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Article

Dirichlet Averages of Generalized Mittag-Leffler Type Function

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Abstract: Since Gösta Magus Mittag-Leffler introduced the so-called Mittag-Leffler function in 1903 and studied its features in five subsequent notes, passing the first half of the 20th century during which the majority of scientists remained almost unaware of the function, the Mittag-Leffler function and its various extensions (referred to as Mittag-Leffler type functions) have been researched and applied to a wide range of problems in physics, biology, chemistry, and engineering. In the context of fractional calculus, Mittag-Leffler type functions have been widely studied. Since Carlson established the notion of Dirichlet average and its different variations, these averages have been explored and used in a variety of fields. This paper aims to investigate the Dirichlet and modified Dirichlet averages of the R-function (an extended Mittag-Leffler type function), which are provided in terms of Riemann-Liouville integrals and hypergeometric functions of several variables. Principal findings in this article are (possibly) applicable. This article concludes by addressing an open problem.

Keywords: Dirichlet averages; B-splines; dirichlet splines; Riemann–Liouville fractional integrals; hypergeometric functions of one and several variables; generalized Mittag-Leffler type function; Srivastava–Daoust generalized Lauricella hypergeometric function

MSC: 26A33; 33C20; 33E12; 33E20

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1. Introduction and Preliminaries

The Mittag-Leffler function $E_\alpha(z)$ (see [1])

$$E_\alpha(z) = \sum_{\ell=0}^{\infty} \frac{z^\ell}{\Gamma(\alpha\ell+1)} \quad (\Re(\alpha) > 0), \tag{1}$$

Γ being the familiar Gamma function (see, for example, Section 1.1 in [2]), is named after the eminent Swedish mathematician Gösta Magus Mittag-Leffler (1846–1927), who explored its features in 1902–1905 in five notes (consult, for instance, [1]) related to his summation technique for divergent series (see also Chapter 1, [3]). Because $\Gamma(\ell+1) = \ell!$ $(\ell \in \mathbb{N}_0)$ and therefore $E_1(z) = e^z$, this function gives a straightforward extension of the exponential function. Here and elsewhere, let $\mathbb{N}$, $\mathbb{Z}_0^-$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ be the sets of positive integers, non-positive integers, real numbers, positive real numbers, and complex numbers, respectively, and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Passing the first half of the 20th century during which the majority of scientists remained almost unaware of the function, the Mittag-Leffler function and its various extensions (referred to as Mittag-Leffler type functions) have been studied and applied to a wide range of problems in physics, biology, chemistry, engineering, etc. This function's most significant features are described in Chapter XVIII [4], which is dedicated to so-called miscellaneous functions. The Mittag-Leffler function was categorized as miscellaneous because it was not until the 1960s that it was discovered as belonging to a broader class of higher transcendental functions known as Fox H-functions, thus the term "miscellaneous" (consult, for instance, [5]). In reality, this class was not well-established until Fox's landmark study (see [6]). The simplest (and most crucial for applications) extension of the Mittag-Leffler function, notably the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\Gamma(\alpha\ell+\beta)} \quad (\alpha,\, \beta \in \mathbb{C},\, \Re(\alpha) > 0) \tag{2}$$

was separately studied by Humbert and Agarwal in 1953 (see, for example, [7]) and by Dzherbashyan in 1954 (see, for example, [8]). However, it first appeared formally in Wiman's article [9]. Prabhakar [10] introduced the following three-parametric Mittag-Leffler function:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{\ell=0}^{\infty} \frac{(\gamma)_{\ell}}{\ell!\,\Gamma(\alpha\ell+\beta)} z^{\ell} \quad (\alpha,\, \beta,\, \gamma \in \mathbb{C},\, \Re(\alpha) > 0,\, \Re(\gamma) > 0), \tag{3}$$

where $(\lambda)_{\nu}$ denotes the Pochhammer symbol defined (for $\lambda,\, \nu \in \mathbb{C}$) by

$$(\lambda)_{\nu} := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\; \lambda \in \mathbb{C}), \end{cases} \tag{4}$$

it being accepted conventionally that $(0)_0 := 1$. This Function (3) is being used for a variety of applicable issues. Scientists, engineers, and statisticians recognize the significance of the aforementioned H-function due to its great potential for applications in several scientific and technical domains. In addition to the Mittag-Leffler Functions (1)–(3), the H-function includes a variety of functions (see, for example, [5]). Among several monographs on the H-function, monograph [5] discusses the theory of the H-function with a focus on its applications. The H-function (or Fox's H-function [6]) is defined by means of a Mellin–Barnes type integral in the following manner (consult also [5]):

$$\begin{aligned} H(z) = H^{m,n}_{p,q}(z) &= H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right] \\ &= H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right] = \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \Omega(s)\, z^{-s}\, ds, \end{aligned} \tag{5}$$

where $\omega = \sqrt{-1}$, and

$$\Omega(s) := \frac{\prod_{j=1}^{m} \Gamma\big(b_j + \beta_j s\big) \cdot \prod_{j=1}^{n} \Gamma\big(1 - a_j - \alpha_j s\big)}{\prod_{j=m+1}^{q} \Gamma\big(1 - b_j - \beta_j s\big) \cdot \prod_{j=n+1}^{p} \Gamma\big(a_j + \alpha_j s\big)}. \tag{6}$$

We also assume the following: $z^{-s} = \exp[-s\{\ln|z| + i \arg z\}]$, where $\ln|z|$ is the natural logarithm, and $\eta < \arg z < \eta + 2\pi$ for some $\eta \in \mathbb{R}$. The integration path $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty} (\gamma \in \mathbb{R})$ extends from $\gamma - i\infty$ to $\gamma + i\infty$ with indentations, if necessary, so that the poles of $\Gamma(1 - a_j - \alpha_j s)$ $(1 \le j \le n \in \mathbb{N}_0)$ can be separated from those of $\Gamma(b_j + \beta_j s)$ $(1 \le j \le m \in \mathbb{N}_0)$ and has no those poles on it. The parameters $p,\, q \in \mathbb{N}_0$ satisfy the conditions $0 \le n \le p,\; 0 \le m \le q$; the parameters $\alpha_j,\, \beta_j \in \mathbb{R}^+$ and $a_j,\, b_j \in \mathbb{C}$. The empty product in (6) (and elsewhere) is (as usual) understood to be unity.

For the existence conditions of the H-function, one may refer to Appendix F.4 [3], Section 1.2 [5]. Here it is recalled that the three-parametric Mittag-Leffler function (Prabhakar function) (3) is represented by the following Mellin–Barnes integral (see p.10, Example 1.5 in [5]):

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{2\pi\omega\,\Gamma(\gamma)} \int_{\xi-\omega\infty}^{\xi+\omega\infty} \frac{\Gamma(s)\,\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}\,(-z)^{-s}\,ds \tag{7}$$

$$\left(|\arg z| < 2\pi,\ \xi \in \mathbb{R}\ (\text{fixed}),\ \alpha \in \mathbb{R}^+,\ \Re(\beta) > 0,\ \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

We find from (5) and (7) that

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)}\, H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{array}{l} (1-\gamma,1) \\ (0,1),\ (1-\beta,\alpha) \end{array}\right]. \tag{8}$$

Using (8) in the relation $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$, we get (consult, for example, p.9, Equation (1.50) in [5])

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{array}{l} (0,1) \\ (0,1),\ (1-\beta,\alpha) \end{array}\right]. \tag{9}$$

Indeed, the Mittag-Leffler type functions in association with the fractional calculus have been actively researched (see, for example, [11,12]).

Carlson developed the notion of the Dirichlet average in his work [13] (see also [14–18]). Carlson also provided a full and thorough analysis of the numerous varieties of Dirichlet averages. A function's so-called Dirichlet average is the integral mean of the function with regard to the Dirichlet measure. Subsequently and more recently, this study topic has been explored in publications such as [19–28]. Neuman and Van Fleet [19] defined Dirichlet averages of multivariate functions and demonstrated their recurrence formula. Daiya and Kumar [20] researched the double Dirichlet averages of S-functions. Saxena and Daiya [29] proposed and explored the S-functions. Kilbas and Kattuveettill [22] investigated Dirichlet averages of the three-parametric Mittag-Leffler Function (3), whose representations are provided in terms of the Riemann–Liouville fractional integrals and the hypergeometric functions with multiple variables. Saxena et al. [25] explored Dirichlet averages of the generalized multi-index Mittag-Leffler functions (see, for instance, [30]), whose representations are expressed in terms of Riemann–Liouville integrals and hypergeometric functions of several variables. Using Riemann–Liouville fractional integral operators, Vyas [31] investigated the solution of the Euler–Darboux equation in terms of Dirichlet averages of boundary conditions on Hölder space and weighted Hölder spaces of continuous functions. For further Dirichlet averages in connection with fractional calculus, one may consult [21,24,32–36]. These Dirichlet averages were used in a number of studies, in particular, Dirichlet splines (see [19]), B-splines (see [18,23]), and Stolarsky means (see [37]).

In this work, we propose to investigate the Dirichlet and modified Dirichlet averages of the R-function (an extended Mittag-Leffler type function) (see, for details, Section 2). Main results stated in this paper, which are presented in terms of Riemann–Liouville integrals and hypergeometric functions of several variables, are (potentially) useful.

Let Ω be a convex set in $\mathbb{C}$ and $z := (z_1, \ldots, z_n) \in \Omega^n$ $(n \in \mathbb{N} \setminus \{1\})$. Suppose that f is a measurable function on Ω. Then the general Dirichlet average of the function f is defined as follows (see [15]):

$$F(b;z) = \int_{E_{n-1}} f(u \circ z)\, d\mu_b(u), \tag{10}$$

where b and u denote the arrays of n parameters $b_1, \ldots, b_n$ and $u_1, \ldots, u_n$, respectively, and $d\mu_b(u)$ is the Dirichlet measure defined by

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \cdots u_{n-1}^{b_{n-1}-1} (1 - u_1 - \cdots - u_{n-1})^{b_n - 1} du_1 \cdots du_{n-1}, \tag{11}$$

and E_{n-1} is the Euclidean simplex in $\mathbb{R}^{n-1}$ $(n \in \mathbb{N} \setminus \{1, 2\})$ given by

$$E_{n-1} = \left\{(u_1, \ldots, u_{n-1}) : u_j \geq 0\ (j \in \overline{1, n-1}),\ u_1 + \cdots + u_{n-1} \leq 1\right\}, \tag{12}$$

and $B(b)$ is the multivariate Beta-function defined by

$$B(b) := \frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(b_1+\cdots+b_n)} \quad (\Re(b_j) > 0 \ (j \in \overline{1,n})),$$

and

$$u \circ z := \sum_{j=1}^{n-1} u_j z_j + (1-u_1-\cdots-u_{n-1})z_n.$$

Here and throughout this paper, the notation $\overline{1,p} := \{1, \ldots, p\}$ $(p \in \mathbb{N})$ is used. The special case of (11) when $n = 2$ reduces to the following form:

$$d\mu_{\beta,\beta'}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1}\, du. \tag{13}$$

Carlson [15] investigated the average (10) for the function $f(z) = z^k$ $(k \in \mathbb{R})$ in the following form:

$$R_k(b;z) = \int_{E_{n-1}} (u \circ z)^k \, d\mu_b, \tag{14}$$

whose special case $n = 2$ was given as follows (see [13,15]):

$$R_k(\beta,\beta';x,y) = \frac{1}{B(\beta,\beta')} \int_0^1 [ux+(1-u)y]^k\, u^{\beta-1}(1-u)^{\beta'-1} du, \tag{15}$$

where β, $\beta' \in \mathbb{C}$ with $\min\{\Re(\beta), \Re(\beta')\} > 0$, and x, $y \in \mathbb{R}$, $B(\beta,\beta')$ is the familiar Beta function (consult, for instance, Chapter 1, [2]).

The Riemann–Liouville fractional integral of a function f is defined as follows (consult, for instance, (p. 69) [38]): For $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $a \in \mathbb{R}$,

$$(I^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \quad (x > a). \tag{16}$$

The Srivastava–Daoust generalization $F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}$ of the Lauricella hypergeometric function F_D in n variables is defined by (see (p. 454) [39]; see also (p. 37) [40], (p. 209) [5])

$$\begin{aligned} &F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}\left(\begin{matrix} \left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right]: & \left[\left(b^{(1)}\right):\varphi^{(1)}\right];\ldots;\left[\left(b^{(n)}\right):\varphi^{(n)}\right]; \\ \left[(c):\psi^{(1)},\ldots,\psi^{(n)}\right]: & \left[\left(d^{(1)}\right):\delta^{(1)}\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]; \end{matrix}\ x_1,\ldots,x_n \right) \\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{m_1\varphi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{m_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}} \\ &\quad \times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \end{aligned} \tag{17}$$

where the coefficients, for all $k \in \overline{1,n}$,

$$\theta_j^{(k)} \ (j \in \overline{1,A});\ \varphi_j^{(k)} \ \left(j \in \overline{1,B^{(k)}}\right);\ \psi_j^{(k)} \ (j \in \overline{1,C});\ \delta_j^{(k)} \ \left(j \in \overline{1,D^{(k)}}\right)$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$ $\left(j \in \overline{1,B^{(k)}}\right)$ for all $k \in \overline{1,n}$, with similar interpretations for (c) and $\left(d^{(k)}\right)$ $(k \in \overline{1,n})$; et cetera.

One may refer to Srivastava and Daoust [41] for the specific convergence requirements of the multiple series (17).

2. The Generalized Mittag-Leffler Type Function (the *R*-Function)

The *R*-function, which Kumar and Kumar [42] proposed and Kumar and Purohit [43] studied, is defined as follows:

$$ {}_p^{\kappa}R_q^{\alpha,\beta;\gamma}(z) = {}_p^{\kappa}R_q^{\alpha,\beta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{18} $$
$$ (\alpha,\ \beta,\ \gamma \in \mathbb{C};\ \Re(\alpha) > \max\{0, \Re(\kappa)-1\};\ \Re(\kappa) > 0), $$

where $(a_j)_n$ $(j \in \overline{1,p})$ and $(b_j)_n$ $(j \in \overline{1,q})$ are the Pochhammer symbols in (4).

The series (18) is defined when

$$ b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j \in \overline{1,q}). \tag{19} $$

If any parameter a_j is a negative integer or zero, then the series (18) terminates to become a polynomial in z.

Assuming that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise) and with the restriction given by (19), the ${}_p^{\kappa}R_q^{\alpha,\beta;\gamma}$ series in (18)

(i) converges for $|z| < \infty$, if $p < q+1$,
(ii) converges for $|z| < 1$, if $p = q+1$, and
(iii) diverges for all $z \in \mathbb{C} \setminus \{0\}$ if $p > q+1$.

Furthermore, if we set

$$ \omega := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j, \tag{20} $$

then it is seen that the ${}_p^{\kappa}R_q^{\alpha,\beta;\gamma}$ series in (18), with $p = q+1$, is

(a) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,
(b) conditionally convergent for $|z| = 1$ $(z \neq 1)$, if $-1 < \Re(\omega) \leqq 0$, and
(c) divergent for $|z| = 1$, if $\Re(\omega) \leqq -1$.

Remark 1. *The R-function in (18) is general enough to include, as its special cases, such functions as (for example) the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ introduced by Srivastava and Tomovski [44]:*

$$ {}_1^{\kappa}R_1^{\alpha,\beta;\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma,\kappa}(z) \tag{21} $$

as well as the Mittag-Leffler function $E_\alpha(z)$ (see [1]):

$$ {}_1^{1}R_1^{\alpha,1;1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_\alpha(z). \tag{22} $$

3. Bivariate Dirichlet Averages

The Dirichlet average of the generalized Mittag-Leffler type Function (18) is denoted and defined as follows:

$$ \begin{aligned} &{}_p^{\kappa}\mathcal{M}_q^{\alpha,\delta;\gamma}\left[(\beta,\beta';x,y)\right]_{(b)_{1,q}}^{(a)_{1,p}} \\ &:= \int_{E_1} \left[{}_p^{\kappa}R_q^{\alpha,\delta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;(u\circ z))\right] d\mu_{\beta,\beta'}(u), \end{aligned} \tag{23} $$

where $(a)_{1,n}$ and $(b)_{1,n}$ $(n \in \mathbb{N})$ denote the horizontal arrays $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively; $z = (x,y) \in \mathbb{R}^2$ and $\min\{\Re(\beta), \Re(\beta')\} > 0$. In fact, it is shown that the Dirichlet average of the R-function (18) is stated in terms of the Riemann–Liouville fractional integrals (16) claimed by Theorems 1 and 2.

Theorem 1. *Let z, α, β, β', δ, γ, $\kappa \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0, \Re(\kappa) - 1\}$ and $\min\{\Re(\kappa), \Re(\beta), \Re(\beta')\} > 0$. Also let x, $y \in \mathbb{R}$ with $x > y$ and $I_{0+}^{\beta'}$ be the Riemann–Liouville fractional integral given in (16). Then the Dirichlet average of the generalized Mittag-Leffler type function (18) is given by the following formula:*

$$ {}_{p}^{\kappa}\mathcal{M}_{q}^{\alpha,\delta;\gamma}\left[(\beta,\beta';x,y)\right]_{(b)_{1,q}}^{(a)_{1,p}} = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}} \left(I_{0+}^{\beta'} f\right)(x-y), \tag{24} $$

where the function f is given by

$$ f(t) = t^{\beta-1}\, {}_{p}^{\kappa}R_{q}^{\alpha,\delta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;y+t). \tag{25} $$

Proof. With the aid of (10) to (13), by applying the R-function (18) to (23), we find that

$$ \begin{aligned} \mathcal{D}_1 &:= {}_{p}^{\kappa}M_{q}^{\alpha,\delta;\gamma}[(\beta,\beta';x,y)]_{(b)_{1,q}}^{(a)_{1,p}} \\ &= \tfrac{1}{B(\beta,\beta')} \int_0^1 u^{\beta-1}(1-u)^{\beta'-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \tfrac{(\gamma)_{\kappa n}[y+u(x-y)]^n}{\Gamma(\alpha n+\delta)\,n!}\, du. \end{aligned} \tag{26} $$

By changing the order of integration and summation, which is verified under the stated conditions, we get

$$ \mathcal{D}_1 = \frac{1}{B(\beta,\beta')} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\delta)\,n!} \int_0^1 u^{\beta-1}(1-u)^{\beta'-1}[y+u(x-y)]^n du. $$

Setting $t := u(x-y)$, we find that

$$ \begin{aligned} \mathcal{D}_1 &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\delta)\,n!} \left(\frac{1}{x-y}\right)^{\beta+\beta'-1} \\ &\quad \times \int_0^{x-y} t^{\beta-1}(x-y-t)^{\beta'-1}(y+t)^n dt \\ &= \left(\frac{1}{x-y}\right)^{\beta+\beta'-1} \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} \\ &\quad \times \left[\frac{1}{\Gamma(\beta')} \int_0^{x-y} \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{\kappa n}\,(y+t)^n}{\Gamma(\alpha n+\delta)\,n!} \right\} t^{\beta-1}(x-y-t)^{\beta'-1} dt\right]. \end{aligned} $$

Then, using (16) and (18), we arrive at the desired result in (24). This completes the proof. □

We take into account the following modification to the Dirichlet average in (23):

$$ \begin{aligned} & {}_{p}^{\kappa,\lambda}\mathcal{M}_{q}^{\alpha,\delta;\gamma}\left[(\beta,\beta';x,y)\right]_{(b)_{1,q}}^{(a)_{1,p}} \\ &= \int_{E_1} (u\circ z)^{\lambda-1}\left[{}_{p}^{\kappa}R_{q}^{\alpha,\delta;\gamma}(a_1,\ldots,a_p;b_1,\ldots,b_q;(u\circ z)^{\gamma})\right] d\mu_{\beta,\beta'}(u), \end{aligned} \tag{27} $$

where $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $z = (x,y)$.

Theorem 2. *Let z, α, β, β', δ, $\gamma \in \mathbb{C}$ with $\min\{\Re(\beta), \Re(\beta')\} > 0$ and $\kappa \in \mathbb{N}$. Furthermore, let x, $y \in \mathbb{R}$ with $x > y$ and the convergence conditions of the R-function be satisfied. Then the following formula holds true: For $\Re(\lambda) > 0$,*

$$ {}^{\kappa,\lambda}_{p}\mathcal{M}^{\alpha,\delta;\gamma}_{q}\big[(\beta,\beta';x,y)\big]^{(a)_{1,p}}_{(b)_{1,q}} = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left(I^{\beta'}_{y+}g\right)(x), \tag{28}$$

where the function g is given by

$$ g(t) = t^{\lambda-1}(t-y)^{\beta-1}\,{}^{\kappa}_{p}R^{\alpha,\delta;\gamma}_{q}\big(a_1,\ldots,a_p;b_1,\ldots,b_q;t^{\gamma}\big). \tag{29}$$

Proof. With the aid of (10)–(13), by applying the R-function (18)–(27), we find that

$$\begin{aligned}
\mathcal{D}_2 :&= {}^{\kappa,\lambda}_{p}\mathcal{M}^{\alpha,\delta;\gamma}_{q}\big[(\beta,\beta';x,y)\big]^{(a)_{1,p}}_{(b)_{1,q}} \\
&= \frac{1}{B(\beta,\beta')}\int_0^1 u^{\beta-1}(1-u)^{\beta'-1}[y+u(x-y)]^{\lambda-1} \\
&\quad\times \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}[y+u(x-y)]^{n\gamma}}{n!\Gamma(\alpha n+\delta)}\,du \\
&= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}}{n!\Gamma(\alpha n+\delta)} \\
&\quad\times \int_0^1 u^{\beta-1}(1-u)^{\beta'-1}[y+u(x-y)]^{n\gamma+\lambda-1}du.
\end{aligned}$$

Then, setting $t := y + u(x-y)$, we obtain

$$\begin{aligned}
\mathcal{D}_2 &= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\delta)\,n!}\left(\frac{1}{x-y}\right)^{\beta+\beta'-1} \\
&\quad\times \int_y^x t^{n\gamma+\lambda-1}(t-y)^{\beta-1}(x-t)^{\beta'-1}dt \\
&= \left(\frac{1}{x-y}\right)^{\beta+\beta'-1}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} \\
&\quad\times\left[\frac{1}{\Gamma(\beta')}\int_y^x\left\{\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}t^{n\gamma}}{\Gamma(\alpha n+\delta)\,n!}\right\}t^{\lambda-1}\,(t-y)^{\beta-1}(x-t)^{\beta'-1}dt\right].
\end{aligned}$$

Finally, using (16), we are led to the desired result (28). This complete the proof. □

4. Dirichlet Average Expressed in Terms of Srivastava–Daoust Function

This section discusses an alternative formulation of the modified Dirichlet averages of the R-function.

Theorem 3. *Let β, β', δ, $\lambda \in \mathbb{C}$ with $\min\{\Re(\beta), \Re(\beta'), \Re(\lambda)\} > 0$ and $x, y, \kappa, \alpha, \gamma \in \mathbb{R}$ with $x > y$ and $\min\{\kappa, \alpha, -\gamma\} > 0$. The convergence conditions of the R-function are supposed to be satisfied. Then the following formula holds true:*

$$\begin{aligned}
&{}^{\kappa,\lambda}_{p}\mathcal{M}^{\alpha,\delta;\gamma}_{q}\big[(\beta,\beta';x,y)\big]^{(a)_{1,p}}_{(b)_{1,q}} \\
&= \frac{y^{\lambda-1}}{\Gamma(\delta)}F^{1:p+1;1}_{0:q+2;1}\left(\begin{array}{ccc} [1-\kappa:-\gamma,1] & : & [(a),\gamma:1_{(p)},\kappa]; \\ \text{———} & : & [(b),\delta,1-\kappa:1_{(q)},\alpha,-\gamma]; \end{array}\right. \\
&\qquad\qquad\left.\begin{array}{c}[\beta:1];\\ [\beta+\beta':1];\end{array}\; y^{\gamma}, 1-\frac{x}{y}\right),
\end{aligned}\tag{30}$$

where $a_{(\ell)}$, here and throughout this paper, abbreviates the array of ℓ times repetition of the same parameter a's, $a,\ldots,a$, and (a) and (b) abbreviate the arrays of p and q parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$, respectively.

Proof. In view of (28) and (15), we have

$$\begin{aligned}\mathcal{D}_3 :=& \ {}^{\kappa,\lambda}_{p}\mathcal{M}^{\alpha,\delta,\gamma}_{q}\left[(\beta,\beta';x,y)\right]^{(a)_{1,p}}_{(b)_{1,q}}\\ =& \ \frac{1}{B(\beta,\beta')}\int_0^1 u^{\beta-1}(1-u)^{\beta'-1}[y+u(x-y)]^{\lambda-1}\\ &\times\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}[y+u(x-y)]^{n\gamma}}{n!\Gamma(\alpha n+\delta)}\,du.\end{aligned}$$

Exchanging the order of integral and summation and using the generalized binomial series

$$(1-z)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n\,z^n}{n!}\quad(|z|<1;\ a\in\mathbb{C})$$

and the Beta function, we obtain

$$\begin{aligned}\mathcal{D}_3 :=& \ \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}\,y^{n\gamma+\lambda-1}}{n!\Gamma(\alpha n+\delta)}\\ &\times\int_0^1 u^{\beta-1}(1-u)^{\beta'-1}\left[1-\left(1-\frac{x}{y}\right)u\right]^{n\gamma+\lambda-1}du\\ =& \ y^{\lambda-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{(\gamma)_{\kappa n}\,y^{n\gamma}}{n!\Gamma(\alpha n+\delta)}\,{}_2F_1\left(\begin{matrix}\beta,\,1-\gamma n-\lambda;\\ \beta+\beta';\end{matrix}\ 1-\frac{x}{y}\right).\end{aligned}$$

Applying $\Gamma(\lambda+\nu)=\Gamma(\lambda)\,(\lambda)_\nu$ $(\lambda,\,\nu\in\mathbb{C})$ and

$$(1-\lambda-\gamma n)_r=\frac{\Gamma(1-\lambda-\gamma n+r)}{\Gamma(1-\lambda-\gamma n)}=\frac{(1-\lambda)_{-\gamma n+r}}{(1-\lambda)_{-\gamma n}},$$

we find

$$\mathcal{D}_3=\frac{y^{\lambda-1}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n(\gamma)_{\kappa n}(1-\kappa)_{-\gamma n+r}\,(\beta)_r}{\prod_{j=1}^{q}(b_j)_n\,(\delta)_{\alpha n}(1-\kappa)_{-\gamma n}\,(\beta+\beta')_r}\,\frac{(y^{\gamma})^n\left(1-\frac{x}{y}\right)^r}{n!\,r!},$$

which, in view of (17), leads to the right-hand side of (30). This completes the proof. □

5. Multivariate Dirichlet Averages

Consider the Dirichlet average (23) and its modification (27) where $(z):=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and $d_1,\ldots,d_n$ are parameters. Our finding is predicated on the following basic premise in Lemma 1 (see [22]).

Lemma 1. *Let $d_j,\ r_j\in\mathbb{C}$ $(j\in\overline{1,n};\ n\in\mathbb{N})$ such that $\min\{\Re(d_j),\Re(r_j)\}>-1$. Furthermore, let E_{n-1} denote the Euclidean simplex in (12) and $d\mu_d(u)$ stand for the Dirichlet measure in (11). Then the following formula holds true:*

$$\int_{E_{n-1}}u_1^{r_1}\cdots u_{n-1}^{r_{n-1}}(1-u_1-\cdots-u_{n-1})^{r_n}\,d\mu_d(u)=\frac{(d_1)_{r_1}\cdots(d_n)_{r_n}}{(d_1+\cdots+d_n)_{r_1+\cdots+r_n}}\tag{31}$$

(see Equation (52) [22]).

The Lauricella function F_D defined for complex parameters $d = (d_1, \ldots, d_n) \in \mathbb{C}^n$ is defined as follows (consult, for example, Section 1.4 in [40]):

$$F_D(a, (d); c; z) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (d_1)_{m_1} \cdots (d_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{z_1^{m_1} \cdots z_n^{m_n}}{m_1! \cdots m_n!}. \tag{32}$$

The series (32) converges for all variables inside unit circle $\max\limits_{1 \le j \le n} |z_j| < 1$.

Here we investigate the following Dirichlet average:

$$\begin{aligned} &{}_p^{\kappa,\eta}\mathcal{M}_q^{\alpha,\delta;\gamma}[(d); (1-z)]_{(b)_{1,q}}^{(a)_{1,p}} \\ &= \int_{E_{n-1}} (1 - u \circ z)^{\eta - 1} \left[{}_p^{\kappa}R_q^{\alpha,\delta;\gamma}\big(a_1, \ldots, a_p; b_1, \ldots, b_q; (1 - u \circ z)^{\gamma}\big) \right] d\mu_d(u). \end{aligned} \tag{33}$$

We also need the following multinomial expansion:

$$(1 - z_1 - \cdots - z_n)^{\rho} = \sum_{r_1, \ldots, r_n = 0}^{\infty} (-\rho)_{r_1+\cdots+r_n} \frac{z_1^{r_1} \cdots z_n^{r_n}}{r_1! \cdots r_n!} \qquad (|z_1 + \cdots + z_n| < 1). \tag{34}$$

Theorem 4. *Let κ, α, $\gamma \in \mathbb{R}$ with $\min\{\kappa, \alpha, \gamma\} > 0$ and δ, η, d_j, $z_j \in \mathbb{C}$ with $\Re(\eta) > 0$ and $\Re(d_j) > 0$ $\big(j \in \overline{1, n}\big)$. Convergence conditions of the R-function are assumed to be satisfied. Then the following result holds true:*

$$\begin{aligned} &{}_p^{\kappa,\eta}\mathcal{M}_q^{\alpha,\delta;\gamma}[d_1, \ldots, d_n; 1 - z_1, \ldots, 1 - z_n]_{(b)_{1,q}}^{(a)_{1,p}} \\ &= \frac{1}{\Gamma(\delta)} F_{2:q+1;0;\ldots;0}^{0:p+2;1;\ldots;1} \left(\begin{matrix} \text{——} & : & [(a), \gamma, \eta : 1_{(p)}, \kappa, \gamma]; \\ \left[\eta, \sum\limits_{j=1}^{n} d_j : \theta^{(1)}, \theta^{(2)}\right] & : & [(b), \delta : 1_{(q)}, \alpha]; \end{matrix} \right. \\ &\qquad\qquad \left. \begin{matrix} [d_1 : 1]; & \ldots; & [d_n : 1]; \\ \text{——}\,; & \ldots; & \text{——}\,; \end{matrix} \quad 1, -z_1, \ldots, -z_n \right), \end{aligned} \tag{35}$$

where (a) and (b) abbreviate the arrays of p and q parameters a_1, $\ldots$, a_p and b_1, $\ldots$, b_q, respectively, $\theta^{(1)}$ and $\theta^{(2)}$ abbreviate the arrays of $n+1$ parameters γ, $(-1)_{(n)}$ and 0, $1_{(n)}$, respectively.

Proof. Considering the multivariate Dirichlet average (33), we have

$$\begin{aligned} \mathcal{D}_4 : &= {}_p^{\kappa,\eta}\mathcal{M}_q^{\alpha,\delta;\gamma}[(d); (1-z)]_{(b)_{1,q}}^{(a)_{1,p}} \\ &= \int_{E_{n-1}} (1 - u \circ z)^{\eta - 1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (1 - u \circ z)^{\gamma n} (\gamma)_{\kappa n}}{\prod_{j=1}^{q} (b_j)_n \Gamma(\alpha n + \delta)\, n!} \, d\mu_d(u) \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (\gamma)_{\kappa n}}{\prod_{j=1}^{q} (b_j)_n \Gamma(\alpha n + \delta)\, n!} \int_{E_{n-1}} (1 - u \circ z)^{\gamma n + \eta - 1} \, d\mu_d(u). \end{aligned}$$

Applying Lemma 1 and the polynomial expansion (34), and assuming $|u_1 z_1 + \cdots + u_n z_n| < 1$, we arrive at

$$\begin{aligned}
\mathcal{D}_4 &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (\gamma)_{\kappa n}}{\prod_{j=1}^{q} (b_j)_n \Gamma(\alpha n + \delta)\, n!} \sum_{r_1,\cdots,r_n=0}^{\infty} (1-\gamma n-\eta)_{r_1+\cdots+r_n} \frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!} \\
&\quad \times \int_{E_{n-1}} u_1^{r_1}\cdots u_n^{r_n} (1-u_1-\cdots-u_{n-1})^{r_n}\, d\mu_d(u) \\
&= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (\gamma)_{\kappa n}}{\prod_{j=1}^{q} (b_j)_n \Gamma(\alpha n + \delta)\, n!} \\
&\quad \times \sum_{r_1,\cdots,r_n=0}^{\infty} \frac{(1-\gamma n-\eta)_{r_1+\cdots+r_n} (d_1)_{r_1}\cdots (d_n)_{r_n}}{(d_1+\cdots+d_n)_{r_1+\cdots+r_n}} \frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!}.
\end{aligned}$$

The n-fold inner sum (with respect to $r_1,\cdots,r_n$) forms a Lauricella $F_D^{(n)}$ function in n variables (see, for instance, (p. 33) [40]), we have

$$\begin{aligned}
\mathcal{D}_4 &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (\gamma)_{\kappa n}}{\prod_{j=1}^{q} (b_j)_n \Gamma(\alpha n + \delta)\, n!} \\
&\quad \times F_D^{(n)}[1-\gamma n-\eta; d_1,\ldots,d_n;\, d_1+\cdots+d_n;\, z_1,\ldots,z_n].
\end{aligned}$$

Using $\Gamma(\delta+\alpha n) = \Gamma(\delta)\,(\delta)_{\alpha n}$ and

$$(1-\gamma n-\eta)_{r_1+\cdots+r_n} = (-1)^{r_1+\cdots+r_n} \frac{(\eta)_{\gamma n}}{(\eta)_{\gamma n-r_1-\cdots-r_n}},$$

we obtain

$$\begin{aligned}
\mathcal{D}_4 = \frac{1}{\Gamma(\delta)} \sum_{n,r_1,\ldots,r_n=0}^{\infty} & \frac{\left(\prod_{j=1}^{p} (a_j)_n\right) (\gamma)_{\kappa n} (\eta)_{\gamma n} (d_1)_{r_1}\cdots (d_n)_{r_n}}{\left(\prod_{j=1}^{q} (b_j)_n\right) (\delta)_{\alpha n} (\eta)_{\gamma n-r_1-\cdots-r_n} (d_1+\cdots+d_n)_{r_1+\cdots+r_n}} \\
& \times \frac{(-z_1)^{r_1}\cdots(-z_n)^{r_n}}{n!\, r_1!\cdots r_n!},
\end{aligned}$$

which, in view of (17), is easily seen to yield the expression of the right-hand side of (35). □

6. Concluding Remarks

The Dirichlet and modified Dirichlet averages of the R-function in (18) (a generalized Mittag-Leffler type function) were explored. In Theorems 1 and 2, the bivariate Dirichlet averages of the R-function (18) were expressed in terms of the Riemann–Liouville fractional integrals whose kernel functions are products of some elementary functions and the R-function (18). In Theorem 3, the bivariate Dirichlet average of the R-function (18) (see Theorem 2) was shown to be expressed in terms of the Srivastava–Daoust generalization (17) of the Lauricella hypergeometric function. In Theorem 4, the multivariate Dirichlet average of the R-function (18) was proven to be expressed in terms of the Srivastava–Daoust generalization (17) of the Lauricella hypergeometric function. The main results in Theorems 1–4 are believed to be useful.

The Mittag-Leffler function $E_\alpha(z)$ in (1), the two-parametric Mittag-Leffler function $E_{\alpha,\beta}(z)$ in (2), the three-parametric Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ in (3), and the R-function in (18) are obviously contained as special cases in the well-known Fox–Wright function ${}_p\Psi_q$ (see, for details, p. 21 [40]; see also p. 56 [38]). Because the R-function in (18) is of general character, all results in Theorem 1–Theorem 4 are seen to be able to yield a large number of particular instances. The following corollary demonstrates just a particular instance of Theorem 1:

Corollary 1. *Let the conditions in* Theorem 1 *be satisfied and set* $p = q = 1$ *and* $a_j = b_j = 1$ *in* (24)*. Then the Dirichlet average for the generalized Mittag-Leffler function holds true:*

$$ {}_{1}^{\kappa}\mathcal{M}_{1}^{\alpha,\delta;\gamma}\left[(\beta,\beta';x,y)\right] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left\{I_{0+}^{\beta'}\left(t^{\beta-1}\,E_{\alpha,\delta}^{\gamma,\kappa}(y+t)\right)\right\}(x-y), \tag{36} $$

where $E_{\alpha,\delta}^{\gamma,\kappa}$ *is given in* (21)*.*

As with the H-function of the single variable in (5), the H-function of multiple variables is generated using multiple contour integrals of the Mellin–Barnes type (see pp. 205–207, Appendix A.1 in [5]). This article concludes with the questions posed: Like (8),

- Express (possibly) the Srivastava–Daoust generalization (17) of the Lauricella hypergeometric function in terms of the multivariate H-function;
- Express (possibly) the right members of Theorems 3 and 4 in terms of the multivariate H-function.

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