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Even-Order Neutral Delay Differential Equations with Noncanonical Operator: New Oscillation Criteria

Osama Moaaz [1,2,*], Barakah Almarri [3], Fahd Masood [2] and Doaa Atta [1,2]

1 Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia; doaaatta44@gmail.com
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; fahdmasoud22@std.mans.edu.eg
3 Department of Mathematical Sciences, College of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; bjalmarri@pnu.edu.sa
* Correspondence: o_moaaz@mans.edu.eg

Abstract: The main objective of our paper is to investigate the oscillatory properties of solutions of differential equations of neutral type and in the noncanonical case. We follow an approach that simplifies and extends the related previous results. Our results are an extension and reflection of developments in the study of second-order equations. We also derive criteria for improving conditions that exclude the decreasing positive solutions of the considered equation.

Keywords: differential equations of neutral type; non-canonical case; even-order; oscillation

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1. Introduction

The aim of this work is to provide new criteria for testing the oscillation of solutions to the even-order neutral differential equation (NDE)

$$\left(r(\mathfrak{f})\left(\left(v(\mathfrak{f})+p(\mathfrak{f})(v(\tau(\mathfrak{f})))\right)^{(n-1)}\right)^{\gamma}\right)'+q(\mathfrak{f})v^{\gamma}(\rho(\mathfrak{f}))=0, \tag{1}$$

in the noncanonical case, that is,

$$\mathfrak{y}_0(\mathfrak{f}_0):=\int_{\mathfrak{f}_0}^{\infty} r^{-1/\gamma}(\zeta)\mathrm{d}\zeta<\infty,$$

where $\mathfrak{f}\geq\mathfrak{f}_0$, $n\geq 4$ is an even integer, and γ is a quotient of odd positive integers. The following assumptions are taken into consideration during the study:

(H1) $r\in C^1[\mathfrak{f}_0,\infty)$, $r(\mathfrak{f})>0$, $r'(\mathfrak{f})\geq 0$ and $\mathfrak{y}_k(\mathfrak{f}_0)<\infty$, where

$$\mathfrak{y}_k(\mathfrak{f}):=\int_{\mathfrak{f}}^{\infty}\mathfrak{y}_{k-1}(\zeta)\mathrm{d}\zeta \ \text{ for } k=1,2,\ldots,n-2;$$

(H2) $p,q\in C[\mathfrak{f}_0,\infty)$, $0\leq p<1$, $q\geq 0$, q does not vanish identically on any half-line $[\mathfrak{f}_*,\infty)$ for all $\mathfrak{f}_*\geq\mathfrak{f}_0$;

(H3) $\tau,\rho\in C[\mathfrak{f}_0,\infty)$, $\tau(\mathfrak{f})\leq\mathfrak{f}$, $\rho(\mathfrak{f})\leq\mathfrak{f}$, $\rho'(\mathfrak{f})\geq 0$, $\lim_{\mathfrak{f}\to\infty}\tau(\mathfrak{f})=\infty$, and $\lim_{\mathfrak{f}\to\infty}\rho(\mathfrak{f})=\infty$.

Since the time of Newton, differential equations (DE) are still used to understand and model phenomena. Applied phenomena and problems are constantly increasing as a result of the development of all different branches of science. Delay differential equations (DDE) are DE that take into account the time memory of a solution, and therefore are a more suitable method for modeling different phenomena. However, the problem of finding solutions to the equations resulting from the modeling of phenomena stands as an obstacle to understanding and studying these phenomena. Therefore, the qualitative theory contributes greatly to solving this problem, so that the qualitative properties of equations can be studied without finding their solutions. One of the branches of the qualitative theory is oscillation theory, which is the theory that deals with the oscillatory, non-oscillatory, asymptotic behavior and the distribution of zeros for solutions of DE.

Fourth-order DEs often appear in mathematical models of many biological, chemical, and physical phenomena. Structure deformation, elasticity issues, and soil settlement are examples of these uses. The existence or nonexistence of oscillatory solutions is one of the most crucial topics that arise when investigating mechanical problems. Oscillatory muscle movement is one of the models given by a fourth-order equation with a delay that might arise as a result of a muscle's inertial pregnancy, see [1]. Fourth-order equations arise in the theory of numbers, which is unusual, see [2].

NDEs appear as models for many phenomena and problems in applied science, including automatic control, population dynamics, mixing liquids, and vibrating masses attached to an elastic bar (see [3]). In particular, neutral equations of second order are of considerable interest in biology to study the self-balance of the human body and in robotics to construct biped robots (see [4]). The applications of these equations have largely stimulated the study of qualitative properties, in particular the theory of oscillation.

Recently, there has been a great interest in the oscillation theory of DDEs, see [5–25]. This interest greatly contributed to the development of different new techniques and approaches that significantly improved the oscillation criteria, especially for second-order DE. Furthermore, this development has also resulted in many open and interesting issues.

We find that the study of the oscillatory behavior of the NDE in the canonical case ($\mathfrak{y}_0(\mathfrak{f}_0) = \infty$) has attracted the greatest interest regarding the verification and study by many techniques and methods. However, it is possible to find some studies on the oscillation of even-order delay (nonneutral) DE with a noncanonical operator in the works by Baculikova et al. [5], Zhang et al. [6–8], and, recently, Moaaz et al. [9,10].

In 2011, Zhang et al. [6] presented the conditions of oscillation for DDE

$$\left(r(\mathfrak{f})\left(v^{(n-1)}(\mathfrak{f})\right)^{\gamma}\right)' + q(\mathfrak{f})v^{\alpha}(\rho(\mathfrak{f})) = 0 \tag{2}$$

converge to zero, where $\alpha \leq \gamma$ and α is a ratio of odd positive integers. Later, in 2013, Zhang et al. [7] established three independent conditions for oscillation of (2). In [8], under four independent conditions, Zhang et al. obtained some oscillation criteria for (2) when $n = 4$ and $\alpha = \gamma$. The results in [8] differ from the results in [6,7] in that they apply to ordinary DE. By using the comparison theorems, Baculikova et al. [5] investigated the oscillatory propertiws of (2).

Reflecting the great development in studying the oscillation of DDEs of second-order (see, for example, [11–13]), Muhammad and Ahmed established extended and improved results for the oscillation of higher-order DDEs, see [14–17].

The two most common methods for studying oscillation of DDEs are: Riccati substitution and comparison with first-order equations. For NDEs in the noncanonical case, Li and Rogovchenko [18] obtained criteria for oscillation of the NDE (1) using the principle of comparison. They mainly relied on the assumption of the existence of the functions ξ_i that fulfill the restrictions:

$$\begin{cases} \xi_1, \xi_2, \xi_3 \in C([\mathfrak{f}_0, \infty), R), \\ \xi_1(\mathfrak{f}) \leq \rho(\mathfrak{f}) \leq \xi_2(\mathfrak{f}),\ \xi_1(\mathfrak{f}) \leq \tau(\mathfrak{f}) \leq \mathfrak{f} < \xi_2(\mathfrak{f}), \\ \xi_3(\mathfrak{f}) \geq \rho(\mathfrak{f}),\ \xi_3(\mathfrak{f}) > \mathfrak{f} \text{ and } \lim_{\mathfrak{f}\to\infty} \xi_1(\mathfrak{f}) = \infty, \end{cases}$$

and in addition the delay functions were restricted by the conditions

$$\tau' \geq \tau_* > 0 \text{ and } \tau \circ \rho = \rho \circ \tau. \tag{3}$$

Using these conditions, they were able to compare (1) with three DDEs of first order, and thus obtained three independent conditions to check oscillation.

NDEs of higher order have not received the attention of researchers compared to equations with delay (not neutral) or equations of the second order. The studies that deal with the oscillatory behavior of higher-order neutral equations have many interesting open issues as well as additional restrictions on the functions.

This paper aims to study the oscillatory behavior of solutions to the NDE (1) by using several different methods. The motive of this study is to develop and improve the oscillation criteria for solutions of higher-order DE by:

- obtaining oscillation criteria for (1) without requiring the existence of the unknown functions ξ_i, and without requiring the condition in (3);
- reducing the number of conditions that are sufficient to verify the oscillation of all solutions;
- obtaining criteria that apply to ordinary DE.

2. Preliminaries

In this part, we provide definitions and elementary results in the literature that help us to present the main results.

Definition 1. *A solution v of (1) means a real-valued function in $C[\mathfrak{f}_*, \infty)$, $\mathfrak{f}_* \geq \mathfrak{f}_0$, which satisfies*

$$v + p \cdot (v \circ \tau) \in C^{(n-1)}[\mathfrak{f}_*, \infty), \;\; r \cdot \left((v + p \cdot (v \circ \tau))^{(n-1)} \right)^{\gamma} \in C^1[\mathfrak{f}_*, \infty),$$

and v satisfies (1) on $[\mathfrak{f}_*, \infty)$. To facilitate calculations, we will denote the corresponding function of the solution v by $z := v + p \cdot (v \circ \tau)$.

Definition 2. *A solution of any DDE is called oscillatory if it is neither positive nor negative, ultimately; otherwise, it is called nonoscillatory.*

Lemma 1. *[Lemma 2.2.1, [19]] Suppose that $w \in C^n([\mathfrak{f}_0, \infty), (0, \infty))$, and $w^{(n)}(\mathfrak{f})$ is of constant sign for $\mathfrak{f} \geq \mathfrak{f}_1 \geq \mathfrak{f}_0$. Then, there is a nonnegative integer $\mu \leq n$, with $(-1)^{n+\mu} w^{(n)}(\mathfrak{f}) \geq 0$, such that*

$$\mu > 0 \text{ implies } w^{(\mu)}(\mathfrak{f}) > 0 \text{ for } \mu = 0, 1, \ldots, \mu - 1$$

and

$$\mu \leq n - 1 \text{ implies } (-1)^{\mu+m} w^{(m)}(\mathfrak{f}) > 0 \text{ for } m = \mu, \mu + 1, \ldots, n - 1,$$

for $\mathfrak{f} \geq \mathfrak{f}_1$.

Lemma 2. *[Lemma 2.2.3, [19]] Suppose that $w \in C^{\kappa}([\mathfrak{f}_0, \infty), (0, \infty))$ with derivatives up to order $\kappa - 1$ of constant sign, and $w^{(\kappa-1)}(\mathfrak{f}) w^{(\kappa)}(\mathfrak{f}) \leq 0$ for $\mathfrak{f} \geq \mathfrak{f}_1 \geq \mathfrak{f}_0$, and $\lim_{\mathfrak{f} \to \infty} w(\mathfrak{f}) \neq 0$. Then, for every $\lambda \in (0, 1)$, there is a $\mathfrak{f}_\lambda \geq \mathfrak{f}_1$ such that*

$$w(\mathfrak{f}) \geq \frac{\lambda}{(\kappa - 1)!} t^{\kappa-1} \left| w^{(\kappa-1)}(\mathfrak{f}) \right|,$$

for $\mathfrak{f} \geq \mathfrak{f}_\lambda$.

Lemma 3. *Let $H(v) = L_1 v - L_2 (v - L_3)^{(\gamma+1)/\gamma}$, where $L_2 > 0$, L_1 and L_3 are constants. Then, the maximum value of H on $\mathbb{R}$ at $v^* = L_3 + (\gamma L_1 / ((\gamma + 1) L_2))^{\gamma}$ is*

$$\max_{v \in R} H(v) = H(v^*) = L_1 L_3 + \frac{\gamma^{\gamma}}{(\gamma + 1)^{(\gamma+1)}} \frac{L_1^{\gamma+1}}{L_2^{\gamma}}.$$

Remark 1. *In this work, we consider only the solutions which are not identically zero, eventually. Further, all functional equations and inequalities are supposed to hold eventually.*

3. Simplified Criteria for Oscillation

When studying the oscillatory properties of solutions of DDEs, it is known that positive solutions must be classified according to the sign of their derivatives. Now, we assume that υ is an eventually positive solution of (1). Then, from the definition of z, we have that $z(\mathfrak{f}) > 0$, eventually. From the DE in (1) and taking into account that $q(\mathfrak{f}) > 0$, we have that $r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}$ is a nonincreasing function. Furthermore, according to Lemma 1, we obtain the following three cases, eventually:

(**1**): $z^{(m)}(\mathfrak{f}) > 0$ for $m = 0, 1, n-1$ and $z^{(n)}(\mathfrak{f}) < 0$;
(**2**): $z^{(m)}(\mathfrak{f}) > 0$ for $m = 0, 1, n-2$ and $z^{(n-1)}(\mathfrak{f}) < 0$;
(**3**): $(-1)^m z^{(m)}(\mathfrak{f}) > 0$ for $m = 0, 1, \ldots, n-1$.

In order to facilitate the presentation of results, we will use the following notations:

$\mathbf{U}^+$: The set of all eventually positive solutions of (1);
$\mathbf{U}_i^+$: The set of all solutions $\upsilon \in \mathbf{U}^+$ with z satisfying case (**i**) above, $i = 1, 2, 3$;
and

$$\widetilde{p}(\mathfrak{f}) := 1 - p(\mathfrak{f})\frac{\mathfrak{y}_{n-2}(\tau(\mathfrak{f}))}{\mathfrak{y}_{n-2}(\mathfrak{f})}.$$

Lemma 4. *If $\upsilon \in \mathbf{U}_3^+$, then*

$$(-1)^{k+1} z^{(n-k-2)}(\mathfrak{f}) \leq r^{1/\gamma}(\mathfrak{f}) z^{(n-1)}(\mathfrak{f}) \mathfrak{y}_k(\mathfrak{f}), \tag{4}$$

and

$$(-1)^k \frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z^{(n-k-2)}(\mathfrak{f})}{\mathfrak{y}_k(\mathfrak{f})}\right) \geq 0, \tag{5}$$

for $k = 0, 1, \ldots, n-2$, eventually.

Proof. Assume that $\upsilon \in \mathbf{U}_3^+$. From (1), we have that $r \cdot \left(z^{(n-1)}\right)^{\gamma}$ is nonincreasing, and hence

$$\begin{aligned} r^{1/\gamma}(\mathfrak{f}) z^{(n-1)}(\mathfrak{f}) \int_{\mathfrak{f}}^{\infty} \frac{1}{r^{1/\gamma}(\zeta)} \mathrm{d}\zeta &\geq \int_{\mathfrak{f}}^{\infty} \frac{1}{r^{1/\gamma}(\zeta)} r^{1/\gamma}(\zeta) z^{(n-1)}(\zeta) \mathrm{d}\zeta \\ &= \lim_{\zeta \to \infty} z^{(n-2)}(\zeta) - z^{(n-2)}(\mathfrak{f}). \end{aligned} \tag{6}$$

Since $z^{(n-2)}(\mathfrak{f})$ is a positive decreasing function, we have that $z^{(n-2)}(\mathfrak{f})$ converges to a nonnegative constant when $\mathfrak{f} \to \infty$. Thus, (6) becomes

$$-z^{(n-2)}(\mathfrak{f}) \leq r^{1/\gamma}(\mathfrak{f}) z^{(n-1)}(\mathfrak{f}) \mathfrak{y}_0(\mathfrak{f}). \tag{7}$$

Using the fact that $(-1)^m z^{(m)}(\mathfrak{f}) > 0$ for $m = 0, 1, \ldots, n-1$, and integrating the inequality in (7) and the successive inequalities that result a total of $n-2$ times from $\mathfrak{f}$ to ∞, we obtain

$$(-1)^{k+1} z^{(n-k-2)}(\mathfrak{f}) \leq r^{1/\gamma}(\mathfrak{f}) z^{(n-1)}(\mathfrak{f}) \mathfrak{y}_k(\mathfrak{f}),$$

for $k = 1, \ldots, n-2$. Thus, we arrive at (4).

Now, from (7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z^{(n-2)}(\mathfrak{f})}{\mathfrak{y}_0(\mathfrak{f})}\right) = \frac{1}{\mathfrak{y}_0^2(\mathfrak{f})}\left(z^{(n-1)}(\mathfrak{f})\mathfrak{y}_0(\mathfrak{f}) + z^{(n-2)}(\mathfrak{f}) r^{-1/\gamma}(\mathfrak{f})\right) \geq 0.$$

which leads to

$$-z^{(n-3)}(\mathfrak{f}) \geq \int_{\mathfrak{f}}^{\infty} \eta_0(\zeta)\frac{z^{(n-2)}(\zeta)}{\eta_0(\zeta)}\mathrm{d}\zeta \geq \frac{z^{(n-2)}(\mathfrak{f})}{\eta_0(\mathfrak{f})}\eta_1(\mathfrak{f}).$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z^{(n-3)}(\mathfrak{f})}{\eta_1(\mathfrak{f})}\right) = \frac{1}{\eta_1^2(\mathfrak{f})}\left(z^{(n-2)}(\mathfrak{f})\eta_1(\mathfrak{f}) + z^{(n-3)}(\mathfrak{f})\eta_0(\mathfrak{f})\right) \leq 0.$$

By repeating a similar approach, we can obtain (5). Thus, the proof is complete. □

Theorem 1. *Assume that* $\widetilde{p}(\mathfrak{f}) > 0$. *If*

$$\limsup_{\mathfrak{f}\to\infty} \int_{\mathfrak{f}_0}^{\mathfrak{f}} \left(q(s)\left((1-p(\rho(s)))\eta_0(s)\frac{\lambda_0\rho^{n-2}(s)}{(n-2)!}\right)^{\gamma} - \frac{(\gamma/(\gamma+1))^{\gamma+1}}{r^{1/\gamma}(s)\eta_0(s)}\right)\mathrm{d}s = \infty, \tag{8}$$

for some constant $\lambda_0 \in (0,1)$, *and*

$$\limsup_{\mathfrak{f}\to\infty} \int_{\mathfrak{f}_1}^{\mathfrak{f}} \frac{1}{r^{1/\gamma}(\xi)}\left(\int_{\mathfrak{f}_1}^{\xi} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\eta_{n-2}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\xi = \infty, \tag{9}$$

then all solutions of (1) are oscillatory.

Proof. Assume, for the sake of contradiction, that (1) has a nonoscillatory solution. Without loss of generality, we let $v \in \mathbf{U}^+$. Thus, v belongs to one of the classes $\mathbf{U}_i^+$, $i = 1,2,3$. Using Theorem 2.1 in [20], we find that the condition (8) is in opposition to $v \in \mathbf{U}_2^+$. So, v belongs to either $\mathbf{U}_1^+$ or $\mathbf{U}_3^+$.

First, let us assume that $v \in \mathbf{U}_3^+$. From Lemma 4, (4) and (5) hold. Using (5) with $k = n-2$, we obtain that

$$z(\tau(\mathfrak{f})) \leq \frac{\eta_{n-2}(\tau(\mathfrak{f}))}{\eta_{n-2}(\mathfrak{f})}z(\mathfrak{f}).$$

Hence, from the definition of z and the above inequality, we see that

$$v(\mathfrak{f}) \geq \left(1 - p(\mathfrak{f})\frac{\eta_{n-2}(\tau(\mathfrak{f}))}{\eta_{n-2}(\mathfrak{f})}\right)z(\mathfrak{f}) = \widetilde{p}^{\gamma}(\mathfrak{f})z(\mathfrak{f}).$$

Then, from this inequality and (1), we obtain

$$\left(r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}\right)' + q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))z^{\gamma}(\rho(\mathfrak{f})) \leq 0. \tag{10}$$

Now, by integrating (10) twice from $\mathfrak{f}_1$ to $\mathfrak{f}$, we have

$$z^{(n-2)}(\mathfrak{f}) \leq z^{(n-2)}(\mathfrak{f}_1) - \int_{\mathfrak{f}_1}^{\mathfrak{f}} \frac{1}{r^{1/\gamma}(\xi)}\left(\int_{\mathfrak{f}_1}^{\xi} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))z^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\xi. \tag{11}$$

Taking into account the monotonicity of $r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}$, we obtain that

$$r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \leq r(\mathfrak{f}_1)\left(z^{(n-1)}(\mathfrak{f}_1)\right)^{\gamma} := -L < 0.$$

Then, from (4) with $k = n-2$, we arrive at

$$-z^{\gamma}(\mathfrak{f}) \leq r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}\eta_{n-2}^{\gamma}(\mathfrak{f}) \leq -L\eta_{n-2}^{\gamma}(\mathfrak{f}),$$

which in view of (11) implies that

$$z^{(n-2)}(\mathfrak{f}) \le z^{(n-2)}(\mathfrak{f}_1) - L^{1/\gamma} \int_{\mathfrak{f}_1}^{\mathfrak{f}} \frac{1}{r^{1/\gamma}(\xi)} \left(\int_{\mathfrak{f}_1}^{\xi} q(\zeta) \widetilde{p}^{\gamma}(\rho(\zeta)) \mathfrak{y}_{n-2}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta \right)^{1/\gamma} \mathrm{d}\xi.$$

From (9) and the above inequality, we obtain that $z^{(n-2)}(\mathfrak{f}) \to -\infty$ when $\mathfrak{f} \to \infty$. This is a contradiction with the positivity of $z^{(n-2)}(\mathfrak{f})$.

Finally, let us assume that $v \in \mathbf{U}_1^+$. Since $z'(\mathfrak{f}) > 0$ and $\mathfrak{y}_{n-2}(\tau(\mathfrak{f})) \ge \mathfrak{y}_{n-2}(\mathfrak{f})$, we obtain that

$$v(\mathfrak{f}) > (1 - p(\mathfrak{f}))z(\mathfrak{f}) \ge \left(1 - p(\mathfrak{f}) \frac{\mathfrak{y}_{n-2}(\tau(\mathfrak{f}))}{\mathfrak{y}_{n-2}(\mathfrak{f})}\right) z(\mathfrak{f}),$$

which with (1) gives again (10). Integrating (10) from $\mathfrak{f}_1$ to $\mathfrak{f}$, we find

$$\begin{aligned} r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} &\le r(\mathfrak{f}_1)\left(z^{(n-1)}(\mathfrak{f}_1)\right)^{\gamma} - \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) z^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta \\ &\le r(\mathfrak{f}_1)\left(z^{(n-1)}(\mathfrak{f}_1)\right)^{\gamma} \\ &\quad - z^{\gamma}(\rho(\mathfrak{f}_1)) \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta. \end{aligned} \tag{12}$$

Using (9) and taking into account that $\mathfrak{y}_0(\mathfrak{f}_0) < \infty$ and $\mathfrak{y}_{n-2}'(\mathfrak{f}) < 0$, we have that

$$\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta \to \infty \ \text{ as } \mathfrak{f} \to \infty. \tag{13}$$

Letting $\mathfrak{f} \to \infty$ in (12) and using (13), we arrive at $r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \to -\infty$ as $\mathfrak{f} \to \infty$, which contradicts the positivity of $z^{(n-1)}(\mathfrak{f})$. Thus, the proof is complete. □

Theorem 2. *Assume that* $\widetilde{p}(\mathfrak{f}) > 0$. *If (8) holds for some constant* $\lambda_0 \in (0,1)$ *and*

$$\limsup_{\mathfrak{f} \to \infty} \left(\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f}) \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta \right) > 1, \tag{14}$$

then all solutions of (1) are oscillatory.

Proof. Assume, for the sake of contradiction, that (1) has a nonoscillatory solution. Without loss of generality, we let $v \in \mathbf{U}^+$. As in the proof of Theorem 1, using the condition (8), we have that v belongs to either $\mathbf{U}_1^+$ or $\mathbf{U}_3^+$.

First, let us assume that $v \in \mathbf{U}_3^+$. In a similar fashion as in the proof of Theorem 1, we can arrive at (10). Then, integrating (10) from $\mathfrak{f}_1$ to $\mathfrak{f}$, we find

$$\begin{aligned} r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} &\le -\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) z^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta \\ &\le -z^{\gamma}(\rho(\mathfrak{f})) \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta. \end{aligned} \tag{15}$$

Since $z'(\mathfrak{f}) < 0$ and $\rho(\mathfrak{f}) \le \mathfrak{f}$, we find that

$$r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \le -z^{\gamma}(\mathfrak{f}) \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta)) \mathrm{d}\zeta. \tag{16}$$

From (4), with $k = n-2$, in Lemma 4, we have that

$$-z^{\gamma}(\mathfrak{f}) \le r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f}). \tag{17}$$

Combining (16) and (17), we conclude that

$$r(\mathfrak{f})\Big(z^{(n-1)}(\mathfrak{f})\Big)^{\gamma} \leq r(\mathfrak{f})\Big(z^{(n-1)}(\mathfrak{f})\Big)^{\gamma}\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta.$$

This implies that

$$\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta \leq 1,$$

which contradicts (14).

Now, let us assume that $v \in \mathbf{U}_1^+$ on $[\mathfrak{f}_1, \infty)$, where $\mathfrak{f}_1 \geq \mathfrak{f}_0$. From (14), there exists a $\mathfrak{f}_2 \geq \mathfrak{f}_1$ such that

$$\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta > 1$$

for all $\mathfrak{f} \geq \mathfrak{f}_2$, and thus

$$\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta > \frac{1}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}.$$

Letting $\mathfrak{f} \to \infty$, we obtain that (13). As in the proof of Theorem 1, we arrive at (12). Letting $\mathfrak{f} \to \infty$ in (12) and using (13), we arrive at $r(\mathfrak{f})\Big(z^{(n-1)}(\mathfrak{f})\Big)^{\gamma} \to -\infty$ as $\mathfrak{f} \to \infty$, which contradicts the positivity of $z^{(n-1)}(\mathfrak{f})$. Thus, the proof is complete. □

Theorem 3. *Assume that* $\widetilde{p}(\mathfrak{f}) > 0$. *If (8) holds for some* $\lambda \in (0,1)$, *and*

$$\liminf_{\mathfrak{f}\to\infty} \int_{\rho(\mathfrak{f})}^{\mathfrak{f}} \mathfrak{y}_{n-3}(\xi)\left(\int_{\mathfrak{f}_1}^{\xi} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\xi > \frac{1}{\mathrm{e}}, \tag{18}$$

then all solutions of (1) are oscillatory.

Proof. Assume, for the sake of contradiction, that (1) has a nonoscillatory solution. Without loss of generality, we let $v \in \mathbf{U}^+$. As in the proof of Theorem 1, using the condition (8), we have that v belongs to either $\mathbf{U}_1^+$ or $\mathbf{U}_3^+$.

First, let us assume that $v \in \mathbf{U}_3^+$. In a similar fashion as in the proof of Theorem 1, we can arrive at (15). From Lemma 4, (4) holds. By (4) with $k = n-3$ and (15), we find that

$$\frac{(z'(\mathfrak{f}))^{\gamma}}{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})} \leq -z^{\gamma}(\rho(\mathfrak{f}))\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta.$$

Hence, we note that $w := z$ is a positive solution of the delay differential inequality

$$w'(\mathfrak{f}) + \mathfrak{y}_{n-3}(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} w(\rho(\mathfrak{f})) \leq 0.$$

It follows from Theorem 1 in [21] that there is also a positive solution of the DDE

$$w'(\mathfrak{f}) + \mathfrak{y}_{n-3}(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} w(\rho(\mathfrak{f})) = 0. \tag{19}$$

By Theorem 2 in [22], we find that the condition (18) guarantees oscillation of all solutions of (19), which is a contradiction.

Finally, let us assume that $v \in \mathbf{U}_1^+$. Since the condition (13) is necessary for the validity of (18), the proof of this part is exactly the same as in Theorem 1. Thus, the proof is complete. □

4. Improved Criteria Ensure That $\mathbf{U}_3^+ = \varnothing$

It is clear that Theorem 2 is not applicable if

$$\mathfrak{y}_{n-2}(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} \leq 1. \tag{20}$$

In what follows, we present a finer estimate for the ratio $z(\rho(\mathfrak{f}))/z(\mathfrak{f})$ that will allow us to improve the previous results and apply the new results when (20) holds.

Lemma 5. *Assume that $v \in \mathbf{U}_3^+$. If there exist a $\mathfrak{f}_1 \geq \mathfrak{f}_0$ and a constant $\beta \geq 0$ such that*

$$\mathfrak{y}_{n-2}(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} \geq \beta, \tag{21}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})}\right) \leq 0 \ \text{ for } \mathfrak{f} \geq \mathfrak{f}_1. \tag{22}$$

Proof. Assume that $v \in \mathbf{U}_3^+$. Proceeding as in the proof of Theorem 2, we arrive at (16). From Lemma 4, we have that (4) holds for $k = n-3$, that is,

$$z'(\mathfrak{f}) \leq r^{1/\gamma}(\mathfrak{f})z^{(n-1)}(\mathfrak{f})\mathfrak{y}_{n-3}(\mathfrak{f}). \tag{23}$$

From (16) and (23), we obtain

$$z'(\mathfrak{f}) \leq -\mathfrak{y}_{n-3}(\mathfrak{f})z(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma}. \tag{24}$$

Now, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})}\right) &= \frac{1}{\mathfrak{y}_{n-2}^{2\beta}(\mathfrak{f})}\left(\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})z'(\mathfrak{f}) - \beta\mathfrak{y}_{n-2}^{\beta-1}(\mathfrak{f})\mathfrak{y}_{n-2}'(\mathfrak{f})z(\mathfrak{f})\right)\\ &= \frac{1}{\mathfrak{y}_{n-2}^{\beta+1}(\mathfrak{f})}\left(\mathfrak{y}_{n-2}(\mathfrak{f})z'(\mathfrak{f}) + \beta\mathfrak{y}_{n-3}(\mathfrak{f})z(\mathfrak{f})\right),\end{aligned}$$

which, with (24), gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})}\right) &\leq \frac{1}{\mathfrak{y}_{n-2}^{\beta+1}(\mathfrak{f})}\left(-\mathfrak{y}_{n-2}(\mathfrak{f})\mathfrak{y}_{n-3}(\mathfrak{f})z(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} + \beta\mathfrak{y}_{n-3}(\mathfrak{f})z(\mathfrak{f})\right)\\ &= \frac{1}{\mathfrak{y}_{n-2}^{\beta+1}(\mathfrak{f})}\left(-\mathfrak{y}_{n-2}(\mathfrak{f})\left(\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta\right)^{1/\gamma} + \beta\right)\mathfrak{y}_{n-3}(\mathfrak{f})z(\mathfrak{f}).\end{aligned}$$

It follows from (21) that

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{f}}\left(\frac{z(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})}\right) \leq 0.$$

This completes the proof. □

Theorem 4. *Assume that $v \in \mathbf{U}^+$ and $\widetilde{p}(\mathfrak{f}) > 0$. If there exist a $\mathfrak{f}_1 \geq \mathfrak{f}_0$ and a constant $\beta \geq 0$ such that (21) holds, and*

$$\limsup_{\mathfrak{f}\to\infty} \frac{\mathfrak{y}_{n-2}^{\beta}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta-\gamma}(\mathfrak{f})}\int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta > 1, \tag{25}$$

then $\mathbf{U}_3^+ = \varnothing$.

Proof. Assume the contrary, that $v \in \mathbf{U}_3^+$ for $\mathfrak{f} \geq \mathfrak{f}_1 \geq \mathfrak{f}_0$. Using Lemma 5, we obtain that (22) holds, and then

$$z(\rho(\mathfrak{f})) \geq \frac{\mathfrak{y}_{n-2}^{\beta}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})} z(\mathfrak{f}). \tag{26}$$

As in the proof of Theorem 2, we arrive at (15). From (15) and (26), we get that

$$r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \leq -\frac{\mathfrak{y}_{n-2}^{\beta}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta}(\mathfrak{f})} z(\mathfrak{f}) \int_{\mathfrak{f}_1}^{\mathfrak{f}} q(\zeta)\widetilde{p}^{\gamma}(\rho(\zeta))\mathrm{d}\zeta. \tag{27}$$

By replacing (16) by (27), and completing the proof as in Theorem 2, we arrive at a contradiction with (25). This completes the proof. □

In what follows, using the comparison principle with a second-order DDE, we infer a criterion which ensures that $\mathbf{U}_3^+ = \varnothing$.

Theorem 5. *Assume that* $v \in \mathbf{U}^+$ *and* $\widetilde{p}(\mathfrak{f}) > 0$. *If the DE*

$$\left(\mathfrak{y}_{n-3}^{-\gamma}(\mathfrak{f})\left(y'(\mathfrak{f})\right)^{\gamma}\right)' + q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\left(\frac{\mathfrak{y}_{n-2}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}(\mathfrak{f})}\right)^{\beta\gamma} y^{\gamma}(\mathfrak{f}) = 0 \tag{28}$$

is oscillatory, then $\mathbf{U}_3^+ = \varnothing$.

Proof. Assume the contrary that $v \in \mathbf{U}_3^+$ for $\mathfrak{f} \geq \mathfrak{f}_1 \geq \mathfrak{f}_0$. As in the proof of Theorem 1, we have that (10) holds. Now, we define the function

$$\omega(\mathfrak{f}) := r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} z^{-\gamma}(\mathfrak{f}).$$

Then, $\omega(\mathfrak{f}) < 0$ for $\mathfrak{f} \geq \mathfrak{f}_1$. Differentiating ω, and using (10), we obtain

$$\begin{aligned} \omega'(\mathfrak{f}) &= \left(r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}\right)' z^{-\gamma}(\mathfrak{f}) - \gamma z'(\mathfrak{f}) z^{-(\gamma+1)}(\mathfrak{f}) r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma} \\ &\leq -q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f})) z^{\gamma}(\rho(\mathfrak{f})) z^{-\gamma}(\mathfrak{f}) \\ &\quad -\gamma z'(\mathfrak{f}) z^{-(\gamma+1)}(\mathfrak{f}) r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}. \end{aligned} \tag{29}$$

Using Lemma 4, we have that (4) holds. Further, as in the proof of Theorem 4, we arrive at (26). From ((4) with $k = n-3$) and (26), the inequality (29) becomes

$$\omega'(\mathfrak{f}) \leq -q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\left(\frac{\mathfrak{y}_{n-2}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}(\mathfrak{f})}\right)^{\beta\gamma} - \gamma \mathfrak{y}_{n-3}(\mathfrak{f}) r^{1+1/\gamma}(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma+1} z^{-(\gamma+1)}(\mathfrak{f}),$$

and thus

$$\omega'(\mathfrak{f}) + q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\left(\frac{\mathfrak{y}_{n-2}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}(\mathfrak{f})}\right)^{\beta\gamma} + \gamma \mathfrak{y}_{n-3}(\mathfrak{f}) \omega^{1+1/\gamma}(\mathfrak{f}) \leq 0. \tag{30}$$

In view of [23,24], the DE (28) is nonoscillatory if and only if there exists a function $\omega \in C([\mathfrak{f}_1, \infty), \mathbb{R})$ satisfying the inequality (30), which is a contradiction. This completes the proof. □

Corollary 1. *Assume that* $v \in \mathbf{U}^{+}$ *and* $\widetilde{p}(\mathfrak{f}) > 0$. *If*

$$\liminf_{\mathfrak{f}\to\infty} \frac{1}{\mathfrak{y}_{n-2}(\mathfrak{f})} \int_{\mathfrak{f}}^{\infty} \mathfrak{y}_{n-2}^{\gamma+1}(\xi) q(\xi) \widetilde{p}^{\gamma}(\rho(\xi)) \left(\frac{\mathfrak{y}_{n-2}(\rho(\xi))}{\mathfrak{y}_{n-2}(\xi)} \right)^{\beta\gamma} \mathrm{d}\xi > \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}, \tag{31}$$

then $\mathbf{U}_3^{+} = \varnothing$.

Proof. Using Theorem 2.3 in [25], we see that the condition (31) ensures that (28) is oscillatory. □

By introducing a generalized Riccati substitution, we provide, in the following, a criterion which ensures that $\mathbf{U}_3^{+} = \varnothing$.

Theorem 6. *Assume that* $v \in \mathbf{U}^{+}$ *and* $\widetilde{p}(\mathfrak{f}) > 0$. *If there exists a function* $w \in C([\mathfrak{f}_0, \infty), \mathbb{R}^{+})$ *such that*

$$\limsup_{\mathfrak{f}\to\infty} \frac{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}{w(\mathfrak{f})} \int_{\mathfrak{f}_1}^{\mathfrak{f}} \left(q(\xi) w(\xi) \widetilde{p}^{\gamma}(\rho(\xi)) \frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\xi))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\xi)} - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(w'(\xi))^{\gamma+1}}{w^{\gamma}(\xi) \mathfrak{y}_{n-3}^{\gamma}(\xi)} \right) \mathrm{d}\xi > 1, \tag{32}$$

then $\mathbf{U}_3^{+} = \varnothing$.

Proof. Assume the contrary that $v \in \mathbf{U}_3^{+}$ for $\mathfrak{f} \geq \mathfrak{f}_1 \geq \mathfrak{f}_0$. As in the proof of Theorems 1 and 4, we arrive, respectively, at (10) and (26). Now, we define the function

$$\Phi(\mathfrak{f}) := w(\mathfrak{f}) \left(\frac{r(\mathfrak{f}) \left(z^{(n-1)}(\mathfrak{f}) \right)^{\gamma}}{z^{\gamma}(\mathfrak{f})} + \frac{1}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})} \right). \tag{33}$$

From Lemma 4, we see that (4) holds, and thus $\Phi(\mathfrak{f}) > 0$ for $\mathfrak{f} \geq \mathfrak{f}_1$. Differentiating Φ and using (10) and (26), we obtain that

$$\begin{aligned} \Phi'(\mathfrak{f}) &= \frac{w'(\mathfrak{f})}{w(\mathfrak{f})} \Phi(\mathfrak{f}) + w(\mathfrak{f}) \left(\frac{\left(r(\mathfrak{f}) \left(z^{(n-1)}(\mathfrak{f}) \right)^{\gamma} \right)'}{z^{\gamma}(\mathfrak{f})} \right. \\ &\quad \left. - \gamma \frac{r(\mathfrak{f}) \left(z^{(n-1)}(\mathfrak{f}) \right)^{\gamma}}{z^{\gamma+1}(\mathfrak{f})} z'(\mathfrak{f}) + \gamma \frac{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma+1}(\mathfrak{f})} \right) \\ &\leq \frac{w'(\mathfrak{f})}{w(\mathfrak{f})} \Phi(\mathfrak{f}) + w(\mathfrak{f}) \left(-q(\mathfrak{f}) \widetilde{p}^{\gamma}(\rho(\mathfrak{f})) \frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\mathfrak{f})} \right. \\ &\quad \left. - \gamma \frac{r(\mathfrak{f}) \left(z^{(n-1)}(\mathfrak{f}) \right)^{\gamma}}{z^{\gamma+1}(\mathfrak{f})} z'(\mathfrak{f}) + \gamma \frac{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma+1}(\mathfrak{f})} \right), \end{aligned}$$

which, with ((4) when $k = n-3$), gives

$$\begin{aligned}
\Phi'(\mathfrak{f}) \quad \leq \quad & \frac{w'(\mathfrak{f})}{w(\mathfrak{f})}\Phi(\mathfrak{f}) + w(\mathfrak{f})\left(-q(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\mathfrak{f})}\right. \\
& \left. -\gamma r^{1+1/\gamma}(\mathfrak{f})\mathfrak{y}_{n-3}(\mathfrak{f})\frac{\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma+1}}{z^{\gamma+1}(\mathfrak{f})} + \gamma\frac{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma+1}(\mathfrak{f})}\right). \\
\leq \quad & \frac{w'(\mathfrak{f})}{w(\mathfrak{f})}\Phi(\mathfrak{f}) - q(\mathfrak{f})w(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\mathfrak{f})} \\
& -\frac{\gamma\mathfrak{y}_{n-3}(\mathfrak{f})}{w^{1/\gamma}(\mathfrak{f})}\left(\Phi(\mathfrak{f}) - \frac{w(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}\right)^{1+1/\gamma} + \gamma w(\mathfrak{f})\frac{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma+1}(\mathfrak{f})}. \qquad (34)
\end{aligned}$$

Using Lemma 3 with $L_1 = w'/w$, $L_2 = \gamma\mathfrak{y}_{n-3}w^{-1/\gamma}$, $L_3 = w\mathfrak{y}_{n-2}^{-\gamma}(\mathfrak{f})$, and $H = \Phi$, we obtain that

$$\frac{w'}{w}\Phi - \frac{\gamma\mathfrak{y}_{n-3}}{w^{1/\gamma}}\left(\Phi - \frac{w}{\mathfrak{y}_{n-2}^{\gamma}}\right)^{1+1/\gamma} \leq \frac{w'}{\mathfrak{y}_{n-2}^{\gamma}} + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(w')^{\gamma+1}}{w^{\gamma}\mathfrak{y}_{n-3}^{\gamma}}. \qquad (35)$$

Combining (34) and (35), we obtain

$$\begin{aligned}
\Phi'(\mathfrak{f}) \quad \leq \quad & -q(\mathfrak{f})w(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\mathfrak{f})} + \frac{w'(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})} \\
& + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(w'(\mathfrak{f}))^{\gamma+1}}{w^{\gamma}(\mathfrak{f})\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})} + \gamma w(\mathfrak{f})\frac{\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma+1}(\mathfrak{f})} \\
= \quad & -q(\mathfrak{f})w(\mathfrak{f})\widetilde{p}^{\gamma}(\rho(\mathfrak{f}))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\mathfrak{f}))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\mathfrak{f})} + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(w'(\mathfrak{f}))^{\gamma+1}}{w^{\gamma}(\mathfrak{f})\mathfrak{y}_{n-3}^{\gamma}(\mathfrak{f})} \\
& + \left(\frac{w(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}\right)'. \qquad (36)
\end{aligned}$$

Integrating (36) from $\mathfrak{f}_1$ to $\mathfrak{f}$, we have

$$\begin{aligned}
& \int_{\mathfrak{f}_1}^{\mathfrak{f}}\left(q(\xi)w(\xi)\widetilde{p}^{\gamma}(\rho(\xi))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\xi))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\xi)} - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(w'(\xi))^{\gamma+1}}{w^{\gamma}(\xi)\mathfrak{y}_{n-3}^{\gamma}(\xi)}\right)\mathrm{d}\xi \\
& \qquad \leq \left(\frac{w(\xi)}{\mathfrak{y}_{n-2}^{\gamma}(\xi)} - \Phi(\xi)\right)\Bigg|_{\mathfrak{f}_1}^{\mathfrak{f}}. \qquad (37)
\end{aligned}$$

From (4) and (33), we obtain that

$$-\frac{w(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})} \leq w(\mathfrak{f})\frac{r(\mathfrak{f})\left(z^{(n-1)}(\mathfrak{f})\right)^{\gamma}}{z^{\gamma}(\mathfrak{f})} = \Phi(\mathfrak{f}) - \frac{w(\mathfrak{f})}{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}. \qquad (38)$$

From (37) and (38), we find

$$\frac{\mathfrak{y}_{n-2}^{\gamma}(\mathfrak{f})}{w(\mathfrak{f})}\int_{\mathfrak{f}_1}^{\mathfrak{f}}\left(q(\xi)w(\xi)\widetilde{p}^{\gamma}(\rho(\xi))\frac{\mathfrak{y}_{n-2}^{\beta\gamma}(\rho(\xi))}{\mathfrak{y}_{n-2}^{\beta\gamma}(\xi)} - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(w'(\xi))^{\gamma+1}}{w^{\gamma}(\xi)\mathfrak{y}_{n-3}^{\gamma}(\xi)}\right)\mathrm{d}\xi \leq 1,$$

which is a contradiction. This completes the proof. □

5. Conclusions

The oscillatory properties of a class of NDEs were studied. To extend the evolution of the study of the second-order DDEs, we presented criteria with only two conditions that ensure the oscillation of the considered equation. Due to the importance of the conditions that exclude solutions in class $\mathbf{U}_3^+$, we improve these conditions by finding a better estimate of the ratio $z(\rho(\mathfrak{f}))/z(\mathfrak{f})$. We establish more than one criterion for oscillation, and these new criteria are distinguished by taking into consideration the impact of delay functions, as well as not needing to define additional functions and conditions. Extension of our results to neutral equations of odd order will be interesting as one of the future issues. There are several studies concerned with the oscillatory properties of solutions of fractional DE, see [26–29]. It is also interesting to extend the approach taken in this work to study the oscillatory properties of fractional DEs.

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