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# Regularization for a Sideways Problem of the Non-Homogeneous Fractional Diffusion Equation

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**Abstract:** In this article, we investigate a sideways problem of the non-homogeneous time-fractional diffusion equation, which is highly ill-posed. Such a model is obtained from the classical non-homogeneous sideways heat equation by replacing the first-order time derivative by the Caputo fractional derivative. We achieve the result of conditional stability under an a priori assumption. Two regularization strategies, based on the truncation of high frequency components, are constructed for solving the inverse problem in the presence of noisy data, and the corresponding error estimates are proved.

**Keywords:** sideways problem; non-homogeneous fractional diffusion equation; ill-posedness; stability estimate; regularization method



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## 1. Introduction

Fractional partial differential equations arose from the studies of Lévy motion [1], continuous random walk [2] and high-frequency financial data [3], which has a wide range of applications in some scientific fields, such as chemistry, physics, mechanical engineering, fluid mechanics, signal processing and systems identification, control theory, electron transportation, viscoelasticity, image processing, and so on [4–13]. Moreover, fractional derivatives have been found to be more flexible in describing some practical phenomena than the traditional integer-order derivatives. In particular, fractional diffusion equations play an extremely important role in the study of some anomalous diffusion processes. These equations can describe the dynamics of various non-local and complex systems. Kinds of anomalous diffusion can be modeled by the following time-fractional diffusion equation: find the temperature  $u(x, t)$  from known boundary temperature  $u(1, t) = \psi(t)$  measurements satisfying the following system

$$\begin{cases} \frac{\partial^\nu u}{\partial t^\nu} - u_{xx} = 0, & x > 0, t > 0, \\ u(x, 0) = 0, & x > 0, \\ u(1, t) = \psi(t), & t > 0, \\ u(x, t) |_{x \rightarrow \infty} \text{ bounded}, \end{cases} \quad (1)$$

where  $\psi(t)$  is given function (usually in  $L^2(\mathbb{R})$ ),  $\frac{\partial^\nu u}{\partial t^\nu}$  is the Caputo fractional derivative of order  $\nu$  ( $0 < \nu \leq 1$ ) defined by [14]

$$\frac{\partial^\nu u}{\partial t^\nu} = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\nu}, \quad 0 < \nu < 1, \quad (2)$$

$$\frac{\partial^\nu u}{\partial t^\nu} = \frac{\partial u(x, t)}{\partial t}, \quad \nu = 1. \quad (3)$$

The problem (1) in the case of  $\nu = 1$ , i.e., the following problem

$$\begin{cases} u_t - u_{xx} = 0, & x > 0, t > 0, \\ u(x, 0) = 0, & x > 0, \\ u(1, t) = \varphi(t), & t > 0, \\ u(x, t) |_{x \rightarrow \infty} \text{ bounded}, \end{cases} \quad (4)$$

has been studied extensively in recent decades by many methods [15–28]. Tuan et al. [29] and Triet et al. [30] extended this work to the non-linear case.

When  $0 < \nu < 1$ , Xiong et al. [31,32] proposed an optimal filtering regularization method for calculating an approximate solution of the fractional sideways heat equation where the spatial domain is the interval  $[0, 1]$ . Li et al. [33] tackled the inverse problem of recovering the temperature and flux distribution in the domain  $0 \leq x < 1$  for (1) from the boundary data at  $x = 1$ , but the conditional stability result is not given. Zheng et al. [34–36] obtains a stable estimate of temperature distribution by utilizing the spectral regularization method, and numerical example shows that the computational effect of their methods are satisfactory. Zhang [37] applied a Tikhonov-type regularized method to construct an approximate solution and overcome the ill-posedness of (1). The a-posteriori convergence estimates of logarithmic and double logarithmic types for the regularized method are derived. Moreover, the authors verify the effectiveness of their method by doing the numerical experiments. Furthermore, there are also some articles that discuss the fractional sideways heat equation in 2-dimensional and higher-dimensions in space (see, e.g., [38–43] and the references therein).

To the best of our knowledge, few investigations has been performed with respect to a sideways problem of the non-homogeneous diffusion equation, and estimating the heat flux at the inaccessible surface is more difficult than estimating temperature. Liu and Chang in [44] addressed a three-dimensional non-homogeneous sideways heat equation in a cuboid by a Fourier sine series method, and the analysis of the regularization parameter and the stability of solution was worked out. According to them, this method is quite accurate. Luan in [45] discussed the two-dimensional non-homogeneous heat equation in the presence of a general source term, and proposed a kernel regularization method to recover the temperature and heat flux distribution from the given data. However, the above two articles only consider the case of integer order. Hence, in contrast to the previous work, we consider a sideways problem of the non-homogeneous time-fractional diffusion equation, which occurs in many applications related to reaction-diffusion

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} - u_{xx} = f(x, t), & x > 0, t > 0, \\ u(x, 0) = 0, & x > 0, \\ u(1, t) = g(t), & t > 0, \\ u(x, t) |_{x \rightarrow \infty} \text{ bounded}, \end{cases} \quad (5)$$

where the function  $f(x, t)$  is the heat source density. We first obtain an analytical solution to (5) via Fourier transform, and give the result of conditional stability under an a priori assumption. Due to the problem considered is severely ill-posed, it is impossible to solve it using classical numerical methods. Therefore, we propose dynamic spectral and Fourier regularization method, the goal here consists of recovering not only the temperature but also the heat flux distribution from the given data. Furthermore, for both regularization strategies, in the presence of noisy data, we establish and prove the stability and convergence estimates in the whole domain, i.e., including the case  $0 < x < 1$  and the case  $x = 1$ .

The remainder of the paper is organized as follows: in Section 2, we give an analysis on the ill-posedness of the non-homogeneous fractional sideways heat equation. The conditional stability result is then given in Section 3. In Sections 4 and 5, error estimates for determination of temperature and flux distribute are derived. Finally, we draw a conclusion to our method.

## 2. Mathematical Analysis of the Problem

In order to simplify the discussion, our theoretical analysis will be performed in  $L^2(\mathbb{R})$  and define all functions to be zero for  $t < 0$ . Let  $\hat{g}$  denote the Fourier transform of  $g(t)$  defined by

$$\hat{g}(\xi) =: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{-i\xi t} dt,$$

and  $\|\cdot\|_p$  denotes the norm in Sobolev space  $H_p(\mathbb{R})$  defined by

$$\|u(0, \cdot)\|_p := \left( \int_{-\infty}^{\infty} (1 + \xi^2)^p |\hat{u}(0, \xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

When  $p = 0$ ,  $\|\cdot\|_p = \|\cdot\|$  denotes the  $L^2(\mathbb{R})$  norm. Furthermore, we introduce the following norm

$$\|f(x, \cdot)\|_{L^2(0,1;H_p(\mathbb{R}))} = \left( \int_0^1 \|f(x, \cdot)\|_p^2 dx \right)^{\frac{1}{2}}.$$

Applying the Fourier transform with respect to  $t$  to both sides of (1), we obtain in the frequency space the following second order ordinary differential equation

$$\begin{cases} \hat{u}_{xx}(x, \xi) - (i\xi)^\alpha \hat{u}(x, \xi) = -\hat{f}(x, \xi), & \xi \in \mathbb{R}, \\ \hat{u}(1, \xi) = \hat{g}(\xi), & \xi \in \mathbb{R}, \\ \hat{u}(x, \xi) |_{x \rightarrow \infty} \text{ bounded}, & \xi \in \mathbb{R}. \end{cases} \quad (6)$$

The standard calculation procedure yields the solution of (6) as

$$\hat{u}(x, \xi) = e^{\tau(\xi)(1-x)} \hat{g}(\xi) + \int_x^1 \hat{f}(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, \quad 0 \leq x < 1, \quad (7)$$

and equivalently

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( e^{\tau(\xi)(1-x)} \hat{g}(\xi) + \int_x^1 \hat{f}(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds \right) e^{i\xi t} d\xi, \quad 0 \leq x < 1, \quad (8)$$

where

$$\tau(\xi) := (i\xi)^{\frac{\alpha}{2}} = |\xi|^{\frac{\alpha}{2}} \left( \cos\left(\frac{\alpha\pi}{4}\right) + i \operatorname{sign}(\xi) \sin\left(\frac{\alpha\pi}{4}\right) \right), \quad \forall \xi \in \mathbb{R}. \quad (9)$$

Note that the real part of  $\tau(\xi)$  is increasing positive function of  $\xi$ . Hence, the term  $|e^{(1-x)\tau(\xi)}|$  and  $|\sinh(\tau(\xi)(s-x))|$  increase rather quickly when  $|\xi| \rightarrow \infty$ , small errors in the data can blow up and ultimately destroy the solution for  $x \in [0, 1)$ . Comparing this with homogeneous fractional sideways heat equation [31], it is no doubt that the problem (5) is much more ill-posed, and some regularization methods are in order.

**Remark 1.** We do not consider the case  $f(x, t) = 0$  in this paper. In fact, if  $f(x, t) = 0$ , our problem is a homogeneous time-fractional sideways heat problem. We only note that, using our method, we obtain again the results of [33].

**Remark 2.** By using the Fourier transform, the solution of general problem (5) where the data  $g$  is fixed at an specific point  $x_0 \in (0, 1]$ , can be expressed as

$$\hat{u}(x, \xi) = e^{\tau(\xi)(x_0-x)} \hat{g}(\xi) + \int_x^{x_0} \hat{f}(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, \quad 0 \leq x < 1. \quad (10)$$

If we put  $x_0 = 1$  in (10), we will obtain (7). In this context, the similar property can be acquired for this general problem and it is also an ill-posed problem. Furthermore, the similarity in (10) and (7) indicates that the methods using in the present paper are also applicable to solve the general problem.

**Lemma 1** ([45]). For arbitrary  $z \in \mathbb{C}$ ,  $x \in [0, 1]$  and  $\eta \in (x, 1]$ , we have

$$\left| \frac{\sinh((\eta - x)z)}{z} \right| \leq \frac{e^{(\eta-x)\Re(z)}}{|z|}, \tag{11}$$

$$\left| \frac{\sinh((\eta - x)z)}{z} \right| \leq (\eta - x)e^{(\eta-x)|z|}, \tag{12}$$

$$|\cosh(xz)| \leq e^{x\Re(z)} \leq e^{x|z|}, \tag{13}$$

where  $\Re(z)$  denotes the real parts of  $z$ .

**Lemma 2** ([45]). For arbitrary  $c, d, p > 0$ , the following inequality holds

$$(c + d)^p \leq \begin{cases} c^p + d^p, & 0 < p \leq 1, \\ 2^{p-1}(c^p + d^p), & p > 1. \end{cases}$$

**Lemma 3.** If  $s \neq 0$ , then the function  $h(s) = \frac{e^{(1-x)s}}{s}$  gets its minimum  $h_{\min} = (1 - x)e$  at  $s = \frac{1}{1-x}$ .

So as to acquire a more sharp convergence, we use the following a priori condition

$$\|u(0, \cdot)\| \leq E. \tag{14}$$

Furthermore, since the  $\sinh(\cdot)$  function is exponentially increasing, we must find a sharply decreasing function to suppress its growth. Therefore, we also give the following assumption

$$\int_0^1 |\hat{f}(s, \xi)|^2 ds < e^{-3|\xi|^{\frac{\alpha}{2}}}, \quad \forall \xi \in \mathbb{R}. \tag{15}$$

and the measured data  $(g_\delta, f_\delta)$  satisfy

$$\|g - g^\delta\| + \|f - f^\delta\|_{L^2(0,1;L^2(\mathbb{R}))} \leq \delta. \tag{16}$$

Throughout this paper, we denote the real part and imaginary part of  $\tau(\xi)$  as follows

$$a := \Re(\tau(\xi)), \quad b := \Im(\tau(\xi)). \tag{17}$$

### 3. A Conditional Stability Estimate

The object of stability estimates is to describe how much the development of solution from data magnifies errors, when noise contaminated the data. Next, we give the main results of this part.

**Theorem 1.** Suppose that  $\hat{u}(x, \xi)$  given by (7) be the exact solution of problem (5) in the frequency space, and (16) is satisfied, then the following estimate holds for  $0 \leq x < 1$

$$\begin{aligned} \|u(x, t)\| \leq & \sqrt{C_1(\|\hat{g}\|^2 + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2)} \\ & + \sqrt{2^{x+2}\|\hat{u}(0, \xi)\|^{2-2x}(\|\hat{g}\|^{2x} + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x}) + C_2\|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2}. \end{aligned}$$

where  $C_1$  and  $C_2$  are constants that only depends on  $x$ .

**Proof.** By the Parseval’s identity, we have

$$\begin{aligned} \|u(x, t)\|^2 &= \|\hat{u}(x, \xi)\|^2 \\ &= \underbrace{\int_{|\xi| \leq 1} \left| e^{\tau(1-x)} \hat{g} + \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right|^2 d\xi}_{A_1} \\ &\quad + \underbrace{\int_{|\xi| > 1} \left| e^{\tau(1-x)} \hat{g} + \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right|^2 d\xi}_{A_2}. \end{aligned} \tag{18}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $A_1$  in (18). By Lemma 2, we have

$$A_1 \leq \underbrace{2 \int_{|\xi| \leq 1} |e^{\tau(1-x)} \hat{g}|^2 d\xi}_{A_{11}} + \underbrace{2 \int_{|\xi| \leq 1} \left| \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right|^2 d\xi}_{A_{12}}. \tag{19}$$

Note that

$$|\tau| = |\xi|^{\frac{\alpha}{2}} \leq 1, \tag{20}$$

we obtain

$$A_{11} \leq 2 \int_{|\xi| \leq 1} e^{2|\tau|(1-x)} |\hat{g}|^2 d\xi \leq 2e^{2(1-x)} \|\hat{g}\|^2. \tag{21}$$

Using Cauchy–Schwarz integral inequality, (12) yields

$$\begin{aligned} A_{12} &\leq 2 \int_{|\xi| \leq 1} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 \left| \frac{\sinh(\tau(s-x))}{\tau} \right|^2 ds \right) d\xi \\ &\leq 2 \int_{|\xi| \leq 1} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) d\xi \\ &\leq e^{2(1-x)} \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2. \end{aligned} \tag{22}$$

Substituting (21) and (22) into (19), we obtain

$$A_1 \leq C_1 (\|\hat{g}\|^2 + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2), \tag{23}$$

where

$$C_1 = 2e^{2(1-x)}. \tag{24}$$

Step 2. Estimate the term  $A_2$  in (18). Again, in view of Lemma 2, we have

$$A_2 \leq \underbrace{2 \int_{|\xi| > 1} |e^{\tau(1-x)} \hat{g}|^2 d\xi}_{A_{21}} + \underbrace{2 \int_{|\xi| > 1} \left| \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right|^2 d\xi}_{A_{22}}. \tag{25}$$

We first estimate  $A_{21}$ . Using (7) and Lemma 2, we have

$$\begin{aligned} A_{21} &= 2 \int_{|\xi| > 1} \left| e^{-\tau x} \left[ \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] \right|^2 d\xi \\ &\leq \underbrace{4 \int_{|\xi| > 1} |e^{-\tau x}|^2 |\hat{u}(0, \xi)|^2 d\xi}_{\hat{A}_{21}} + \underbrace{4 \int_{|\xi| > 1} |e^{-\tau x}|^2 \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 d\xi}_{\hat{A}_{22}}. \end{aligned}$$

By (17), Lemma 2, Hölder inequality, we have

$$\begin{aligned}
 \tilde{A}_{21} &= \int_{|\xi|>1} (|\hat{u}(0, \xi)|^2)^{1-x} \left[ e^{-2a} \left| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds + \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 \right]^x d\xi \\
 &\leq \int_{|\xi|>1} (|\hat{u}(0, \xi)|^2)^{1-x} \left[ \left( 2e^{-2a} \left| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 \right)^x \right. \\
 &\quad \left. + \left( 2e^{-2a} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 \right)^x \right] d\xi \\
 &= \int_{|\xi|>1} (|\hat{u}(0, \xi)|^2)^{1-x} \left( 2e^{-2a} \left| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 \right)^x d\xi \\
 &\quad + \int_{|\xi|>1} (|\hat{u}(0, \xi)|^2)^{1-x} \left( 2e^{-2a} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 \right)^x d\xi \\
 &\leq \left( \int_{|\xi|>1} |\hat{u}(0, \xi)|^2 d\xi \right)^{1-x} \left( \int_{|\xi|>1} 2e^{-2a} \left| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 d\xi \right)^x \\
 &\quad + \left( \int_{|\xi|>1} |\hat{u}(0, \xi)|^2 d\xi \right)^{1-x} \left( \int_{|\xi|>1} 2e^{-2a} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 d\xi \right)^x.
 \end{aligned}$$

By Cauchy–Schwarz integral inequality and (11), we obtain

$$\begin{aligned}
 2e^{-2a} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 &\leq 2e^{-2a} \left( \int_0^1 |\hat{f}|^2 ds \right) \left( \int_0^1 \left| \frac{\sinh(\tau s)}{\tau} \right|^2 ds \right) \\
 &\leq 2e^{-2a} \left( \int_0^1 |\hat{f}|^2 ds \right) \left( \int_0^1 \frac{e^{2sa}}{|\tau|^2} ds \right) \\
 &\leq 2 \left( \int_0^1 |\hat{f}|^2 ds \right) \frac{1}{2a|\tau|^2}.
 \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}
 \tilde{A}_{21} &\leq \left( \int_{|\xi|>1} |\hat{u}(0, \xi)|^2 d\xi \right)^{1-x} \left( \int_{|\xi|>1} |\sqrt{2}\hat{g}|^2 d\xi \right)^x \\
 &\quad + \left( \int_{|\xi|>1} |\hat{u}(0, \xi)|^2 d\xi \right)^{1-x} \left( \int_{|\xi|>1} 2 \left( \int_0^1 |\hat{f}|^2 ds \right) \frac{1}{2a|\tau|^2} d\xi \right)^x \\
 &\leq \|\hat{u}(0, \xi)\|^{2-2x} \|\sqrt{2}\hat{g}\|^{2x} + 2^x \|\hat{u}(0, \xi)\|^{2-2x} \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x} \\
 &= 2^x \|\hat{u}(0, \xi)\|^{2-2x} (\|\hat{g}\|^{2x} + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x}).
 \end{aligned}$$

Likewise, we have

$$\begin{aligned}
 \tilde{A}_{22} &\leq \left( \int_{|\xi|>1} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 d\xi \right)^{1-x} \left( \int_{|\xi|>1} e^{-2a} \left| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right|^2 d\xi \right)^x \\
 &\leq \left[ \int_{|\xi|>1} \left( \int_0^1 |\hat{f}|^2 ds \right) \left( \int_0^1 e^{2|\tau|s} ds \right) \right]^{1-x} \left( \int_{|\xi|>1} \left( \int_0^1 |\hat{f}|^2 ds \right) \frac{1}{2a|\tau|^2} d\xi \right)^x \\
 &\leq \left[ \int_{|\xi|>1} \left( \int_0^1 |\hat{f}|^2 ds \right) \frac{e^{2|\tau|}}{2|\tau|} \right]^{1-x} \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x} \\
 &\leq e^{1-x} \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2.
 \end{aligned}$$

Combining the estimates of  $\tilde{A}_{21}$  with  $\tilde{A}_{22}$ , we obtain

$$A_{21} \leq 2^{x+2} \|\hat{u}(0, \xi)\|^{2-2x} (\|\hat{g}\|^{2x} + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x}) + 4e^{1-x} \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2. \tag{27}$$

Next, we estimate  $A_{22}$ . By Cauchy–Schwarz integral inequality, (12) and Lemma 3, we obtain

$$\begin{aligned} A_{22} &\leq 2 \int_{|\xi|>1} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 \left| \frac{\sinh(\tau(s-x))}{\tau} \right|^2 ds \right) d\xi \\ &\leq 2 \int_{|\xi|>1} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) d\xi \\ &\leq 2 \int_{|\xi|>1} \left( \int_x^1 |\hat{f}|^2 ds \right) \frac{e^{2|\tau|(1-x)}}{2|\tau|} d\xi \\ &\leq 2(1-x)e \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2. \end{aligned} \tag{28}$$

Inserting (27) and (28) into (25), we have

$$A_2 \leq 2^{x+2} \|\hat{u}(0, \xi)\|^{2-2x} (\|\hat{g}\|^{2x} + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x}) + C_2 \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2, \tag{29}$$

where

$$C_2 = 4e^{1-x} + 2(1-x)e. \tag{30}$$

Substituting (23) and (29) into (18), and using Lemma 2, we obtain

$$\begin{aligned} \|\hat{u}(x, \xi)\| &\leq \sqrt{A_1} + \sqrt{A_2} \\ &\leq \sqrt{C_1 (\|\hat{g}\|^2 + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2)} \\ &\quad + \sqrt{2^{x+2} \|\hat{u}(0, \xi)\|^{2-2x} (\|\hat{g}\|^{2x} + \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^{2x}) + C_2 \|\hat{f}\|_{L^2(0,1;L^2(\mathbb{R}))}^2}, \end{aligned}$$

where  $C_1$  and  $C_2$  is given by (24) and (30), respectively.  $\square$

#### 4. Determination of the Temperature Distribution

In this part, we use the dynamic spectral method to recover the temperature distribution from the measured data. Since the matter of instability lies in the noise of data in the high frequency components, naturally a ‘‘corrector’’ is added to these in order to control their growth. As a result, one may obtain a stable approximation. Suppose  $\beta$  is the regularization parameter, motivated by [31], we contemplate the following regularized solutions in the frequency domain:

Method 1

$$\hat{u}_\beta^\delta(x, \xi) = \begin{cases} e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, & e^{-a(1-x)} \geq \sqrt{\beta}, \\ \frac{e^{-2a(1-x)}}{\beta} [e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds], & e^{-a(1-x)} < \sqrt{\beta}. \end{cases} \tag{31}$$

Method 2

$$\hat{v}_\beta^\delta(x, \xi) = \begin{cases} e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, & e^{-a(1-x)} \geq \sqrt{\beta}, \\ \frac{e^{-a(1-x)}}{\sqrt{\beta}} [e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds], & e^{-a(1-x)} < \sqrt{\beta}. \end{cases} \tag{32}$$

Method 3

$$\hat{w}_\beta^\delta(x, \xi) = \begin{cases} e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, & e^{-a(1-x)} \geq \sqrt{\beta}, \\ \frac{e^{-\frac{a}{2}(1-x)}}{\beta^{\frac{1}{4}}} [e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds], & e^{-a(1-x)} < \sqrt{\beta}. \end{cases} \tag{33}$$

Generally,

$$\hat{\mu}_\beta^\delta(x, \xi) = \begin{cases} e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds, & e^{-a(1-x)} \geq \sqrt{\beta}, \\ \frac{e^{-\gamma a(1-x)}}{\sqrt{\beta}^\gamma} [e^{\tau(\xi)(1-x)} \hat{g}^\delta(\xi) + \int_x^1 \hat{f}^\delta(s, \xi) \frac{\sinh(\tau(\xi)(s-x))}{\tau(\xi)} ds], & e^{-a(1-x)} < \sqrt{\beta}, \end{cases} \tag{34}$$

where  $\gamma > 0$  is a real number. Because the three spectral methods are very similar, then we only give the properties of the first two methods.

**Remark 3.** It is apparently that the regularization solutions approach the exact solution if  $\beta \rightarrow 0$  as  $\delta \rightarrow 0$ .

**Lemma 4.** If condition (14) and (15) hold,  $B(\xi) = \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds$ , then

$$\|B(\xi)\| \leq E + N_1,$$

where  $N_1$  is a constant.

**Proof.** Successively using the triangle inequality, (14), Cauchy–Schwarz integral inequality, (12) and (15), we obtain

$$\begin{aligned} \|B(\xi)\| &\leq \|\hat{u}(0, \xi)\| + \left\| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right\| \\ &\leq E + \left[ \int_{-\infty}^{\infty} \left( \int_0^1 \left| \frac{\sinh(\tau s)}{\tau} \right|^2 ds \int_0^1 |\hat{f}|^2 ds \right) d\xi \right]^{\frac{1}{2}} \\ &\leq E + \left[ \int_{-\infty}^{\infty} \left( \int_0^1 |s e^{s|\tau|}|^2 ds \int_0^1 |\hat{f}|^2 ds \right) d\xi \right]^{\frac{1}{2}} \\ &\leq E + \left( \int_{-\infty}^{\infty} \frac{1}{2|\xi|^{\frac{\alpha}{2}}} e^{-|\xi|^{\frac{\alpha}{2}}} d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

It is easy to know that the generalized integral on the right-hand side of the last inequality converges, here we introduce the notation

$$N_1 := \left( \int_{-\infty}^{\infty} \frac{1}{2|\xi|^{\frac{\alpha}{2}}} e^{-|\xi|^{\frac{\alpha}{2}}} d\xi \right)^{\frac{1}{2}}. \tag{35}$$

Therefore,

$$\|B(\xi)\| \leq E + N_1,$$

where  $N_1$  is a constant.  $\square$

**Theorem 2.** Let  $\hat{u}(x, \xi)$  given by (7) be the exact solution of problem (5) in the frequency space,  $\hat{u}_\beta^\delta(x, \xi)$  given by (31) be the regularized solution, condition (14)–(16) hold. If the regularization parameter  $\beta$  is selected dynamically

$$\beta(x) = 2^{2x-2} \left( \frac{x}{2-x} \right)^{x-2} \left( \frac{\delta}{E + 2N_1} \right)^{2(1-x)}. \tag{36}$$

Then, for a fixed  $x \in (0, 1)$ , we have

$$\|u_\beta^\delta(x, \cdot) - u(x, \cdot)\| \leq \frac{2^{1-x}}{2-x} \left(\frac{x}{2-x}\right)^{-\frac{x}{2}} \delta^x (E + 2N_1)^{1-x} + \delta \sqrt{(1-x)e}. \tag{37}$$

**Proof.** By the triangle inequality, we have

$$\|u_\beta^\delta(x, \cdot) - u(x, \cdot)\| \leq \underbrace{\|u_\beta^\delta(x, \cdot) - u_\beta(x, \cdot)\|}_{\mathcal{I}_1} + \underbrace{\|u_\beta(x, \cdot) - u(x, \cdot)\|}_{\mathcal{I}_2}. \tag{38}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $\mathcal{I}_1$  in (38). It follows immediately from Parseval’s equality and the triangle inequality that

$$\begin{aligned} \mathcal{I}_1 &= \|\hat{u}_\beta^\delta(x, \cdot) - \hat{u}_\beta(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \left[ e^{\tau(1-x)} (\hat{g}^\delta - \hat{g}) + \int_x^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau(s-x))}{\tau} ds \right] \right\| \\ &\leq \underbrace{\left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} e^{\tau(1-x)} (\hat{g}^\delta - \hat{g}) \right\|}_{\bar{\mathcal{I}}_1} \\ &\quad + \underbrace{\left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \int_x^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau(s-x))}{\tau} ds \right\|}_{\bar{\mathcal{I}}_2}. \end{aligned}$$

By (17) and (16), we obtain

$$\bar{\mathcal{I}}_1 \leq \left\| \frac{e^{-2a(1-x)}}{\beta} e^{(a+bi)(1-x)} (\hat{g}^\delta - \hat{g}) \right\|_{e^{-a(1-x)} < \sqrt{\beta}} = \left\| \frac{e^{(-a+bi)(1-x)}}{\beta} (\hat{g}^\delta - \hat{g}) \right\| \leq \delta \beta^{-\frac{1}{2}}.$$

Using Cauchy–Schwarz integral inequality, (12), Lemma 3 and (16) yields

$$\begin{aligned} \bar{\mathcal{I}}_2 &\leq \left\| \int_x^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau(x-s))}{\tau} ds \right\| \\ &\leq \left\| \left( \int_x^1 |\hat{f}^\delta - \hat{f}|^2 ds \right) \left( \int_x^1 e^{2(s-x)|\tau|} ds \right) \right\| \\ &\leq \left\| \left( \int_x^1 |\hat{f}^\delta - \hat{f}|^2 ds \right) \frac{e^{2(1-x)|\tau|}}{2|\tau|} \right\| \\ &\leq \delta \sqrt{(1-x)e}. \end{aligned}$$

Hence,

$$\mathcal{I}_1 \leq \delta \beta^{-\frac{1}{2}} + \delta \sqrt{(1-x)e}. \tag{39}$$

Step 2. Estimate the term  $\mathcal{I}_2$  in (38). Again, by the Parseval’s identity and the triangle inequality,

$$\begin{aligned} \mathcal{I}_2 &= \|\hat{u}_\beta(x, \cdot) - \hat{u}(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \cdot \left( e^{\tau(1-x)} \hat{\xi} + \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right) - \left( e^{\tau(1-x)} \hat{\xi} + \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right) \right\| \\ &\leq \underbrace{\left\| \left( 1 - \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \right) e^{\tau(1-x)} \hat{\xi} \right\|}_{\tilde{\mathcal{I}}_1} \\ &\quad + \underbrace{\left\| \left( 1 - \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \right) \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right\|}_{\tilde{\mathcal{I}}_2}. \end{aligned}$$

We start by estimating the first term above. Let

$$B_1(a) = \left( 1 - \frac{e^{-2a(1-x)}}{\beta} \right) e^{-ax}.$$

Using (7) and (17), we obtain

$$\begin{aligned} \tilde{\mathcal{I}}_1 &= \left\| \left( 1 - \min \left\{ 1, \frac{e^{-2a(1-x)}}{\beta} \right\} \right) e^{-\tau x} \left[ \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] \right\| \\ &\leq \sup_{e^{-2a(1-x)} \leq \beta} B_1(a) \left\| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right\|. \end{aligned}$$

By elementary calculations, it is easy to find the zero point  $a^*$  of  $B'_1(a)$  satisfies

$$e^{-2a^*(1-x)} = \frac{\beta x}{2-x},$$

and  $a^*$  maximize the function  $B_1(a)$ . Thus,

$$B_1(a) \leq B_1(a^*) = \left( 1 - \frac{x}{2-x} \right) \left( \frac{\beta x}{2-x} \right)^{\frac{x}{2(1-x)}}. \tag{40}$$

Using Lemma 4, we have

$$\tilde{\mathcal{I}}_1 \leq \left( 1 - \frac{x}{2-x} \right) \left( \frac{\beta x}{2-x} \right)^{\frac{x}{2(1-x)}} (E + N_1).$$

Now we estimate  $\tilde{\mathcal{I}}_2$ . Using Cauchy–Schwarz integral inequality, (12), (17), (15), (40), (9) and (10) yields

$$\begin{aligned} \tilde{\mathcal{I}}_2 &\leq \left\| \left( 1 - \frac{e^{-2a(1-x)}}{\beta} \right) e^{-ax} \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds e^{ax} \right\| \\ &\leq \sup_{e^{-2a(1-x)} \leq \beta} B_1(a) \left[ \int_{-\infty}^{\infty} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) e^{2ax} d\xi \right]^{\frac{1}{2}} \\ &\leq \sup_{e^{-2a(1-x)} \leq \beta} B_1(a) \left[ \int_{-\infty}^{\infty} \left( \int_x^1 |\hat{f}|^2 ds \right) \frac{e^{2|\tau|(1-x)}}{2|\tau|} e^{2|\tau|x} d\xi \right]^{\frac{1}{2}} \\ &\leq \left( 1 - \frac{x}{2-x} \right) \left( \frac{\beta x}{2-x} \right)^{\frac{x}{2(1-x)}} N_1. \end{aligned}$$

Therefore,

$$\mathcal{I}_2 \leq \left( 1 - \frac{x}{2-x} \right) \left( \frac{\beta x}{2-x} \right)^{\frac{x}{2(1-x)}} (E + 2N_1). \tag{41}$$

Substituting (39) and (40) into (38), we obtain

$$\|u_\beta^\delta(x, \cdot) - u(x, \cdot)\| \tag{42}$$

$$\leq \delta \beta^{-\frac{1}{2}} + \delta \sqrt{(1-x)e} + \left( 1 - \frac{x}{2-x} \right) \left( \frac{\beta x}{2-x} \right)^{\frac{x}{2(1-x)}} (E + 2N_1) := h_1(\beta). \tag{43}$$

Minimizing the right-hand side of (42) with respect to  $\beta$ , we can obtain (36). Hence, (37) hold.  $\square$

**Theorem 3.** Let  $\hat{u}(x, \xi)$  given by (7) be the exact solution of problem (5) in the frequency space,  $\hat{v}_\beta^\delta(x, \xi)$  given by (32) be the regularized solution, condition (14)–(16) hold. If the regularization parameter  $\beta$  is selected dynamically

$$\beta(x) = x^{-2} \left( \frac{\delta}{E + 2N_1} \right)^{2(1-x)}. \tag{44}$$

Then, for a fixed  $x \in (0, 1)$ , we have

$$\|v_\beta^\delta(x, \cdot) - u(x, \cdot)\| \leq \delta^x (E + 2N_1)^{1-x} + \delta \sqrt{(1-x)e}. \tag{45}$$

**Proof.** By the triangle inequality, we have

$$\|v_\beta^\delta(x, \cdot) - u(x, \cdot)\| \leq \underbrace{\|v_\beta^\delta(x, \cdot) - v_\beta(x, \cdot)\|}_{\mathcal{I}_3} + \underbrace{\|v_\beta(x, \cdot) - u(x, \cdot)\|}_{\mathcal{I}_4}. \tag{46}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $\mathcal{I}_3$  in (45). Taking a similar procedure of the estimate of  $\mathcal{I}_1$ , we have

$$\begin{aligned} \mathcal{I}_3 &= \|\hat{v}_\beta^\delta(x, \cdot) - \hat{v}_\beta(x, \cdot)\| \\ &\leq \left\| \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right\} e^{\tau(1-x)} (\hat{g}^\delta - \hat{g}) \right\| \\ &\quad + \left\| \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right\} \int_x^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau(s-x))}{\tau} ds \right\| \\ &\leq \left\| \frac{e^{-a(1-x)}}{\sqrt{\beta}} e^{(a+bi)(1-x)} (\hat{g}^\delta - \hat{g}) \right\|_{e^{-a(1-x)} < \sqrt{\beta}} + \left\| \int_x^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau(s-x))}{\tau} ds \right\| \\ &\leq \delta \beta^{-\frac{1}{2}} + \delta \sqrt{(1-x)e}. \end{aligned} \tag{47}$$

Step 2. Estimate the term  $\mathcal{I}_4$  in (45). By the Parseval’s identity, we have

$$\begin{aligned} \mathcal{I}_4 &= \|\hat{v}_\beta(x, \cdot) - \hat{u}(x, \cdot)\| \\ &\leq \underbrace{\left\| \left( 1 - \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right\} \right) e^{\tau(1-x)} \hat{g} \right\|}_{\tilde{\mathcal{I}}_3} \\ &\quad + \underbrace{\left\| \left( 1 - \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right\} \right) \int_x^1 \hat{f} \frac{\sinh(\tau(s-x))}{\tau} ds \right\|}_{\tilde{\mathcal{I}}_4}. \end{aligned}$$

We start by estimating the first term above. Let

$$B_2(a) = \left( 1 - \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right) e^{-ax}.$$

Using (7) and (17), we obtain

$$\begin{aligned} \tilde{\mathcal{I}}_3 &= \left\| \left( 1 - \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\beta}} \right\} \right) e^{-\tau x} \left[ \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] \right\| \\ &\leq \sup_{e^{-a(1-x)} \leq \sqrt{\beta}} B_2(a) \left\| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right\|. \end{aligned}$$

By elementary calculations, it is easy to find the zero point  $a^*$  of  $B_2'(a)$  satisfies

$$e^{-a^*(1-x)} = \sqrt{\beta}x,$$

and  $a^*$  maximize the function  $B_2(a)$ . Thus,

$$B_2(a) \leq B_2(a^*) = (1-x)(\sqrt{\beta}x)^{\frac{x}{1-x}}. \tag{48}$$

By Lemma 4, we have

$$\tilde{\mathcal{I}}_3 \leq (1-x)(\sqrt{\beta}x)^{\frac{x}{1-x}} (E + N_1).$$

Now we estimate  $\tilde{\mathcal{I}}_4$ . Using Cauchy–Schwarz integral inequality, (12), (17), (15), (9), (47) and (35) yields

$$\begin{aligned} \tilde{\mathcal{I}}_4 &\leq \left\| \left(1 - \frac{e^{-a(1-x)}}{\sqrt{\beta}}\right) e^{-ax} \int_x^1 \hat{f} \frac{\sinh(\tau(x-s))}{\tau} ds e^{ax} \right\| \\ &\leq \sup_{e^{-a(1-x)} \leq \sqrt{\beta}} B_2(a) \left[ \int_{-\infty}^{\infty} \left( \int_x^1 |\hat{f}|^2 ds \right) \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) e^{2ax} d\xi \right]^{\frac{1}{2}} \\ &\leq (1-x)(\sqrt{\beta x})^{\frac{x}{1-x}} N_1. \end{aligned}$$

Therefore,

$$\mathcal{I}_4 \leq (1-x)(\sqrt{\beta x})^{\frac{x}{1-x}} (E + 2N_1). \tag{49}$$

Substituting (46) and (48) into (45), we obtain

$$\|v_{\beta}^{\delta}(x, \cdot) - u(x, \cdot)\| \leq \delta \beta^{-\frac{1}{2}} + \delta \sqrt{(1-x)e} + (1-x)(\sqrt{\beta x})^{\frac{x}{1-x}} (E + 2N_1) := h_2(\beta). \tag{50}$$

Minimizing the right-hand side of (49) with respect to  $\beta$ , we can obtain (43). Hence, (44) hold.  $\square$

**Remark 4.** In Theorems 2 and 3, we choose the regularization parameter  $\beta$  to depend on the position of  $x$ , which will justify our use of the phrase “dynamic spectral”. Moreover, we can find that the estimate of Theorem 3 is better than the estimate of Theorem 2.

It is easy to see that two errors in Theorems 2 and 3 are not near to zero, if  $\delta$  fixed and  $x$  tend to zero. Hence, the convergence of the approximate solution is very slow when  $x$  is in a neighborhood of zero. In addition, considering that the  $\sinh(\cdot)$  function is exponentially increasing, to retain the continuous dependence of the solution at  $x = 0$ , we have to introduce some stronger a priori assumptions

$$\|u(0, \cdot)\|_p \leq E, \quad p > 0, \tag{51}$$

$$(1 + \zeta^2)^p \int_0^1 |\hat{f}(s, \zeta)|^2 ds < e^{-3|\zeta|^{\frac{p}{2}}}, \quad \forall \zeta \in \mathbb{R}. \tag{52}$$

Next, we only give error estimate at  $x = 0$  for (32).

**Lemma 5.** Let condition (50) and (51) hold,  $\tilde{B}(\zeta) = (1 + \zeta^2)^{\frac{p}{2}} [\hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds]$ , then

$$\|\tilde{B}(\zeta)\| \leq E + N_1,$$

where  $N_1$  is a constant.

**Theorem 4.** Let  $\hat{u}(x, \zeta)$  given by (7) be the exact solution of problem (5) in the frequency space,  $\hat{v}_{\beta}^{\delta}(x, \zeta)$  given by (32) be the regularized solution, condition (16), (50), (51) hold. The regularization parameter  $\beta$  is chosen as

$$\beta = \frac{1}{(C(a^*)\delta^{-r})^2}, \tag{53}$$

where  $0 < r < 1, C(a^*) = \frac{2p}{a^* \alpha + 2p} < 1, a^*$  is a constant. Then, the following inequality hold

$$\|v_{\beta}^{\delta}(0, \cdot) - u(0, \cdot)\| \leq C(a^*)\delta^{1-r} + \delta \sqrt{e} + [1 - C(a^*)] \left(r \ln \frac{1}{\delta}\right)^{-\frac{2p}{\alpha}} (E + 2N_1), \quad p > 0. \tag{54}$$

**Proof.** By the triangle inequality, we have

$$\|v_\beta^\delta(0, \cdot) - u(0, \cdot)\| \leq \underbrace{\|v_\beta^\delta(0, \cdot) - v_\beta(0, \cdot)\|}_{\mathcal{I}_5} + \underbrace{\|v_\beta(0, \cdot) - u(0, \cdot)\|}_{\mathcal{I}_6}. \tag{55}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $\mathcal{I}_5$  in (54). In view of the Parseval’s equality, the triangle inequality, Cauchy–Schwarz integral inequality, (12), Lemma 3 and (16), we have

$$\begin{aligned} \mathcal{I}_5 &= \|\hat{v}_\beta^\delta(0, \cdot) - \hat{v}_\beta(0, \cdot)\| \\ &\leq \left\| \min\left\{1, \frac{e^{-a}}{\sqrt{\beta}}\right\} e^\tau (\hat{g}^\delta - \hat{g}) \right\| + \left\| \min\left\{1, \frac{e^{-a}}{\sqrt{\beta}}\right\} \int_0^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau s)}{\tau} ds \right\| \\ &\leq \left\| \frac{e^{-a}}{\sqrt{\beta}} e^{(a+bi)} (\hat{g}^\delta - \hat{g}) \right\|_{e^{-a} < \sqrt{\beta}} + \left\| \int_0^1 (\hat{f}^\delta - \hat{f}) \frac{\sinh(\tau s)}{\tau} ds \right\| \\ &\leq \delta \beta^{-\frac{1}{2}} + \left[ \int_{-\infty}^\infty \left( \int_0^1 |\hat{f}^\delta - \hat{f}|^2 ds \right) \left( \int_0^1 e^{2s|\tau|} ds \right) d\zeta \right]^{\frac{1}{2}} \\ &\leq \delta \beta^{-\frac{1}{2}} + \delta \sqrt{e}. \end{aligned}$$

Step 2. Estimate the term  $\mathcal{I}_6$  in (54). By the Parseval’s equality and the triangle inequality, we obtain

$$\begin{aligned} \mathcal{I}_6 &= \|\hat{v}_\beta(0, \cdot) - \hat{u}(0, \cdot)\| \\ &\leq \underbrace{\left\| \left(1 - \min\left\{1, \frac{e^{-a}}{\sqrt{\beta}}\right\}\right) e^\tau \hat{g} \right\|}_{\tilde{\mathcal{I}}_5} + \underbrace{\left\| \left(1 - \min\left\{1, \frac{e^{-a}}{\sqrt{\beta}}\right\}\right) \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right\|}_{\tilde{\mathcal{I}}_6}. \end{aligned}$$

We start by estimating the first term above. Let

$$B_3(a) = \left(1 - \frac{e^{-a}}{\sqrt{\beta}}\right) a^{-\frac{2p}{a}}.$$

Using (7), and note that  $a \leq |\zeta|^{\frac{\alpha}{2}}$ , we obtain

$$\begin{aligned} \tilde{\mathcal{I}}_5 &= \left\| \left(1 - \min\left\{1, \frac{e^{-a}}{\sqrt{\beta}}\right\}\right) \left[ \hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] \right\| \\ &\leq \sup_{e^{-a} \leq \sqrt{\beta}} \left(1 - \frac{e^{-a}}{\sqrt{\beta}}\right) (1 + \zeta^2)^{-\frac{p}{2}} \left\| \left[ \hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] (1 + \zeta^2)^{\frac{p}{2}} \right\| \\ &\leq \sup_{e^{-a} \leq \sqrt{\beta}} B_3(a) \left\| \left[ \hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right] (1 + \zeta^2)^{\frac{p}{2}} \right\|. \end{aligned}$$

By elementary calculations, it is easy to find the zero point  $a^*$  of  $B_3'(a)$  satisfies

$$\frac{e^{-a^*}}{\sqrt{\beta}} = C(a^*),$$

where

$$C(a^*) = \frac{2p}{a^* \alpha + 2p} < 1, \tag{56}$$

and  $a^*$  maximize the function  $B_3(a)$ . Thus,

$$B_3(a) \leq B_3(a^*) = [1 - C(a^*)] \left( \ln \frac{1}{\sqrt{\beta}C(a^*)} \right)^{-\frac{2p}{\alpha}}. \tag{57}$$

By Lemma 5, we obtain

$$\tilde{\mathcal{I}}_5 \leq [1 - C(a^*)] \left( \ln \frac{1}{\sqrt{\beta}C(a^*)} \right)^{-\frac{2p}{\alpha}} (E + N_1).$$

Now we estimate  $\tilde{\mathcal{I}}_6$ . Using (56) and Lemma 5 yields

$$\begin{aligned} \tilde{\mathcal{I}}_6 &\leq \left\| \left( 1 - \frac{e^{-a}}{\sqrt{\beta}} \right) (1 + \zeta^2)^{-\frac{p}{2}} \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds (1 + \zeta^2)^{\frac{p}{2}} \right\| \\ &\leq \sup_{e^{-a(1-x)} \leq \sqrt{\beta}} B_3(a) \left\| \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds (1 + \zeta^2)^{\frac{p}{2}} \right\| \\ &\leq [1 - C(a^*)] \left( \ln \frac{1}{\sqrt{\beta}C(a^*)} \right)^{-\frac{2p}{\alpha}} N_1. \end{aligned}$$

Therefore,

$$\mathcal{I}_6 \leq [1 - C(a^*)] \left( \ln \frac{1}{\sqrt{\beta}C(a^*)} \right)^{-\frac{2p}{\alpha}} (E + 2N_1). \tag{58}$$

Then, by (54), we have

$$\begin{aligned} \|v_\beta^\delta(x, \cdot) - u(x, \cdot)\| &\leq \delta\beta^{-\frac{1}{2}} + \delta\sqrt{e} + [1 - C(a^*)] \left( \ln \frac{1}{\sqrt{\beta}C(a^*)} \right)^{-\frac{2p}{\alpha}} (E + 2N_1) \\ &= C(a^*)\delta^{1-r} + \delta\sqrt{e} + [1 - C(a^*)] \left( r \ln \frac{1}{\delta} \right)^{-\frac{2p}{\alpha}} (E + 2N_1). \end{aligned}$$

where  $C(a^*)$  is given by (55).  $\square$

**Remark 5.** If we replace the assumption (14) and (15) by (50) and (51), then the convergence  $\|u_\beta^\delta(x, \cdot) - u(x, \cdot)\|_p$  and  $\|v_\beta^\delta(x, \cdot) - u(x, \cdot)\|_p$  is also hold.

**Remark 6.** From a theoretical point of view, Theorem 4 has obtained the stability estimate for the endpoint  $x = 0$ , since  $\lim_{\delta \rightarrow 0} \|v_\beta^\delta(0, \cdot) - u(0, \cdot)\| = 0$ .

**Remark 7.** In 1987, Eldén [19] proved that it is impossible to obtain the error asymptotically better than logarithmic rate at  $x = 0$ . So our estimates is reasonable, although the logarithmic term  $\ln \frac{1}{\delta}$  implies the convergence rate is very slow.

### 5. Determination of Flux Structure and Error Estimate

In this section, we use the Fourier regularization method to recover the flux distribution from the measure data. Differentiating the variable  $x$  on the right-hand side of (7), we obtain the following formula for the heat flux, denoted by

$$\hat{u}_x(x, \zeta) = -\tau(\zeta)e^{(1-x)\tau(\zeta)}\hat{g}^\delta - \int_x^1 \hat{f}^\delta \cosh(\tau(\zeta)(s-x))ds, \quad 0 \leq x < 1.$$

The method we adopt is to eliminate all high frequencies from the solution, and instead consider (5) for  $|\xi| \leq \xi_{\max}$ . Then, we obtain a regularized solution

$$\hat{u}_x^{\delta, \xi_{\max}}(x, \xi) = \left[ -\tau(\xi)e^{(1-x)\tau(\xi)}\hat{g}^\delta - \int_x^1 \hat{f}^\delta \cosh(\tau(\xi)(s-x))ds \right] \chi_{\max}. \tag{59}$$

where  $\xi_{\max}$  is the regularization parameter,  $\chi_{\max}$  is the characteristic function of the interval  $[-\xi_{\max}, \xi_{\max}]$ .

**Theorem 5.** Let  $\hat{u}(x, \xi)$  given by (7) be the exact solution of problem (5) in the frequency space,  $\hat{u}_x^{\delta, \xi_{\max}}(x, \xi)$  given by (58) be the regularized solution, condition (14)–(16) hold. If the regularization parameter  $\xi_{\max}$  is selected by

$$\xi_{\max} = \left( \ln \frac{E}{\delta} \right)^{\frac{2}{\alpha}}. \tag{60}$$

Then for a fixed  $x \in (0, 1)$ , we have

$$\begin{aligned} & \|u_x^{\delta, \xi_{\max}}(x, \cdot) - u_x(x, \cdot)\| \\ & \leq \left( 2 \ln \frac{E}{\delta} + \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} \right) E^{1-x} \delta^x + \epsilon_1 \sqrt{2} E^{-x} \delta^x (E + N_1) + \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} N_2, \end{aligned}$$

where  $\epsilon_1 = \max\{\frac{1}{x}, \ln \frac{E}{\delta}\}$ ,  $N_1$  and  $N_2$  are some constants.

**Proof.** By the triangle inequality, we have

$$\|u_x^{\delta, \xi_{\max}}(x, \cdot) - u_x(x, \cdot)\| \leq \underbrace{\|u_x^{\delta, \xi_{\max}}(x, \cdot) - u_x^{\xi_{\max}}(x, \cdot)\|}_{\mathcal{J}_1} + \underbrace{\|u_x^{\xi_{\max}}(x, \cdot) - u_x(x, \cdot)\|}_{\mathcal{J}_2}. \tag{61}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $\mathcal{J}_1$  in (60). It follows immediately from Parseval’s equality and Lemma 2 that

$$\begin{aligned} \mathcal{J}_1 &= \|\hat{u}_x^{\delta, \xi_{\max}}(x, \cdot) - \hat{u}_x^{\xi_{\max}}(x, \cdot)\| \\ &= \left[ \int_{|\xi| \leq \xi_{\max}} \left| \tau e^{\tau(1-x)} (\hat{g} - \hat{g}^\delta) + \int_x^1 (\hat{f} - \hat{f}^\delta) \cosh(\tau(s-x))ds \right|^2 d\xi \right]^{\frac{1}{2}} \\ &\leq \underbrace{\left[ \int_{|\xi| \leq \xi_{\max}} 2 \left| \tau e^{\tau(1-x)} (\hat{g} - \hat{g}^\delta) \right|^2 d\xi \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_1} \\ &\quad + \underbrace{\left[ \int_{|\xi| \leq \xi_{\max}} 2 \left| \int_x^1 (\hat{f} - \hat{f}^\delta) \cosh(\tau(s-x))ds \right|^2 d\xi \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_2}. \end{aligned}$$

By (16) and (59), we obtain

$$\tilde{\mathcal{J}}_1 \leq 2\delta \sup_{|\xi| \leq \xi_{\max}} |\tau e^{\tau(1-x)}| \leq 2\delta |\tau| e^{|\tau|(1-x)} \leq 2\delta \xi_{\max}^{\frac{\alpha}{2}} e^{\xi_{\max}^{\frac{\alpha}{2}}(1-x)} = 2E^{1-x} \delta^x \ln \frac{E}{\delta}. \tag{62}$$

Using Cauchy–Schwarz integral inequality, (13), (59) and (16) yields

$$\begin{aligned} \tilde{\mathcal{J}}_2 &\leq \left[ \int_{|\xi| \leq \xi_{\max}} 2 \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) \left( \int_x^1 |\hat{f} - \hat{f}^\delta|^2 ds \right) d\xi \right]^{\frac{1}{2}} \\ &\leq \xi_{\max}^{-\frac{\alpha}{4}} e^{(1-x)\xi_{\max}^{\frac{\alpha}{2}}} \|\hat{f} - \hat{f}^\delta\|_{L^2(0,1;L^2(\mathbb{R}))} \\ &\leq \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} E^{1-x} \delta^x. \end{aligned} \tag{63}$$

Thus, by (61) and (62)

$$\mathcal{J}_1 \leq \left( 2 \ln \frac{E}{\delta} + \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} \right) E^{1-x} \delta^x. \tag{64}$$

Step 2. Estimate the term  $\mathcal{J}_2$  in (60). Again, using the Parseval’s identity and Lemma 2, we have

$$\begin{aligned} \mathcal{J}_2 &= \|\hat{u}_x^{\xi_{\max}}(x, \cdot) - \hat{u}_x(x, \cdot)\| \\ &= \left[ \int_{|\xi| > \xi_{\max}} \left| \tau e^{\tau(1-x)} \hat{g} + \int_x^1 \hat{f} \cosh(\tau(s-x)) ds \right|^2 d\xi \right]^{\frac{1}{2}} \\ &\leq \underbrace{\left[ \int_{|\xi| > \xi_{\max}} 2 |\tau e^{\tau(1-x)} \hat{g}|^2 d\xi \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_1} + \underbrace{\left[ \int_{|\xi| > \xi_{\max}} 2 \left| \int_x^1 \hat{f} \cosh(\tau(s-x)) ds \right|^2 d\xi \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_2}. \end{aligned}$$

We first estimate  $\tilde{\mathcal{J}}_1$ . Let

$$B_4(|\xi|) = \sqrt{2} |\xi|^{\frac{\alpha}{2}} e^{-x|\xi|^{\frac{\alpha}{2}}}.$$

By (7), we have

$$\begin{aligned} \tilde{\mathcal{J}}_1 &= \left[ \int_{|\xi| > \xi_{\max}} 2 \left| \tau e^{-\tau x} \left( \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right) \right|^2 d\xi \right]^{\frac{1}{2}} \\ &\leq \sup_{|\xi| > \xi_{\max}} B_4(|\xi|) \left\| \hat{u}(0, \xi) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right\|. \end{aligned}$$

By elementary calculations, it is easy to find the unique zero point  $|\xi^*|$  of  $B'_4(|\xi|)$  is

$$|\xi^*| = \left( \frac{1}{x} \right)^{\frac{2}{\alpha}},$$

and  $|\xi^*|$  maximize the function  $B_4(|\xi|)$ . Thus,

$$\sup_{|\xi| > \xi_{\max}} B_4(|\xi|) = \begin{cases} \sqrt{2} |\xi^*|^{\frac{\alpha}{2}} e^{-x|\xi^*|^{\frac{\alpha}{2}}} \leq \sqrt{2} |\xi^*|^{\frac{\alpha}{2}} e^{-x\xi_{\max}^{\frac{\alpha}{2}}}, & \xi_{\max} < |\xi^*|, \\ \sqrt{2} \xi_{\max}^{\frac{\alpha}{2}} e^{-x\xi_{\max}^{\frac{\alpha}{2}}}, & \xi_{\max} \geq |\xi^*|. \end{cases}$$

By (59), we have

$$\sup_{|\xi| > \xi_{\max}} B_4(|\xi|) \leq \left\{ \frac{1}{x}, \ln \frac{E}{\delta} \right\} \sqrt{2} e^{-x \ln \frac{E}{\delta}} := \epsilon_1 \sqrt{2} E^{-x} \delta^x,$$

where

$$\epsilon_1 = \max \left\{ \frac{1}{x}, \ln \frac{E}{\delta} \right\}. \tag{65}$$

Therefore, by Lemma 4, we obtain

$$\tilde{\mathcal{J}}_1 \leq \epsilon_1 \sqrt{2} E^{-x} \delta^x (E + N_1).$$

Now we estimate  $\tilde{\mathcal{J}}_2$ . Using Cauchy–Schwarz integral inequality, (13) and (15) yields

$$\begin{aligned} \tilde{\mathcal{J}}_2 &\leq \left[ \int_{|\xi| > \tilde{\xi}_{\max}} 2 \left( \int_x^1 e^{2|\tau|(s-x)} ds \right) \left( \int_x^1 |\hat{f}|^2 ds \right) d\xi \right]^{\frac{1}{2}} \\ &\leq \left[ \int_{|\xi| > \tilde{\xi}_{\max}} \left( \frac{1}{|\xi|^{\frac{\alpha}{2}}} e^{2|\xi|^{\frac{\alpha}{2}}(1-x)} \right) \left( \int_x^1 |\hat{f}|^2 ds \right) d\xi \right]^{\frac{1}{2}} \\ &\leq \tilde{\xi}_{\max}^{-\frac{\alpha}{4}} \left[ \int_{|\xi| > \tilde{\xi}_{\max}} e^{2|\xi|^{\frac{\alpha}{2}}(1-x)} e^{-3|\xi|^{\frac{\alpha}{2}}} d\xi \right]^{\frac{1}{2}}. \end{aligned}$$

Since the generalized integral on the right side of the last inequality converges for  $0 < x < 1$ , we introduce the notation

$$N_2 = \left[ \int_{|\xi| > \tilde{\xi}_{\max}} e^{-|\xi|^{\frac{\alpha}{2}}} d\xi \right]^{\frac{1}{2}}. \tag{66}$$

Using (59), we have

$$\tilde{\mathcal{J}}_2 \leq \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} N_2.$$

Thus,

$$\mathcal{J}_2 \leq \epsilon_1 \sqrt{2} E^{-x} \delta^x (E + N_1) + \left( \ln \frac{E}{\delta} \right)^{-\frac{1}{2}} N_2. \tag{67}$$

where  $\epsilon_1$  is given by (64). By substituting (63) and (66) into (60), we arrive at the final conclusion.  $\square$

**Remark 8.** If we replace assumptions (14) and (15) by (50) and (51), then the convergence  $\|u_x^{\delta, \tilde{\xi}_{\max}}(x, \cdot) - u_x(x, \cdot)\|_p$  also holds.

Similarly, the accuracy of the regularized solution becomes progressively lower as  $x \rightarrow 0$ , and then we use the condition (50) and (51) to give convergence estimate at  $x = 0$ .

**Theorem 6.** Let  $\hat{u}(x, \xi)$  given by (7) be the exact solution of problem (5) in the frequency space,  $\hat{u}_x^{\delta, \tilde{\xi}_{\max}}(x, \xi)$  given by (58) be the regularized solution, condition (16), (50) and (51) hold. If the regularization parameter  $\tilde{\xi}_{\max}$  is selected by

$$\tilde{\xi}_{\max} = \left( \ln \left( \frac{E}{\delta} \left( \ln \frac{E}{\delta} \right)^{-\frac{2p}{\alpha}} \right) \right)^{\frac{2}{\alpha}}. \tag{68}$$

Then, for  $p > \frac{\alpha}{2}$ , we have

$$\|u_x^{\delta, \tilde{\xi}_{\max}}(0, \cdot) - u_x(0, \cdot)\| \leq (2\epsilon_2^{-1} + \epsilon_2^{\frac{1}{2}}) E \left( \ln \frac{E}{\delta} \right)^{-\frac{2p}{\alpha}} + \sqrt{2} \epsilon_2^{\frac{2p}{\alpha}-1} (E + N_1) + \epsilon_2^{\frac{1}{2} + \frac{2p}{\alpha}} N_2, \tag{69}$$

where  $\epsilon_2 = \left( \ln \left( \frac{E}{\delta} \left( \ln \frac{E}{\delta} \right)^{-\frac{2p}{\alpha}} \right) \right)^{-1}$ ,  $N_1$  and  $N_2$  are some constants.

**Proof.** By the triangle inequality, we have

$$\|u_x^{\delta, \zeta_{\max}}(0, \cdot) - u_x(0, \cdot)\| \leq \underbrace{\|u_x^{\delta, \zeta_{\max}}(0, \cdot) - u_x^{\zeta_{\max}}(0, \cdot)\|}_{\mathcal{J}_3} + \underbrace{\|u_x^{\zeta_{\max}}(0, \cdot) - u_x(0, \cdot)\|}_{\mathcal{J}_4}. \tag{70}$$

Next, we divide the argument into two steps.

Step 1. Estimate the term  $\mathcal{J}_3$  in (69). Taking a similar procedure of the estimate of  $\mathcal{J}_1$ , and by (67), we obtain

$$\begin{aligned} \mathcal{J}_3 &= \|\hat{u}_x^{\delta, \zeta_{\max}}(0, \cdot) - \hat{u}_x^{\zeta_{\max}}(0, \cdot)\| \\ &\leq \left[ \int_{|\zeta| \leq \zeta_{\max}} 2|\tau e^\tau (\hat{g} - g^\delta)|^2 d\zeta \right]^{\frac{1}{2}} + \left[ \int_{|\zeta| \leq \zeta_{\max}} 2 \left| \int_0^1 (\hat{f} - f^\delta) \cosh(\tau s) ds \right|^2 d\zeta \right]^{\frac{1}{2}} \\ &\leq 2\delta \sup_{|\zeta| \leq \zeta_{\max}} |\tau e^\tau| + \left[ \int_{|\zeta| \leq \zeta_{\max}} 2 \left( \int_0^1 e^{2|\tau|s} ds \right) \left( \int_0^1 |\hat{f} - f^\delta|^2 ds \right) d\zeta \right]^{\frac{1}{2}} \\ &\leq 2\delta \zeta_{\max}^{\frac{\alpha}{2}} e^{\zeta_{\max}^{\frac{\alpha}{2}}} + \zeta_{\max}^{-\frac{\alpha}{4}} e^{\zeta_{\max}^{\frac{\alpha}{2}}} \|\hat{f}^\delta - \hat{f}\|_{L^2(0,1;H_p(\mathbb{R}))} \\ &\leq (2\epsilon_2^{-1} + \epsilon_2^{\frac{1}{2}}) E \left( \ln \frac{E}{\delta} \right)^{-\frac{2p}{\alpha}}, \end{aligned} \tag{71}$$

where

$$\epsilon_2 = \left( \ln \left( \frac{E}{\delta} \left( \ln \frac{E}{\delta} \right)^{-\frac{2p}{\alpha}} \right) \right)^{-1}. \tag{72}$$

Step 2. Estimate the term  $\mathcal{J}_4$  in (69). By Lemma 2, we have

$$\begin{aligned} \mathcal{J}_4 &= \|\hat{u}_x^{\zeta_{\max}}(0, \cdot) - \hat{u}_x(0, \cdot)\| \\ &\leq \underbrace{\left[ \int_{|\zeta| > \zeta_{\max}} 2|\tau e^\tau \hat{g}|^2 d\zeta \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_3} + \underbrace{\left[ \int_{|\zeta| > \zeta_{\max}} 2 \left| \int_0^1 \hat{f} \cosh(\tau s) ds \right|^2 d\zeta \right]^{\frac{1}{2}}}_{\tilde{\mathcal{J}}_4}. \end{aligned}$$

We first estimate  $\tilde{\mathcal{J}}_3$ . By (7), (9), (67) and Lemma 5, we have

$$\begin{aligned} \tilde{\mathcal{J}}_3 &= \left[ \int_{|\zeta| > \zeta_{\max}} 2 \left( 1 + \zeta^2 \right)^{-\frac{p}{2}} \tau \left( \hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right) \left( 1 + \zeta^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{2}} \\ &\leq \sqrt{2} \zeta_{\max}^{\frac{\alpha}{2} - p} \left\| \left( 1 + \zeta^2 \right)^{\frac{p}{2}} \left( \hat{u}(0, \zeta) - \int_0^1 \hat{f} \frac{\sinh(\tau s)}{\tau} ds \right) \right\| \\ &\leq \sqrt{2} \epsilon_2^{\frac{2p}{\alpha} - 1} (E + N_1). \end{aligned}$$

Now we estimate  $\tilde{\mathcal{J}}_4$ . Using Cauchy–Schwarz integral inequality, (13), (9), (51), (67) and (65) yields

$$\begin{aligned} \tilde{\mathcal{J}}_4 &\leq \left[ \int_{|\zeta| > \zeta_{\max}} 2 \left( 1 + \zeta^2 \right)^{-p} \left( \int_0^1 e^{2|\tau|s} ds \right) \left( \int_0^1 |\hat{f}|^2 ds \right) \left( 1 + \zeta^2 \right)^p d\zeta \right]^{\frac{1}{2}} \\ &\leq \left[ \int_{|\zeta| > \zeta_{\max}} |\zeta|^{-2p} \left( \frac{1}{|\zeta|^{\frac{\alpha}{2}}} e^{2|\zeta|^{\frac{\alpha}{2}}} \right) \left( \int_0^1 |\hat{f}|^2 ds \right) \left( 1 + \zeta^2 \right)^p d\zeta \right]^{\frac{1}{2}} \\ &\leq \zeta_{\max}^{-\left(\frac{\alpha}{4} + p\right)} \left[ \int_{|\zeta| > \zeta_{\max}} e^{-|\zeta|^{\frac{\alpha}{2}}} d\zeta \right]^{\frac{1}{2}} \\ &= \epsilon_2^{\frac{1}{2} + \frac{2p}{\alpha}} N_2. \end{aligned}$$

Then

$$\mathcal{J}_4 \leq \sqrt{2}\epsilon_2^{\frac{2p}{\alpha}-1}(E + N_1) + \epsilon_2^{\frac{1}{2}+\frac{2p}{\alpha}}N_2, \quad (73)$$

where  $\epsilon_2$  is given by (71). The Theorem now follows from equations (69)–(72).  $\square$

**Remark 9.** Since the regularization parameter  $\zeta_{\max} \rightarrow \infty$  as  $\delta \rightarrow 0$ , we can easily find that, for  $p > \frac{\alpha}{2}$ ,  $\epsilon_2 \rightarrow 0$  ( $\delta \rightarrow 0$ ). In addition, note that for  $p > \frac{\alpha}{2}$  there hold

$$\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-\frac{2p}{\alpha}}\right)\left(\ln\frac{E}{\delta}\right)^{-\frac{2p}{\alpha}} = \left(\ln\frac{E}{\delta}\right)^{1-\frac{2p}{\alpha}} - \frac{2p}{\alpha}\left[\ln\left(\ln\frac{E}{\delta}\right)\right]\left(\ln\frac{E}{\delta}\right)^{-\frac{2p}{\alpha}} \rightarrow 0, \quad \delta \rightarrow 0.$$

Therefore,

$$\lim_{\delta \rightarrow 0} \|u_x^{\delta, \zeta_{\max}}(0, \cdot) - u_x(0, \cdot)\| = 0, \quad p > \frac{\alpha}{2}.$$

**Remark 10.** In 2007, Qian [46] proved that it is impossible to obtain the error asymptotically better than logarithmic rate at  $x = 0$ . So our estimates is reasonable.

## 6. Conclusions

In this paper, we have considered the problem of finding a function  $u(x, t)$  satisfying (5). This is a sideways problem for non-homogeneous fractional heat equation, and the problem is ill-posed. To regularize the problem, we propose the dynamic spectral method and Fourier method, which are rather simple and convenient for dealing with some ill-posed problems. Error estimations between the approximate solution and the exact one, established from noise data  $g_\delta$  and  $f_\delta$ , are given. In fact, the paper extends the work in [33]. It is worth noting that the obtained estimates are sufficient to prove the results, but most of them are quite rough and can be improved.

As we all know, the most common regularization methods are the Tikhonov method, iterative method, quasi-reversibility method, truncation method, quasi-boundary value method and spectral method. The main difference between these methods is their convergence order. We can compare the convergence rate of errors by using different methods to discuss the problem. In addition, the dynamic spectral method and Fourier method can easily be extended to multi-dimensional case, which needs further study.

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