1\max\{\mathcal{B}(\beta(1-\gamma),1-\beta(1-\gamma)),\mathcal{B}(\beta(1-\alpha),1-\beta(1-\gamma))\}, \\
\mathcal{L} &\geq \mathcal{M}\mathcal{C}_1\max\Big\{\mathcal{B}(\beta(1-\gamma),1-2\beta(\alpha-\gamma)),\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\gamma))\Big\},
\end{aligned}$$

where $\mathcal{M}$ is given later.

Theorem 1. *Assume $1 < \wp < \infty, 0 < \gamma < 1$ and $(\mathfrak{g})$ hold. For every $\beta \in \mathbf{H}^{\gamma,\wp}$, let that*

$$\mathcal{C}_1 D_{\mathbf{H}^{\gamma,\wp}}(a) + \mathcal{B}_1\mathcal{M}_{\infty} < \frac{1}{4\mathcal{L}}, \tag{6}$$

where $\mathcal{M}_{\infty} := \underset{s\in(0,\infty)}{sup}\{s^{\beta(1-\gamma)}P\mathfrak{g}(s)\}$. If $\frac{n}{2q}-\frac{1}{2} < \gamma$, then there is a $\alpha > \max\{\gamma,\frac{1}{2}\}$ and unique function $U : [0,\infty) \to \mathbf{H}^{\gamma,\wp}$ satisfying:

(i) *$U : [0,\infty) \to \mathbf{H}^{\gamma,\wp}$ is a continuous and $U(0) = a$;*
(ii) *$U : [0,\infty) \to \mathbf{H}^{\alpha,\wp}$ is a continuous and $\lim_{\omega\to 0}\omega^{\beta(\alpha-\gamma)}D_{\mathbf{H}^{\gamma,\wp}}(U(\omega)) = 0$;*
(iii) *U satisfies (5) for $\omega \in [0,\infty)$.*

Proof. Let $\alpha = \frac{(1+\gamma)}{2}$. Define $\chi_{\infty} = \chi[\infty]$ as the space of all curves $U : (0,\infty) \to \mathbf{H}^{\gamma,\wp}$ such that:

(1) $U : [0,\infty) \to \mathbf{H}^{\gamma,\wp}$ is bounded and a continuous;

(2) $U : [0,\infty) \to \mathbf{H}^{\alpha,\wp}$ is bounded and a continuous, in addition, $\lim_{\omega\to 0} \omega^{\beta(\alpha-\gamma)}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U(\omega)) = 0$; with its natural norm.

$$\mathcal{D}_{X_\infty}(U) = \max\left\{ \sup_{\omega\geq 0} \mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(U(\omega)), \sup_{\omega\geq 0} \omega^{\beta(\alpha-\gamma)}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U(\omega)) \right\}.$$

It is obvious that χ_∞ is a non-empty complete metric space.

We know that $\mathfrak{G} : \mathbf{H}^{\alpha,\wp} \times H^{\alpha,\wp} \to \mathbf{S}_\wp$ is bounded bilinear map because of Weissler [36], so there exists $\mathcal{M}$ such that for $u, v \in \mathbf{H}^{\alpha,\wp}$.

$$\begin{aligned} \mathcal{D}_\wp(\mathfrak{G}(U,v)) &\leq \mathcal{M}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U,v), \\ \mathcal{D}_\wp(\mathfrak{G}(U,U) - \mathfrak{G}(v,v)) &\leq \mathcal{M}(\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U) + \mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(v))\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U-v). \end{aligned} \tag{7}$$

Step 1. Suppose $U, v \in \chi_\infty$. We demonstrate that the operator $\mathfrak{g}(U(\omega), v(\omega))$ belongs to $\mathcal{C}([0,\infty), \mathbf{H}^{\gamma,\wp})$ as well as $\mathcal{C}((0,\infty), \mathbf{H}^{\gamma,\wp})$. For arbitrary $\omega_0 \geq 0$ fixed and $\epsilon > 0$ enough small, consider $\omega > \omega_0$, we have:

$$\begin{aligned} & \mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(\mathfrak{g}(U(\omega), v(\omega)) - \mathfrak{g}(U(\omega_0), v(\omega_0))) \\ \leq\ & \int_0^\omega (\omega - s)^{\beta-1}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(U(s), v(s)))ds \\ & + \int_0^{\omega_0} \mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(((\omega-s)^{\beta-1} - (\omega_0 - s)^{\beta-1})E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(U(s), v(s)))ds \\ & + \int_0^{\omega_0-\epsilon} (\omega_0 - s)^{\beta-1}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A}) - E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A}))\mathfrak{G}(U(s), v(s)))ds \\ & + \int_{\omega_0-\epsilon}^{\omega_0} (\omega_0 - s)^{\beta-1}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A}) - E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A}))\mathfrak{G}(U(s), v(s)))ds \\ :=\ & I + I_{11}(\omega) + I_{12}(\omega) + I_{13}(\omega) + I_{14}(\omega). \end{aligned}$$

Each of the four terms is estimated separately. In view of Lemma 6, we derive $I_{11}(\omega)$,

$$\begin{aligned} I_{11} &\leq \mathcal{C}_1 \int_{\omega_0}^{\omega} (\omega - s)^{\beta(1-\gamma)-1}\mathcal{D}_\wp(\mathfrak{G}(U(s), v(s)))ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_{\omega_0}^{\omega} (\omega - s)^{\beta(1-\gamma)-1}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(U(s))\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(v(s))ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_{\omega_0}^{\omega} (\omega - s)^{\beta(1-\gamma)-1}s^{-2\beta(\alpha-\gamma)}ds \sup_{s\in[0,\omega]} \{s^{2\beta(\alpha-\gamma)}|U(s)|_{\mathbf{H}^{\alpha,\wp}}|v(s)|_{\mathbf{H}^{\alpha,\wp}}\} \\ &= \mathcal{M}\mathcal{C}_1 \int_{\frac{\omega_0}{\omega}}^{1} (\omega - s)^{\beta(1-\gamma)-1}s^{-2\beta(\alpha-\gamma)}ds \sup_{s\in[0,\omega]} \{s^{2\beta(\alpha-\gamma)}|U(s)|_{\mathbf{H}^{\alpha,\wp}}|v(s)|_{\mathbf{H}^{\alpha,\wp}}\}. \end{aligned}$$

According to the characteristics of the β function, there exists a $\delta > 0$ small enough that for $0 < \omega - \omega_0 < \delta$,

$$\int_{\frac{\omega_0}{\omega}}^{1} (\omega - s)^{\beta(1-\gamma)-1}s^{-2\beta(\alpha-\gamma)}ds \to 0;$$

as a result, $I_{11}(\omega)$ tends to zero as $\omega - \omega_0 \to 0$. For $I_{12}(\omega)$, since:

$$\begin{aligned} I_{12}(\omega) &\leq \mathcal{C}_1 \int_0^{\omega_0} ((\omega_0 - s)^{\beta-1} - (\omega - s)^{\beta-1})(\omega - s)^{-\beta\gamma}\mathcal{D}_\wp(\mathfrak{G}(U(s), v(s)))ds \\ &\leq \mathcal{M}\mathcal{C}_1 \int_0^{\omega_0} ((\omega_0 - s)^{\beta-1} - (\omega - s)^{\beta-1})(\omega - s)^{-\beta\gamma}s^{-2\beta(\alpha-\gamma)}ds \sup_{s\in[0,\omega_0]} \{s^{2\beta(\alpha-\gamma)}|U(s)|_{\mathbf{H}^{\gamma,\wp}}|v(s)|_{\mathbf{H}^{\gamma,\wp}}\}, \end{aligned}$$

noting that

$$\begin{aligned}
& \int_0^{\omega_0} \mathcal{D}((\omega_0 - s)^{\beta-1} - (\omega - s)^{\beta-1})(\omega - s)^{-\beta\gamma} s^{-2\beta(\alpha-\gamma)} ds \\
\leq & \int_0^{\omega_0} (\omega - s)^{\beta-1}(\omega - s)^{-\beta\gamma} s^{-2\beta(\alpha-\gamma)} ds + \int_0^{\omega_0} (\omega_0 - s)^{\beta-1}(\omega - s)^{-\beta\gamma} s^{-2\beta(\alpha-\gamma)} ds \\
\leq & \; 2\int_0^{\omega_0} (\omega_0 - s)^{\beta(1-\gamma)-1} s^{-2\beta(\alpha-\gamma)} ds \\
= & \; 2\mathcal{B}(\beta(1-\gamma), 1 - 2\beta(\alpha-\gamma)).
\end{aligned}$$

Then, we get the dominated convergence theorem of Lebesgue:

$$\int_0^{\omega_0} ((\omega_0 - s)^{\beta-1} - (\omega - s)^{\beta-1})(\omega - s)^{-\beta\gamma} s^{-2\beta(\alpha-\gamma)} ds \to 0, \; as \; \omega \to \omega_0;$$

one can deduce $\lim_{\omega\to\omega_0} I_{12}(\omega) = 0$. For $I_{13}(\omega)$, since:

$$\begin{aligned}
I_{13}(\omega) & \leq \int_0^{\omega_0-\epsilon} (\omega_0 - s)^{\beta-1} \mathcal{D}_{H^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega-s)^\beta \mathcal{A}) + E_{\beta,\beta}(-(\omega_0 - s)^\beta \mathcal{A}))\mathfrak{G}(U(s), v(s)))ds \\
& \leq \int_0^{\omega_0-\epsilon} (\omega_0 - s)^{\beta-1}((\omega - s)^{-\beta\gamma} + (\omega_0 - s)^{-\beta\gamma})\mathcal{D}_{\wp}(\mathfrak{G}(U(s), v(s)))ds \\
& \leq 2\mathcal{M}\mathcal{C}_1 \int_0^{\omega_0-\epsilon} (\omega_0 - s)^{\beta(1-\gamma)-1} s^{-2\beta(\alpha-\gamma)} ds \sup_{s\in[0,\omega_0]} \{ s^{2\beta(\alpha-\gamma)} |U(s)|_{\mathbf{H}^{\alpha,\wp}} |v(s)|_{\mathbf{H}^{\alpha,\wp}} \}.
\end{aligned}$$

Using Lebesgue's dominated convergence theorem once more, the fact that the operator $E_{\beta,\beta}(-\omega^\beta \mathcal{A})$ has uniform continuity owing to Lemma 1 indicates:

$$\begin{aligned}
\lim_{\omega\to\omega_0} I_{13} & = \int_0^{\omega_0-\epsilon} (\omega_0 - s)^{\beta-1} \lim_{\omega\to\omega_0} \mathcal{D}_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega - s)^\beta \mathcal{A}) - E_{\beta,\beta}(-(\omega_0 - s)^\beta \mathcal{A}))\mathfrak{G}(U(s), v(s)))ds \\
& = 0.
\end{aligned}$$

For $I_{14}(\omega)$, by immediate calculation, we estimate:

$$\begin{aligned}
I_{14}(\omega) & \leq \int_{\omega_0-\epsilon}^{\omega_0} (\omega_0 - s)^{\beta-1}((\omega - s)^{-\beta\gamma} + (\omega_0 - s)^{-\beta\gamma})\mathcal{D}_{\wp}(\mathfrak{G}(U(s), v(s)))ds \\
& \leq 2\mathcal{M}\mathcal{C}_1 \int_{\omega_0-\epsilon}^{\omega_0} (\omega_0 - s)^{\beta(1-\gamma)-1} s^{-2\beta(\alpha-\gamma)} ds \sup_{s\in[0,\omega_0]} \{ s^{2\beta(\alpha-\gamma)} |U(s)|_{\mathbf{H}^{\alpha,\wp}} |v(s)|_{H^{\alpha,\wp}} \} \to 0 \; as \; \epsilon \to 0,
\end{aligned}$$

according to the β function's properties. As a result, it follows:

$$|\mathfrak{g}(U(\omega), v(\omega)) - \mathfrak{g}(U(\omega_0), v(\omega_0))|_{\mathbf{H}^{\gamma,\wp}} \to 0, \; as \; \omega \to \omega_0.$$

The continuity of the operator $\mathfrak{g}(U, v)$ evaluated in $\mathcal{C}((0,\infty), \mathbf{H}^{\alpha,\wp})$ follows by a similar discussion to that above. So, we omit the details.

Step 2. We show that the operation $\mathfrak{g} : \chi_\infty \times \chi_\infty \to \chi_\infty$ is continuous bilinear operator. By Lemma 6, we have:

$$\begin{aligned}
\mathcal{D}(\mathfrak{g}(U(\omega), v(\omega)))_{\mathbf{H}^{\gamma,\wp}} & \leq \mathcal{D}_{\mathbf{H}^{\gamma,\wp}} \left(\int_0^\omega (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^\beta \mathcal{A})\mathfrak{G}(U(s), v(s))ds \right) \\
& \leq \mathcal{C}_1 \int_0^\omega (\omega - s)^{\beta(1-\gamma)-1} \mathcal{D}_{\wp}(\mathfrak{G}(U(s), v(s)))ds \\
& \leq \mathcal{M}\mathcal{C}_1 \int_0^\omega (\omega - s)^{\beta(1-\gamma)-1} s^{-2\beta(\alpha-\gamma)} ds \sup_{s\in[0,\omega_0]} \{ s^{2\beta(\alpha-\gamma)} |U(s)|_{\mathbf{H}^{\alpha,\wp}} |v(s)|_{\mathbf{H}^{\alpha,\wp}} \} \\
& = \mathcal{M}\mathcal{C}_1 \mathcal{B}(\beta(1-\gamma), 1 - 2\beta(\alpha-\gamma)) ||U||_{\chi_\infty} ||v||_{\chi_\infty};
\end{aligned}$$

it follows that

$$\sup_{\omega\in[0,\infty)} \omega^{\beta(\alpha-\gamma)}|\mathfrak{g}(U(\omega),v(\omega))|_{\mathbf{H}^{\alpha,\wp}} \leq \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha),1-2\beta(\alpha-\gamma))||U||_{\chi_\infty}||v||_{\chi_\infty}.$$

More precisely,

$$\lim_{\omega\to 0} \omega^{\beta(\alpha-\gamma)}D_{\mathbf{H}^{\alpha,\wp}}(\mathfrak{g}(U(\omega),v(\omega))) = 0.$$

Hence, $\mathfrak{g}(U,v)\in\chi_\infty$ and $||\mathfrak{g}(U(\omega),v(\omega))||_{\chi_\infty} \leq L||u||_{\chi_\infty}||v||_{\chi_\infty}$.
Step 3. We verify that (c) holds. Let $0<\omega_0<\omega$. Since:

$$\begin{aligned}
\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(\varpi(\omega)-\varpi(\omega_0)) \leq\ & \int_{\omega_0}^{\omega}(\omega-s)^{\beta-1}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s))ds \\
& + \int_0^{\omega_0}((\omega_0-s)^{\alpha-1}-(\omega-s)^{\alpha-1}-(\omega-s)^{\alpha-1})\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s))ds \\
& + \int_0^{\omega_0-\epsilon}(\omega_0-s)^{\beta-1}D_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})-E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A}))P\mathfrak{g}(s))ds \\
& + \int_{\omega_0-\epsilon}^{\omega_0}(\omega_0-s)^{\beta-1}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})-E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A}))P\mathfrak{g}(s))ds \\
\leq\ & \mathcal{C}_1\mathcal{M}(\omega)\int_{\omega_0}^{\omega}(\omega-s)^{\beta(1-\gamma)-1}s^{-\beta(1-\gamma)}ds + \mathcal{C}_1\mathcal{M}(\omega)\int_0^{\omega_0}((\omega-s)^{\beta-1}-(\omega_0-s)^{\beta-1}) \\
& s^{-\beta(1-\gamma)}ds + \mathcal{C}_1\mathcal{M}(\omega)\int_0^{\omega_0-\epsilon}(\omega_0-s)^{\beta-1}D_{\mathbf{H}^{\gamma,\wp}}((E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A})- \\
& E_{\beta,\beta}(-(\omega_0-s)^\beta\mathcal{A}))P\mathfrak{g}(s))ds + 2\mathcal{C}_1\mathcal{M}(\omega)\int_{\omega_0-\epsilon}^{\omega_0}(\omega_0-s)^{\beta(1-\gamma)-1}s^{-\beta(1-\gamma)}ds.
\end{aligned}$$

The first two integrals and the last integral trend to 0 as $\omega\to\omega_0$ and $\epsilon\to 0$ due to the properties of the β function. In light of Lemma 1, the third integral also equals 0 as $\omega\to\omega_0$, implying that:

$$\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(\varpi(\omega)-\varpi(\omega_0)) \to 0 \ as\ \omega\to\omega_0.$$

The same argument applies to the evaluation of $\varpi(\omega)$ in $\mathbf{H}^{\alpha,\wp}$.
However, we have:

$$\begin{aligned}
\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(\varpi(\omega)) &\leq \mathcal{D}_{\mathbf{H}^{\gamma,q}}\left(\int_0^\omega(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds\right) \\
&\leq \mathcal{C}_1\int_0^\omega(\omega-s)^{\beta(1-\gamma)-1}\mathcal{D}_\wp(P\mathfrak{g}(s))ds \\
&\leq \mathcal{C}_1\mathcal{M}(\omega)\int_0^\omega(\omega-s)^{\beta(1-\gamma)-1}s^{-\beta(1-\gamma)}ds \\
&= \mathcal{C}_1\mathcal{M}(\omega)\mathcal{B}(\beta(1-\gamma),1-\beta(1-\gamma)),
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(\varpi(\omega)) &\leq \mathcal{D}_{\mathbf{H}^{\alpha,\wp}}\left(\int_0^\omega(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds\right) \\
&\leq \mathcal{C}_1\int_0^\omega(\omega-s)^{\beta(1-\alpha)-1}\mathcal{D}_\wp(P\mathfrak{g}(s))ds \\
&\leq \mathcal{C}_1\mathcal{M}(\omega)\int_0^\omega(\omega-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\gamma)}ds \\
&= \omega^{-\beta(\alpha-\gamma)}\mathcal{C}_1\mathcal{M}(\omega)\mathcal{B}(\beta(1-\alpha),1-\beta(1-\gamma)).
\end{aligned}$$

More precisely,

$$\omega^{\beta(\alpha-\gamma)}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(\phi(\omega)) \leq \mathcal{C}_1\mathcal{M}(\omega)\mathcal{B}(\beta(1-\alpha),1-\beta(1-\gamma)) \to 0,\ as\ \omega\to 0.$$

$\mathcal{M}(\omega) \to 0$ as $\omega \to 0$ owing to assumption ($\mathfrak{g}$). This implies that $\varpi(\omega) \in \chi_\infty$ and $\mathcal{D}_\infty(\phi(\omega)) \leq \mathcal{B}_1\mathcal{M}_\infty$,
$E_\beta(-\omega^\beta\mathcal{A})by(\omega)\sin\gamma \in \mathcal{C}([0,\infty),\mathbf{H}^{\gamma,\wp})$ and $E_\beta(-\omega^\beta A)b\sin\gamma \in \mathcal{C}([0,\infty),\mathbf{H}^{\alpha,\wp})$.
This, together with Lemma 6, implies that for all $\omega \in (0,\Im]$,

$$\begin{aligned} E_\beta(-\omega^\beta\mathcal{A})b\sin\gamma &\in \chi_\infty, \\ \omega^{\beta(\alpha-\gamma)}E_\beta(-\omega^\beta\mathcal{A})b\sin\gamma &\in \mathcal{C}([0,\infty),\mathbf{H}^{\alpha,\wp}), \\ \mathcal{D}_\infty(E_\beta(-\omega^\beta\mathcal{A})b\sin\gamma) &\leq \mathcal{C}_1\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(by(\omega)\sin\gamma). \end{aligned}$$

According to (6), the inequality,

$$\begin{aligned} D_{\chi_\infty}(E_\beta(-\omega^\beta\mathcal{A})b\sin\gamma+\varpi(\omega)) &\leq D_{\chi_\infty}(E_\beta(-\omega^\beta\mathcal{A})b\sin\gamma)+\mathcal{D}_{X_\infty}(\varpi(\omega)) \\ &\leq \frac{1}{4\mathcal{L}}, \end{aligned}$$

holds, which yields that $\mathfrak{G}$ has a unique fixed point.

Step 4. To demonstrate that $u(\omega) \to by(\omega)\sin\gamma$ in $\mathbf{H}^{\gamma,\wp}$ as $\omega \to 0$. We need to verify:

$$\begin{aligned} \lim_{\omega\to 0}\int_0^\omega(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds &= 0, \\ \lim_{\omega\to 0}\int_0^\omega(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(U(s),U(s))ds &= 0, \end{aligned}$$

in $\mathbf{H}^{\gamma,\wp}$. It is obvious that $\lim_{\omega\to 0}\varpi(\omega) = 0(\lim_{\omega\to 0}\mathcal{M}(\omega) = 0)$ owing to (8). In addition,

$$\begin{aligned} &\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}\left(\int_0^\infty(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})\mathfrak{G}(U(s),U(s))ds\right) \\ \leq\;& \mathcal{C}_1\int_0^\omega(\omega-s)^{\beta(1-\gamma)-1}\mathcal{D}_\wp(\mathfrak{G}(U(s),U(s)))ds \\ \leq\;& \mathcal{M}\mathcal{C}_1\int_0^\omega(\omega-s)^{\beta(1-\gamma)-1}\mathcal{D}^2_{\mathbf{H}^{\alpha,\wp}}(U(s))ds \end{aligned}$$

$$\begin{aligned} \leq\;& \mathcal{M}\mathcal{C}_1\int_0^\omega(\omega-s)^{\beta(1-\gamma)-1}s^{-2\beta(\alpha-\gamma)}ds\sup_{s\in[0,\omega]}\{s^{2\beta(\alpha-\gamma)}\mathcal{D}^2_{\mathbf{H}^{\alpha,\wp}}(U(s))\} \\ =\;& \mathcal{M}\mathcal{C}_1B(\beta(1-\gamma),1-2\beta(\alpha-\gamma))\sup_{s\in[0,\omega]}\{s^{2\beta(\alpha-\gamma)}\mathcal{D}^2_{\mathbf{H}^{\alpha,\wp}}(U(s))\to 0 \text{ as } \omega\to 0. \end{aligned}$$

□

Local Existence in $\mathbf{H}^{\gamma,\wp}$

In this section, the local mild solution of the problem (3) in $\mathbf{H}^{\gamma,\wp}$ is investigated.

Theorem 2. *Assume* $1 < \wp < \infty, 0 < \gamma < 1$ *and* $\mathfrak{g}$ *hold. Let*

$$\frac{n}{2q}-\frac{1}{2} < \gamma.$$

Then there is a $\alpha > \max\{\gamma,\frac{1}{2}\}$ *such that for every* $\beta \in \mathbf{H}^{\gamma,\wp}$ *there exists* $T_* > 0$ *and a unique function* $U : [0,\Im_*] \to \mathbf{H}^{\gamma,\wp}$ *satisfying:*

(i) $U : [0,\Im_*) \to \mathbf{H}^{\gamma,\wp}$ *is a continuous and* $U(0) = by\sin\gamma$*;*
(ii) $U : (0,\Im_*] \to \mathbf{H}^{\alpha,\wp}$ *is a continuous and* $\lim_{\omega\to 0}\omega^{\beta(\alpha-\gamma)}|U(\omega)|_{\mathbf{H}^{\alpha,\wp}} = 0$*;*
(iii) U *satisfies (5) for* $\omega \in [0,\Im_*]$.

Proof. Let $\alpha = \frac{(1+\gamma)}{2}$. Assume that $\chi = \chi[\Im]$ is the space containing all curves $U : (0, \Im] \to \mathbf{H}^{\gamma,\wp}$, and that:

(1) $U : [0, \infty) \to \mathbf{H}^{\gamma,\wp}$ is a continuous;

(2) $U : [0, \infty) \to \mathbf{H}^{\alpha,\wp}$ is a continuous and $\lim_{\omega \to 0} \omega^{\beta(\alpha-\gamma)}|u(\omega)|_{\mathbf{H}^{\alpha,\wp}} = 0$;

with its natural norm:

$$||u||_{\chi} = \sup_{\omega \in [0,\Im]} \{\omega^{\beta(\alpha-\gamma)}|U(\omega)|_{\mathbf{H}^{\alpha,\wp}}\}.$$

It is simple to show that $\mathfrak{g} : \chi \times \chi \to \chi$ is a continuous linear map and $\phi(\omega) \in \chi$, similar to the proof of Theorem 1. It is clear from Lemma 1 that for all $\omega \in (0, \Im]$,

$$\begin{aligned} E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma &\in \mathcal{C}([0, \Im], \mathbf{H}^{\gamma,\wp}), \\ E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma &\in \mathcal{C}([0, \Im], \mathbf{H}^{\alpha,\wp}). \end{aligned}$$

It follows from Lemma 6 that:

$$\begin{aligned} E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma &\in \chi, \\ \omega^{\beta(\alpha-\gamma)} E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma &\in \mathcal{C}([0, \Im], \mathbf{H}^{\alpha,\Im}). \end{aligned}$$

As a result, suppose $\Im_* > 0$ is sufficiently small such that:

$$\mathcal{D}_{X[\Im_*]}(E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma + \phi(\omega)) \le D_{X[\Im_*]}(E_\beta(-\omega^\beta \mathcal{A}) by(\omega) \sin\gamma) + \mathcal{D}_{X[\Im_*]}(\phi(\omega)) < \frac{1}{4\mathcal{L}},$$

which implies that, due to Lemma 4, $\mathfrak{G}$ has a unique fixed point. □

5. Local Existence in $\mathbf{S}_\wp$

In this section, the iteration method is used to consider a local mild solution to the problem (3) in $\mathbf{S}_\wp$. Let $\alpha = \frac{(1+\gamma)}{2}$.

Theorem 3. *Suppose $1 < \wp < \infty, 0 < \gamma < 1$ and (f) hold. Assume that: $by \sin\gamma \in \mathbf{H}^{\gamma,\wp}$ with $\frac{n}{2\wp} - \frac{1}{2} < \gamma$.*

Then problem (3) has a unique mild solution u in $\mathbf{S}_\wp$ for $by \sin\gamma \in \mathbf{H}^{\gamma,\wp}$. Furthermore, u is a continuous in $[0, \Im]$, $\mathcal{A}^\alpha u$ is continuous in $(0, \Im]$ and $\omega^{\beta(\alpha-\gamma)} \mathcal{A}^\alpha u(\omega)$ is bounded as $\omega \to 0$.

Proof. Step 1. Set

$$\mathcal{K}(\omega) := \sup_{s \in (0,\omega]} s^{\beta(\alpha-\gamma)} |\mathcal{A}^\alpha U(s)|_\wp$$

and

$$\psi(\omega) := \mathfrak{g}(U, U)(\omega) = \int_0^\omega (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^\beta A) \mathfrak{G}(U(s), U(s)) ds.$$

As an immediate consequence of step 2 in Theorem 1, $\psi(\omega)$ is continuous in $[0, \Im]$, $\mathcal{A}^\alpha \psi(\omega)$ exists and is continuous in $(0, \Im]$ with

$$\mathcal{D}_\Im(\mathcal{A}^\alpha \psi(\omega)) \le \mathcal{M}\mathcal{C}_1 \mathcal{B}(\beta(1-\alpha), 1 - 2\beta(\alpha-\gamma)) \mathcal{K}^2(\omega) \omega^{-\beta(\alpha-\gamma)}. \tag{9}$$

The integral $\phi(\omega)$ is also considered. The inequality exists because $(\mathfrak{g})$ is true.

$$\mathcal{D}(p\mathfrak{g}(s)) \le \mathcal{M}(\omega) s^{\beta(1-\gamma)}$$

is satisfied with a continuous function $\mathcal{M}(\omega)$. From Step 3 in Theorem 1, we derive that $\mathcal{A}^\alpha \varpi(\omega)$ is continuous in $(0, \Im]$ with:

$$\mathcal{D}_\wp(\mathcal{A}^\alpha \varpi(\omega)) \le \mathcal{C}_1 \mathcal{M}(\omega) \mathcal{B}(\beta(1-\alpha), 1 - \beta(1-\gamma)) \omega^{-\beta(\alpha-\gamma)}. \tag{10}$$

For $\mathcal{D}_{\wp}(P\mathfrak{g}(\omega)) = o(\omega^{-\beta(1-\gamma)})$ as $\omega \to 0$, we have $\mathcal{M}(\omega) = 0$. Here (10) means $\mathcal{D}_{\wp}(\mathcal{A}^{\alpha}\varpi(\omega)) = o(\omega^{-\beta(\alpha-\gamma)})$ as $\omega \to 0$. We show that ϖ is continuous in $\mathbf{S}_{\wp}$. In fact, take $0 \le \omega_0 < \omega < \Im$, we get that:

$$
\begin{aligned}
\mathcal{D}_{\wp}(\varpi(\omega) - \varpi(\omega_0)) \le\ & \mathcal{C}_3 \int_{\omega_0}^{\omega} (\omega - s)^{\beta-1}\mathcal{D}_{\wp}(P\mathfrak{g}(s))ds + \mathcal{C}_3 \int_0^{\omega_0} ((\omega_0 - s)^{\beta-1} - (\omega - s)^{\beta-1})\mathcal{D}_{\wp}(P\mathfrak{g}(s))ds \\
& + \mathcal{C}_3 \int_0^{\omega_0-\xi} (\omega_0 - s)^{\beta-1}\mathcal{D}(E_{\beta,\beta}(-(\omega-s)^{\beta}A) - E_{\beta,\beta}(-(\omega_0-s)^{\beta}\mathcal{A}))\mathcal{D}_{\wp}(P\mathfrak{g}(s))ds \\
& + 2\mathcal{C}_3 \int_{\omega_0-\xi}^{\omega_0} (\omega_0 - s)^{\beta-1}\mathcal{D}_{\wp}(P\mathfrak{g}(s))ds \\
\le\ & \mathcal{C}_3\mathcal{M}(\omega) \int_{\omega_0}^{\omega} (\omega - s)^{\beta-1}s^{-\beta(1-\gamma)}ds + \mathcal{C}_3\mathcal{M}(\omega) \int_0^{\omega_0} ((\omega - s)^{\beta-1} - (\omega_0 - s)^{\beta-1})s^{-\beta(1-\gamma)}ds \\
& + \mathcal{C}_3\mathcal{M}(\omega) \int_0^{\omega_0-\xi} (\omega_0 - s)^{\beta-1}s^{-\beta(1-\gamma)}ds \sup_{s\in[0,\omega-\xi]} D(E_{\beta,\beta}(-(\omega-s)^{\beta}A) - \\
& E_{\beta,\beta}(-(\omega_0-s)^{\beta}\mathcal{A})) + 2\mathcal{C}_3\mathcal{M}(\omega) \int_{\omega_0-\xi}^{\omega_0} (\omega_0 - s)^{\beta-1}s^{-\beta(1-\gamma)}ds \to 0,\ as\ \omega \to \omega_0,
\end{aligned}
$$

from previous discussions.

Moreover, we will look at the function $E_{\beta}(-\omega^{\beta}\mathcal{A})by(\omega)\sin\gamma$. It is clear by Lemma 6 that:

$$
\begin{aligned}
\mathcal{D}_{\wp}(\mathcal{A}^{\alpha}E_{\beta}(-\omega^{\beta}\mathcal{A})by(\omega)\sin\gamma|_{\wp} &\le \mathcal{C}_1\omega^{-\beta(\alpha-\gamma)}\mathcal{D}_q(\mathcal{A}^{\gamma}by(\omega)\sin\gamma) \\
&= \mathcal{C}_1\omega^{-\beta(\alpha-\gamma)}\mathcal{D}_{\mathbf{H}^{\gamma,\wp}}(by(\omega)\sin\gamma), \\
\lim_{\omega\to 0}\omega^{\beta(\alpha-\gamma)}\mathcal{D}_{\wp}(\mathcal{A}^{\alpha}E_{\beta}(-\omega^{\beta}\mathcal{A})by(\omega)\sin\gamma) &= \lim_{\omega\to 0}\omega^{\beta(\alpha-\gamma)}\mathcal{D}_{\mathbf{H}^{\alpha,\wp}}(E_{\beta}(-\omega^{\beta}\mathcal{A})by(\omega)\sin\gamma) \\
&= 0.
\end{aligned}
$$

Step 2. We now establish the result using consecutive approximations:

$$
\begin{aligned}
u_0(\omega) &= E_{\beta}(-\omega^{\beta}\mathcal{A})by(\omega)\sin\gamma + \varpi(\omega), \\
U_{n+1}(\omega) &= U_0(\omega) + \mathfrak{g}(U_n, U_n)(\omega),\ n = 0,1,2,\ldots.
\end{aligned} \tag{11}
$$

Using the results presented above, we can deduce that:

$$
\mathcal{K}_n(\omega) := \sup_{s\in(0,\Im]} s^{\beta(\alpha-\gamma)}|\mathcal{A}^{\alpha}U_n(s)|_{\wp}
$$

are continuous and increasing functions on $[0,\Im]$ with $\mathcal{K}_n(0) = 0$. In addition, $\mathcal{K}_n(\omega)$ fulfills the following inequality as a result of (10) and (11),

$$
\mathcal{K}_{n+1}(\omega) \le \mathcal{K}_0(\omega) + \mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 2\beta(\alpha-\gamma))\mathcal{K}_n^2(\omega). \tag{12}
$$

We choose a $T > 0$ for $\mathcal{K}_0(0) = 0$, therefore:

$$
4M\mathcal{C}_1\mathcal{B}(\alpha(1-\gamma), 1-2\alpha(\gamma-\beta))\mathcal{K}_0(T) < 1. \tag{13}
$$

The sequence $\{\mathcal{K}_n(\Im)\}$ is then bounded, as a result of a fundamental consideration of (12),

$$
\mathcal{K}_n(\Im) \le \rho(\Im),\ n = 0,1,2,\ldots,
$$

where

$$
\rho(\omega) = \frac{1 - \sqrt{1 - 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1-2\beta(\alpha-\gamma))\mathcal{K}_0(\omega)}}{2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1-2\beta(\alpha-\gamma))}.
$$

Similarly, $\mathcal{K}_n(\omega) \leq \rho(\omega)$ holds for any $\omega \in (0, \Im]$. Similarly, we can see that $\rho \leq 2\mathcal{K}_0(\omega)$. Consider the equality

$$\omega_{n+1}(\omega) = \int_0^{\omega} (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^{\beta}\mathcal{A})[\mathfrak{G}(U_{n+1}(s), U_{n+1}(s)) - \mathfrak{G}(U_n(s), U_n(s))]ds,$$

where $\omega_n = U_{n+1} - U_n, n = 0, 1, \ldots,$ and $\omega \in (0, \Im]$. Writing

$$\mathcal{W}_n(\omega) := \sup_{s\in(0,\Im]} s^{\beta(\alpha-\gamma)} |\mathcal{A}^{\alpha} w_n(s)|_{\wp}.$$

According to (6), we have:

$$\mathcal{D}_{\wp}(\mathfrak{G}(U + 1(s), U_{n+1}(s)) - \mathfrak{G}(U_n(s), U_n(s))) \leq \mathcal{M}(\mathcal{K}_{n+1}(s) + \mathcal{K}_n(\omega))\mathcal{W}_n(s)s^{-2\beta(\alpha-\gamma)},$$

which follows from Step 2 in Theorem 1 that:

$$\omega^{\beta(\alpha-\gamma)} \mathcal{D}_{\wp}(\mathcal{A}^{\alpha}\omega_{n+1}(\omega)) \leq 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1 - \beta(1-\gamma))\rho(\omega)\mathcal{W}_n(\omega).$$

This inequality results in

$$\begin{aligned} \mathcal{W}_{n+1}(\Im) &\leq 2\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1 - 2\beta(\alpha-\gamma)\rho(\Im)\mathcal{W}_n(\Im) \\ &\leq 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1 - 2\beta(1-\alpha), 1 - 2\beta(\alpha-\gamma))\mathcal{K}_0(\Im)\mathcal{W}_n(\Im). \end{aligned} \tag{14}$$

According to (13) and (14), it is easy to see that:

$$\lim_{n\to 0} \frac{\mathcal{W}_{n+1}(\Im)}{\mathcal{W}_n(\Im)} < 4\mathcal{M}\mathcal{C}_1\mathcal{B}(\beta(1-\alpha), 1 - 2\beta(\alpha-\gamma)) < 1.$$

As a result, the series $\sum_{n=0}^{\infty} W_n(T)$ converges. It shows that the series $\sum_{n=0}^{\infty} \omega^{\beta(\alpha-\gamma)} \mathcal{A}^{\gamma}\Omega_n(\omega)$ converges uniformly for $\omega \in (0, \Im]$; therefore, the sequence $\{\omega^{\beta(\alpha-\gamma)}\mathcal{A}^{\alpha}U_n(\omega)\}$ converges uniformly in $(0, \Im]$. This shows that:

$$\lim_{n\to\infty} U_n(\omega) = U(\omega) \in \mathcal{D}(\mathcal{A}^{\alpha})$$

and

$$\lim_{n\to\infty} \omega^{\beta(\alpha-\gamma)}\mathcal{A}^{\alpha}U_n(\omega) = \omega^{\beta(\alpha-\gamma)}\mathcal{A}^{\alpha}U(\omega)$$

uniformly,
since $\mathcal{A}^{-\alpha}$ is bounded and $\mathcal{A}^{\alpha}$ is closed. As a result, the function $\mathcal{K}(\omega) = \sup_{s\in(0,\omega]} s^{\beta(\alpha-\gamma)}|\mathcal{A}^{\alpha}u(s)|_{\wp}$ also satisfies

$$\mathcal{K}(\omega) \leq \rho(\omega) \leq 2\mathcal{K}_0(\omega),\ \omega \in (0, \omega], \tag{15}$$

and

$$\begin{aligned} \zeta_n &:= \sup_{s\in(0,\Im]} s^{2\beta(\alpha-\gamma)}|\mathfrak{G}(U_n(s), U_n(s)) - \mathfrak{G}(U(s), U(s))|_{\wp} \\ &\leq \mathcal{M}(\mathcal{K}_n(\Im) + \mathcal{K}(\Im)) \sup_{s\in(0,\Im]} s^{\beta(\alpha-\gamma)}|\mathcal{A}^{\alpha}(U_n(s) - U(s))|_{\wp} \to 0,\ as\ n \to \infty. \end{aligned}$$

Finally, u must be verified as a mild solution of the problem (3) in $[0, \Im]$. Since

$$\mathcal{D}_{\wp}(\mathfrak{g}(U_n, U_n)(\omega) - \mathfrak{g}(U, U)(\omega)) \leq \int_0^{\omega} (\omega - s)^{\beta-1}\zeta_n s^{-2\beta(\alpha-\gamma)} ds = \omega^{\beta\gamma}\zeta_n \to 0, (n \to \infty),$$

we have $\mathfrak{g}(U_n, U_n)(\omega) \to \mathfrak{g}(U, U)(\omega)$. By taking the limits on both sides of (10), we get

$$U(\omega) = U_0(\omega) + \mathfrak{g}(U, U)(\omega). \tag{16}$$

Assuming that $U(0) = by \sin\gamma$, (16) holds for $\omega \in [0, \Im]$ and $U \in \mathcal{C}([0, \Im], \mathbf{S}_\wp)$. Furthermore, uniform convergence of $\omega^{\beta(\alpha-\gamma)}\mathcal{A}^\alpha U_n(\omega)$ to $\omega^{\beta(\alpha-\gamma)}\mathcal{A}^\alpha U(\omega)$ results in $\mathcal{A}^\alpha U(\omega)$ continuity on $(0, \Im)$. We can see that $\mathcal{D}_\wp(\mathcal{A}^\gamma U_n(\omega)) = o(\omega^{-\beta(\alpha-\gamma)})$ is clear from (15) and $\mathcal{K}_0(0) = 0$.

Step 3: We demonstrate that the mild solution is one-of-a-kind. Assume that U and v are mild solutions to the (3) problem. We consider the equality if $\Omega = U - v$.

$$\Omega(\omega) = \int_0^\omega (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^{\beta-1}\mathcal{A})[\mathfrak{G}(U(s), U(s)) - \mathfrak{G}(v(s), v(s))]ds.$$

Defining the functions:

$$\tilde{\mathcal{K}}(\omega) := \max\{\sup_{s\in(0,\Im]} s^{\beta(\alpha-\gamma)}|\mathcal{A}^\alpha u(s)|_\wp, \sup_{s\in(0,\Im]} s^{\beta(\alpha-\gamma)}|\mathcal{A}^\alpha v(s)|_\wp\}.$$

By (6) and Lemma 6, we have:

$$\mathcal{D}_\wp(\mathcal{A}^\alpha\Omega(\omega)) \le \mathcal{M}\mathcal{C}_1\tilde{\mathcal{K}}(\omega)\int_0^\omega (\omega - s)^{\beta(1-\alpha)-1} s^{-\beta(\alpha-\gamma)}\mathcal{D}_\wp(\mathcal{A}^\alpha\Omega(s))ds.$$

Gronwall inequality shows that $\mathcal{A}^\alpha\Omega(\omega) = 0$ for $\omega \in (0, \Im]$. This implies that $\Omega(\omega) = u(\omega) - v(\omega) \equiv 0$ for $\omega \in [0, \Im]$. As a result, there is a unique mild solution. □

6. Regularity

In this section, the regularity of solution u that fulfils the problem (3) is examined. We will assume the following throughout this section:

(g_1) $P\mathfrak{g}(\omega)$ is Hölder continuous with an exponent $\vartheta \in (0, \beta(1-\alpha))$, which means:

$$\mathcal{D}_\wp(P\mathfrak{g}(\omega) - P\mathfrak{g}(s)) \le L\mathcal{D}^\vartheta(\omega - s),\ \forall\, 0 < \omega, s \le \Im.$$

Definition 4. *A function $U : [0, \Im] \to \mathbf{S}_\wp$, if $u \in \mathcal{C}([0, \Im], \mathbf{S}_\wp)$ with ${}^C\mathcal{D}^\beta_\omega u(\omega) \in \mathcal{C}((0, \Im], \mathbf{S}_\wp)$, which takes values in $\mathcal{D}(\mathcal{A})$ and satisfies (3) for all $\omega \in (0, \Im]$ is termed a classical solution of problem (3).*

Lemma 7. *Let (f_1) be satisfied. If*

$$\varpi_1(\omega) := \int_0^\omega (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^\beta\mathcal{A})(p\mathfrak{g}(s) - p\mathfrak{g}(\omega))ds,\ for\ \omega \in (0, \Im],$$

then $\varpi_1(\omega) \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\varpi_1(\omega) \in \mathcal{C}^\vartheta([0, \Im], \mathbf{S}_\wp)$.

Proof. From Lemma 6 and (f_1), we have for fixed $\omega \in (0, \Im]$:

$$\begin{aligned}(\omega - s)^{\beta-1}\mathcal{D}_\wp(\mathcal{A}E_{\beta,\beta}(-(\omega - s)^\beta\mathcal{A})(P\mathfrak{g}(\omega) - P\mathfrak{g}(\omega)) &\le (\omega - s)^{-1}D_\wp(P\mathfrak{g}(s) - P\mathfrak{g}(\omega)) \\ &\le \mathcal{C}_1 L(\omega - s)^{\vartheta-1} \in L^1([0, \Im], \mathbf{S}_\wp),\end{aligned} \tag{17}$$

then

$$\begin{aligned}\mathcal{D}_\wp(\mathcal{A}\varpi_1(\omega)) &\le \int_0^\omega (\omega - s)^{\beta-1}\mathcal{D}_\wp(\mathcal{A}E_{\beta,\beta}(-(\omega - s)^\beta\mathcal{A})(P\mathfrak{g}(s) - P\mathfrak{g}(\omega)))ds \\ &\le \mathcal{C}_1 L\int_0^\omega (\omega - s)^{\vartheta-1}ds \\ &\le \frac{\mathcal{C}_1\mathcal{K}}{\vartheta}\omega^\vartheta \\ &< \infty.\end{aligned}$$

We get $\varpi_1(\omega) \in \mathcal{D}(\mathcal{A})$ by the closeness of $\mathcal{A}$. It is necessary to demonstrate that $\mathcal{A}\varpi_1(\omega)$ is Hölder continuous. Since:

$$\frac{d}{d\omega}(\omega^{\beta-1}E_{\beta,\beta}(-\mu\omega^{\beta})) = \omega^{\beta-2}E_{\beta,\beta-1}(-\mu\omega^{\beta}),$$

then

$$\begin{aligned}
\frac{d}{d\omega}(\omega^{\beta-1}E_{\beta,\beta}(-\omega^{\omega}\mathcal{A})) &= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}\omega^{\beta-2}E_{\beta,\beta-1}(-\mu\omega^{\beta})\mathcal{A}(\mu I+\mathcal{A})^{-1}d\mu \\
&= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}\omega^{\beta-2}E_{\beta,\beta-1}(-\mu\omega^{\beta})d\mu - \frac{1}{2\pi\iota}\int_{\Gamma_\theta}\omega^{\beta-2}\mu E_{\beta,\beta-1}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu \\
&= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}-\omega^{\beta-2}E_{\beta,\beta-1}(\xi)\frac{1}{\omega^{\beta}}d\xi - \frac{1}{2\pi\iota}\int_{\Gamma'_\theta}\omega^{\beta-2}E_{\beta,\beta-1}(\xi)\frac{\xi}{\omega^{\beta}}\left(-\frac{\xi}{\omega^{\beta}}I+\mathcal{A}\right)^{-1}\frac{1}{\omega^{\beta}}d\xi.
\end{aligned}$$

Given that $\mathcal{D}((\mu I+\mathcal{A})-1) \leq \frac{\mathcal{C}}{|\mu|}$, we can deduce that:

$$\mathcal{D}\left(\frac{d}{d\omega}(\omega^{\beta-1}\mathcal{A}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}))\right) \leq \mathcal{C}_{\beta}\omega^{-2}, 0<\omega\leq\Im.$$

According to mean value theorem, we have, for every $0<s<\omega\leq\Im$,

$$\begin{aligned}
\mathcal{D}(\omega^{\beta-1}\mathcal{A}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}) - s^{\beta-1}\mathcal{A}E_{\beta,\beta}(-s^{\beta}\mathcal{A})) &= \mathcal{D}\left(\int_s^{\omega}\frac{d}{d\omega}(\tau^{\beta-1}\mathcal{A}E_{\beta,\beta}(-\tau^{\beta}\mathcal{A}))d\tau\right) \\
&\leq \int_s^{\omega} D\left(\frac{d}{d\omega}(\tau^{\beta-1}\mathcal{A}E_{\beta,\beta}(-\tau^{\beta}\mathcal{A}))\right)d\tau \\
&\leq \mathcal{C}_{\beta}\int_s^{\omega}\tau^{-2}d\tau \\
&= \mathcal{C}_{\beta}(s^{-1}-\omega^{-1}).
\end{aligned} \tag{18}$$

Assume that $h>0$, $0<\omega<\omega+h\leq\Im$, then

$$\begin{aligned}
\mathcal{A}\varpi_1(\omega+h) - \mathcal{A}\varpi_1(\omega) &= \int_0^{\omega}((\omega+h-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A}) \\
&\quad -(\omega-s)^{\gamma-1}\mathcal{A}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A}))(P\mathfrak{g}(s)-P\mathfrak{g}(\omega)ds \\
&\quad +\int_0^{\omega}(\omega+h-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})(P\mathfrak{g}(\omega)-P\mathfrak{g}(\omega+h))ds \\
&\quad +\int_{\omega}^{\omega+h}(\omega+h-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})(P\mathfrak{g}(s)-P\mathfrak{g}(\omega+h))ds \\
&= h+h_1(\omega)+h_2(\omega)+h_3(\omega).
\end{aligned} \tag{19}$$

Each of the three terms is evaluated individually. From (18) and (f_1), we have $h_1(\omega)$.

$$\begin{aligned}
\mathcal{D}_{\wp}(h_1(\omega)) &\leq \int_0^{\omega}\mathcal{D}((\omega+h-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A}) - (\omega-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A})) \\
&\quad \mathcal{D}_q(P\mathfrak{g}(s)-P\mathfrak{g}(\omega))ds \\
&\leq \mathcal{C}_{\beta}\mathcal{L}h\int_0^{\omega}(\omega+h-s)^{-1}(\omega-s)^{\vartheta-1}ds \\
&\leq \mathcal{C}_{\beta}\mathcal{L}h\int_0^{\omega}(s+h)^{-1}(\omega-s)^{\vartheta-1}ds \\
&\leq \mathcal{C}_{\beta}\mathcal{L}\int_0^{h}\frac{h}{s+h}s^{\vartheta-1}ds + \mathcal{C}_{\beta}Lh\int_h^{\infty}\frac{s}{s+h}s^{\vartheta-1}ds \\
&\leq \mathcal{C}_{\beta}\mathcal{L}h^{\vartheta}.
\end{aligned} \tag{20}$$

We apply Lemma 6 and (f_1) for $h_2(\omega)$.

$$\begin{aligned}\mathcal{D}_{\wp}(h_2(\omega)) &\leq \int_0^{\omega}(\omega+h-s)^{\omega-1}\mathcal{A}E_{\omega,\omega}(-(\omega+h-s)^{\omega}\mathcal{A})\mathcal{D}_{\wp}(P\mathfrak{g}(\omega)-P\mathfrak{g}(\omega+h))ds \\ &\leq \mathcal{C}_1\int_0^{\omega}(\omega+h-s)^{-1}\mathcal{D}_{\wp}(P\mathfrak{g}(\omega)-P\mathfrak{g}(\omega+h))ds \\ &\leq \mathcal{C}_1\mathcal{L}h^{\vartheta}\int_0^{\omega}(\omega+h-s)^{-1}ds \\ &\leq \mathcal{C}_1\mathcal{L}[lnh-ln(\omega+h)]h^{\vartheta}. \end{aligned} \tag{21}$$

Moreover, for $h_3(\omega)$, by Lemma 1 and (g_1), we now have:

$$\begin{aligned}\mathcal{D}_{\wp}(h_3(\omega)) &\leq \int_{\omega}^{\omega+h}(\omega+h-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})\mathcal{D}_{\wp}((P\mathfrak{g}(s)-P\mathfrak{g}(\omega+h))ds \\ &\leq \mathcal{C}_1\int_{\omega}^{\omega+h}(\omega+h-s)^{-1}\mathcal{D}_{\wp}(P\mathfrak{g}(s)-P\mathfrak{g}(\omega+h))ds \\ &\leq \mathcal{C}_1\mathcal{L}\int_{\omega}^{\omega+h}(\omega+h-s)^{\vartheta-1}ds \\ &\leq \mathcal{C}_1\mathcal{L}\frac{h^{\vartheta}}{\vartheta}. \end{aligned} \tag{22}$$

Combining (20), (21) with (22), we conclude that $A\phi_1(\omega)$ is Hölder continuous. □

Theorem 4. *Assume that assumptions of Theorem 3 are fulfilled. If (f_1) holds, then the mild solution of (3) is a classical one for every $a \in \mathcal{D}(\mathcal{A})$.*

Proof. In the case of $a \in \mathcal{D}(\mathcal{A})$. Then, Lemma 2(ii) ensures that $u(\omega) = E_{\beta}(-\omega^{\beta}\mathcal{A})a(\omega) > 0$ is a classical solution to the below problem:

$$\begin{aligned} {}^{C}\mathcal{D}_{\omega}^{\beta}U = -\mathcal{A}U, \omega > 0, \\ U(0) = by\sin\gamma. \end{aligned}$$

Step 1. We prove that:

$$\varpi(\omega) = \int_0^{\omega}(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A})P\mathfrak{g}(s)ds$$

is a classical solution to the problem:

$$\begin{aligned} {}^{C}\mathcal{D}_{\omega}^{\beta}u = -\mathcal{A}U + Ph(\omega), \omega > 0, \\ U(0) = by\sin\gamma. \end{aligned}$$

Theorem 3 states that $\varpi \in \mathcal{C}([0,\Im],\mathbf{S}_{\wp})$. We rewrite $\varpi(\omega) = \varpi_1(\omega) + \varpi_2(\omega)$, where

$$\begin{aligned}\varpi_1(\omega) &= \int_0^{\omega}(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A})(P\mathfrak{g}(s)-P\mathfrak{g}(\omega))ds \\ \varpi_2(\omega) &= \int_0^{\omega}(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A})P\mathfrak{g}(s)ds.\end{aligned}$$

Lemma 7 states that $\varpi_1(\omega) \in \mathcal{D}(\mathcal{A})$. To show that $\varpi_2(\omega)$ has the same conclusion. We may see from Lemma 2(ii) that:

$$\mathcal{A}\varpi_2(\omega) = P\mathfrak{g}(\omega) - E_{\beta}(-\omega^{\beta}\mathcal{A})P\mathfrak{g}(\omega).$$

Given that (f_1) is true, it follows that:

$$\mathrm{D}_{\wp}(\mathcal{A}\varpi_2(\omega)) \leq (1+\mathcal{C}_1)\mathcal{D}_{\wp}(P\mathfrak{g}(\omega)),$$

thus

$$\varpi_2(\omega) \in \mathcal{D}(\mathcal{A})\ for\ \omega \in (0,\Im]\ and\ \mathcal{A}\varpi_2(\omega) \in \mathcal{C}^v((0,\Im],\mathbf{S}_{\wp}). \tag{23}$$

After that, we verify ${}^{\mathcal{C}}\mathcal{D}^{\beta}_{\omega}\varpi \in \mathcal{C}((0,\Im],\mathbf{S}_{\wp})$. We have $\varpi(0)=0$ because of Lemma 2(iv) and $\varpi(0)=0$.

$$\begin{aligned} {}^{\mathcal{C}}\mathcal{D}^{\beta}_{\omega}\varpi(\omega) &= \frac{d}{d\omega}(I^{1-\beta}_{\omega}\varpi(\omega)) \\ &= \frac{d}{d\omega}(E_\beta(-\omega^\beta\mathcal{A}) * P\mathfrak{g}). \end{aligned}$$

It is still necessary to demonstrate that $E_\beta(\omega^\beta\mathcal{A}) * P\mathfrak{g}$ is continuous differentiable in $\mathbf{S}_{\wp}$. When $0<h\leq T-\omega$ is used, the following results are obtained:

$$\begin{aligned} &\frac{1}{h}(E_\beta(-(\omega+h)^\beta\mathcal{A}) * P\mathfrak{g} - E_\beta(-\omega^\beta\mathcal{A}) * P\mathfrak{g} \\ = \ &\int_0^\omega \frac{1}{h}(E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s) - E_\beta(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds + \frac{1}{h}\int_\omega^{\omega_h} E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds. \end{aligned}$$

Notice that:

$$\begin{aligned} &\frac{1}{h}(E_\beta(-(\omega+h)^\beta\mathcal{A}) * Pg - E_\beta(-\omega^\beta\mathcal{A}) * P\mathfrak{g}) \\ = \ &\int_0^\omega \frac{1}{h}\mathcal{D}_{\wp}\Big(E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s) - E_\beta(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)\Big)ds + \frac{1}{h}\int_\omega^{\omega+h} E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds. \end{aligned}$$

Notice that:

$$\begin{aligned} &\int_0^\omega \frac{1}{h}\mathcal{D}_{\wp}\Big(E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s) - E_\beta(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)\Big)ds \\ \leq \ &\mathcal{C}_1\frac{1}{h}\int_0^\omega \mathcal{D}_{\wp}\Big(E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s)\Big)ds + \mathcal{C}_1\frac{1}{h}\int_0^\omega \mathcal{D}_{\wp}\Big(E_\beta(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)\Big)ds \\ \leq \ &\mathcal{C}_1\mathcal{M}(\omega)\frac{1}{h}\int_0^\omega (\omega+h-s)^{-\beta}s^{-\beta(1-\gamma)}ds + \mathcal{C}_1\mathcal{M}(\omega)\frac{1}{h}\int_0^\omega (\omega-s)^{-\beta}s^{-\beta(1-\gamma)}ds \\ \leq \ &\mathcal{C}_1\mathcal{M}(\omega)\frac{1}{h}((\omega+h)^{1-\beta}+\omega^{1-\beta})\mathcal{B}(1-\beta,1-\beta(1-\gamma)). \end{aligned}$$

The dominated convergence theorem is then used to obtain:

$$\begin{aligned} &\lim_{h\to0}\int_0^\omega \frac{1}{h}(E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s) - E_\beta(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s))ds \\ = \ &\int_0^\omega (\omega-s)^{\beta-1}\mathcal{A}E_{\beta,\beta}(-(\omega-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds \\ = \ &\mathcal{A}\varpi(\omega). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\frac{1}{h}\int_{\omega}^{\omega+h} E_\beta(-(\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds &= \frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})P\mathfrak{g}(\omega+h-s)ds \\
&= \frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})(P\mathfrak{g}(\omega+h-s)-P\mathfrak{g}(\omega-s))ds \\
&\quad +\frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})(P\mathfrak{g}(\omega-s)-P\mathfrak{g}(\omega))ds \\
&\quad +\frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})P\mathfrak{g}(\omega)ds.
\end{aligned}
$$

From Lemmas 1 and 6 and (g_1),

$$
\begin{aligned}
\mathcal{D}_\wp\left(\frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})(P\mathfrak{g}(\omega+h-s)-P\mathfrak{g}(\omega-s))ds\right) &\leq \mathcal{C}_1 L h^\vartheta, \\
\mathcal{D}_\wp\left(\frac{1}{h}\int_0^h E_\beta(-s^\beta\mathcal{A})(P\mathfrak{g}(\omega-s)-P\mathfrak{g}(\omega))ds\right) &\leq \mathcal{C}_1 L\frac{h^\vartheta}{\vartheta+1}.
\end{aligned}
$$

Additionally, Lemma 2(i) gives that $\lim\limits_{h\to 0}\frac{1}{h}\int_0^h E_\beta(s^\beta\mathcal{A})P\mathfrak{g}(\omega)ds = P\mathfrak{g}(\omega)$. Hence,

$$
\lim_{h\to 0}\frac{1}{h}\int_{\omega}^{\omega+h} E_\beta((\omega+h-s)^\beta\mathcal{A})P\mathfrak{g}(s)ds = P\mathfrak{g}(\omega).
$$

We deduce that $E_\beta(\omega^\beta\mathcal{A}) * P\mathfrak{g}$ is differentiable at ω_+ and $\frac{d}{d\omega}(E_\beta(\omega^\beta\mathcal{A}) * P\mathfrak{g})_+ = \mathcal{A}\phi(\omega)+P\mathfrak{g}(\omega)$. Similarly, $E_\beta(\omega^\beta\mathcal{A}) * P\mathfrak{g}$ is differentiable at ω_- and $\frac{d}{d\omega}(E_\beta(\omega^\beta\mathcal{A}) * P\mathfrak{g})_- = \mathcal{A}\phi(\omega)+P\mathfrak{g}(\omega)$.

We demonstrate that $\mathcal{A}\varpi = \mathcal{A}\varpi_1 + \mathcal{A}\varpi_2 \in \mathcal{C}((0,\Im],\mathbf{S}_\wp)$. In fact, it is obvious that $\varpi_2(\omega) = P\mathfrak{g}(\omega) - E_\beta(\omega^\beta\mathcal{A})P\mathfrak{g}(\omega)$ due to Lemma 2(iii), which is continuous in view of Lemma 1. Moreover, according to Lemma 7 we know that $\mathcal{A}\varpi_1(\omega)$ is also continuous. Consequently, ${}^{C}\mathcal{D}^\beta_\omega\phi \in \mathcal{C}((0,\Im],\mathbf{S}_\wp)$.

Step 2. Assume u be a mild solution of (3). To demonstrate that $\mathfrak{G}(u,u) \in \mathcal{C}^\vartheta((0,\Im],\mathbf{S}_\wp)$, in view of (6), we prove that $\mathcal{A}^\gamma u$ is Hölder continuous in $\mathbf{S}_\wp$. Take $h > 0$ such that $0 < \omega < \omega + h$.

Denote $\varpi(\omega) := E_\beta(-\omega^\beta\mathcal{A})by\sin\gamma$, by Lemmas 2(iv) and 7, then:

$$
\begin{aligned}
\mathcal{D}_\wp(\mathcal{A}^\alpha\varphi(\omega+h)-\mathcal{A}^\alpha\varphi(\omega)) &= \mathrm{D}_\wp\left(\int_\omega^{\omega+h} -s^{\beta-1}\mathcal{A}^\alpha E_{\beta,\beta}(-s^\beta\mathcal{A})by(\omega)\sin\gamma ds\right) \\
&\leq \int_\omega^{\omega+h} s^{\beta-1}\mathcal{D}_\wp(\mathcal{A}^{\alpha-\gamma}E_{\beta,\beta}(-s^\beta\mathcal{A})\mathcal{A}^\gamma by(\omega)\sin\gamma)ds \\
&\leq \mathcal{C}_1\int_\omega^{\omega+h} s^{\beta(1+\gamma-\alpha)-1}ds\mathcal{D}_\wp(\mathcal{A}^\gamma by(\omega)\sin\gamma) \\
&= \frac{\mathcal{C}_1|by(\omega)\sin\gamma|_{\mathbf{H}^{\gamma,\wp}}}{\beta(1+\gamma-\alpha)}((\omega+h)^{\beta(1+\gamma-\alpha)}-\omega^{\beta(1+\gamma-\alpha)}) \\
&\leq \frac{\mathcal{C}_1|by(\omega)\sin\gamma|_{\mathbf{H}^{\gamma,\wp}}}{\beta(1+\gamma-\alpha)}h^{\beta(1+\gamma-\alpha)}.
\end{aligned}
$$

Thus, $\mathcal{A}^\alpha\varphi \in \mathcal{C}^\vartheta((0,\Im],\mathbf{S}_\wp)$.

Take h for every small $\epsilon > 0$, such that $\epsilon \leq \omega < \omega + h \leq \Im$, so:

$$
\begin{aligned}
& \mathcal{D}_{\wp}(\mathcal{A}^{\alpha}\varpi(\omega+h)-\mathcal{A}^{\alpha}\varpi(\omega)) \\
\leq\; & \mathcal{D}_{\wp}\bigg(\int_{\omega}^{\omega+h}(\omega+h-s)^{\beta-1}\mathcal{A}^{\alpha}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})P\mathfrak{g}(s)ds\bigg) \\
& +\mathcal{D}_{\wp}\bigg(\int_{0}^{\omega}\mathcal{A}^{\alpha}((\omega+h-s)^{\beta-1}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})-(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A}))P\mathfrak{g}(s)ds\bigg) \\
=\; & \varpi_1(\omega)+\varpi_2(\omega).
\end{aligned}
$$

Using Lemma 6 and (g), we have:

$$
\begin{aligned}
\varpi_1(\omega) &\leq \mathcal{C}_1\int_{\omega}^{\omega+h}(\omega+h-s)^{\beta(1-\alpha)-1}|P\mathfrak{g}(s)|_{\wp}ds \\
&\leq \mathcal{C}_1\mathcal{M}(\omega)\int_{\omega}^{\omega+h}(\omega+h-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\gamma)}ds \\
&\leq \mathcal{M}(\omega)\frac{\mathcal{C}_1}{\beta(1-\alpha)}h^{\beta(1-\alpha)}\omega^{-\beta(1-\gamma)} \\
&\leq \mathcal{M}(\omega)\frac{\mathcal{C}_1}{\beta(1-\alpha)}h^{\beta(1-\alpha)}\epsilon^{-\beta(1-\gamma)}.
\end{aligned}
$$

We use the inequality to estimate ϖ_2.

$$
\begin{aligned}
\frac{d}{d\omega}(\omega^{\beta-1}E_{\beta,\beta}(-\omega^{\beta})) &= \frac{1}{2\pi\iota}\int_{\Gamma}\mu^{\alpha}\omega^{\beta-2}E_{\beta,\beta-1}(-\mu\omega^{\beta})(\mu I+\mathcal{A})^{-1}d\mu \\
&= \frac{1}{2\pi\iota}\int_{\Gamma}-\bigg(-\frac{\xi}{\omega^{\beta}}\bigg)\omega^{\beta-2}E_{\beta,\beta-1}(\xi)\bigg(-\frac{\xi}{\omega^{\beta}}I+\mathcal{A}\bigg)^{-1}\frac{1}{\omega^{\beta}}d\xi
\end{aligned}
$$

this yields that

$$
\begin{aligned}
\mathcal{D}\bigg(\frac{d}{d\omega}(\omega^{\beta-1}\mathcal{A}^{\alpha}E_{\beta,\beta}(-\omega^{\beta}\mathcal{A}))\bigg) &\leq \int_{s}^{\omega}\bigg\|\frac{d}{d\omega}(\tau^{\beta-1}\mathcal{A}^{\alpha}E_{\beta,\beta}(-\tau^{\beta}\mathcal{A}))\bigg\|d\tau \\
&\leq \mathcal{C}_{\beta}\int_{s}^{\omega}\tau^{\beta(1-\alpha)-2}d\tau \\
&= \mathcal{C}_{\beta}(s^{\beta(1-\alpha)-1}-\omega^{\beta(1-\alpha)-1}),
\end{aligned}
$$

thus

$$
\begin{aligned}
\varpi_2(\omega) &\leq \int_{0}^{\omega}\mathcal{D}_{\wp}(\mathcal{A}^{\alpha}((\omega+h-s)^{\beta-1}E_{\beta,\beta}(-(\omega+h-s)^{\beta}\mathcal{A})-(\omega-s)^{\beta-1}E_{\beta,\beta}(-(\omega-s)^{\beta}\mathcal{A}))P\mathfrak{g}(s))ds \\
&\leq \int_{0}^{\omega}\mathcal{D}_{\wp}(((\omega-s)^{\beta(1-\alpha)-1}-(\omega+h-s)^{\beta(1-\alpha)-1})P\mathfrak{g}(s))ds \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(\omega)\bigg(\int_{0}^{\omega}(\omega-s)^{\beta(1-\alpha)-1}s^{-\beta(1-\gamma)}ds-\int_{0}^{\omega+h}(\omega-s+h)^{\beta(1-\alpha)-1}s^{-\beta(1-\gamma)}ds\bigg) \\
&\quad +\mathcal{C}_{\beta}\mathcal{M}(\omega)\int_{\omega}^{\omega+h}(\omega-s+h)^{\beta(1-\alpha)-1}s^{-\beta(1-\gamma)}ds \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(\omega)(\omega^{\beta(\gamma-\alpha)}-(\omega+h)^{\beta(\gamma-\alpha)}\mathcal{B}(\beta(1-\alpha),1-\beta(1-\gamma))+\mathcal{C}_{\beta}\mathcal{M}(\omega)h^{\beta(1-\alpha)}\omega^{-\beta(1-\gamma)} \\
&\leq \mathcal{C}_{\beta}\mathcal{M}(\omega)h^{\beta(\alpha-\alpha)}[\epsilon(\epsilon+h)]^{\beta(\gamma-\alpha)}+\mathcal{C}_{\beta}\mathcal{M}(\omega)h^{\beta(1-\alpha)}\epsilon^{-\beta(1-\gamma)},
\end{aligned}
$$

which ensures that $\mathcal{A}^{\alpha}\phi \in \mathcal{C}^{\vartheta}([\epsilon,\Im],\mathbf{S}_{\wp})$. Therefore, $\mathcal{A}^{\alpha}\phi \in \mathcal{C}^{\vartheta}((0,\Im],\mathbf{S}_{\wp})$ due to arbitrary ϵ.

Recall that:

$$\psi(\omega) = \int_0^\omega (\omega - s)^{\beta-1} E_{\beta,\beta}(-(\omega - s)^\beta)\mathfrak{G}(U(s), U(s))ds.$$

Since $\mathcal{D}_\wp(\mathfrak{G}(U(s), U(s)) \leq \mathcal{M}\mathcal{K}^2(\omega)s^{-2\beta(\alpha-\gamma)}$, where $\mathcal{K}(\omega) := \sup_{s\in[0,\Im]} s^{\beta(\alpha-\gamma)}|U(s)|\mathbf{H}^{\alpha,\wp}$ is continuous and bounded in $(0, \Im]$. We can get the Hölder continuity of $\mathcal{A}^\alpha\psi$ in $\mathcal{C}^\vartheta((0, \Im], \mathbf{S}_\wp)$ using a similar argument. Therefore, we have $\mathcal{A}^\alpha u(\omega) = \mathcal{A}^\alpha\varphi(\omega) + \mathcal{A}^\beta\varpi(\omega) + \mathcal{A}^\alpha\psi(\omega) \in \mathcal{C}^\vartheta((0, \Im], \mathbf{S}_\wp)$.

Since $\mathfrak{G}(U, U) \in \mathcal{C}^\vartheta((0, \Im], \mathbf{S}_\wp)$ is proved, according to Step 2, this yields that ${}^C\mathcal{D}^\beta_\omega\psi \in \mathcal{C}((0, \Im], \mathbf{S}_\wp)$ and ${}^C\mathcal{D}^\beta_\omega\psi = -\mathcal{A}\psi + \mathfrak{G}(U, U)$. As a result, we are able to achieve that ${}^C\mathcal{D}^\beta_\omega u \in \mathcal{C}((0, \Im], \mathbf{S}_\wp)$ and ${}^C\mathcal{D}^\beta_\omega U = -\mathcal{A}U + \mathfrak{G}(U, U) + P\mathfrak{g}$. We reach the conclusion that U is a classical solution. □

Theorem 5. *Suppose that* (f_1) *holds. If* U *is a classical solution of* (3)*, then* $\mathcal{A}U \in \mathcal{C}^v((0, \Im], \mathbf{S}_\wp)$ *and* $(\mathcal{C})\mathcal{D}^\beta_\omega U \in \mathcal{C}^v((0, \Im], \mathbf{S}_\wp)$.

Proof. If U is a classical solution of (3), then $U(\omega) = \phi(\omega) + \psi(\omega)$. It is still necessary to demonstrate that $\mathcal{A}\varphi \in \mathcal{C}^{\beta(1-\gamma)}((0, \Im, \mathbf{S}_\wp)$; it suffices to show that $\mathcal{A}\varphi \in \mathcal{C}^{\beta(1-\gamma)}([\xi, \Im], \mathbf{S}_\wp)$ for every $\xi > 0$. In fact, take h that is $\xi \leq \omega < \omega + h \leq \Im$, by Lemma 2(iii):

$$\begin{aligned}
\mathcal{D}_\wp(\mathcal{A}\varphi(\omega + h) - \mathcal{A}\varphi(\omega)) &= \mathcal{D}_\wp\left(\int_\omega^{\omega+h} -s^{\beta-1}\mathcal{A}^2 E_{\beta,\beta}(-s^\beta\mathcal{A})ads\right) \\
&\leq \mathcal{C}_1 \int_\omega^{\omega+h} s^{-\beta(1-\gamma)-1}ds|by(\omega)\sin\gamma|_{\mathbf{H}^{\gamma,\wp}} \\
&= \frac{\mathcal{C}_1|by(\omega)\sin\gamma|_{\mathbf{H}^{\gamma,\wp}}}{\beta}(\omega^{-\beta(1-\gamma)} - (\omega + h)^{-\beta(1-\gamma)}) \\
&\leq \frac{\mathcal{C}_1|by(\omega)\sin\gamma|_{\mathbf{H}^{\gamma,\wp}}}{\beta} \frac{h^{\beta(1-\gamma)}}{[\xi(\xi + h)]^{\beta(1-\gamma)}}.
\end{aligned}$$

We write $\phi(\omega)$ in the same way as Lemma 7,

$$\begin{aligned}
\varpi(\omega) &= \varpi(\omega) + \varpi(\omega) \\
&= \int_0^\omega (\omega - s)^{\beta-1}E_{\beta,\beta}(-(\omega - s)^\beta\mathcal{A})(P\mathfrak{g}(s) - P\mathfrak{g}(\omega)) + \int_0^\omega (\omega - s)^{\beta-1}E_{\beta,\beta}(-(\omega - s)^\beta\mathcal{A})(P\mathfrak{g}(\omega)ds,
\end{aligned}$$

for $\omega \in (0, \wp]$. From Lemma 7 and Equation (23), it follows that $A\varpi_1(\omega) \in \mathcal{C}^v([0, \Im], \mathbf{S}_\wp)$ $\mathcal{A}\varpi_2(\omega) \in \mathcal{C}^\vartheta((0, \Im], \mathbf{S}_\wp)$, respectively.

Since $\mathfrak{G}(U, U) \in \mathcal{C}^\vartheta((0, \Im], \mathbf{S}_\wp)$, the result for function $\varpi(\omega)$ is also proven by a similar argument, implying that $\mathcal{A}\varpi \in \mathcal{C}^v((0, \Im], \mathbf{S}_\wp)$. As a result, $\mathcal{A}U \in \mathcal{C}^v(0, \Im], \mathbf{S}_\wp)$ and ${}^C\mathcal{D}^\beta_\omega U = \mathcal{A}U + \mathfrak{G}(U, U) + P\mathfrak{g} \in \mathcal{C}^v((0, \Im], \mathbf{S}_\wp)$. The proof is completed. □

7. Conclusions

The aim of this paper is to prove the existence-uniqueness of local and global mild solutions by using fuzzy techniques. Meanwhile, in $\mathbf{S}_\wp$, we provide a local moderate solution. Anomaly diffusion in fractal media is simulated using the Navier–Stokes equations (NSEs) with time-fractional derivatives of order $\beta \in (0, 1)$. We further show that classical solutions to such equations exist and are regular in $\mathbf{S}_\wp$. Future work could include extending this concept by incorporating MHD effects, expanding on the concept proposed in this mission, including observability and generalizing other activities. This is an interesting area with a lot of study going on that could lead to a lot of different applications and theories. This is a path in which we want to invest considerable resources.

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