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Existence of Solutions and Relative Controllability of a Stochastic System with Nonpermutable Matrix Coefficients

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Abstract: In this study, time-delayed stochastic dynamical systems of linear and nonlinear equations are discussed. The existence and uniqueness of the stochastic semilinear time-delay system in finite dimensional space is investigated. Introducing the delay Gramian matrix, we establish some sufficient and necessary conditions for the relative approximate controllability of time-delayed linear stochastic dynamical systems. In addition, by applying the Banach fixed point theorem, we establish some sufficient relative approximate controllability conditions for semilinear time-delayed stochastic differential systems. Finally, concrete examples are given to illustrate the main results.

Keywords: stochastic equation; existence; uniqueness; relative controllability; approximate controllability



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1. Introduction

One of the main problems of the qualitative theory of linear time-delay differential equations is associated with obtaining explicit representation of solutions. In 2003, the authors of [1] introduced a new concept of the delayed matrix exponential function to represent solutions of linear delayed differential equations with permutable linear coefficient matrices. In [2], the results are transferred to linear fractional time-delay differential equations using delayed fractional matrix function. In [3], the concept of delayed matrix exponential function is extended to nonpermutable matrices by introducing the concept of delay perturbation of matrix exponential/Mittag–Leffler function. Two pioneering papers [1,4] led to new results in fractional/nonfractional differential systems.

Stochastic differential equations are progressively used to model mathematical problems in control theory, dynamics of complex systems in engineering, epidemiology, infectious disease, and other areas. Existence and relative exact controllability notions for different deterministic/stochastic semilinear evolution systems in finite dimensional setting have been studied in many publications by using different methods. There are many different notions of controllability for linear deterministic/stochastic time-delayed equations: relative exact controllability, relative exact null controllability, space controllability, and so on. Among the many scientific articles on existence and uniqueness and relative controllability, we will mention only a few that motivate this work.

- (i) The existence and uniqueness of solutions to the stochastic finite dimensional systems has been studied by many authors, see [5–7] and references therein. In particular, existence and uniqueness results of solutions to stochastic differential equations have achieved a great deal of attention. Anh et al. [8] and Taniguchi [9] considered the existence and uniqueness of mild solutions to stochastic partial differential equations under the Lipschitz condition, respectively. Govindan [10] established the existence and uniqueness results for stochastic evolution differential equations with variable delay under the global Lipschitz condition. Ahmadova and Mahmudov [11] investigated the existence and uniqueness of mild solutions to stochastic neutral differential equations.

- (ii) There are several approaches to study approximate controllability of stochastic or deterministic evolution systems. In [12], a resolvent approach have used, introduced in [13] to study approximate controllability for linear evolution equations, and obtained some sufficient conditions for the approximate controllability of deterministic or stochastic semilinear systems. Later, this method was adapted to study the approximate controllability of fractional semilinear evolution systems in [14]. Later, several researchers, i.e., Bora and Roy [15], Dhayal and Malik [16], Kavitha et al. [17], Haq and Sukavanam [18], Aimene [19], Bedi [20], Matar [21], Ge et al. [22], Grudka and Rykaczewski [23], Ke et al. [24], Kumar and Sukavanam [25,26], Liu and Li [27], Sakthivel et al. [28], Wang et al. [29], Yan [30], Yang and Wang [31], Rykaczewski [32], Mahmudov and McKibben [33,34], and Ndambomve and Ezzinbi [35], have used different methods to study approximate controllability for several fractional differential and integro-differential systems.
- (iii) The relative exact controllability notion for first-order linear time-delay deterministic systems with commutative matrices was established in [36], see also [37–39]. Some authors have studied the relative exact controllability for linear/semilinear time-delayed stochastic differential systems, see [40–50] and the references therein. In [45], the authors studied the relative and approximate controllability of the nonlinear stochastic differential systems with delays in control. The paper [46] is concerned with the relative controllability for a class of nonlinear dynamical control systems described by fractional stochastic differential equations with nonlocal conditions. In [47], the relative controllability of a fractional stochastic system with pure delay in finite dimensional stochastic spaces is investigated. In [48], both linear and semilinear stochastic impulsive control systems modeled by finite-dimensional Itô stochastic differential equations with time-varying multiple delays in admissible controls are considered.

However, according to authors' knowledge, the results on the relative approximate controllability of linear/semilinear time-delayed stochastic differential systems in finite/infinite dimensional spaces have not yet been studied. The proposed work on the relatively approximate controllability of finite dimensional linear/semilinear time-delayed stochastic differential systems is new.

We give the notations needed to provide our principal results:

- $(\mathfrak{F}_t)_{t \geq 0}$ is a normal filtration and $(\Omega, \mathfrak{F}, P)$ is a probability space;
- $\mathbb{R}^d$ is a d -dimensional Euclidean space;
- $L^2(\mathfrak{F}_T, \mathbb{R}^d)$ is the Hilbert space of all $\mathfrak{F}_T$ -measurable functions $f : \Omega \rightarrow \mathbb{R}^d$;
- $L^2_{\mathfrak{F}}(0, T; \mathbb{R}^d)$ is the Hilbert space of all square integrable and $\mathfrak{F}_t$ -adapted functions $f : [0, T] \times \Omega \rightarrow \mathbb{R}^d$;
- $C([0, T]; \mathbb{R}^d)$ is the Banach space of all continuous functions $f : [0, T] \rightarrow \mathbb{R}^d$ endowed with the norm $\|x\|_C := \sup\{\|x(t)\|_{\mathbb{R}^d} : t \in [0, T]\}$;
- $\mathfrak{H}_2 \subset L^2(\mathfrak{F}; C([0, T]; \mathbb{R}^d))$ is the closed subspace of measurable and $\mathfrak{F}_t$ -adapted processes

$$\varphi \in L^2(\mathfrak{F}; C([0, T]; \mathbb{R}^d)) \text{ furnished with the norm } \|\varphi\|_{\mathfrak{H}_2} = \left(\mathbf{E} \sup_{0 \leq t \leq T} \|\varphi(t)\|_{\mathbb{R}^d}^2 \right)^{\frac{1}{2}}.$$

We consider the following semilinear stochastic time-delay control system of differential equations:

$$\begin{cases} \zeta'(t) = A\zeta(t) + B\zeta(t-h) + Du(t) + P(t, \zeta(t), \zeta(t-h)) + Q(t, \zeta(t), \zeta(t-h)) \frac{dw(t)}{dt}, \\ \zeta(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \quad (1)$$

where $A, B \in \mathbb{R}^{d \times d}$, $D \in \mathbb{R}^{d \times m}$ are matrices, the control $u \in L^2_{\mathfrak{F}}(0, T; \mathbb{R}^m)$, $(P, Q) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$, $h > 0$ is delay, and $\varphi : [0, T] \rightarrow \mathbb{R}^d$ is a continuously differentiable function. Moreover, $w(t)$ is a one-dimensional Wiener process.

It should be noticed that, say at time $t = T$, the solution $\zeta(T) \in L^2(\mathfrak{F}_T, \mathbb{R}^d)$ and the space $L^2(\mathfrak{F}_T, \mathbb{R}^d)$ is infinite dimensional. Thus, one can consider the relative exact controllability and the relative approximate controllability concepts for the mild solution to the semilinear stochastic time-delay system of differential Equation (1).

The main contribution and advantage of this article can be highlighted as follows:

- (i) The existence and uniqueness of the linear/semilinear time-delayed stochastic differential system in finite dimensional space is investigated;
- (ii) The relative approximate controllability of the of the semilinear time-delay stochastic system in finite dimensional space is studied under the suitable sufficient conditions that for the corresponding linear time-delay deterministic system is relatively exact controllable;
- (iii) The delayed controllability Grammian operator, defined using the delayed perturbation of the matrix exponential function, is used to derive sufficient conditions at stochastic settings to guarantee that the time-delayed semilinear stochastic differential system is relatively approximate controllable;
- (iv) Examples are given to verify the proposed theoretical results.

2. Preliminaries

Recall some definitions and lemmas for more details.

Definition 1. $\mathbb{R}$ -valued process w is called a standard one-dimensional Wiener process over $[0, \infty)$ if it is $\mathfrak{F}_t$ -adapted and for all $0 < s < t$, $w(t) - w(s)$ is independent of $\mathfrak{F}_s$ and is normally distributed with mean 0 and covariance $t - s$. Namely, for any $0 \leq s \leq t$, $P(w(0) = 0) = 1$:

$$\begin{aligned}\mathbb{E}\{w(t) - w(s) \mid \mathfrak{F}_s\} &= 0, \quad P\text{-a.s.}, \\ \mathbb{E}\{(w(t) - w(s))^2 \mid \mathfrak{F}_s\} &= t - s, \quad P\text{-a.s.}\end{aligned}$$

Definition 2 ([17]). Delay matrix exponential function e_h^{Bt} generated by B is defined by:

$$e_h^{Bt} = \begin{cases} \Theta, & -\infty \leq t < -h, \\ I, & -h \leq t < 0, \\ \sum_{j=0}^p B^j \frac{(t - (j-1)h)^j}{j!}, & (p-1)h < t \leq ph. \end{cases} \quad (2)$$

The delayed extension of exponential matrix function, the so called purely delayed exponential matrix function, is defined in [1]. The delayed extension of exponential matrix functions has been defined and studied more recently in [3], where the delayed exponential matrix functions is defined by means of the following determining matrix equation for $Q_k(s)$, $k = 1, 2, \dots$:

$$\begin{aligned}Q_{k+1}(s) &= AQ_k(s) + BQ_k(s-h), \\ Q_0(s) &= Q_k(-h) = \Theta, \quad Q_1(0) = I, \\ k &= 0, 1, 2, \dots, \quad s = 0, h, 2h, \dots\end{aligned} \quad (3)$$

where Θ is a zero matrix and I is an identity matrix.

Definition 3. The delay exponential matrix function $\mathfrak{Y}_h^{A,B}$ [3] generated by A, B is defined as:

$$\mathfrak{Y}_h^{A,B}(t) := \begin{cases} \Theta, & -h \leq t < 0, \\ I, & t = 0, \\ \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{t^i}{i!} + \sum_{i=1}^{\infty} Q_{i+1}(h) \frac{(t-h)^i}{i!} \\ + \cdots + \sum_{i=p}^{\infty} Q_{i+1}(ph) \frac{(t-ph)^i}{i!}, & ph < t \leq (p+1)h. \end{cases}$$

Remark 1. It is clear that $Q_{k+1}(s)$ plays a role of kernel in this definition. If $B = \Theta$ then $Q_{i+1}(0) = A^i$ and $Q_{i+1}(ih) = \Theta$, $i = 1, \dots, p$, $\mathfrak{Y}_h^{A,B}(t)$ becomes exponential matrix function: $\mathfrak{Y}_h^{A,B}(t) = \exp(At)$. If $A = \Theta$ then $Q_{i+1}(ih) = B^i$, $Q_{i+1}(jh) = \Theta$ for $j \neq i+1$ (see [3]) and $\mathfrak{Y}_h^{A,B}(t)$ becomes purely delayed exponential matrix function.

Remark 2. It is clear that for any $t \geq 0$:

$$\begin{aligned} \|\mathfrak{Y}_h^{A,B}(t)\| &\leq \sum_{k=0}^{\infty} \sum_{m=0}^k \|Q_{k+1}(mh)\| \frac{(t-mh)^k}{k!} \\ &\leq \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} \|A\|^{k-m} \|B\|^m \frac{(t-mh)^k}{k!} \\ &\leq \sum_{k=0}^{\infty} (\|A\| + \|B\|)^k \frac{(t-mh)^k}{k!} \leq t^{\beta-1} \exp((\|A\| + \|B\|)t). \end{aligned} \quad (4)$$

Lemma 1 ([5]). Let $\{M_t : t \geq 0\}$ be $\mathbb{R}^d$ -valued martingale. Then:

(i) If $q \geq 1$ and $M_t \in L^q(\Omega; \mathbb{R}^d)$, then:

$$P\left\{\omega : \sup_{0 \leq t \leq T} |M_t(\omega)| \geq c\right\} \leq \frac{1}{c^q} E(|M_T|^q)$$

for all positive $c > 0$.

(ii) If $q > 1$ and $M_t \in L^q(\Omega; \mathbb{R}^d)$, then:

$$E\left(\sup_{0 \leq t \leq T} |M_t(\omega)|^q\right) \leq \left(\frac{q}{q-1}\right)^q E(|M_T|^q).$$

Theorem 1 ([51]). Suppose that $\zeta(\tau)$ and $\nu(\tau)$ are and locally integrable nonnegative functions defined on $[0, T]$, $\mu(\tau) \geq 0$ is a continuous nondecreasing function on $[0, T]$. Assume that:

$$\zeta(\tau) \leq \nu(\tau) + \mu(\tau) \int_0^\tau \zeta(s) ds, \quad \tau \in [0, T],$$

then:

$$\zeta(\tau) \leq \nu(\tau) + \mu(\tau) \int_0^\tau \nu(s) \exp\left(\int_s^\tau \mu(r) dr\right) ds, \quad \tau \in [0, T].$$

Corollary 1 ([51]). If $\nu(r)$ is a nondecreasing function, then under the hypothesis of Theorem 1 we have:

$$\zeta(r) \leq \nu(r) \exp(r\mu(r)) \quad r \in [0, T].$$

3. Existence and Uniqueness Result

We formulate and prove the existence and uniqueness results for solution to the semilinear stochastic delayed differential system (5). To this purpose, we apply the method of Picard iteration. Firstly, we give the definition of a mild solution.

Definition 4. A stochastic process $\zeta : [-h, T] \times \Omega \rightarrow \mathbb{R}^d$ is called a mild solution of (1) if it satisfies the following conditions:

- (i) $\zeta(t) \in \mathfrak{H}_2$;
- (ii) $(P, Q) \in L^2([0, T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \times L^2([0, T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$;
- (iii) For each $t \in [0, T]$, $\zeta(t)$ satisfies the integral equation almost surely:

$$\begin{aligned} \zeta(t) &= \mathfrak{V}(t+h)\varphi(-h) + \int_{-h}^0 \mathfrak{V}(t-s)[\varphi'(s) - A\varphi(s)]ds \\ &+ \int_0^t \mathfrak{V}(t-s)[Du(s) + P(s, \zeta(s), \zeta(s-h))]ds + \int_0^t \mathfrak{V}(t-s)Q(s, \zeta(s), \zeta(s-h))dw(s). \end{aligned} \quad (5)$$

Consider the following two assumptions:

Hypothesis 1 (H1). For all $t \in [0, T]$, $x_1, x_2, z_1, z_2 \in \mathbb{R}^d$, the function $(P, Q) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ satisfies the global Lipschitz condition:

$$\|P(t, x_1, x_2) - P(t, z_1, z_2)\|^2 \leq L\|x_1 - z_1\|^2 + M\|x_2 - z_2\|^2, \quad (6)$$

$$\|Q(t, x_1, x_2) - Q(t, z_1, z_2)\|^2 \leq L\|x_1 - z_1\|^2 + M\|x_2 - z_2\|^2, \quad (7)$$

Hypothesis 2 (H2). The function $(P, Q) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ satisfies the following condition:

$$\|P(t, 0, 0)\| \leq N, \quad \|Q(t, 0, 0)\| \leq N, \quad t \in [0, T].$$

Next, we prove the a priori estimate. It is based on the Picard iterations, Doob's inequality, and the Gronwall inequality.

Theorem 2. Assume that the coefficients P and Q obey the Lipschitz conditions (H1) and the condition (H2). Then, any solution of (1) satisfies:

$$\begin{aligned} &\mathbb{E} \left(\sup_{s \in [-h, T]} \|\zeta(s)\|^2 \right) \leq \mathbb{E} \|\vartheta(T)\|^2 \\ &+ \left\{ (3 + 6N^2(T+4)T) \mathbb{E} \|\vartheta(T)\|^2 + 6N^2(T+4)T \exp(2(\|A\| + \|B\|)T) \right\} \\ &\times \exp(6(T+4)(L+M)T), \end{aligned}$$

where:

$$\vartheta(t) := \mathfrak{V}(t+h)\varphi(-h) + \int_{-h}^0 \mathfrak{V}(t-s)[\varphi'(s) - A\varphi(s)]ds.$$

Proof. We introduce the stopping time $\eta_j := T \wedge \inf\{t \in [0, T] : \|\zeta(t-h)\| \geq j\}$ for each $j \geq 1$. Obviously, $\eta_j \uparrow T$ a.s. as $j \rightarrow \infty$, setting $\zeta^j(t) := \zeta(t \wedge \eta_j)$ for $t \in [0, T]$. Equation (5) can be written as:

$$\begin{aligned} \zeta^j(t) &= \vartheta(t) + \int_0^{t \wedge \eta_j} \mathfrak{V}(t \wedge \eta_j - s)P(s, \zeta(s), \zeta(s-h))ds \\ &+ \int_0^{t \wedge \eta_j} \mathfrak{V}(t \wedge \eta_j - s)Q(s, \zeta(s), \zeta(s-h))dw(s). \end{aligned} \quad (8)$$

Having in mind the Doob's martingale inequality, one can get:

$$\begin{aligned}\mathbb{E}\left(\sup_{s \in [0, t]} \|\zeta^j(s)\|^2\right) &\leq 3\mathbb{E}\|\vartheta(t)\|^2 \\ &\quad + 3T\mathbb{E}\int_0^{t \wedge \eta_j} \|\mathfrak{Y}(t \wedge \eta_j - s)\|^2 \|P(s, \zeta^j(s), \zeta^j(s-h))\|^2 ds \\ &\quad + 12\mathbb{E}\int_0^{t \wedge \eta_j} \|\mathfrak{Y}(t \wedge \eta_j - s)\|^2 \|Q(s, \zeta^j(s), \zeta^j(s-h))\|^2 ds.\end{aligned}$$

From (6) and (7), we can easily derive:

$$\begin{aligned}&\|P(s, \zeta^j(s), \zeta^j(s-h))\|^2 \vee \|Q(s, \zeta^j(s), \zeta^j(s-h))\|^2 \\ &\leq 2L\|\zeta^j(s)\|^2 + 2M\|\zeta^j(s-h)\|^2 + 2N^2.\end{aligned}$$

On the other hand:

$$\|\mathfrak{Y}(t \wedge \eta_j - s)\| \leq \exp((\|A\| + \|B\|)(t \wedge \eta_j - s)).$$

Therefore:

$$\begin{aligned}&\mathbb{E}\left(\sup_{s \in [0, t]} \|\zeta^j(s)\|^2\right) \leq 3\mathbb{E}\|\vartheta(t)\|^2 \\ &\quad + 6T\mathbb{E}\int_0^{t \wedge \eta_j} \exp 2((\|A\| + \|B\|)(t \wedge \eta_j - s)) (L\|\zeta^j(s)\|^2 + M\|\zeta^j(s-h)\|^2 + N^2) ds \\ &\quad + 24\mathbb{E}\int_0^{t \wedge \eta_j} \exp 2((\|A\| + \|B\|)(t \wedge \eta_j - s)) (L\|\zeta^j(s)\|^2 + M\|\zeta^j(s-h)\|^2 + N^2) ds \\ &\quad \leq 3\mathbb{E}\|\vartheta(t)\|^2 + 6N^2(T+4)T \exp 2((\|A\| + \|B\|)T) \\ &\quad + 6(T+4)\mathbb{E}\int_0^{t \wedge \eta_j} \exp 2((\|A\| + \|B\|)(t \wedge \eta_j - s)) (L\|\zeta^j(s)\|^2 + M\|\zeta^j(s-h)\|^2) ds \\ &\quad \leq (3 + 6N^2(T+4)T)\mathbb{E}\|\vartheta(T)\|^2 + 6N^2(T+4)T \exp(2(\|A\| + \|B\|)T) \\ &\quad \quad + 6(T+4)(L+M)\mathbb{E}\int_0^{t \wedge \eta_j} \mathbb{E}\left(\sup_{r \in [0, s]} \|\zeta^j(r)\|^2\right) ds.\end{aligned}$$

The application of Lemma 1 to the above inequality gives:

$$\begin{aligned}&\mathbb{E}\left(\sup_{s \in [0, t]} \|\zeta^j(s)\|^2\right) \\ &\leq \left\{ (3 + 6N^2(T+4)T)\mathbb{E}\|\vartheta(T)\|^2 + 6N^2(T+4)T \exp(2(\|A\| + \|B\|)T) \right\} \\ &\quad \times \exp(6(T+4)(L+M)T).\end{aligned}$$

Since $\zeta^j(s) = \varphi(s)$ for any $-h \leq s \leq 0$, we deduce that:

$$\begin{aligned}&\mathbb{E}\left(\sup_{s \in [-h, t]} \|\zeta^j(s \wedge \eta_j)\|^2\right) = \mathbb{E}\left(\sup_{s \in [-h, t]} \|\zeta^j(s)\|^2\right) \leq \mathbb{E}\|\vartheta(t)\|^2 + \mathbb{E}\left(\sup_{s \in [0, t]} \|\zeta^j(s)\|^2\right) \\ &\leq \mathbb{E}\|\vartheta(t)\|^2 + \left\{ (3 + 6N^2(T+4)T)\mathbb{E}\|\vartheta(T)\|^2 + 6N^2(T+4)T \exp(2(\|A\| + \|B\|)T) \right\} \\ &\quad \times \exp(6(T+4)(L+M)T).\end{aligned}$$

Next we take the limit as $j \rightarrow \infty$ to obtain:

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [-h, T]} \|\zeta(s)\|^2 \right) \leq \mathbb{E} \|\vartheta(t)\|^2 \\ & + \left\{ (3 + 6N^2(T + 4)T) \mathbb{E} \|\vartheta(T)\|^2 + 6N^2(T + 4)T \exp(2(\|A\| + \|B\|)T) \right\} \\ & \times \exp(6(T + 4)(L + M)T). \end{aligned}$$

□

Using the Theorem 2, we prove the first main result of the paper on the uniqueness and existence solution.

Theorem 3. Assuming that $\varphi \in C([-h, 0]; \mathbb{R}^d)$ and P, Q satisfy the Lipschitz condition, then Equation (5) has a unique solution in $\mathfrak{H}_2$.

Proof. Uniqueness: Let $\zeta(t), \widehat{\zeta}(t) \in \mathfrak{H}_2$ represent two mild solutions of the Equation (1). Then:

$$\begin{aligned} \zeta(t) - \widehat{\zeta}(t) &= \int_0^t \mathfrak{V}(t-s) \left[P(s, \zeta(s), \zeta(s-h)) - P(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right] ds \\ &+ \int_0^t \mathfrak{V}(t-s) \left[Q(s, \zeta(s), \zeta(s-h)) - Q(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right] dw(s). \end{aligned}$$

By the Holder's and the Dood's inequalities due to the Lipchitz condition, we have:

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [-h, t]} \|\zeta(s) - \widehat{\zeta}(s)\|^2 \right) \\ & \leq 2T \exp 2((\|A\| + \|B\|)T) \int_0^t \left\| P(s, \zeta(s), \zeta(s-h)) - P(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right\|^2 ds \\ & + 8 \exp 2((\|A\| + \|B\|)T) \int_0^t \left\| Q(s, \zeta(s), \zeta(s-h)) - Q(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right\|^2 ds \\ & \leq (L + M)(2T + 8) \exp^2((\|A\| + \|B\|)T) \int_0^t \mathbb{E} \left(\sup_{r \in [-h, s]} \|\zeta(r) - \widehat{\zeta}(r)\|^2 \right) ds. \end{aligned}$$

It follows from the Gronwall inequality (Corollary 1) that:

$$\mathbb{E} \left(\sup_{s \in [-h, T]} \|\zeta(s) - \widehat{\zeta}(s)\|^2 \right) = 0.$$

Hence, the mild solution for Equation (1) is unique in $\mathfrak{H}_2$.

Existence: Without loss of generality, it is assumed that $T > 0$ is sufficiently small, such that:

$$\gamma := (L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T)T < 1. \quad (9)$$

Let $\zeta^0(t) := \vartheta(0), 0 < t \leq T$. For each natural $j = 1, 2, \dots$, we set $\zeta_0^0 = \vartheta$ and define the following Picard iteration sequence for $0 < t \leq T$:

$$\begin{aligned} \zeta^j(t) &= \vartheta(t) + \int_0^t \mathfrak{V}(t-s) P(s, \zeta^{j-1}(s), \zeta^{j-1}(s-h)) ds \\ &+ \int_0^t \mathfrak{V}(t-s) Q(s, \zeta^{j-1}(s), \zeta^{j-1}(s-h)) dw(s). \end{aligned} \quad (10)$$

By (10), we have:

$$\begin{aligned} \mathbb{E}\|\zeta^j(t)\|^2 &\leq 3\mathbb{E}\|\vartheta(t)\|^2 \\ +6(T+1)\mathbb{E}\int_0^t \exp 2((\|A\| + \|B\|)(t-s)^\alpha) &\left(L\|\zeta^{j-1}(s)\|^2 + M\|\zeta^{j-1}(s-h)\|^2 + N\right)ds \\ &\leq (3+6M(T+4)T)\mathbb{E}\|\vartheta(t)\|^2 + 6N(T+4)T \exp(2(\|A\| + \|B\|)T) \\ &\quad +6(T+1)(L+M)\mathbb{E}\int_0^t \mathbb{E}\left(\|\zeta^{j-1}(s)\|^2\right)ds. \end{aligned}$$

For a natural number $k \geq 1$, we deduce from the above inequality that:

$$\begin{aligned} \max_{1 \leq j \leq k} \mathbb{E}\|\zeta^j(t)\|^2 &\leq (3+6M(T+4)T)\mathbb{E}\|\vartheta(t)\|^2 + 6N(T+4)TE^2((\|A\| + \|B\|)T) \\ &\quad +6(T+1)(L+M)\mathbb{E}\int_0^t \max_{1 \leq j \leq k} \mathbb{E}\left(\|\zeta^{j-1}(s)\|^2\right)ds. \end{aligned}$$

We should notice that:

$$\max_{1 \leq j \leq k} \mathbb{E}\left(\|\zeta^{j-1}(s)\|^2\right) \leq \mathbb{E}\|\vartheta(0)\|^2 + \max_{1 \leq j \leq k} \mathbb{E}\|\zeta^j(t)\|^2.$$

Therefore:

$$\begin{aligned} \max_{1 \leq j \leq k} \mathbb{E}\|\zeta^j(t)\|^2 &\leq (3+6M(T+1)T)\mathbb{E}\|\vartheta(0)\|^2 + 6N(T+4)T \exp^2((\|A\| + \|B\|)T) \\ &\quad +6(T+1)(L+M) \exp(2(\|A\| + \|B\|)T^\alpha) \mathbb{E}\int_0^t \left(\mathbb{E}\|\vartheta(0)\|^2 + \max_{1 \leq j \leq k} \mathbb{E}\left(\|\zeta^j(s)\|^2\right)\right)ds \\ &\leq (3+6(2M+1)(T+1)T)\mathbb{E}\|\vartheta(0)\|^2 + 6N(T+4)T \exp(2(\|A\| + \|B\|)T) \\ &\quad +6(T+1)(L+M) \exp(2(\|A\| + \|B\|)T) \mathbb{E}\int_0^t \max_{1 \leq j \leq k} \mathbb{E}\left(\|\zeta^j(s)\|^2\right)ds. \end{aligned}$$

By the Gronwall Lemma 1 inequality, we get:

$$\max_{1 \leq j \leq k} \mathbb{E}\|\zeta^j(t)\|^2 \leq C_1 \exp(C_2 T),$$

where:

$$\begin{aligned} C_1 &:= (3+6(2M+1)(T+1)T)\mathbb{E}\|\vartheta(0)\|^2 + 6N(T+4)T \exp 2((\|A\| + \|B\|)T), \\ C_2 &:= 6(T+1)(L+M) \exp(2(\|A\| + \|B\|)T). \end{aligned}$$

It follows that:

$$\mathbb{E}\|\zeta^j(t)\|^2 \leq C_1 \exp(C_2 T), \quad j \geq 1, \quad 0 < t \leq T. \quad (11)$$

Next, integral representation (10) implies that:

$$\begin{aligned} \zeta^1(t) - \zeta^0(0) &= \int_0^t \mathfrak{Y}(t-s)P(s, \zeta^0(s), \zeta^0(s-h))ds \\ &\quad + \int_0^t \mathfrak{Y}(t-s)Q(s, \zeta^0(s), \zeta^0(s-h))dw(s). \end{aligned}$$

Similar to the proof of uniqueness, one can have:

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [0, t]} \left\| \zeta^{j+1}(s) - \zeta^j(s) \right\|^2 \right) \\ & \leq (L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T^\alpha) \\ & \times \int_0^t \mathbb{E} \left(\sup_{r \in [0, s]} \left\| \zeta^j(r) - \zeta^{j-1}(r) \right\|^2 \right) ds \\ & \leq (L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T) T \mathbb{E} \left(\sup_{r \in [0, t]} \left\| \zeta^j(r) - \zeta^{j-1}(r) \right\|^2 \right). \end{aligned} \quad (12)$$

Considering mathematical induction by j , one may have:

$$\mathbb{E} \left(\sup_{s \in [0, t]} \left\| \zeta^{j+1}(s) - \zeta^j(s) \right\|^2 \right) \leq \mathbb{E} \left(\sup_{s \in [0, t]} \left\| \zeta^1(s) - \zeta^0(s) \right\|^2 \right) \gamma^j.$$

We assumed that:

$$\gamma = (L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T)T < 1.$$

It follows that there exists: $\zeta \in \mathfrak{H}_2$ such that

$$\lim_{j \rightarrow \infty} \zeta^j(t) = \zeta(t) \quad \text{in } \mathfrak{H}_2.$$

Taking limit in (10), we get:

$$\begin{aligned} \zeta(t) &= \vartheta(t) + \int_0^t \mathfrak{Y}(t-s)P(s, \zeta(s), \zeta(s-h))ds \\ &+ \int_0^t \mathfrak{Y}(t-s)Q(s, \zeta(s), \zeta(s-h))dw(s). \end{aligned}$$

This shows that $\zeta(t)$ is a mild solution of (1).

In order to eliminate the restriction posed by (9), we choose $\tau > 0$ sufficiently small such that:

$$\gamma = (L + M)(2\tau + 8) \exp(2(\|A\| + \|B\|)\tau)\tau < 1.$$

Then, the system (1) has a unique mild solution $[-h, \tau]$. Considering the solution on $[\tau, 2\tau]$ and repeating the same argument as before, we can prove that the system (1) has a unique mild solution on $[-h, T]$. $\square$

4. Relative Approximate Controllability

Consider the related linear time-delay stochastic differential control system of (1):

$$\begin{cases} \zeta'(t) = A\zeta(t) + B\zeta(t-h) + Du(t) + Q(t)\frac{dw(t)}{dt}, \\ \zeta(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \quad (13)$$

We know that the solution has the following representation:

$$\begin{aligned} \zeta(t) &= \mathfrak{Y}(t+h)\varphi(-h) + \int_{-h}^0 \mathfrak{Y}(t-s)[\varphi'(s) - A\varphi(s)]ds \\ &+ \int_0^t \mathfrak{Y}(t-s)Du(s)ds + \int_0^t \mathfrak{Y}(t-s)Q(s)dw(s). \end{aligned}$$

Next, we consider the linear time-delay deterministic control system of (13):

$$\begin{cases} \zeta'(t) = A\zeta(t) + B\zeta(t-h) + Du(t), \\ \zeta(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \quad (14)$$

We define the deterministic and stochastic controllability operators (see [12]):

$$\begin{aligned} (\hat{L})_0^T u &:= \int_0^T \mathfrak{Y}(T-s)Du(s)ds : L^2(0, T; \mathbb{R}^m) \rightarrow \mathbb{R}^d, \\ L_0^T u &:= \int_0^T \mathfrak{Y}(T-s)Du(s)ds : L_{\mathfrak{F}}^2(0, T; \mathbb{R}^m) \rightarrow L^2(\mathfrak{F}_T, \mathbb{R}^d). \end{aligned}$$

Similar to the Gramian matrix, we give the following deterministic delayed Gramian matrix and stochastic Gramian operator:

$$\begin{aligned} W_0^T &:= \int_0^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s)ds : \mathbb{R}^d \rightarrow \mathbb{R}^d, \\ \Gamma_0^T \{\cdot\} &:= L_0^T (L_0^T)^* h = \int_0^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s) \mathbb{E}\{\cdot \mid \mathfrak{F}_s\} ds : L^2(\mathfrak{F}_T, \mathbb{R}^d) \rightarrow L^2(\mathfrak{F}_T, \mathbb{R}^d). \end{aligned}$$

Definition 5. The system (14) is said to be relatively exactly controllable on $[0, T]$ if $\text{Im}(\hat{L})_0^T = \mathbb{R}^d$.

Definition 6. The system (13) is said to be relatively approximate controllable on $[0, T]$ if $\text{Im} L_0^T = L^2(\mathfrak{F}_T, \mathbb{R}^d)$.

Theorem 4. If the deterministic system (14) corresponding to (13) is relatively exact controllable on $[0, T]$, then the system (13) is relatively approximate controllable on $[0, T]$.

Proof. Assume that the deterministic system (14) is relatively exact controllable on $[0, T]$. Then, it is known that the following:

$$W_r^T := \int_r^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s)ds, \quad 0 \leq r < T,$$

is positive. On the other hand, for any $h \in L^2(\mathfrak{F}_T, \mathbb{R}^d)$, there exists a stochastic process $\psi \in L_{\mathfrak{F}}^2(0, T; \mathbb{R}^d)$ such that:

$$\mathbb{E}\{h \mid \mathfrak{F}_t\} = \mathbb{E}h + \int_0^t \psi(s)dw(s).$$

Then, we can write Γ_0^T in terms of matrix W_r^T :

$$\begin{aligned} \Gamma_0^T h &= \int_0^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s) \mathbb{E}\{h \mid \mathfrak{F}_s\} ds \\ &= \int_0^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s) \left(\mathbb{E}h + \int_0^s \psi(r)dw(r) \right) ds \\ &= W_0^T \mathbb{E}h + \int_0^T \int_r^T \mathfrak{Y}(T-s)DD^\top \mathfrak{Y}^\top(T-s) ds \psi(r) dw(r) \\ &= W_0^T \mathbb{E}h + \int_0^T W_r^T \psi(r) dw(r). \end{aligned}$$

It follows that for any nonzero $h \in L^2(\mathfrak{F}_T, \mathbb{R}^d)$

$$\mathbb{E}\langle \Gamma_0^T h, h \rangle = \langle W_0^T \mathbb{E}h, \mathbb{E}h \rangle + \mathbb{E} \int_0^T \langle W_r^T \psi(r), \psi(r) \rangle dr > 0.$$

Thus, $\Gamma_0^T = L_0^T (L_0^T)^* : L^2(\mathfrak{F}_T, \mathbb{R}^d) \rightarrow L^2(\mathfrak{F}_T, \mathbb{R}^d)$ is a positive operator. It follows that:

$$\overline{\text{Im } L_0^T} = L^2(\mathfrak{F}_T, \mathbb{R}^d).$$

This means that the system (13) is relatively approximate controllable on $[0, T]$. $\square$

Next, we investigate the relative approximate controllability for semilinear stochastic time-delay systems. We derive sufficient conditions of relatively approximate controllability for the following semilinear time-delay fractional stochastic system:

$$\begin{cases} \zeta'(t) = A\zeta(t) + B\zeta(t-h) + Du(t) + P(t, \zeta(t), \zeta(t-h)) + Q(t, \zeta(t), \zeta(t-h)) \frac{dw(t)}{dt}, \\ \zeta(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \quad (15)$$

Consider the following relatively exact controllability assumption:

Hypothesis 3 (H3). *The linear deterministic system corresponding to (13) is relatively exact controllable $[0, T]$.*

The mild solution of (15) is expressed in the following form:

$$\begin{aligned} \zeta(t) &= \mathfrak{V}(t+h)\varphi(-h) + \int_{-h}^0 \mathfrak{V}(t-s)[\varphi'(s) - A\varphi(s)]ds \\ &+ \int_0^t \mathfrak{V}(t-s)Du(s)ds + \int_0^t \mathfrak{V}(t-s)P(s, \zeta(s), \zeta(s-h))ds \\ &+ \int_0^t \mathfrak{V}(t-s)Q(s, \zeta(s), \zeta(s-h))dw(s). \end{aligned} \quad (16)$$

We define the control $u(s)$ as follows:

$$\begin{aligned} u(s, \zeta) &:= D^\top \mathfrak{V}^\top(T-s)(\alpha I + \Gamma_0^T)^{-1} \left(\mathbb{E}\zeta_T - \mathfrak{V}(T+h)\varphi(-h) - \int_{-h}^0 \mathfrak{V}(T-s)[\varphi'(s) - A\varphi(s)]ds \right) \\ &+ D^\top \mathfrak{V}^\top(T-s) \int_0^s (\alpha I + \Gamma_r^T)^{-1} \mathfrak{V}(T-r)\psi(r)dw(r) \\ &- D^\top \mathfrak{V}^\top(T-s) \int_0^s (\alpha I + \Gamma_r^T)^{-1} \mathfrak{V}(T-r)P(r, \zeta(r), \zeta(r-h))dr \\ &- D^\top \mathfrak{V}^\top(T-s) \int_0^s (\alpha I + \Gamma_r^T)^{-1} \mathfrak{V}(T-r)Q(r, \zeta(r), \zeta(r-h))dw(r), \end{aligned} \quad (17)$$

where $\psi \in L_{\mathfrak{F}}^2(0, T; \mathbb{R}^d)$ from the representation:

$$\zeta_T = \mathbb{E}\zeta_T + \int_0^T \psi(r)dw(r)$$

of $\zeta_T \in L^2(\mathfrak{F}_T, \mathbb{R}^d)$, see [12].

Next we define the nonlinear operator $\mathfrak{R} : \mathfrak{H}_2 \rightarrow \mathfrak{H}_2$:

$$\begin{aligned} (\mathfrak{R}\zeta)(t) &:= \mathfrak{Y}(t+h)\varphi(-h) + \int_{-h}^0 \mathfrak{Y}(t-s)[\varphi'(s) - A\varphi(s)]ds \\ &+ \int_0^t \mathfrak{Y}(t-s)Du(s, \zeta)ds + \int_0^t \mathfrak{Y}(t-s)P(s, \zeta(s), \zeta(s-h))ds \\ &+ \int_0^t \mathfrak{Y}(t-s)Q(s, \zeta(s), \zeta(s-h))dw(s). \end{aligned}$$

Thus, the relative approximate controllability problem has transformed into the existence of a fixed point for $\mathfrak{R}$.

Lemma 2. *There exists a positive constant $L_u > 0$ such that for all $\zeta, \widehat{\zeta} \in \mathfrak{H}_2$ the control function u satisfies the following inequalities:*

$$\begin{aligned} \mathbb{E} \left(\sup_{s \in [-h, t]} \|u(s, \zeta) - u(s, \widehat{\zeta})\|^2 \right) &\leq \frac{1}{\alpha^2} L_u \int_0^t \mathbb{E} \left(\sup_{r \in [-h, s]} \|\zeta(r) - \widehat{\zeta}(r)\|^2 \right) ds, \\ \mathbb{E} \sup_{s \in [-h, t]} \|u(s, \zeta)\|^2 &\leq \frac{1}{\alpha^2} L_u \left(1 + \int_0^s \mathbb{E} \sup_{r \in [-h, s]} \|\zeta(r)\|^2 dr \right). \end{aligned}$$

Proof. We will prove the first inequality, since the proof of the second inequality is similar.

$$\begin{aligned} &\mathbb{E} \left(\sup_{s \in [-h, t]} \|u(s, \zeta) - u(s, \widehat{\zeta})\|^2 \right) \\ &\leq 2\mathbb{E} \sup_{s \in [-h, t]} \left\| D^\top \mathfrak{Y}^\top(T-s) \int_0^s (\alpha I + \Gamma_r^\top)^{-1} \mathfrak{Y}(T-r) \right. \\ &\quad \times \left(P(s, \zeta(s), \zeta(s-h)) - P(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right) ds \left. \right\|^2 \\ &\quad + 2\mathbb{E} \sup_{s \in [-h, t]} \left\| D^\top \mathfrak{Y}^\top(T-s) \int_0^s (\alpha I + \Gamma_r^\top)^{-1} \mathfrak{Y}(T-r) \right. \\ &\quad \times \left(Q(s, \zeta(s), \zeta(s-h)) - Q(s, \widehat{\zeta}(s), \widehat{\zeta}(s-h)) \right) dw(s) \left. \right\|^2 \\ &\leq \frac{1}{\alpha^2} (L + M)(2T + 8) \|D^\top\| \exp(2(\|A\| + \|B\|)T) \int_0^t \mathbb{E} \left(\sup_{r \in [-h, s]} \|\zeta(r) - \widehat{\zeta}(r)\|^2 \right) ds. \end{aligned}$$

□

Lemma 3. *Under assumptions (H1)–(H3), the operator $\mathfrak{R}$ maps $\mathfrak{H}_2$ into itself.*

Proof. The proof goes in similar lines to that of Theorem 2. In addition, we need to use Lemma 2. □

Lemma 4. *Under assumptions (H1)–(H3), the operator $\mathfrak{R}^n$, $n \geq 1$, is contractive on $\mathfrak{H}_2$ and $\mathfrak{R}$ has a unique fixed point in $\mathfrak{H}_2$.*

Proof. We prove that for any fixed $\alpha > 0$, the operator $\mathfrak{R}$ has a unique fixed point in $\mathfrak{H}_2$. To do so, we use the classical Banach fixed point theorem. By Lemma 3, $\mathfrak{R}$ sends $\mathfrak{H}_2$ into

$\mathfrak{H}_2$. We prove that there exists $n > 0$ such that $\mathfrak{R}^n$ is a contraction. Take $\zeta, \widehat{\zeta} \in \mathfrak{H}_2$. Using Lemma 2 and the estimations used in the proof of inequality (12), one can easily get:

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [0, t]} \left\| (\mathfrak{R}\zeta)(s) - (\mathfrak{R}\widehat{\zeta})(s) \right\|^2 \right) \\ & \leq 2(L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T) \int_0^t \mathbb{E} \left(\sup_{r \in [0, s]} \left\| \zeta(r) - \widehat{\zeta}(r) \right\|^2 \right) ds \\ & \quad + 2\mathbb{E} \sup_{s \in [0, t]} \left\| \int_0^s \mathfrak{Y}(s-r) D \left(u(r, \zeta) - u(r, \widehat{\zeta}) \right) dr \right\|^2 \\ & \leq 2(L + M)(2T + 8) \exp(2(\|A\| + \|B\|)T) T \int_0^t \mathbb{E} \left(\sup_{r \in [0, s]} \left\| \zeta(r) - \widehat{\zeta}(r) \right\|^2 \right) dr \\ & \quad + 2 \exp(2(\|A\| + \|B\|)T) \|D\|^2 \frac{1}{\alpha^2} L_u \int_0^t \mathbb{E} \left(\sup_{r \in [-h, s]} \left\| \zeta(r) - \widehat{\zeta}(r) \right\|^2 \right) ds. \end{aligned}$$

Hence, there exists a positive number $k(\alpha) > 0$ such that:

$$\mathbb{E} \left(\sup_{s \in [-h, t]} \left\| (\mathfrak{R}\zeta)(s) - (\mathfrak{R}\widehat{\zeta})(s) \right\|^2 \right) \leq k(\alpha) \int_0^t \mathbb{E} \left(\sup_{r \in [-h, s]} \left\| \zeta(r) - \widehat{\zeta}(r) \right\|^2 \right) ds. \quad (18)$$

For any natural number $n \geq 1$, by iteration, it follows from (18) that:

$$\mathbb{E} \left(\sup_{s \in [-h, t]} \left\| (\mathfrak{R}^n \zeta)(s) - (\mathfrak{R}^n \widehat{\zeta})(s) \right\|^2 \right) \leq \frac{(Tk(\alpha))^n}{n!} \mathbb{E} \left(\sup_{r \in [-h, t]} \left\| \zeta(r) - \widehat{\zeta}(r) \right\|^2 \right).$$

Since for sufficiently large n , $\frac{(Tk(\alpha))^n}{n!}$, $\mathfrak{R}^n$ is a contraction map on $\mathfrak{H}_2$, and therefore, $\mathfrak{R}$ itself has a unique fixed point in $\mathfrak{H}_2$. The theorem is proved. $\square$

Theorem 5. Assume that the functions $(P, Q) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{E}^d \times \mathbb{R}^d$ are uniformly bounded. Under assumptions (H1) and (H3), the mild solution of the system (16) is relatively approximate controllable on $[0, T]$.

Proof. Let ζ^α be a fixed point of $\mathfrak{R}$ in $\mathfrak{H}_2$ and $\zeta_T \in \mathbb{R}^d$. Using the Fubini theorem, one can show that any fixed point of $\mathfrak{R}$ satisfies:

$$\begin{aligned} \zeta^\alpha(T) &= \zeta_T - \alpha(\alpha I + \Gamma_0^T)^{-1} \left(\mathbb{E} \zeta_T - \mathfrak{Y}(T+h) \varphi(-h) - \int_{-h}^0 \mathfrak{Y}(T-s) [\varphi'(s) - A\varphi(s)] ds \right) \\ &\quad - \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1} \psi(r) dw(r) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1} \mathfrak{Y}(T-r) P(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) dr \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1} \mathfrak{Y}(T-r) Q(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) dw(r). \end{aligned} \quad (19)$$

By our assumption, P and Q are uniformly bounded on $[0, T] \times \Omega$, that is, there exists $L > 0$ such that:

$$\|P(r, \zeta^\alpha(r), \zeta^\alpha(r-h))\|^2 + \|Q(r, \zeta^\alpha(r), \zeta^\alpha(r-h))\|^2 \leq L.$$

Then, there is a subsequence, still denoted by $\{P(r, \zeta^\alpha(r), \zeta^\alpha(r-h)), Q(r, \zeta^\alpha(r), \zeta^\alpha(r-h))\}$, weakly converging to, say, $\{P(r), Q(r)\}$.

Thus, by the Lebesgue dominated convergence theorem, we have:

$$\begin{aligned}\mathbb{E} \int_0^T \|\mathfrak{Y}(T-r)(P(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) - P(r))\|^2 ds &\rightarrow 0, \\ \mathbb{E} \int_0^T \|\mathfrak{Y}(T-r)(Q(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) - Q(r))\|^2 ds &\rightarrow 0,\end{aligned}$$

as $\alpha \rightarrow 0^+$.

Then, having in mind $\|\alpha(\alpha I + \Gamma_r^T)^{-1}\| \leq 1$ and $\|\alpha(\alpha I + \Gamma_r^T)^{-1}\| \rightarrow 0$ for all $0 \leq r < T$, from (19) we obtain:

$$\begin{aligned}&\mathbb{E} \|\zeta^\alpha(T) - \zeta_T\|^2 \\&\leq 4 \left\| \alpha(\alpha I + \Gamma_0^T)^{-1} \left(\mathbb{E} \zeta_T - \mathfrak{Y}(T+h)\varphi(-h) - \int_{-h}^0 \mathfrak{Y}(T-s)[\varphi'(s) - A\varphi(s)]ds \right) \right\|^2 \\&+ 4 \int_0^T \left\| \alpha(\alpha I + \Gamma_r^T)^{-1} \psi(r) \right\|^2 dr \\&+ \int_0^T \left\| \alpha(\alpha I + \Gamma_r^T)^{-1} \right\|^2 \|\mathfrak{Y}(T-r)(P(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) - P(r)) + \mathfrak{Y}(T-r)P(r)\|^2 dr \\&+ \int_0^T \left\| \alpha(\alpha I + \Gamma_r^T)^{-1} \right\|^2 \|\mathfrak{Y}(T-r)(Q(r, \zeta^\alpha(r), \zeta^\alpha(r-h)) - Q(r)) + \mathfrak{Y}(T-r)Q(r)\|^2 dr \\&\rightarrow 0,\end{aligned}$$

as $\alpha \rightarrow 0^+$. This gives the relative approximate controllability. The theorem is proved. $\square$

5. Example

Example 1. We consider linear stochastic delayed dynamic differential control system of the form (13):

$$\begin{cases} \zeta'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \zeta(t) + \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \zeta(t-h) \\ \quad + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \sigma(t, \zeta(t)) \frac{dw(t)}{dt}, \\ \zeta(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \quad (20)$$

defined in a given time interval $[0, T]$, $T > 1$, with one constant point delay $h = 1$, with an arbitrary 3 dimensional vector σ . Hence, $d = 3$, $m = 2$ and:

$$Q(T) := \{Q_0(t), Q_1(t), Q_2(t) : t \in [0, T]\}.$$

Moreover, using the notation given in [41], we have:

$$\begin{aligned}Q(T) &= [Q_1(0) \ Q_2(0) \ Q_2(h) \ Q_3(0) \ Q_3(h) \ Q_3(2h)] \\ &= [D \ AD \ BD \ A^2D \ (AB + BA)D \ B^2D].\end{aligned}$$

Substituting the matrices A , B , and D given above, we easily obtain:

$$\text{rank} Q(T) = \text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 4 & 2 & 1 \end{bmatrix} = 3.$$

Hence, by [52] the linear deterministic system associated with (20) is relatively exactly controllable in each time interval $[0, T]$ for $T > 1$. Thus, by Theorem 3, the system (20) is relatively approximate controllable, provided that $\sigma(t, \zeta)$ satisfies the Lipschitz condition and uniformly bounded.

Example 2. Now, we present a stochastic model of population dynamics with delayed birthrates and delayed logistic terms given by the system. The deterministic analogue of the following system used in [53] was used to model the population growth:

$$\begin{aligned} \zeta'(t) = & \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \zeta_1(t-h) \\ \zeta_2(t-h) \end{bmatrix} \\ & + \begin{bmatrix} P_1(t, \zeta_1(t), \zeta_2(t-h)) \\ P_2(t, \zeta_2(t), \zeta_1(t-h)) \end{bmatrix} + \begin{bmatrix} \psi_1(t) \frac{dw(t)}{dt} \\ \psi_2(t) \frac{dw(t)}{dt} \end{bmatrix}, \quad 0 \leq t \leq T. \end{aligned} \quad (21)$$

Then, (21) can be turned into:

$$\zeta'(t) = A\zeta(t) + B\zeta(t-h) + P(t, \zeta(t), \zeta(t-h)) + Q(t) \frac{dw(t)}{dt}.$$

The deterministic linear system corresponding to (21) is controllable, see [53]. Thus, for any Lipschitz continuous and uniformly bounded P , by Theorem 5, the system (21) is relatively approximate controllable on $[0, T]$.

Example 3. Let $h = 0.2$, $T = 0.8$, and $d = 2$. Consider the relative approximate controllability of the following linear time-delay differential controlled system:

$$\begin{cases} y'(t) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} y(t-0.2) + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} u(t) + \sigma(t)dw(t), & 0 \leq t \leq 0.8 \\ y(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}, & -0.2 \leq t \leq 0. \end{cases} \quad (22)$$

In this case, $\mathfrak{Y}_h^{A,B}(t) = e_h^{B(t-h)}$, which is defined in Definition 2. The delayed Grammian matrix of system (22) has the following explicit form:

$$\begin{aligned} W_0^{0.8} &= \int_0^{0.8} \mathfrak{Y}(0.8-s) D D^T \mathfrak{Y}^T(0.8-s) ds = \int_0^{0.8} e_h^{B(0.6-s)} D D^T e_h^{B^T(0.6-s)} ds \\ &= \int_0^{0.8} e_h^{B(0.6-s)} D^2 e_h^{B^T(0.6-s)} ds \\ &= W_1 + W_2 + W_3 + W_4. \end{aligned}$$

Here:

$$\begin{aligned} W_1 &= \int_0^{0.2} \left(I + B \frac{(0.6-s)}{1!} + B^2 \frac{(0.4-s)^2}{2!} + B^3 \frac{(0.2-s)^3}{3!} \right) \\ &\quad \times D^2 \left(I + B^T \frac{(0.6-s)}{1!} + (B^T)^2 \frac{(0.4-s)^2}{2!} + (B^T)^3 \frac{(0.2-s)^3}{3!} \right) ds, \\ W_2 &= \int_{0.2}^{0.4} \left(I + B \frac{(0.6-s)}{1!} + B^2 \frac{(0.4-s)^2}{2!} \right) \\ &\quad \times D^2 \left(I + B^T \frac{(0.6-s)}{1!} + (B^T)^2 \frac{(0.4-s)^2}{2!} \right) ds, \\ W_3 &= \int_{0.4}^{0.6} \left(I + B \frac{(0.6-s)}{1!} \right) D^2 \left(I + B^T \frac{(0.6-s)}{1!} \right) ds, \\ W_4 &= \int_{0.6}^{0.8} D^2 ds. \end{aligned}$$

By elementary computation, one can get:

$$\begin{aligned} W_1 &= \begin{pmatrix} 0.1923 & 9.2648 \times 10^{-2} \\ 9.2648 \times 10^{-2} & 6.9945 \times 10^{-2} \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.10617 & 4.1383 \times 10^{-2} \\ 4.1383 \times 10^{-2} & 8.5571 \times 10^{-2} \end{pmatrix} \\ W_3 &= \begin{pmatrix} 6.3333 \times 10^{-2} & 1.1333 \times 10^{-2} \\ 1.1333 \times 10^{-2} & 6.0667 \times 10^{-2} \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}. \end{aligned}$$

Therefore, we obtain:

$$W = \begin{pmatrix} 0.4118 & 0.14536 \\ 0.14536 & 0.26618 \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} 3.0082 & -1.6428 \\ -1.6428 & 4.6539 \end{pmatrix}.$$

This means that the linear deterministic delay system corresponding to (22) is relatively exact controllable on $[0, 0.8]$. By Theorem 4 the system (22) is relatively approximate controllable on $[0, 0.8]$.

6. Conclusions

The purpose of this contribution is to discuss the existence uniqueness and relative approximate controllability of time-delayed stochastic dynamical systems of linear and nonlinear equations with linear parts defined by noncommutative matrix coefficients. Introducing the delayed Gramian matrix, we establish some sufficient and necessary conditions for the relative approximate controllability of linear time-delayed stochastic dynamical systems. In addition, by applying the generalized Banach fixed point theorem, we establish sufficient the relative approximate controllability conditions for semilinear time-delayed stochastic differential systems.

One possible direction in which to extend the results of this paper is toward fractional time-delay and conformable fractional time-delay differential systems, as well as fractional time-delay stochastic differential systems. Another challenge is to find out if similar results can be derived in the infinite dimensional cases for various type of time-delay fractional systems.

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