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Article

An Averaging Principle for Stochastic Fractional Differential Equations Driven by fBm Involving Impulses

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Abstract: In contrast to previous research on periodic averaging principles for various types of impulsive stochastic differential equations (ISDEs), we establish an averaging principle without periodic assumptions of coefficients and impulses for impulsive stochastic fractional differential equations (ISFDEs) excited by fractional Brownian motion (fBm). Under appropriate conditions, we demonstrate that the mild solution of the original equation is approximately equivalent to that of the reduced averaged equation without impulses. The obtained convergence result guarantees that one can study the complex system through the simplified system. Better yet, our techniques dealing with multi-time scales and impulsive terms can be applied to improve some existing results. As for application, three examples are worked out to explain the procedure and validity of the proposed averaging principles.

Keywords: stochastic fractional differential equations; impulsive dynamical systems; averaging principle; fractional Brownian motion; multi-time scale; convergence results

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1. Introduction

In the past few decades, fractional calculus and fractional dynamical systems have gained widespread concern from scholars in various research fields. Different researchers explored fractional dynamical systems from both theoretical and practical aspects due to their applications and prospects in many scientific areas [1–3]. Specifically, taking into account the inevitable uncertainties and random factors, massive research achievements on stochastic fractional differential equations have emerged one after another [4–6]. Besides, some real-world systems are affected by instantaneous disturbances or undergo sudden changes, and impulsive differential equations are naturally used to characterize them. A wide range of mathematical models in the study of control theory, telecommunications, biology, ecology, epidemiology, finance and economics can be described by integer or fractional-order impulsive stochastic differential equations (ISDEs) [7,8].

As we know, random perturbations with LRD (long-range dependence) exist in various application fields; for instance, hydrology, meteorology, physical chemistry, bioengineering, mathematical finance, etc. [9]. Fractional Brownian motion has a long correlation time and can be used to model these stochastic excitations. Therefore, various types of stochastic fractional differential equations (SFDEs) excited by fBm are being emphasized and some preliminary study results [10–12] have been acquired. Meanwhile, some recent research [13,14] considered the well-posedness and controllability of different classes of ISFDEs. In particular, Pedjeu and Ladde [15] introduce a stochastic model under multi-time scales, which can model the complex dynamical processes in sciences, engineering, ecology and epidemiology. Correspondingly, considering that the correlated noises can not be ignored, Abouagwa, Cheng and Li [16] establish the existence and uniqueness results of SFDEs under fBm involving impulses based on Carathéodory successive approximation method.

On the other hand, the averaging method is an effective approach to explore different types of non-linear dynamical systems. This is because you can concentrate on the reduced autonomous averaged equation instead of the original complex time-varying one, providing an approximate way to remove some of the complexity. The core of demonstrating an averaging principle is examining the conditions when the averaged system is equivalent to the original system in some sense. Since Khasminskii's contribution [17], stochastic averaging methods for SDEs have been widely developed and applied [18,19]. However, most of the noises they had considered are uncorrelated. Such results ruled out these stochastic differential systems under the random perturbations with long-term dependence. Fortunately, Pei, Xu and Guo [20,21] have recently obtained the averaging principles for several different kinds of SDEs under fBm with Hurst index $H \in (\frac{1}{2}, 1)$. With the development of research on ISDEs, in two recent years, a few study achievements on periodic averaging methods of ISDEs have appeared [22–25]. There are two disadvantages of this research. One is that the periodic assumptions of the coefficient functions and the impulses are strict; the other is the estimate of the difference between the impulsive term of the original equation and the impulsive integral term of the averaged equation is flawed. Here, it is noted that Liu and Xu [26] proved an averaging theorem for impulsive stochastic partial differential equations without periodic assumptions. In light of complex characteristics of stochastic fractional differential systems, Xu et al. [27] developed stochastic averaging for several classes of SFDEs driven by different noises. Abouagwa and Li [28] considered an averaging method for SFDEs under a non-Lipschitz condition. Luo and Zhu [29] discussed an averaging principle for SFDEs with delay. Shen et al. [30] showed an averaging principle for SFDEs with Lévy noise and Markovian switching. Guo et al. [31] examined an averaging theorem under a weaker condition for a kind of Caputo SFDEs. Liu and Xu [32] proved an averaging principle for neutral ISFDEs. These works offer posible ways to reduce various types of SFDEs. However, these papers do not involve the impulsive effects and the random perturbations with long-range dependence with which we are concerned. Another gap is that they showed no uniformity of time scales. To be precise, the aforementioned results cannot solve the problem if the SFDEs are excited by fBm and involve impulses. In this work, our main objective is to fill these two gaps. We will establish an averaging principle for the SFDEs driven by fBm with impulses in the following form:

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))(dt)^{\beta} + h(t, x(t))d^{-}B^{H}(t), \\ \qquad\qquad t \in [0, T],\ t \neq t_k, \\ \Delta x(t_k) = x(t_k^{+}) - x(t_k^{-}) = I_k\big(x({t_k}^{-})\big),\ t = t_k,\ k \in \mathbb{N}, \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $f, g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are measurable functions; the processes $B^H(t)$ stand for the independent fBms, in which the Hurst parameter $H \in (\frac{1}{2}, 1), 0 < \beta < 1, 0 < T < \infty, I_k : \mathbb{R}^d \to \mathbb{R}^d$ indicate the pulse size of x at time t_k, with $x({t_k}^{+}) = \lim\limits_{\tau \to 0^{+}} x(t_k + \tau), x({t_k}^{-}) = \lim\limits_{\tau \to 0^{-}} x(t_k + \tau), x(t_k) = x({t_k}^{+})$; and the impulsive time sequence satisfies that $0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $\lim\limits_{k \to \infty} t_k = \infty$. The initial value $x_0 \in \mathbb{R}^d$ is a given random variable with $\mathbb{E}|x_0|^2 < \infty$.

Since Equation (1) is introduced under multi-time scales, the time scale of the standard form of system (1) should be uniform in the process of deriving the averaging principle. However, this point was overlooked in the present literature [28–30]. Additionally, the involvement of impulses makes the proof of the main results more technical. The techniques shown in the aforementioned research on periodic averaging methods for ISDEs are not applicable to our case. The innovation of this paper is not only that an averaging theorem for a new class of SFDEs is discussed, but also that some published results are improved. Naturally, we highlight the contributions of this article here.

- We propose an effective approximation for the original system (1); hence, the complexity of ISFDEs under fBm can be reduced.
- The problem of no match on each of the time scales of the standard stochastic fractional differential equations is pointed out and corrected.
- The obtained averaging principle is valid for stochastic fractional differential equations driven by fBm; that is, our results are new even for non-impulsive SFDEs with fBm.
- we present a method to estimate the impulsive terms, which is helpful to develop averaging principles for different types of ISDEs.

The rest of this work is as follows: Some essential notations and preliminaries are presented in Section 2. In Section 3, an averaging principle for system (1) is established under a Non-Lipschitz condition, and some useful remarks are also presented. In the end, Section 4 shows three examples to explain the procedure and validity of the averaging theorem for SFDEs and ISFDEs.

2. Preliminary

Let $\phi(r,t) = H(2H-1)|r-t|^{2H-2}$, in which $H \in (1/2, 1)$ is a constant. Defining

$$L^2_\phi(\mathbb{R}^+) = \left\{ p : \|p\|^2_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} p(r)p(t)\phi(r,t)dtdr < \infty \right\},$$

where $p : \mathbb{R}^+ \to \mathbb{R}$ is a Borel measurable function; $L^2_\phi(\mathbb{R}^+)$ comes to a separable Hilbert space when it is endowed with the following scalar product:

$$\langle p_1, p_2 \rangle_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} p_1(r)p_2(t)\phi(r,t)dtdr, p_1, p_2 \in L^2_\phi(\mathbb{R}^+).$$

Denote $\mathcal{S}$ as the family of cylindrical and smooth random variables:

$$F(\omega) = p\Big(B^H(\psi_1), B^H(\psi_2), \cdots, B^H(\psi_n)\Big),$$

in which $p \in C_b^\infty(\mathbb{R}^n), n \geq 1$ and $\psi_i \in \mathcal{H}$, where $\mathcal{H}$ represents the complete set of all measurable mappings with $\|\psi\|^2_\varphi < \infty$ with $\langle \psi_i, \psi_j \rangle_\varphi = \delta_{ij}$. Now, we can define the space $|\mathcal{H}|$ of p:

$$\|p\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T |p(t)||p(s)|\varphi(t,s)dsdt < \infty.$$

The Malliavin derivative D^H_t of $F \in \mathcal{S}$ is determined as follows:

$$D^H_t F = \sum_{i=1}^n \frac{\partial p}{\partial x_i}\Big(B^H(\psi_1), B^H(\psi_2), \cdots, B^H(\psi_n)\Big)\psi_i(t).$$

Moreover, for any positive integer k, $D^{H,k}_t$ symbolizes the iteration of the above formula. Then $\mathcal{D}^{k,q}$ represents the closure of $\mathcal{S}$ and is normed by:

$$\|F\|^q_{k,q} = \mathbb{E}|F|^q + \mathbb{E}\sum_{j=1}^k \left\| D^{H,j}_t F \right\|^q_{\mathcal{H}\otimes j},$$

in which $q \geq 1, \otimes$ stands for the tensor product. Likewise, let us introduce $\mathcal{D}^{1,q}(|\mathcal{H}|)$ as the subspace of $\mathcal{D}^{1,q}(\mathcal{H})$. Then, the Malliavin ϕ-derivative of F can be given: $D^\phi_t F = \int_{\mathbb{R}^+} \phi(t,s) D^H_s F ds$.

There are three types of pathwise integrals definitions for fBm (i.e., forward pathwise integral, symmetric pathwise integral and backward pathwise integral) [33]. In this paper, we use the forward pathwise integral of fBm. For more details on fBm and stochastic integrals of fBm, one can refer [20,21,33,34].

Definition 1 ([33]). *Assume that $y(t)$ is a stochastic process on $[0,T]$ with integrable trajectories, $H \in (0,1)$. If the limit of*

$$\lim_{\delta \to 0} \int_0^T y(t) \left[\frac{B^H(t+\delta) - B^H(t)}{\delta} \right] dt$$

exists in probability, then the above expression is called as the forward integral of $y(t)$ for $B^H(t)$, denoted as $\int_0^T y(t) d^- B^H(t)$.

Remark 1. *Defining $\mathcal{L}_\phi[0,T]$ as the space of stochastic processes $y(t)$ satisfying $\mathbb{E}\|y(t)\|_\phi^2 < \infty$, in which $y(t)$ is ϕ-differentiable, and $D_s^\phi y(t)$ meet that $\mathbb{E} \int_0^T \int_0^T [D_t^\varphi y(s)]^2 dsdt < \infty$. More details about $\mathcal{L}_\varphi[0,T]$ are given in [21].*

Lemma 1 ([20]). *Let $y(t) \in \mathcal{L}_\phi[0,T] \cap \mathcal{D}^{1,2}(|\mathcal{H}|)$, and $B^H(t)$ is a fBm with $H \in (\frac{1}{2}, 1)$, then for any $0 < T < \infty$, there exists a positive constant C, such that*

$$\begin{aligned} \mathbb{E}\left[\int_0^T y(s) d^- B^H(s) \right]^2 \leq & 2HT^{2H-1} \mathbb{E}\left[\int_0^T |y(s)|^2 ds \right] + 4T\mathbb{E} \int_0^T [D_s^\varphi y(s)]^2 ds \\ \leq & 2HT^{2H-1} \mathbb{E}\left[\int_0^T |y(s)|^2 ds \right] + 4CT^2. \end{aligned}$$

Definition 2 ([15]). *Given $\beta \in (0,1]$, $g(t)$ is a continuous function, then the integral of $g(t)$ with $(dt)^\beta$ is determined as follows:*

$$\int_0^t g(\tau)(d\tau)^\beta = \beta \int_0^t (t-\tau)^{\beta-1} g(\tau) d\tau.$$

Following Refs. [16,35], we show the definition of the mild solution for system (1).

Definition 3. *A stochastic process $x(t)$ defined on $[0,T]$ is called a mild solution of (1) with initial value $x(0) = x_0$, if*

(i) $x(t) \in \mathbb{R}^d$ *is $\mathcal{F}_t$-adapted, and has càdlàg path a.e. on $[0,T]$;*
(ii) *for all $t \in [0,T]$, $x(t)$ meets the integral equation below:*

$$\begin{aligned} x(t) =& x(0) + \int_0^t f(s, x(s)) ds + \beta \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\beta}} ds \\ &+ \int_0^t h(s, x(s)) d^- B^H(s) + \sum_{0 < t_k < t} I_k(x(t_k^-)). \end{aligned} \tag{2}$$

To achieve the approximate theoretical result, the following hypotheses are supposed throughout this work.

Condition 1. *$f(t,\cdot), g(t,\cdot), h(t,\cdot) \in \mathcal{L}_\phi[0,T] \cap \mathcal{D}^{1,2}(|\mathcal{H}|)$ satisfy that*

$$\begin{aligned} |f(t,x) - f(t,y)|^2 + &|g(t,x) - g(t,y)|^2 + |h(t,x) - h(t,y)|^2 \\ &+ \left| D_t^\phi (h(t,x) - h(t,y)) \right|^2 \leq \mathcal{T}(t, |x-y|^2),\ \forall t \in [0,T], \end{aligned}$$

where $\mathcal{T} : [0+\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable with t, given any fixed $t \geq 0$, $\mathcal{T}$ is a continuous, non-decreasing, concave function with respect to x, $\mathcal{T}(t,0) = 0$ and $\int_{0^+} \frac{1}{\mathcal{T}(t,x)} dx = +\infty$. Additionally, for any $t \in \mathbb{R}^+$ and positive constant κ, if the non-negative continuous function $\mathcal{Y}(t)$ fulfills that $\mathcal{Y}(t) \leq \kappa \int_0^t \mathcal{T}(s, \mathcal{Y}(s)) ds$, we have $\mathcal{Y}(t) \equiv 0$.

Condition 2. *$f(t,\cdot), g(t,\cdot), h(t,\cdot) \in \mathcal{L}_\phi[0,T] \cap \mathcal{D}^{1,2}(|\mathcal{H}|)$ satisfy that*

$$|f(t,x)|^2 + |g(t,x)|^2 + |h(t,x)|^2 + \left|D_t^\phi h(t,x)\right|^2 \le \mathcal{M}(t,|x|^2),\ \forall t \in [0,T],$$

where $\mathcal{M} : [0+\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable for t and given every fixed t, $\mathcal{M}$ is continuous, monotone non-decreasing, concave with x. Moreover, given any positive constant κ and initial data x_0, the integral formula $x(t) = x_0 + \kappa \int_0^t \mathcal{M}(s, x(s))ds$ admits a global solution.

Condition 3. *The impulsive functions I_k satisfy the following Lipschitz condition and boundedness condition: there are positive constants c_k, d_k for all $x, y \in \mathbb{R}^d$, such that*

$$|I_k(x) - I_k(y)|^2 \le c_k|x-y|^2, \qquad |I_k(x)|^2 \le d_k.$$

Remark 2. *Following Theorem 3.1 in Ref. [16], the system (1) has a unique mild solution under Conditions 1–3. Additionally, the unique solution of (1) has the property that*

$$\mathbb{E} \sup_{0\le t\le T} |x(t)|^2 \le C,$$

where C is a positive constant.

3. Main Results

In this part, we devote to develop an averaging principle for ISFDEs excited by fBm. That is, an approximate way of simplifying the system is presented.

The standard form of ISFDEs driven by fBm is defined as

$$\begin{aligned} x_\varepsilon(t) =& x(0) + \varepsilon \int_0^t f(s, x_\varepsilon(s))ds + \varepsilon^\beta \beta \int_0^t \frac{g(s, x_\varepsilon(s))}{(t-s)^{1-\beta}} ds \\ &+ \varepsilon^H \int_0^t h(s, x_\varepsilon(s)) d^- B^H(s) + \varepsilon \sum_{0<t_k<t} I_k(x_\varepsilon(t_k^-)), \end{aligned} \tag{3}$$

where $t \in [0,T], \varepsilon \in (0, \varepsilon_1] (0 < \varepsilon_1 \ll 1)$ represents a time-scale parameter; the coefficients f, g, h, I_k meet Conditions 1–3.

Remark 3. *As discussed in Ref. [32], $\beta \int_0^t \frac{g(s,x_\varepsilon(s))}{(t-s)^{1-\beta}} ds$ is gained from the integral of $g(t, x_\varepsilon(t))$ with $(dt)^\beta$, so the time scale of this term should be ε^β.*

Next, we will demonstrate that when ε tends to zero, the solution $x_\varepsilon(t)$ for Equation (1) converges to the solution $z_\varepsilon(t)$ of the following averaged equation:

$$\begin{aligned} z_\varepsilon(t) =& x(0) + \varepsilon \int_0^t \left(\bar{f}(z_\varepsilon(s)) + \bar{I}(z_\varepsilon(s))\right)ds + \varepsilon^\beta \beta \int_0^t \frac{\bar{g}(z_\varepsilon(s))}{(t-s)^{1-\beta}} ds \\ &+ \varepsilon^H \int_0^t \bar{h}(z_\varepsilon(s)) d^- B^H(s), \end{aligned} \tag{4}$$

the coefficients $\bar{f}(\cdot), \bar{g}(\cdot), \bar{h}(\cdot) \in \mathcal{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$ and $\bar{I} : \mathbb{R}^d \to \mathbb{R}^d$ are all measurable, while meeting Conditions 1–3 and the hypotheses below:

Hypothesis 1 (H1):

$$\frac{1}{S}\int_0^S \left|f(s,x) - \bar{f}(x)\right|^2 ds \le \gamma_1(S)\vartheta(|x|^2),$$

Hypothesis 2 (H2):

$$\frac{1}{S}\int_0^S |g(s,x) - \bar{g}(x)|^2 ds \le \gamma_2(S)\vartheta(|x|^2),$$

Hypothesis 3 (H3):

$$\frac{1}{S}\int_0^S \left|h(s,x)-\overline{h}(x)\right|^2 ds \le \gamma_3(S)\vartheta(|x|^2),$$

Hypothesis 4 (H4):

$$\frac{1}{S}\left|\sum_{0<t_j<S} I_j(x)-S\overline{I}(x)\right| \le \gamma_4(S)(1+|x|),$$

where $S \in [0,T], \gamma_i(S)$ are positive bounded functions with $\lim\limits_{S\to\infty}\gamma_i(S)=0, i=1,2,3,4,$ and the function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, continuous and concave.

Remark 4. *As in Remark 2, under Hypotheses H1–H4, it is not difficult to derive that* $\mathbb{E}\sup_{0\le t\le T} |z_\varepsilon(t)|^2 < \infty$.

Now, the main theorem of this paper is presented as follows.

Theorem 1. *Let Conditions 1–3 and Hypotheses H1–H4 hold. For any given small positive number δ, there are $P>0, \varepsilon_2 \in (0,\varepsilon_1]$ and $\gamma \in (0,1)$ such that, for all $0<\varepsilon\le\varepsilon_2, 0\le t\le P\varepsilon^{-H\gamma}$, we have*

$$\mathbb{E}|x_\varepsilon(t)-z_\varepsilon(t)|^2 \le \delta.$$

For the sake of showing Theorem 1, we state the following lemma in advance.

Lemma 2. *Let the conditions of Theorem 1 hold; then, for any $0\le s\le t\le T$, we have*

$$\begin{aligned}\mathbb{E}|z_\varepsilon(t)-z_\varepsilon(s)|^2 \le& \varepsilon^2 Q_1(t-s)^2+\varepsilon^{2\beta}Q_2(t-s)^{2\beta}\\ &+\varepsilon^{2H}Q_3(t-s)^{2H}+\varepsilon^{2H}Q_4(t-s)^2,\end{aligned}$$

in which $Q_i(i=1,2,3,4)$ are four positive constants.

The proof of Lemma 2 is analogous to that of Theorem 3.1 in [16], so we just outline it here.

From Equation (4), with basic inequality, Cauchy–Schwarz inequality and Lemma 1, one can get

$$\begin{aligned}\mathbb{E}|z_\varepsilon(t)-z_\varepsilon(s)|^2 \le& 3\varepsilon^2(t-s)\int_s^t \left|\overline{f}(z_\varepsilon(\theta))+\overline{I}(z_\varepsilon(\theta))\right|^2 d\theta\\ &+3\varepsilon^{2\beta}\beta^2\frac{(t-s)^{2\beta-1}}{2\beta-1}\int_s^t |\overline{g}(z_\varepsilon(\theta))|^2 d\theta\\ &+3\varepsilon^{2H}\left(2H(t-s)^{2H-1}\mathbb{E}\int_s^t \left|\overline{h}(z_\varepsilon(\theta)\right|^2 d\theta+4C(t-s)^2\right),\end{aligned}$$

then, with the aid of Condition 2, $H1$–$H4$ and Remark 4, one can conclude the statement of Lemma 2 without difficulty.

We are now turning back to the proof of Theorem 1.

Proof. Employing the elementary inequality, we obtain from (3) and (4):

$$\begin{aligned}
\mathbb{E}|x_\varepsilon(t)-z_\varepsilon(t)|^2 \leq & 4\varepsilon^2\mathbb{E}\left|\int_0^t \big(f(s,x_\varepsilon(s))-\bar{f}(z_\varepsilon(s))\big)ds\right|^2 \\
& + 4\beta^2\varepsilon^{2\beta}\mathbb{E}\left|\int_0^t \frac{(g(s,x_\varepsilon(s))-\overline{g}(z_\varepsilon(s)))}{(t-s)^{1-\beta}}ds\right|^2 \\
& + 4\varepsilon^{2H}\mathbb{E}\left|\int_0^t \big(h(s,x_\varepsilon(s))-\bar{h}(z_\varepsilon(s))\big)d^-B^H(s)\right|^2 \\
& + 4\varepsilon^2\mathbb{E}\left|\sum_{0<t_k<t} I_k(x_\varepsilon(t_k^-))-\int_0^t \bar{I}(z_\varepsilon(s))ds\right|^2 \\
& := J_1+J_2+J_3+J_4.
\end{aligned}$$

Denote by $[0,t]\subseteq[0,\tilde{u}]\subseteq[0,T]$. Next, we estimate each term separately. For the first term, we have

$$\begin{aligned}
J_1 \leq & 8\varepsilon^2\mathbb{E}\left|\int_0^{\tilde{u}} (f(s,x_\varepsilon(s))-f(s,z_\varepsilon(s)))ds\right|^2 \\
& + 8\varepsilon^2\mathbb{E}\left|\int_0^{\tilde{u}} \big(f(s,z_\varepsilon(s))-\bar{f}(z_\varepsilon(s))\big)ds\right|^2 := J_{11}+J_{12}.
\end{aligned}$$

With the virtue of Cauchy–Schwarz inequality and Condition 1, it follows that

$$J_{11} \leq 8\varepsilon^2\tilde{u}\mathbb{E}\int_0^{\tilde{u}} \mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds. \tag{5}$$

On the basis of Remark 4, the elementary inequality, Condition 1 and $H1$, one has

$$\begin{aligned}
J_{12} \leq & 8\varepsilon^2\tilde{u}^2\mathbb{E}\left[\frac{1}{\tilde{u}}\int_0^{\tilde{u}} \left|f(s,z_\varepsilon(s))-\overline{f}(z_\varepsilon(s))\right|^2ds\right] \\
\leq & 8\varepsilon^2\tilde{u}^2C_{12},
\end{aligned} \tag{6}$$

where C_{12} is a positive constant. Then, we arrive at

$$J_1 \leq 8\varepsilon^2\tilde{u}\mathbb{E}\int_0^{\tilde{u}} \mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds + 8\varepsilon^2\tilde{u}^2C_{12}. \tag{7}$$

Adopting Cauchy–Schwarz inequality to J_2, we get

$$\begin{aligned}
J_2 \leq & 4\beta^2\varepsilon^{2\beta}\int_0^t (t-s)^{2\beta-1}ds\mathbb{E}\int_0^t |g(s,x_\varepsilon(s))-\overline{g}(z_\varepsilon(s))|^2ds \\
\leq & \frac{4\varepsilon^{2\beta}\beta^2t^{2\beta-1}}{2\beta-1}\mathbb{E}\int_0^t |g(s,x_\varepsilon(s))-\overline{g}(z_\varepsilon(s))|^2ds,
\end{aligned}$$

for $\mathbb{E}\int_0^t |g(s,x(s))-\overline{g}(z_\varepsilon(s))|^2ds$, estimating analogously as above, one can easily obtain

$$\begin{aligned}
J_2 \leq & \frac{8\beta^2\varepsilon^{2\beta}\tilde{u}^{2\beta-1}}{2\beta-1}\mathbb{E}\int_0^t \mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds + \frac{8\beta^2\varepsilon^{2\beta}\tilde{u}^{2\beta}}{2\beta-1}\sup_{0\leq t\leq\tilde{u}}\gamma_2(t)\mathbb{E}\left[\vartheta\left(\sup_{0\leq t\leq\tilde{u}}|z_\varepsilon(t)|^2\right)\right] \\
\leq & \frac{8\beta^2\varepsilon^{2\beta}\tilde{u}^{2\beta-1}}{2\beta-1}\mathbb{E}\int_0^{\tilde{u}} \mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds + \frac{8\beta^2\varepsilon^{2\beta}\tilde{u}^{2\beta}}{2\beta-1}C_{22},
\end{aligned} \tag{8}$$

where $\frac{1}{2}<\beta<1$, C_{22} is a positive constant.

To proceed, thanks to Lemma 1, the elementary inequality, Condition 1, Hypothesis $H2$ and Remark 4, one can derive that

$$\begin{aligned} J_3 \le& 8\varepsilon^{2H}Ht^{2H-1}\mathbb{E}\int_0^{\tilde{u}}\left|h(s,x_\varepsilon(s))-\overline{h}(z_\varepsilon(s))\right|^2 ds+16\varepsilon^{2H}Ct^2 \\ \le& 16\varepsilon^{2H}Ht^{2H-1}\mathbb{E}\int_0^{\tilde{u}}|h(s,x_\varepsilon(s))-h(s,z_\varepsilon(s))|^2 ds \\ &+16\varepsilon^{2H}Ht^{2H-1}\mathbb{E}\int_0^{\tilde{u}}\left|h(s,z_\varepsilon(s))-\overline{h}(z_\varepsilon(s))\right|^2 ds+16\varepsilon^{2H}Ct^2 \\ \le& 16\varepsilon^{2H}\tilde{u}^{2H-1}H\mathbb{E}\int_0^{\tilde{u}}\mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds \\ &+16\varepsilon^{2H}H\tilde{u}^{2H}\sup_{0\le t\le\tilde{u}}\gamma_3(t)\mathbb{E}\left[\vartheta\left(\sup_{0\le t\le\tilde{u}}|z_\varepsilon(t)|^2\right)\right]+16\varepsilon^{2H}C\tilde{u}^2 \\ \le& 16\varepsilon^{2H}\tilde{u}^{2H-1}H\mathbb{E}\int_0^{\tilde{u}}\mathcal{T}(s,|x_\varepsilon(s)-z_\varepsilon(s)|^2)ds+16\varepsilon^{2H}\tilde{u}^{2H-1}HC_{32}+16\varepsilon^{2H}C\tilde{u}^2, \end{aligned} \tag{9}$$

where C_{32} is a positive constant.

For the last term J_4, we have

$$\begin{aligned} J_4 \le& 16\varepsilon^2\mathbb{E}\left|\sum_{0<t_k<t}I_k(x_\varepsilon(t_k^-))-\int_0^t\overline{I}(x_\varepsilon(t_k^-))ds\right|^2 \\ &+16\varepsilon^2\mathbb{E}\left|\int_0^t\overline{I}(x_\varepsilon(t_k^-))ds-\int_0^t\overline{I}(z_\varepsilon(t_k^-))ds\right|^2 \\ &+16\varepsilon^2\mathbb{E}\left|\int_0^t\overline{I}(z_\varepsilon(t_k^-))ds-\int_0^t\overline{I}(z_\varepsilon(s))ds\right|^2 \\ :=& J_{41}+J_{42}+J_{43}. \end{aligned} \tag{10}$$

Notice that

$$\sum_{0<t_k<t}I_k(x_\varepsilon(t_k^-))=\sum_{i=1}^m\left(\sum_{0<t_k<t_{i+1}}I_k(x_\varepsilon(t_k^-))-\sum_{0<t_k<t_i}I_k(x_\varepsilon(t_k^-))\right)$$

and

$$\int_0^t\overline{I}(x_\varepsilon(t_k^-))ds=\sum_{i=0}^m(t_{i+1}-t_i)\overline{I}(x_\varepsilon(t_k^-)).$$

Estimating J_{41} in the following manner:

$$\begin{aligned} J_{41}\le 16\varepsilon^2\mathbb{E}\Bigg\{\sum_{i=0}^m\Bigg[t_{i+1}&\left|\frac{1}{t_{i+1}}\sum_{0<t_k<t_{i+1}}I_k(x_\varepsilon(t_k^-))-\overline{I}(x_\varepsilon(t_k^-))\right| \\ &+t_i\left|\frac{1}{t_i}\sum_{0<t_k<t_i}I_k(x_\varepsilon(t_k^-))-\overline{I}(x_\varepsilon(t_k^-))\right|\Bigg]\Bigg\}^2, \end{aligned}$$

then Hypothesis $H4$ yields that

$$\begin{aligned} J_{41} \le& 32\varepsilon^2(m+1)\tilde{u}^2\left(\sup_{0\le t\le\tilde{u}}\gamma_4(t)\right)^2 2\mathbb{E}\left(1+|x_\varepsilon(t_k^-)|^2\right) \\ \le& 64\varepsilon^2(m+1)\tilde{u}^2\left(\sup_{0\le t\le\tilde{u}}\gamma_4(t)\right)^2\left(1+\mathbb{E}\sup_{0\le t\le\tilde{u}}|x_\varepsilon(t)|^2\right) \\ \le& 64\varepsilon^2(m+1)\tilde{u}^2C_{41}, \end{aligned} \tag{11}$$

where C_{41} is a positive constant.

As $\overline{I}$ meets the Lipschitz condition, there is a positive constant N involving with c_k such that

$$\left|\overline{I}(x)-\overline{I}(y)\right|^2 \leq N|x-y|^2,\ \forall x,y\in\mathbb{R}^d.$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}J_{42} \leq & 16\varepsilon^2\tilde{u}N\int_0^{\tilde{u}}\mathbb{E}\left|x_\varepsilon(t_k^-)-z_\varepsilon(t_k^-)\right|^2 ds\\ \leq & 16\varepsilon^2\tilde{u}N\int_0^{\tilde{u}}\sup_{0\leq\theta\leq s}\mathbb{E}|x_\varepsilon(\theta)-z_\varepsilon(\theta)|^2 ds.\end{aligned} \tag{12}$$

With the aid of Lemma 2, one can reach

$$\begin{aligned}J_{43} \leq & 16\varepsilon^2\tilde{u}N\int_0^{t}\mathbb{E}\left|z_\varepsilon(t_k^-)-z_\varepsilon(s)\right|^2 ds\\ \leq & 16\varepsilon^4 NQ_1\tilde{u}^4+16\varepsilon^{2\beta+2}NQ_2\tilde{u}^{2\beta+2}+16\varepsilon^{2H+2}NQ_3\tilde{u}^{2H+2}\\ &+16\varepsilon^{2H+2}NQ_4\tilde{u}^4.\end{aligned} \tag{13}$$

Substituting (11), (12) and (13) together into (10), we get

$$\begin{aligned}J_4 \leq & 64\varepsilon^2(m+1)\tilde{u}^2C_{41}+16\varepsilon^2\tilde{u}N\int_0^{\tilde{u}}\sup_{0\leq\theta\leq s}\mathbb{E}|x_\varepsilon(\theta)-z_\varepsilon(\theta)|^2 ds\\ &+16\varepsilon^4 NQ_1\tilde{u}^4+16\varepsilon^{2\beta+2}NQ_2\tilde{u}^{2\beta+2}+16\varepsilon^{2H+2}NQ_3\tilde{u}^{2H+2}\\ &+16\varepsilon^{2H+2}NQ_4\tilde{u}^4.\end{aligned} \tag{14}$$

Thanks to the fact that $\mathcal{T}$ is concave in x, there is $w(t)>0$ and $e(t)>0$ such that

$$\mathcal{T}(t,x)\leq w(t)+e(t)x,\ \int_0^T w(t)dt<\infty,\ \int_0^T e(t)dt<\infty,$$

then, combining the estimates (7), (8), (9) and (14), we obtain

$$\begin{aligned}\sup_{0\leq t\leq\tilde{u}}\mathbb{E}|x_\varepsilon(t)-z_\varepsilon(t)|^2 \leq & \left(8\varepsilon^2\tilde{u}+16\varepsilon^{2H}\tilde{u}^{2H-1}+\frac{8\varepsilon^{2\beta}\tilde{u}^{2\beta-1}}{2\beta-1}\right)C_1\int_0^{\tilde{u}}\sup_{0\leq\theta\leq s}\mathbb{E}|x_\varepsilon(\theta)-z_\varepsilon(\theta)|^2 ds\\ &+16\varepsilon^2N\tilde{u}\int_0^{\tilde{u}}\sup_{0\leq\theta\leq s}\mathbb{E}|x_\varepsilon(\theta)-z_\varepsilon(\theta)|^2 ds\\ &+80\varepsilon^2\tilde{u}^2C_2+32\varepsilon^{2H}\tilde{u}^{2H}C_3+64\varepsilon^{2H}C_4\tilde{u}^2+\frac{16\varepsilon^{2\beta}\tilde{u}^{2\beta}C_5}{2\beta-1}\\ &+16\varepsilon^4C_6\tilde{u}^4+16\varepsilon^{2\beta+2}C_7\tilde{u}^{2\beta+2}+16\varepsilon^{2H+2}C_8\tilde{u}^{2H+2}\\ &+16\varepsilon^{2H+2}C_9\tilde{u}^4,\end{aligned}$$

where $C_i, i=1,2,\cdots,9$ are positive constants. Gronwall inequality shows that there are $P>0$ and $\gamma\in(0,1)$ fulfilling

$$\sup_{t\in\left[0,P\varepsilon^{-H\gamma}\right]}\mathbb{E}|x_\varepsilon(t)-z_\varepsilon(t)|^2\leq Q\varepsilon^{(1-H\gamma)},$$

where

$$Q = \Big[80C_2P^2\varepsilon^{1-H\gamma} + 32C_3P^{2H}\varepsilon^{(2H-1)(1-H\gamma)} + 64C_4P^2\varepsilon^{(2H-1)(1-\gamma)} + 16C_5P^{2\beta}\varepsilon^{(2\beta-1)(1-H\gamma)}\Big/(2\beta-1) + 16C_6P^4\varepsilon^{3(1-H\gamma)} + 16C_7P^{2\beta+2}\varepsilon^{(2\beta+1)(1-H\gamma)} + 16C_8P^{2H+2}\varepsilon^{(2H+1)(1-H\gamma)} + 16C_9P^4\varepsilon^{2H(1-\gamma)+(1-H\gamma)}\Big]\exp\Big(8P^2C_1\varepsilon^{2(1-H\gamma)} + +16P^{2H}C_1\varepsilon^{2H(1-H\gamma)} + 8P^{2\beta}C_1\varepsilon^{2\beta(1-H\gamma)}\Big/(2\beta-1) + 16NP^2\varepsilon^{2(1-H\gamma)}\Big)$$

is a positive constant.

Therefore, for arbitrarily given small positive number δ, there is $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$\mathbb{E}|x_\varepsilon(t) - z_\varepsilon(t)|^2 \le \delta,$$

for $\forall\varepsilon \in (0, \varepsilon_2], \forall t \in \big[0, P\varepsilon^{-H\gamma}\big] \subseteq [0, T]$. □

Remark 5. *Noted that the order of convergence interval is $\varepsilon^{-H\gamma}$ in this study. It is not stronger as $\varepsilon^{-\gamma}$ of Bm case and Lévy case [27–30] because of the weakness that stochastic integral of fBm is not a martingale. However, the proposed result is still certainly a good theoretical method to simplify non-autonomous ISFDEs driven by fBm with long-term dependence.*

Remark 6. *When $I_k = 0$, system (1) degenerates into stochastic fractional differential equation driven by fBm. Evidently, our obtained result is applicable for SFDEs with fBm.*

4. Example

Example 1. *Consider the SFDE under fBm below:*

$$\begin{aligned} dx_\varepsilon(t) =& -\varepsilon\sin^2(t)x_\varepsilon(t)dt + \varepsilon^\beta(dt)^\beta + \varepsilon^H d^- B^H(t), t \in [0, T] \\ x_\varepsilon(0) =& x_0, \end{aligned} \tag{15}$$

where $\mathbb{E}|x_0|^2 < \infty, f(t, x_\varepsilon(t)) = -\sin^2(t)x_\varepsilon(t), g(t, x_\varepsilon(t)) = 1, h(t, x_\varepsilon(t)) = 1$. Let

$$\bar{f}(z_\varepsilon(t)) = -\frac{1}{2}z_\varepsilon(t),\ \bar{g}(z_\varepsilon(t)) = 1,\ \bar{h}(z_\varepsilon(t)) = 1.$$

Obviously, Conditions 1, 2 and Hypotheses H1–H3 are satisfied. The simplified averaged SFDE is given by

$$dz_\varepsilon(t) = -\frac{1}{2}\varepsilon z_\varepsilon(t)dt + \varepsilon^\beta(dt)^\beta + \varepsilon^H d^- B^H(t). \tag{16}$$

Remark 6 illustrates that $z_\varepsilon(t)$ is approximately equivalent to $x_\varepsilon(t)$ on $[0, T]$. Then, a numerical comparison between the solution $x_\varepsilon(t)$ of original Equation (15) and the solution $z_\varepsilon(t)$ of the averaged Equation (16) is presented. Not surprisingly, good agreement can be observed in Figure 1.

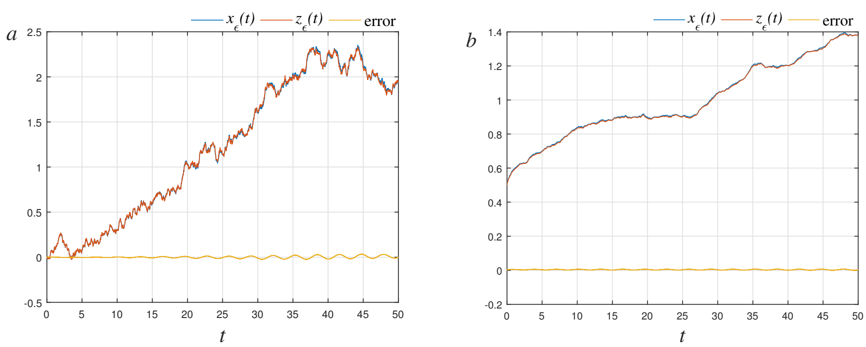

Figure 1. Comparison of $x_\varepsilon(t)$ and $z_\varepsilon(t)$. (**a**) $x_0 = 0, \varepsilon = 0.045, H = 0.65, \beta = 0.8$, (**b**) $x_0 = 0.5$, $\varepsilon = 0.01, H = 0.85, \beta = 0.6$.

Example 2. *Consider the SFDE with fBm below:*

$$\begin{aligned} dx_\varepsilon(t) =&2\varepsilon\sin^2(t)x_\varepsilon(t)dt + 2\varepsilon^\beta\cos^2(t)(dt)^\beta + \varepsilon^H d^- B^H(t), t \in [0, T] \\ x_\varepsilon(0) =&x_0, \end{aligned} \tag{17}$$

where $\mathbb{E}|x_0|^2 < \infty, f(t, x_\varepsilon(t)) = 2\sin^2(t)x_\varepsilon(t), g(t, x_\varepsilon(t)) = 2\cos^2(t), h(t, x_\varepsilon(t)) = 1$. *Let*

$$\bar{f}(z_\varepsilon(t)) = z_\varepsilon(t),\ \bar{g}(z_\varepsilon(t)) = 1,\ \bar{h}(z_\varepsilon(t)) = 1.$$

One can easily verify that Conditions 1, 2 and Hypotheses H1–H3 are satisfied. Meanwhile, a simplified averaged SFDE is presented as

$$dz_\varepsilon(t) = \varepsilon z_\varepsilon(t)dt + \varepsilon^\beta(dt)^\beta + \varepsilon^H d^- B^H(t). \tag{18}$$

Theorem 1 guarantees that $z_\varepsilon(t)$ *can approximate* $x_\varepsilon(t)$ *on* $[0, P\varepsilon^{-H\gamma}] \subseteq [0, T]$. *In succession, we show a numerical comparison between the solution* $x_\varepsilon(t)$ *to the original Equation (17) and the solution* $z_\varepsilon(t)$ *to the averaged Equation (18). It can be found that the two are consistent in a certain range in Figure 2.*

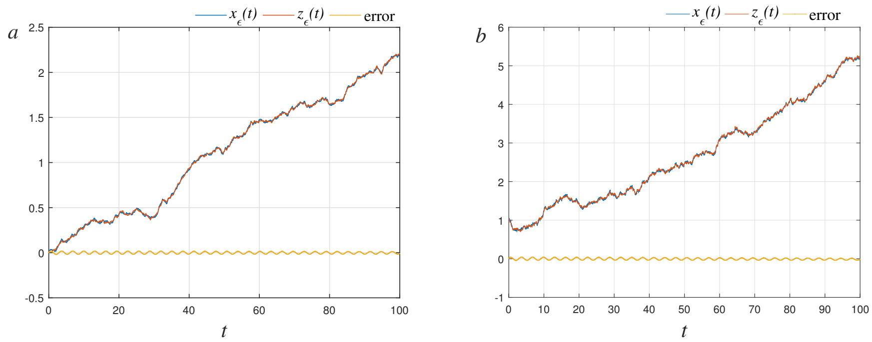

Figure 2. Comparison of $x_\varepsilon(t)$ and $z_\varepsilon(t)$. (**a**) $x_0 = 0, \varepsilon = 0.0045, H = 0.65, \beta = 0.7$, (**b**) $x_0 = 1$, $\varepsilon = 0.01, H = 0.55, \beta = 0.65$.

Example 3. *Consider the ISFDE with fBm below:*

$$\begin{aligned} &dx_\varepsilon(t) = \varepsilon dt + 2\varepsilon^\beta\sin^2(t)(dt)^\beta + \varepsilon^H\cos^2(t)x_\varepsilon(t)d^- B^H(t), t \neq t_k, \\ &\Delta x_\varepsilon(t_k) = x_\varepsilon(t_k^+) - x_\varepsilon(t_k^-) = \varepsilon\arctan(x_\varepsilon(t_k^-)), t = t_k, \\ &x_\varepsilon(0) = x_0, \end{aligned} \tag{19}$$

where $\mathbb{E}|x_0|^2 < \infty, f(t, x_\varepsilon(t)) = 1, g(t, x_\varepsilon(t)) = 2\sin^2(t), h(t, x_\varepsilon(t)) = \cos^2(t)x_\varepsilon(t)$. *Let*

$$\bar{f}(z_\varepsilon(t)) = 1,\ \bar{g}(z_\varepsilon(t)) = 1,\ \bar{h}(z_\varepsilon(t)) = \frac{1}{2}z_\varepsilon(t),\ \bar{I}(z_\varepsilon(t)) = \frac{1}{5}\arctan(z_\varepsilon(t)),$$

then, a new simplified averaged SFDE without impulses is given as

$$dz_\varepsilon(t) = \varepsilon\left(1 + \frac{1}{5}\arctan(z_\varepsilon(t))\right)dt + \varepsilon^\beta(dt)^\beta + \frac{1}{2}\varepsilon^H z_\varepsilon(t)d^- B^H(t). \tag{20}$$

As we can see, the impulses are averaged out and the original system is simplified. Even better, only a small difference is generated in this proximate process. In the end, we carry out a numerical comparison between the solution process $x_\varepsilon(t)$ *of original Equation (19) and the solution process* $z_\varepsilon(t)$ *of the simplified Equation (20). As expected, good agreement can be seen in Figure 3.*

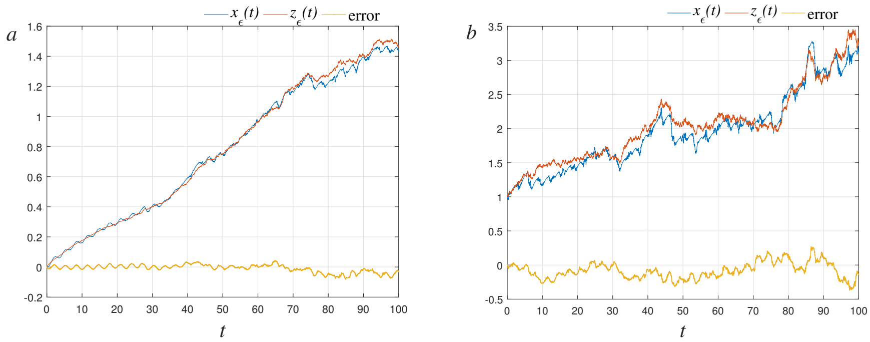

Figure 3. Comparison of $x_\varepsilon(t)$ and $z_\varepsilon(t)$ with $t_k = 4.8k$. (**a**) $x_0 = 0, \varepsilon = 0.005, H = 0.65, \beta = 0.7$, (**b**) $x_0 = 1, \varepsilon = 0.01, H = 0.55, \beta = 0.6$.

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