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Hermite–Hadamard-Type Inequalities for h-Convex Functions Involving New Fractional Integral Operators with Exponential Kernel

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Abstract: In this paper, we use two new fractional integral operators with exponential kernel about the midpoint of the interval to construct some Hermite–Hadamard type fractional integral inequalities for h-convex functions. Taking two integral identities about the first and second derivatives of the function as auxiliary functions, the main results are obtained by using the properties of h-convexity and the module. In order to illustrate the application of the results, we propose four examples and plot function images to intuitively present the meaning of the inequalities in the main results, and we verify the correctness of the conclusion. This study further expands the generalization of Hermite–Hadamard-type inequalities and provides some research references for the study of Hermite–Hadamard-type inequalities.

Keywords: fractional integrals operators; exponential kernel; Hermite–Hadamard-type inequalities; h-convex function

MSC: 26D15; 26A51; 26A33

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1. Introduction

If $g : I \subseteq \mathbb{R} \to \mathbb{R}$, and $m, n \in I$ with $m < n$, then

$$g(\frac{m+n}{2}) \leq \frac{g(m)+g(n)}{2}, \tag{1}$$

which is called Jensen's inequality [1]. Afterward, Hermite and Hadamard insert the integral mean value of convex function g in inequality (1) to obtain the following classical Hermite–Hadamard's inequality [2,3] .

Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $m, n \in I$ with $m < n$, then

$$g(\frac{m+n}{2}) \leq \frac{1}{n-m}\int_m^n g(x)dx \leq \frac{g(m)+g(n)}{2}. \tag{2}$$

If g is concave, the inequalities (2) hold in the reversed direction. We note that this inequality can make a bounded estimation of the integral mean on $[m, n]$, so it has wide applications in numerical integration. For the research on the popularization and application of the Hermite–Hadamard's inequality, the readers can refer to [4–10].

The research shows that the fractional-order phenomenon is widespread, and the fractional calculus modeling method is more accurate and reliable than the traditional integer order method. Therefore, the fractional calculus method has been one of the hot research topics in the academic community. Recently, the research results of fractional order on the Hermite–Hadamard's inequality are also numerous. For example, there are Riemann–Liouville fractional integral inequalities [11–14], conformable fractional integrals inequalities [15], k-Riemann–CLiouville fractional integrals inequalities [16], and local fractional integrals inequalities [17–19]. Because fractional integral operators have relatively convenient applications in some special fields, the research on fractional operator type integral inequalities is becoming more and more abundant. Set et al. [20] used Raina's fractional integral operators to obtain new Hermite–Hadamard-Mercer-type inequalities. Srivastava et al. [21] introduced the generalized left-side and right-side fractional integral operators with a certain modified Mittag–CLeffler kernel and utilized this general family of fractional integral operators to investigate the interesting Chebyshev inequality. In refs. [18,22], Sun presented two local fractional integral operators with a Mittag–Leffler kernel to establish some Hermite–Hadamard-type inequalities for generalized h-convex functions and generalized preinvex functions, respectively; afterward, Xu et al. [23] studied Hermite–Hadamard–Mercer for generalized h-convex functions with the help of the two local fractional integral operators.

In [24], Ahmad et al. proposed two new fractional integral operators with exponential kernels and establish some inequalities related to the right side of the Hermite–Hadamard's inequality. Subsequently, Wu et al. [25] studied the bound for the left side of the Hermite–Hadamard's inequality involving these integral operators. Budak et al. [26] utilized these integral operators with exponential kernels to established some Hermite–CHadamard and Ostrowski type inequalities. On the application of the new integral operators having exponential kernels in Hermite–Hadamard-type inequalities, Du and Zhou et al. extended them to interval-valued and interval-valued co-ordinated, see [27,28]. The new fractional integral operators with exponential kernels are given as follows.

Definition 1 ([24])**.** *Let $g \in L(m,n)$. The fractional integrals $\mathcal{I}^{\beta}_{m^+}g(\xi)$ and $\mathcal{I}^{\beta}_{n^-}g(\xi)$ of order $\beta \in (0,1)$ are, respectively, defined by*

$$\mathcal{I}^{\beta}_{m^+}g(\xi) = \frac{1}{\beta}\int_m^{\xi} exp\Big(-\frac{1-\beta}{\beta}(\xi-u)\Big)g(u)du, \xi > m, \tag{3}$$

and

$$\mathcal{I}^{\beta}_{n^-}g(\xi) = \frac{1}{\beta}\int_{\xi}^{n} exp\Big(-\frac{1-\beta}{\beta}(u-\xi)\Big)g(u)du, \xi < n. \tag{4}$$

Some known results about (3) and (4) in refs [24,25] are stated as follows.

Theorem 1 ([24])**.** *Let $g : [m,n] \to \mathbb{R}$ be a positive function with $0 \le m < n$ and $g \in L(m,n)$. If g is a convex function on $[m,n]$, then the following inequalities about (3) and (4) hold.*

$$g(\frac{m+n}{2}) \le \frac{1-\beta}{2(1-exp(-\rho))}[\mathcal{I}^{\beta}_{m^+}g(n) + \mathcal{I}^{\beta}_{n^-}g(m)] \le \frac{g(m)+g(n)}{2}, \tag{5}$$

where $\rho = \frac{1-\beta}{\beta}(n-m)$.

Theorem 2 ([24])**.** *Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I. If $|g'|$ is convex on $[m,n], m,n \in I$, then the following inequality about (3) and (4) holds.*

$$\left|\frac{g(m)+g(n)}{2} - \frac{1-\beta}{2(1-exp(-\rho))}\Big[\mathcal{I}^{\beta}_{m^+}g(n) + \mathcal{I}^{\beta}_{n^-}g(m)\Big]\right| \le \frac{n-m}{2\rho}\tanh(\frac{\rho}{4})(|g'(m)| + |g'(n)|). \tag{6}$$

Theorem 3 ([25])**.** *Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I. If $|g'|$ is convex on $[m,n], m,n \in I$, then the following inequality about (3) and (4) holds:*

$$\left| \frac{1-\beta}{2(1-exp(-\rho))} \left[\mathcal{I}_{m^+}^{\beta} g(n) + \mathcal{I}_{n^-}^{\beta} g(m) \right] - g\left(\frac{m+n}{2}\right) \right| \leq \frac{n-m}{2} \left[\frac{1}{2} - \frac{\tanh(\frac{\rho}{4})}{\rho} \right] (|g'(m)| + |g'(n)|). \tag{7}$$

These Hermite–Hadamard-type integral inequalities involving the fractional integral operators (3) and (4) are structurally in the form of "$\mathcal{I}_{m^+}^{\beta} g(n) + \mathcal{I}_{n^-}^{\beta} g(m)$" for convex functions. In this paper, our main purpose is to apply the definition of h-convexity and the properties of modules to propose some new Hermite–Hadamard-type fractional integral inequalities about fractional integral operators (3) and (4) for generalized h-convex function, whose integral operators involving the midpoint of the interval $[m, n]$ is in the form of "$\mathcal{I}_{\frac{m+n}{2}^+}^{\beta} g(n) + \mathcal{I}_{\frac{m+n}{2}^-}^{\beta} g(m)$". Some numerical examples are given to illustrate the correctness of the results.

2. Results

In the subsequent text, we denote $\rho = \frac{1-\beta}{\beta}(n-m)$ for $\beta \in (0,1)$. In order to obtain our results, the following definition of h-convex function proposed by Varošanec in [29] will be used in the subsequent text.

Definition 2 ([29]). *Let $h : \Omega \to \mathbb{R}$ be a positive function. We say that $g : \Xi \to \mathbb{R}$ is an h-convex function, if g is nonnegative and for all $u, v \in \Xi$ and $\zeta \in (0,1)$, we have*

$$g(\zeta u + (1-\zeta)v) \leq h(\zeta)g(u) + h(1-\zeta)g(v). \tag{8}$$

If inequality (8) is reversed, then g is said to be h-concave.

Remark 1. *Obviously, if $h(\zeta) = \zeta$, then h-convex function derives the classical convex function.*

Theorem 4. *Let $g : [m, n] \to \mathbb{R}$ be a positive function with $0 \leq m < n$, and $g(x) \in L[m, n]$. If g is an h-convex function on $[m, n]$, then the following inequalities for integral operators (3) and (4) hold.*

$$\begin{aligned} \frac{1-\exp(-\frac{\rho}{2})}{\rho h(\frac{1}{2})} g\left(\frac{m+n}{2}\right) &\leq \frac{\beta}{n-m} \left(\mathcal{I}_{\frac{m+n}{2}^+}^{\beta} g(n) + \mathcal{I}_{\frac{m+n}{2}^-}^{\beta} g(m) \right) \\ &\leq [g(m) + g(n)] \int_0^{\frac{1}{2}} \exp(-\rho t)[h(t) + h(1-t)]dt. \end{aligned} \tag{9}$$

Proof. Since g is an h-convex function on $[m, n]$, we obtain

$$g\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)g(x) + h\left(\frac{1}{2}\right)g(y). \tag{10}$$

For $x = tm + (1-t)n$, $y = (1-t)m + tn$ in (10), $t \in [0,1]$, we have

$$g\left(\frac{m+n}{2}\right) \leq h\left(\frac{1}{2}\right)g\big(tm + (1-t)n\big) + h\left(\frac{1}{2}\right)g\big((1-t)m + tn\big).$$

Multiplying both sides of the above inequality by $\exp(-\rho t)$, and integrating the resulting inequality with respect to t over $[0, \frac{1}{2}]$, we obtain

$$
\begin{aligned}
&\frac{g\left(\frac{m+n}{2}\right)\left(1-\exp(-\frac{\rho}{2})\right)}{\rho h(\frac{1}{2})} = \frac{g\left(\frac{m+n}{2}\right)}{h(\frac{1}{2})}\int_0^{\frac{1}{2}} \exp(-\rho t)dt \\
\leq &\int_0^{\frac{1}{2}} \exp(-\rho t) g\Big(tm+(1-t)n\Big)dt + \int_0^{\frac{1}{2}} \exp(-\rho t) g\Big((1-t)m+tn\Big)dt \\
= &\frac{\beta}{n-m}\frac{1}{\beta}\int_{\frac{m+n}{2}}^{n} \exp\Big(-\frac{1-\beta}{\beta}(n-x)\Big)g(x)dx + \frac{\beta}{n-m}\frac{1}{\beta}\int_m^{\frac{m+n}{2}} \exp\Big(-\frac{1-\beta}{\beta}(y-m)\Big)g(y)dy \\
= &\frac{\beta}{n-m}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big).
\end{aligned} \tag{11}
$$

Thus, the first inequality of (9) holds.

On the other hand, note that g is an h-convex function for $t \in [0,1]$, we get

$$
g\big(tm+(1-t)n\big) \leq h(t)g(m) + h(1-t)g(n)
$$

and

$$
g\big((1-t)m+tn\big) \leq h(1-t)g(m) + h(t)g(n).
$$

Adding the above two inequalities, we have

$$
g\big(tm+(1-t)n\big) + g\big((1-t)m+tn\big) \leq [h(t)+h(1-t)][g(m)+g(n)]. \tag{12}
$$

Multiplying both sides of the inequality (12) by $\exp(-\rho t)$, and integrating the result with respect to t over $[0, \frac{1}{2}]$, we obtain

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \exp(-\rho t) g\Big(tm+(1-t)n\Big)dt + \int_0^{\frac{1}{2}} \exp(-\rho t) g\Big((1-t)m+tn\Big)dt \\
\leq &[g(m)+g(n)]\int_0^{\frac{1}{2}} \exp(-\rho t)[h(t)+h(1-t)]dt.
\end{aligned} \tag{13}
$$

By (11), the inequality (13) becomes

$$
\frac{\beta}{n-m}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \leq [g(m)+g(n)]\int_0^{\frac{1}{2}} \exp(-\rho t)[h(t)+h(1-t)]dt. \tag{14}
$$

Thus, the second inequality of (9) holds. This completes the proof. □

Corollary 1. *Under the conditions of Theorem 4, for $\beta \to 1$, we obtain*

$$
g\Big(\frac{m+n}{2}\Big) \leq \frac{2h(\frac{1}{2})}{n-m}\int_m^n g(x)dx \leq [g(m)+g(n)]2h\Big(\frac{1}{2}\Big)\int_0^{\frac{1}{2}}[h(t)+h(1-t)]dt. \tag{15}
$$

Proof. By (9), that is

$$
\begin{aligned}
g\Big(\frac{m+n}{2}\Big) \leq &\frac{h(\frac{1}{2})(1-\beta)}{1-\exp(-\frac{\rho}{2})}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \\
&\leq [g(m)+g(n)]\frac{\rho h(\frac{1}{2})}{1-\exp(-\frac{\rho}{2})}\int_0^{\frac{1}{2}} \exp(-\rho t)[h(t)+h(1-t)]dt.
\end{aligned} \tag{16}
$$

By calculating, we have

$$
\lim_{\beta\to 1}\frac{1-\beta}{1-\exp(-\frac{\rho}{2})} = \frac{2}{n-m},
$$

and

$$\lim_{\beta\to 1}\frac{\rho\exp(-\rho t)}{1-\exp(-\frac{\rho}{2})}=2.$$

Thus, from the inequality (16) for $\beta \to 1$, we obtain the inequality (15). □

Remark 2. *Taking $h(t) = t$ in (15), we obtain the classical Hermite–Hadamard inequality for convex function (2).*

Corollary 2. *If we take $h(t) = t$ in Theorem 4, then the following fractional integral inequality for the convex function is obtained.*

$$g(\frac{m+n}{2}) \le \frac{1-\beta}{2(1-\exp(-\frac{\rho}{2}))}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \le \frac{g(m)+g(n)}{2}, \tag{17}$$

which is Theorem 2 proved by Budak in ref. [26].

In order to obtain our results, according to Lemma 1 in ref. [26], we can obtain the following identity.

Lemma 1 ([26])**.** *Let $g : [m,n] \to \mathbb{R}$ be a differentiable function with $m < n$. If $g' \in L[m,n]$, then the following identity involving fractional integral operators (3) and (4) holds.*

$$\begin{aligned} &g(\frac{m+n}{2}) - \frac{1-\beta}{2\Big(1-\exp\left(-\frac{\rho}{2}\right)\Big)}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \\ =&\frac{n-m}{2\Big(1-\exp(-\frac{\rho}{2})\Big)}\Bigg[\int_0^{\frac{1}{2}}\left(\exp(-\rho t)-1\right)g'\Big(tm+(1-t)n\Big)dt \\ &+\int_{\frac{1}{2}}^{1}(1-\exp(-\rho(1-t)))g'\Big(tm+(1-t)n\Big)dt\Bigg]. \end{aligned} \tag{18}$$

Theorem 5. *Let $g : [m,n] \to \mathbb{R}$ be a differentiable function with $m < n$. If $g'(u) \in L[m,n]$, and $|g'|$ is h-convex on $[m,n]$, then the following fractional integral inequality holds.*

$$\begin{aligned} &\left|g(\frac{m+n}{2}) - \frac{1-\beta}{2\Big(1-\exp\left(-\frac{\rho}{2}\right)\Big)}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big)\right| \\ \le&\frac{n-m}{2\Big(1-\exp(-\frac{\rho}{2})\Big)}\int_0^{\frac{1}{2}}(1-\exp(-\rho t))\Big(h(t)+h(1-t)\Big)\Big(|g'(m)|+|g'(n)|\Big)dt. \end{aligned} \tag{19}$$

Proof. Since $|g'|$ is h-convex on $[m,n]$ and h is a nonnegative function, by Lemma 1, we obtain

$$
\begin{aligned}
&\left| g\left(\frac{m+n}{2}\right) - \frac{1-\beta}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)} \left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^{+}} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}^{-}} g(m) \right) \right| \\
\leq & \frac{n-m}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)} \left[\int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left| g'\left(tm+(1-t)n\right) \right| dt \right. \\
&\left. + \int_{\frac{1}{2}}^{1} (1-\exp(-\rho(1-t))) \left| g'\left(tm+(1-t)n\right) \right| dt \right] \\
= & \frac{n-m}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)} \left[\int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left| g'\left(tm+(1-t)n\right) \right| dt \right. \\
&\left. + \int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left| g'\left((1-t)m+tn\right) \right| dt \right] \\
\leq & \frac{n-m}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)} \left[\int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left(h(t)|g'(m)| + h(1-t)|g'(n)| \right) dt \right. \\
&\left. + \int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left(h(1-t)|g'(m)| + h(t)|g'(n)| \right) dt \right] \\
= & \frac{n-m}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)} \int_0^{\frac{1}{2}} (1-\exp(-\rho t)) \left(h(t)+h(1-t) \right) \left(|g'(m)| + |g'(n)| \right) dt.
\end{aligned} \tag{20}
$$

This completes the proof. □

Remark 3. *For $\beta \to 1$, by calculating, we obtain*

$$
\lim_{\beta \to 1} \frac{1-\beta}{2[1-\exp(-\frac{\rho}{2})]} = \frac{1}{n-m},
$$

and

$$
\lim_{\beta \to 1} \frac{1-\exp(-\rho t)}{2\left(1-\exp(-\frac{\rho}{2})\right)} = t.
$$

Thus, from the inequality (19) for $\beta \to 1$, we obtain the following inequality

$$
\left| g\left(\frac{m+n}{2}\right) - \frac{1}{n-m} \int_m^n g(u)du \right| \leq (n-m)\left(|g'(m)| + |g'(n)| \right) \int_0^{\frac{1}{2}} t\left(h(t) + h(1-t) \right) dt. \tag{21}
$$

Remark 4. *Taking $h(t) = t$ in (21), we obtain the following inequality*

$$
\left| g\left(\frac{m+n}{2}\right) - \frac{1}{n-m} \int_m^n g(u)du \right| \leq \frac{n-m}{8} \left(|g'(m)| + |g'(n)| \right),
$$

which is Theorem 2.2 proved by U. Kirmaci in ref. [30].

Corollary 3. *If we take $h(t) = t$ in Theorem 5, then the following fractional integral inequality for convex function is obtained.*

$$\begin{aligned}
&\left| g(\frac{m+n}{2}) - \frac{1-\beta}{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big)\right| \\
\leq & \frac{n-m}{2\Big(1-\exp(-\frac{\rho}{2})\Big)}\Big(|g'(m)| + |g'(n)|\Big)\Big(\frac{1}{2} + \frac{\exp(-\frac{\rho}{2}) - 1}{\rho}\Big).
\end{aligned} \tag{22}$$

Proof. If $h(t) = t$, by (19) we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-\exp(-\rho t))\Big(h(t) + h(1-t)\Big)dt \\
=&\int_0^{\frac{1}{2}} (1-\exp(-\rho t))dt \\
=&\frac{1}{2} + \frac{\exp(-\frac{\rho}{2}) - 1}{\rho}.
\end{aligned}$$

This completes the proof. □

Lemma 2. *Let $g : [m,n] \to \mathbb{R}$ be a twice differentiable function on $[m,n]$ with $m < n$. If $g''(x) \in L[m,n]$, then the following identity involving fractional integral operators (3) and (4) holds.*

$$\begin{aligned}
&g(\frac{m+n}{2}) - \frac{1-\beta}{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \\
=&\frac{(n-m)^2}{2\Big(1-\exp(-\frac{\rho}{2})\Big)}\left[\int_0^{\frac{1}{2}}\Big(\frac{1-\exp(-\rho t)}{\rho} - t\Big)g''\Big(tm+(1-t)n\Big)dt\right. \\
&\left. + \int_{\frac{1}{2}}^{1}\Big(\frac{1-\exp(-\rho(1-t))}{\rho} - (1-t)\Big)g''\Big(tm+(1-t)n\Big)dt\right].
\end{aligned} \tag{23}$$

Proof. Using integration by parts, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \big(\exp(-\rho t) - 1\big)g'\Big(tm+(1-t)n\Big)dt \\
=&\int_0^{\frac{1}{2}} \exp(-\rho t)g'\Big(tm+(1-t)n\Big)dt - \int_0^{\frac{1}{2}} g'\Big(tm+(1-t)n\Big)dt \\
=&-\frac{1}{\rho}\left[\int_0^{\frac{1}{2}} g'\Big(tm+(1-t)n\Big)d\big(\exp(-\rho t)\big)\right] - \int_0^{\frac{1}{2}} g'\Big(tm+(1-t)n\Big)dt \\
=&-\frac{1}{\rho}\left[\exp(-\frac{\rho}{2})g'(\frac{m+n}{2}) - g'(n) - (m-n)\int_0^{\frac{1}{2}}\exp(-\rho t)g''\Big(tm+(1-t)n\Big)dt\right] \\
&-\frac{1}{2}g'(\frac{m+n}{2}) + (m-n)\int_0^{\frac{1}{2}} tg''\Big(tm+(1-t)n\Big)dt.
\end{aligned} \tag{24}$$

Similarly, one has

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} \big(1-\exp(-\rho(1-t))\big)g'\Big(tm+(1-t)n\Big)dt \\
=&\int_{\frac{1}{2}}^{1} g'\Big(tm+(1-t)n\Big)dt-\int_{\frac{1}{2}}^{1} \exp(-\rho(1-t))g'\Big(tm+(1-t)n\Big)dt \\
=&\int_{\frac{1}{2}}^{1} g'\Big(tm+(1-t)n\Big)dt-\frac{1}{\rho}\int_{\frac{1}{2}}^{1} g'\Big(tm+(1-t)n\Big)d\big(\exp(-\rho(1-t))\big) \\
=&g'(m)-\frac{1}{2}g'(\frac{m+n}{2})-(m-n)\int_{\frac{1}{2}}^{1} tg''\Big(tm+(1-t)n\Big)dt \\
&-\frac{1}{\rho}\left[g'(m)-\exp(-\frac{\rho}{2})g'(\frac{m+n}{2})-(m-n)\int_{\frac{1}{2}}^{1}\exp(-\rho(1-t))g''\Big(tm+(1-t)n\Big)dt\right].
\end{aligned}
\tag{25}
$$

Adding (24) and (25), we obtain

$$
\begin{aligned}
&\int_{0}^{\frac{1}{2}} \big(\exp(-\rho t)-1\big)g'\Big(tm+(1-t)n\Big)dt+\int_{\frac{1}{2}}^{1} \big(1-\exp(-\rho(1-t))\big)g'\Big(tm+(1-t)n\Big)dt \\
=&g'(m)-g'(\frac{m+n}{2})+(m-n)\int_{0}^{\frac{1}{2}} tg''\Big(tm+(1-t)n\Big)dt-(m-n)\int_{\frac{1}{2}}^{1} tg''\Big(tm+(1-t)n\Big)dt \\
&-\frac{1}{\rho}\Bigg[g'(m)-g'(n)-(m-n)\int_{0}^{\frac{1}{2}}\exp(-\rho t)g''\Big(tm+(1-t)n\Big)dt \\
&-(m-n)\int_{\frac{1}{2}}^{1}\exp(-\rho(1-t))g''\Big(tm+(1-t)n\Big)dt\Bigg] \\
=&(m-n)\int_{\frac{1}{2}}^{1} g''\Big(tm+(1-t)n\Big)dt+(m-n)\int_{0}^{\frac{1}{2}}\Big(t+\frac{\exp(-\rho t)}{\rho}\Big)g''\Big(tm+(1-t)n\Big)dt \\
&-\frac{m-n}{\rho}\int_{0}^{1} g''\Big(tm+(1-t)n\Big)dt-(m-n)\int_{\frac{1}{2}}^{1}\Big(t-\frac{\exp(-\rho(1-t))}{\rho}\Big)g''\Big(tm+(1-t)n\Big)dt \\
=&(m-n)\int_{0}^{\frac{1}{2}}\Big(t-\frac{1-\exp(-\rho t)}{\rho}\Big)g''\Big(tm+(1-t)n\Big)dt \\
&+(m-n)\int_{\frac{1}{2}}^{1}\Big(1-t-\frac{1-\exp(-\rho(1-t))}{\rho}\Big)g''\Big(tm+(1-t)n\Big)dt
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
=&(n-m)\int_{0}^{\frac{1}{2}}\Big(\frac{1-\exp(-\rho t)}{\rho}-t\Big)g''\Big(tm+(1-t)n\Big)dt \\
&+(n-m)\int_{\frac{1}{2}}^{1}\Big(\frac{1-\exp(-\rho(1-t))}{\rho}+t-1\Big)g''\Big(tm+(1-t)n\Big)dt.
\end{aligned}
$$

Substituting (26) into (18), we have

$$
\begin{aligned}
&g(\frac{m+n}{2})-\frac{1-\beta}{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\Big) \\
=&\frac{(n-m)^2}{2\Big(1-\exp(-\frac{\rho}{2})\Big)}\Bigg[\int_{0}^{\frac{1}{2}}\Big(\frac{1-\exp(-\rho t)}{\rho}-t\Big)g''\Big(tm+(1-t)n\Big)dt \\
&+\int_{\frac{1}{2}}^{1}\Big(\frac{1-\exp(-\rho(1-t))}{\rho}+t-1\Big)g''\Big(tm+(1-t)n\Big)dt\Bigg].
\end{aligned}
$$

This completes the proof. □

Theorem 6. *Let $g : [m, n] \to \mathbb{R}$ be a twice differentiable function with $m < n$. If $g''(u) \in L[m, n]$, and $|g''|$ is h-convex on $[m, n]$, then the following fractional integral inequality holds.*

$$\begin{aligned}&\left|g(\frac{m+n}{2}) - \frac{1-\beta}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)}\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\right)\right|\\ \leq&\frac{(n-m)^2\left(|g''(m)|+|g''(n)|\right)}{2\left(1-\exp(-\frac{\rho}{2})\right)}\int_0^{\frac{1}{2}}\left(\frac{1-\exp(-\rho t)}{\rho}+t\right)\left(h(t)+h(1-t)\right)dt. \end{aligned} \tag{27}$$

Proof. Since $|g''|$ is h convex on $[m, n]$, by Lemma 2, we can obtain

$$\begin{aligned}&\left|g(\frac{m+n}{2}) - \frac{1-\beta}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)}\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\right)\right|\\ \leq&\frac{(n-m)^2}{2\left(1-\exp(-\frac{\rho}{2})\right)}\left[\int_0^{\frac{1}{2}}\left|\frac{1-\exp(-\rho t)}{\rho}-t\right|\left|g''\left(tm+(1-t)n\right)\right|dt\right.\\ &\left.+\int_{\frac{1}{2}}^{1}\left|\frac{1-\exp(-\rho(1-t))}{\rho}-(1-t)\right|\left|g''\left(tm+(1-t)n\right)\right|dt\right]\\ =&\frac{(n-m)^2}{2\left(1-\exp(-\frac{\rho}{2})\right)}\left[\int_0^{\frac{1}{2}}\left|\frac{1-\exp(-\rho t)}{\rho}-t\right|\left|g''\left(tm+(1-t)n\right)\right|dt\right.\\ &\left.+\int_0^{\frac{1}{2}}\left|\frac{1-\exp(-\rho t)}{\rho}-t\right|\left|g''\left((1-t)m+tn\right)\right|dt\right]\\ =&\frac{(n-m)^2}{2\left(1-\exp(-\frac{\rho}{2})\right)}\int_0^{\frac{1}{2}}\left|\frac{1-\exp(-\rho t)}{\rho}-t\right|\left(\left|g''\left(tm+(1-t)n\right)\right|+\left|g''\left((1-t)m+tn\right)\right|\right)dt\\ \leq&\frac{(n-m)^2}{2\left(1-\exp(-\frac{\rho}{2})\right)}\int_0^{\frac{1}{2}}\left(\frac{1-\exp(-\rho t)}{\rho}+t\right)\left(h(t)+h(1-t)\right)\left(|g''(m)|+|g''(n)|\right)dt. \end{aligned} \tag{28}$$

This completes the proof. □

Corollary 4. *If we take $h(t) = t$ in Theorem 6, then the following fractional integral inequality for convex function is obtained.*

$$\begin{aligned}&\left|g(\frac{m+n}{2}) - \frac{1-\beta}{2\left(1-\exp\left(-\frac{\rho}{2}\right)\right)}\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}+}g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-}g(m)\right)\right|\\ \leq&\frac{(n-m)^2\left(|g''(m)|+|g''(n)|\right)}{2\left(1-\exp(-\frac{\rho}{2})\right)}\left(\frac{1}{2\rho}+\frac{\exp(-\frac{\rho}{2})-1}{\rho^2}+\frac{1}{8}\right). \end{aligned} \tag{29}$$

Proof. By $h(t) = t$ and the proof of Corollary 3, it is easy to obtain the desired result. □

3. Numerical Examples

To illustrate our main conclusions, we will present four examples to show these conclusions in this section.

Example 1. *From Corollary 2, we get the following inequalities*

$$\frac{2(1-\exp(-\frac{\varrho}{2}))}{1-\beta}g(\frac{m+n}{2}) \leq \left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}^-}g(m)\right) \leq \frac{1-\exp(-\frac{\varrho}{2})}{1-\beta}[g(m)+g(n)].$$

Taking $g(\xi)=e^{2\xi}$, we know that g is an h-convex function for $h(t)=t$. It meets the conditions of Corollary 2 for $\beta\in[0,1]$. If we choose $m=1, n=3$, then the following formulas are drawn.

$$\begin{aligned}
\frac{2(1-\exp(-\frac{\varrho}{2}))}{1-\beta}g(\frac{m+n}{2}) =& \frac{1-\exp(-\frac{\varrho}{2})}{1-\beta}2e^4,\\
\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}^-}g(m)\right) =& \frac{1}{\beta}\Big[\int_2^3 \exp\Big(-\frac{1-\beta}{\beta}(3-u)\Big)e^{2u}du\\
&+\int_1^2 \exp\Big(-\frac{1-\beta}{\beta}(u-1)\Big)e^{2u}du\Big],\\
=& \frac{e^6-e^{5-\frac{1}{\beta}}}{\beta+1}-\frac{e^2-e^{5-\frac{1}{\beta}}}{3\beta-1},\\
\frac{1-\exp(-\frac{\varrho}{2})}{1-\beta}[g(m)+g(n)] =& \frac{1-\exp(-\frac{\varrho}{2})}{1-\beta}(e^2+e^6).
\end{aligned}$$

We plot the function image of the above three functions for $\beta\in[0,1]$, as shown in Figure 1. From the position relationship of the image, we can see that the middle term of the inequalities is just between the left and right images, and the left image is at the bottom, and the right image is at the top. These show that the inequalities relationship in Corollary 2 is tenable.

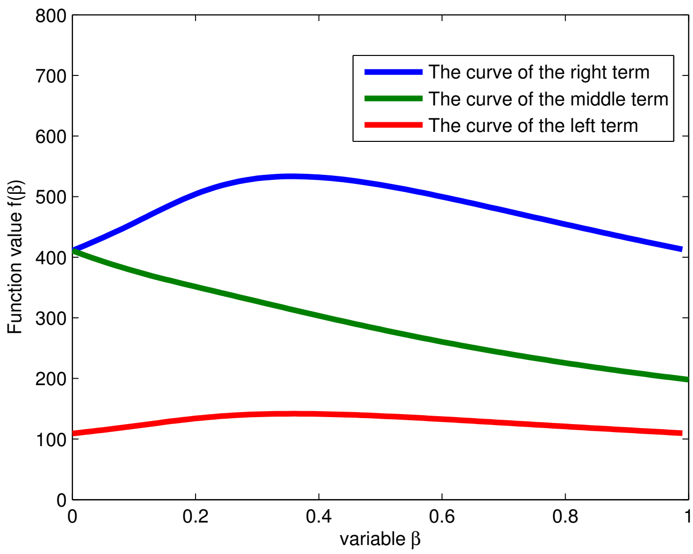

Figure 1. The image description of Corollary 2 for $h(t)=t$ and $g(\xi)=e^{2\xi}$.

Specidically, if we choose $\beta=\frac{1}{2}$, then we have

$$\begin{aligned}
\frac{2(1-\exp(-\frac{\varrho}{2}))}{1-\beta}g(\frac{m+n}{2}) =& 138.0505,\\
\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}^-}g(m)\right) =& 280.9551,\\
\frac{1-\exp(-\frac{\varrho}{2})}{1-\beta}[g(m)+g(n)] =& = 519.3728.
\end{aligned}$$

This further verifies that the conclusion of Corollary 2 is correct.

Example 2. *From Corollary 3, we get the following inequalities*

$$\left|\frac{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}{1-\beta}g\Big(\frac{m+n}{2}\Big)-\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}^-}g(m)\Big)\right|$$
$$\le\frac{n-m}{1-\beta}\Big(|g'(m)|+|g'(n)|\Big)\Big(\frac{1}{2}+\frac{\exp(-\frac{\rho}{2})-1}{\rho}\Big).$$

Taking $g(\xi)=\xi^3$, we know that $|g'|$ is an h-convex function for $h(t)=t$. It meets the conditions of Corollary 3 for $\beta\in[0,1]$. If we choose $m=1,n=3$, then the following formulas are drawn.

$$\begin{aligned}&\left|\frac{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}{1-\beta}g\Big(\frac{m+n}{2}\Big)-\Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+}g(n)+\mathcal{I}^{\beta}_{\frac{m+n}{2}^-}g(m)\Big)\right|\\
=&\left|\frac{2\Big(1-\exp(-\frac{\rho}{2})\Big)}{1-\beta}g(2)-\frac{1}{\beta}\Big[\int_2^3\exp\Big(-\frac{1-\beta}{\beta}(3-u)\Big)u^3du+\int_1^2\exp\Big(-\frac{1-\beta}{\beta}(u-1)\Big)u^3du\Big]\right|\\
=&\left|\frac{16\Big(1-\exp(-\frac{\rho}{2})\Big)}{1-\beta}-\frac{-76\beta^3+156\beta^2-108\beta+28}{(\beta-1)^4}-\frac{\exp(1-\frac{1}{\beta})(40\beta^3-72\beta^2+48\beta-16)}{(\beta-1)^4}\right|,\end{aligned}$$

$$\frac{n-m}{1-\beta}\Big(|g'(m)|+|g'(n)|\Big)\Big(\frac{1}{2}+\frac{\exp(-\frac{\rho}{2})-1}{\rho}\Big)=\frac{60}{1-\beta}\Big(\frac{1}{2}+\frac{\exp(-\frac{\rho}{2})-1}{\rho}\Big).$$

We plot the function image of the above two functions for $\beta\in[0,1]$, as shown in Figure 2. From the position relationship of the image, we can see that the left image is at the bottom, and the right image is at the top. These show that the inequality relationship in Corollary 3 is tenable.

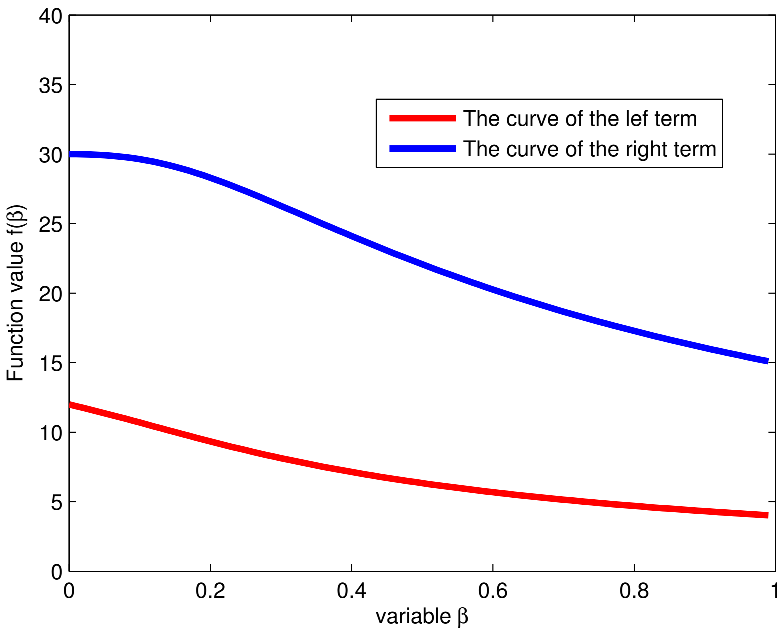

Figure 2. The image description of Corollary 3 for $h(t)=t$ and $g(\xi)=\xi^3$.

Specifically, if we choose $\beta = \frac{1}{3}$, then we have

$$\left| \frac{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}{1-\beta} g\big(\frac{m+n}{2}\big) - \Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-} g(m)\Big) \right| = 7.7820,$$

$$\frac{n-m}{1-\beta}\Big(|g'(m)|+|g'(n)|\Big)\Big(\frac{1}{2}+\frac{\exp(-\frac{\rho}{2})-1}{\rho}\Big) = 25.5450.$$

This further verifies that the conclusion of Corollary 3 is correct.

Example 3. *From Corollary 4, we get the following inequalities*

$$\begin{aligned} &\left| \frac{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}{1-\beta} g\big(\frac{m+n}{2}\big) - \Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-} g(m)\Big) \right| \\ \leq &\frac{(n-m)^2}{1-\beta}\Big(|g''(m)|+|g''(n)|\Big)\Big(\frac{1}{2\rho}+\frac{\exp(-\frac{\rho}{2})-1}{\rho^2}+\frac{1}{8}\Big). \end{aligned}$$

Taking $g(\xi) = e^{3\xi}$, we know that $|g''|$ is an h-convex function for $h(t) = t$. It meets the conditions of Corollary 4 for $\beta \in [0,1]$. If we choose $m = 1, n = 3$, then the following formulas are drawn.

$$\begin{aligned} &\left| \frac{2\Big(1-\exp\big(-\frac{\rho}{2}\big)\Big)}{1-\beta} g\big(\frac{m+n}{2}\big) - \Big(\mathcal{I}^{\beta}_{\frac{m+n}{2}+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}-} g(m)\Big) \right| \\ =&\left| \frac{2\Big(1-\exp(-\frac{\rho}{2})\Big)}{1-\beta} g(2) - \frac{1}{\beta}\Big[\int_2^3 \exp\Big(-\frac{1-\beta}{\beta}(3-u)\Big)e^{3u}du + \int_1^2 \exp\Big(-\frac{1-\beta}{\beta}(u-1)\Big)e^{3u}du\Big] \right| \\ =&\left| \frac{(2e^6)\Big(1-\exp(-\frac{\rho}{2})\Big)}{1-\beta} - \frac{e^9-e^{7-\frac{1}{\beta}}}{2\beta+1} + \frac{e^3-e^{7-\frac{1}{\beta}}}{4\beta-1} \right|, \end{aligned}$$

$$\frac{(n-m)^2}{1-\beta}\Big(|g''(m)|+|g''(n)|\Big)\Big(\frac{1}{2\rho}+\frac{\exp(-\frac{\rho}{2})-1}{\rho^2}+\frac{1}{8}\Big) = \frac{4(9e^9+9e^3)}{1-\beta}\Big(\frac{1}{2\rho}+\frac{\exp(-\frac{\rho}{2})-1}{\rho^2}+\frac{1}{8}\Big).$$

We only plot the function image of the above two functions for $\beta \in [0, 0.5]$, as shown in Figure 3. From the position relationship of the image, we can see that the left image is at the bottom, and the right image is at the top. These show that the inequality relationship in Corollary 4 is tenable for $\beta \in [0,1]$, because the growth rate of the right term is much faster than that of the left term in interval $\beta \in [0.5, 1]$.

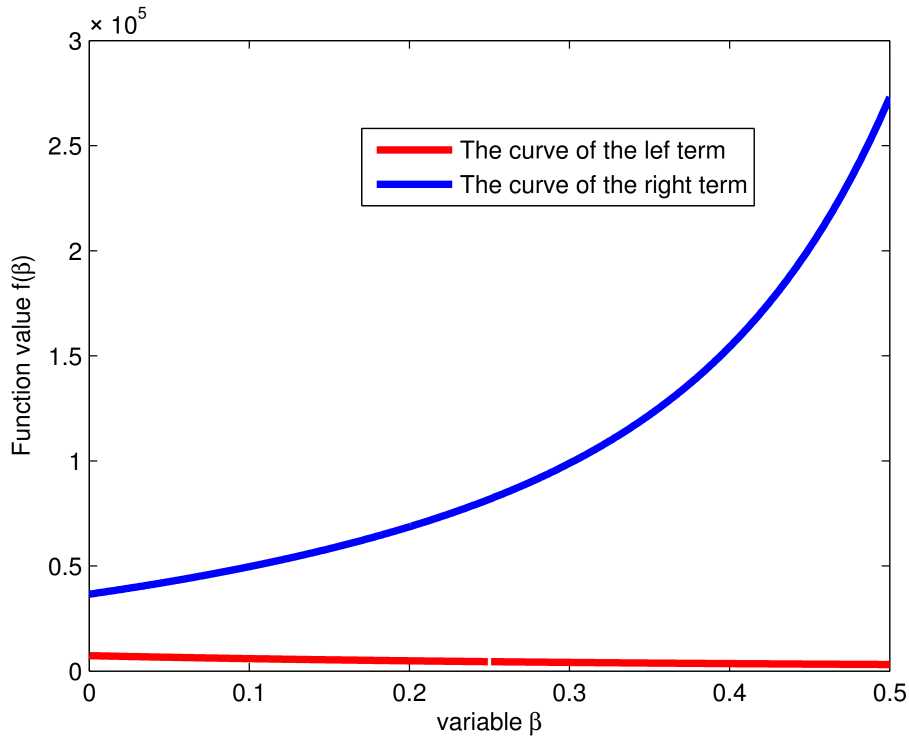

Figure 3. The image description of Corollary 4 for $h(t) = t$ and $g(\xi) = e^{3\xi}$.

Specifically, if we choose $\beta = 0.01$, then we have

$$\left| \frac{2\left(1 - \exp\left(-\frac{\rho}{2}\right)\right)}{1-\beta} g\left(\frac{m+n}{2}\right) - \left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}^-} g(m)\right) \right| = 7150.1,$$

$$\frac{(n-m)^2}{1-\beta}\Big(|g''(m)| + |g''(n)|\Big)\left(\frac{1}{2\rho} + \frac{\exp(-\frac{\rho}{2}) - 1}{\rho^2} + \frac{1}{8}\right) = 37669.$$

This further verifies that the conclusion of Corollary 4 is correct.

Example 4. *Taking $g(\xi) = e^{2\xi}$ and $h(t) = e^t$ for $t \in [0,1]$, we know that g is an h-convex function by Remark 5 in ref. [29]. It meets the conditions of Theorem 4 for $\beta \in [0,1]$. If we choose $m = 2, n = 4$, then the following formulas are drawn.*

$$\begin{aligned}
\frac{1-\exp(-\frac{\rho}{2})}{\rho h(\frac{1}{2})} g\left(\frac{m+n}{2}\right) =& \frac{1-\exp(-\frac{\rho}{2})}{\rho e^{\frac{1}{2}}} e^6 = \frac{\left[1-\exp(-\frac{1-\beta}{\beta})\right]\beta}{2(1-\beta)} e^{\frac{11}{2}},\\
\frac{\beta}{n-m}\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}^-} g(m)\right) =& \frac{1}{2}\Big[\int_3^4 \exp\Big(-\frac{1-\beta}{\beta}(4-u)\Big)e^{2u}du \\
&+ \int_2^3 \exp\Big(-\frac{1-\beta}{\beta}(u-2)\Big)e^{2u}du\Big],\\
=& \frac{\beta}{2}\Big[\frac{e^8 - e^{7-\frac{1}{\beta}}}{\beta+1} - \frac{e^4 - e^{7-\frac{1}{\beta}}}{3\beta-1}\Big],\\
[g(m)+g(n)]\int_0^{\frac{1}{2}} \exp(-\rho t)[h(t)+h(1-t)]dt =& [g(2)+g(4)]\int_0^{\frac{1}{2}} \exp(-\rho t)[e^t + e^{(1-t)}]dt\\
=& (e^4+e^8)\beta\Big[\frac{\exp(\frac{3}{2}-\frac{1}{\beta})-1}{3\beta-2} - \frac{e-\exp(\frac{3}{2}-\frac{1}{\beta})}{\beta-2}\Big].
\end{aligned}$$

We plot the function image of the above three functions for $\beta \in [0,1]$, as shown in Figure 4. From the position relationship of the image, we can see that the middle term of the inequalities is just between the left and right images, and the left image is at the bottom, and the right image is at the top. These show that the inequalities relationship in Theorem 4 for $h(t) = e^t$ is tenable.

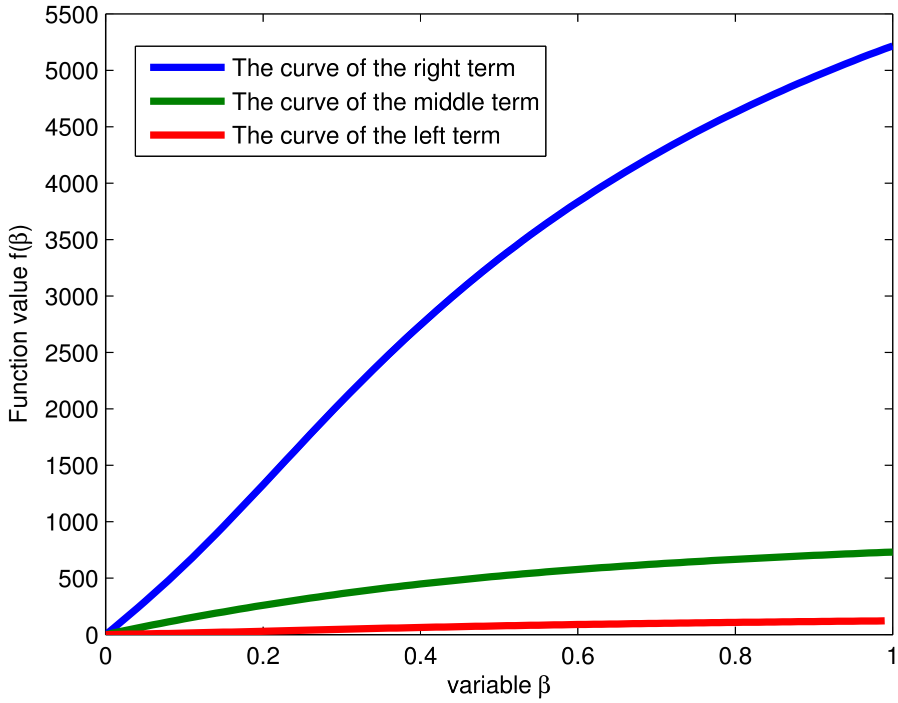

Figure 4. The image description of Theorem 2 for $h(t) = e^t$ and $g(\xi) = e^{2\xi}$.

Specifically, if we choose $\beta = 0.001$, then we have

$$\frac{1-\exp(-\frac{\rho}{2})}{\rho h(\frac{1}{2})} g\Big(\frac{m+n}{2}\Big) = 0.1225,$$

$$\frac{\beta}{n-m}\left(\mathcal{I}^{\beta}_{\frac{m+n}{2}^+} g(n) + \mathcal{I}^{\beta}_{\frac{m+n}{2}^-} g(m)\right) = 1.5164,$$

$$[g(m)+g(n)]\int_0^{\frac{1}{2}} \exp(-\rho t)[h(t)+h(1-t)]dt = [g(2)+g(4)]\int_0^{\frac{1}{2}} \exp(-\rho t)[e^t + e^{(1-t)}]dt = 5.6479.$$

This further verifies that the conclusion of Theorem 4 is correct.

4. Conclusions and Discussion

In this study, using two integral operators with exponential kernel proposed by Ahmad et al. in ref. [24], we establish the new Hermite–Hadamard's integral inequality for h-convex functions. Two midpoint type inequalities are also obtained in which the absolute values of the first derivative and the second derivative of the function are h-convex functions, respectively. The integral operators (3) and (4) involved in the results obtained in this paper are integral operators about the same midpoint of the interval, which is different from the integral operators about the two ends of the interval used in refs [24,25]. For different cases of $h(t) = t$ and $h(t) = e^t$, we construct four numerical examples that intuitively show the size relationship of the function values of the inequalities through the function image, and verify the correctness of the results.

Because fractional integral operators are widely used in the field of engineering technology, such as mathematical models, and different integral operators are suitable for different types of practical problems, our research on the fractional integral operator-type integral inequalities will also expand the practical application scope of Hermite–Hadamard-type inequalities. We know that there are many fractional integral operators involved in other disciplines, which will also inspire us to use other types of integral operators to further study these kinds of inequalities, which also provide a direction for our future research.

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References

1. Jensen, J.L.W.V. Sur les functions convexes et les inegalites entre les valeurs moyennes. *Acta Math.* **1906**, *30*, 175–193. [CrossRef]
2. Hadamard, J. Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
3. Hermite, C.H. Sur deux limites d'une integrale definie. *Mathesis* **1883**, *3*, 82.
4. Kashuri, A.; Iqbal, S.; Liko, R.; Gao, W.; Samraiz, M. Integral inequalities for s-convex functions via generalized conformable fractional integral operators. *Adv. Differ. Equ.* **2020**, *217*, 1–20. [CrossRef]
5. Han, J.; Mohammed, P.O.; Zeng, H. Generalized fractional integral inequalities of Hermite-CHadamard-type for a convex function. *Open Math.* **2020**, *18*, 794–806. [CrossRef]
6. Noor, M.A.; Noor, K.I.; Rashid, S. Some new class of preinvex functions and inequalities. *Mathematics* **2019**, *7*, 29. [CrossRef]
7. Sun, W.B.; Liu, Q. New Hermite-Hadamard type inequalities for (α, m)-convex functions and applications to special means. *J. Math. Inequal* **2017**, *11*, 383–394. [CrossRef]
8. Işcan, I. Hermite-CHadamard type inequalities for harmonically convex functions. *Hacet. J. Math. Stat.* **2014**, *43*, 935–942.
9. Liao, J.G.; Wu, S.H.; Du, T.S. The Sugeno integral with respect to α-preinvex functions. *Fuzzy Sets Syst.* **2020**, *379*, 102–114. [CrossRef]
10. Delavar, M.R.; De La Sen, M. A mapping associated to h-convex version of the Hermite-CHadamard inequality with applications. *J. Math. Inequal.* **2020**, *14*, 329–335. [CrossRef]
11. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
12. Ozdemir, M.E.; Dragomir, S.S.; Yıldız, Ç. The Hadamard inequalities for convex function via fractional integrals. *Acta Math. Sci.* **2013**, *33*, 1293–1299. [CrossRef]
13. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Hamed, Y.S. New Riemann-CLiouville fractional-order inclusions for convex functions via interval-valued settings associated with pseudo-order relations. *Fractal Fract.* **2022**, *6*, 212. [CrossRef]
14. Awan, M.U.; Kashuri, A.; Nisar, K.S.; Javed, M.Z.; Iftikhar, S.; Kumam, P.; Chaipunya, P. New fractional identities, associated novel fractional inequalities with applications to means and error estimations for quadrature formulas. *J. Inequalities Appl.* **2022**, *2022*, 3. [CrossRef]
15. Set, E.; Sarikaya, M.Z.; G̉ozpinar, A. Some Hermite-CHadamard type inequalities for convex functions via conformable fractional integrals and related inequalities. *Creat. Math. Inform.* **2017**, *26*, 221–229. [CrossRef]
16. Du, T.S.; Awan, M.U.; Kashuri, A.; Zhao, S.S. Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m, h)-preinvexity. *Appl. Anal.* **2021**, *100*, 642–662. [CrossRef]
17. Du, T.S.; Wang, H.; Khan, M.A.; Zhang, Y. Certain integral inequalities considering generalized m-convexity on fractal sets and their applications. *Fractals* **2019**, *27*, 1950117. [CrossRef]
18. Sun, W.B. Some new inequalities for generalized h-convex functions involving local fractional integral operators with Mittag-Leffler kernel. *Math. Meth. Appl. Sci.* **2021**, *44*, 4985–4998. [CrossRef]
19. Sun, W.B. Hermite-Hadamard type local fractional integral inequalities for generalized s-preinvex functions and their generalization. *Fractals* **2021**, *29*, 2150098. [CrossRef]
20. Set, E.; Çelik, B.; Ozdemir, M.E.; Aslan, M. Some new results on Hermite-CHadamard-CMercer-type inequalities using a general family of fractional integral operators. *Fractal Fract.* **2021**, *5*, 68. [CrossRef]
21. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Nonlaopon, K. Certain inequalities pertaining to some new generalized fractional integral operators. *Fractal Fract.* **2021**, *5*, 160. [CrossRef]
22. Sun, W.B. Hermite-Hadamard type local fractional integral inequalities with Mittag-Leffler kernel for generalized preinvex functions. *Fractals* **2021**, *29*, 2150253. [CrossRef]

23. Xu, P.; Butt, S.I.; Yousaf, S; Aslam, A; Zia, T.J. Generalized fractal Jensen-CMercer and Hermite-CMercer type inequalities via *h*-convex functions involving Mittag-CLeffler kernel. *Alex. Eng. J.* **2022**, *61*, 4837–4846. [CrossRef]
24. Ahmad, B.; Alsaedi, A.; Kirane, M.; Torebek, B.T. Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. *J. Comput. Appl. Math.* **2019**, *353*, 120–129. [CrossRef]
25. Wu, X.; Wang, J.R.; Zhang, J. Hermite-CHadamard-type inequalities for convex functions via the fractional integrals with exponential kernel. *Mathematics* **2019**, *7*, 845. [CrossRef]
26. Budak, H.; Sarikaya, M.Z.; Usta, F.; Yildirim, H. Some Hermite-CHadamard and Ostrowski type inequalities for fractional integral operators with exponential kernel. *Acta Comment. Univ. Tartu. Math.* **2019**, *23*, 25–36.
27. Zhou, T.C.; Yuan, Z.R.; Du, T.S. On the fractional integral inclusions having exponential kernels for interval-valued convex functions. *Math. Sci.* **2021**. [CrossRef]
28. Du, T.S.; Zhou, T.C. On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings. *Chaos Solitons Fractals* **2022**, *156*, 111846. [CrossRef]
29. Varošanec, S. On *h*-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
30. Kirmaci, U.S. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.* **2004**, *147*, 137–146. [CrossRef]