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$(p(x), q(x))$-Kirchhoff-Type Problems Involving Logarithmic Nonlinearity with Variable Exponent and Convection Term

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Abstract: In the present article, we study a class of Kirchhoff-type equations driven by the $(p(x), q(x))$-Laplacian. Due to the lack of a variational structure, ellipticity, and monotonicity, the well-known variational methods are not applicable. With the help of the Galerkin method and Brezis theorem, we obtain the existence of finite-dimensional approximate solutions and weak solutions. One of the main difficulties and innovations of the present article is that we consider competing $(p(x), q(x))$-Laplacian, convective terms, and logarithmic nonlinearity with variable exponents, another one is the weaker assumptions on nonlocal term $M_{v(x)}$ and nonlinear term g.

Keywords: Kirchhoff-type equations; logarithmic nonlinearity; convection term; Galerkin method; Brezis theorem

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1. Introduction

The purpose of the present article is to investigate the following $(p(x), q(x))$-Kirchhoff-type equations involving logarithmic nonlinearity and convection terms:

$$\begin{cases} -M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\Delta_{p(x)}\eta - \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\Delta_{q(x)}\eta \\ \qquad = \lambda|\eta|^{r(x)-2}\eta\ln|\eta| + g(x,\eta,\nabla\eta), \text{ in } \Omega, \\ \eta|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $r(x) \in C_+(\Omega)$, μ, λ are real parameters, and Ω is an open bounded domain in $\mathbb{R}^N$ with a smooth boundary.

Here, $\Delta_{\gamma(x)}$ is a $\gamma(x)$-Laplace operator, defined by

$$\Delta_{\gamma(x)}\eta = div(|\nabla\eta|^{\gamma(x)-2}\nabla\eta) = \sum_{i=1}^{N}\left(|\nabla\eta|^{\gamma(x)-2}\frac{\partial\eta}{\partial x_i}\right), \ \gamma(x) \in \{p(x), q(x)\}, \tag{2}$$

for all $x \in \Omega$ and $\eta \in C_0^\infty(\mathbb{R}^N)$, and denote

$$\delta_{s(x)}(\eta) = \int_\Omega \frac{1}{s(x)}|\nabla\eta|^{s(x)}dx, \ s(x) \in \{p(x), q(x)\}. \tag{3}$$

From now on, we briefly state some major features of problem (1). One of the significant characteristics of the problem (1) is the presence of double non-local Kirchhoff terms, which were introduced in [1] as follows:

$$\rho \frac{\partial^2 \eta(x)}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial \eta(x)}{\partial t} \right|^2 dx \right) \frac{\partial^2 \eta(x)}{\partial x^2} = 0, \tag{4}$$

where parameters ρ, p_0, h, E, and L are real positive constants. Equation (4) is a nonlocal problem, which contains a nonlocal coefficient $\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial \eta(x)}{\partial t} \right|^2 dx$, and has a wide range of applications and research in physical systems, such as non-homogeneous Kirchhoff-type equations in $\mathbb{R}^N$ [2], nonlocal Kirchhoff equations of elliptic type [3], Kirchhoff–Schrödinger type equations [4], $p(x)$-Laplacian Dirichlet problem [5,6], Kirchhoff–Choquard equations involving variable-order [7,8], fractional $p(\cdot)$-Kirchhoff type problem in $\mathbb{R}^N$ [9], Kirchhoff-type equations involving the fractional $p_1(x)\&p_2(x)$-Laplace operator [10], fractional $p(x,\cdot)$-Kirchhoff-type problems in $\mathbb{R}^N$ [11], and fractional Sobolev space and applications to nonlocal variational problems [12]. For more Kirchhoff-type problems, we also mention that [13] studied a class of Kirchhoff nonlocal fractional equations and obtained the existence of three solutions, Ref. [14] discussed a class of p-Kirchhoff equations via the fountain theorem and dual fountain theorem, and Ref. [15] researched the existence of non-negative solutions for a Kirchhoff type problem driven by a non-local integro-differential operator.

Let $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$ and $p(x), q(x) : \mathbb{R}^N \to (1, +\infty)$ be continuous functions, which satisfy the following conditions:

H_m: There are some constants $m_{v(x)} = m_{v(x)}(\iota) > 0 (v(x) \in \{p(x), q(x)\})$ for all $\iota > 0$ such that

$$M_{v(x)}(t) \geq m_{v(x)}, \ \text{ for any } t > \iota.$$

H_{pq}: The conditions that we impose on $p(x), q(x)$ are as follows:

$$1 < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < +\infty,$$

$$1 < q^- := \inf_{x \in \overline{\Omega}} q(x) \leq q^+ := \sup_{x \in \overline{\Omega}} q(x) < +\infty.$$

Another significant characteristic of the problem (1) is the presence of double operators, which comes from the following system

$$\eta_t = \operatorname{div}[D\eta \nabla \eta] + c(x, \eta), \tag{5}$$

where $D\eta = |\nabla \eta|^{p-2} + |\nabla \eta|^{q-2}$ and $c(x, \eta)$ is a polynomial of η. System (5) had a wide range of applications in the field of physics and related sciences, for example, on the stationary solutions of generalized reaction diffusion equations [16], elliptic problems with critical growth in $\mathbb{R}^N$ [17], nontrivial solutions to nonlinear elliptic equation in $\mathbb{R}^N$ [18], and fractional Choquard problems with variable order [19]. The function η in (5) describes a concentration, and the first term corresponds to the diffusion with a (generally non-constant) diffusion coefficient $D\eta$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, \eta)$ in (5) has a polynomial form with respect to the concentration η.

When $M_{v(x)} = 1 (v(x) \in \{p(x), q(x)\})$ and $\mu = 1$, Chung et al. in [20] devoted to the study of equations involving both $p_1(x)$-Laplacian and $p_2(x)$-Laplacian

$$\begin{cases} (-\Delta)^s_{p_1(\cdot)} \eta(x) + (-\Delta)^s_{p_2(\cdot)} \eta(x) + |\eta(x)|^{q(x)-2} \eta(x) \\ \qquad = \lambda V_1(x) |\eta(x)|^{r_1(x)-2} \eta(x) - \lambda V_2(x) |\eta(x)|^{r_2(x)-2} \eta(x), \ x \in \Omega, \\ \eta(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{6}$$

where p_1, p_2, q, r_1, and r_2 are different continuous functions, while V_1, V_2 are suitable weights. Equation (6) considered the local double Laplace operators, whose results differed from those of the single Laplace operator.

When $M_{v(x)} = 1(v(x) \in \{p(x), q(x)\})$ and $\mu = -1$, we mention that Motreanu in [21] considered Dirichlet problems with competing operators

$$\begin{cases} -\Delta_p \eta(x) + \Delta_q \eta(x) = g(x, \eta(x), \nabla\eta(x)), \text{ in } \Omega, \\ \eta(x) = 0, \text{ on } \partial\Omega, \end{cases} \tag{7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Equation (7) includes the sum $-\Delta_p + \Delta_q$ of the negative p-Laplacian Δ_p and of the q-Laplacian Δ_q, due to competition between $-\Delta_p$ and Δ_q, and the operator $-\Delta_p + \Delta_q$ has a different behavior in comparison to the operator $-\Delta_p + \Delta_q$. Moreover, the ellipticity and monotonicity property of the operator $-\Delta_p + \Delta_q$ are lost.

The third significant characteristic of the problem (1) is the presence of convection term $g(x, \eta, \nabla\eta)$, depending on the function η and on its gradient $\nabla\eta$, which makes the problem (1) non-variational, plays an important role in science and technology fields, and is widely used to describe physical phenomena. For example, due to convection and diffusion processes, particles or energy are converted and transferred inside physical systems. For the work related to this topic, we cite the interesting work [21–24] and their references.

The work in [25] focused on the p-Kirchhoff-type equations with gradient dependence in the reaction that is

$$\begin{cases} -M\left(\int_\Omega |\nabla\eta(x)|^p dx\right)\Delta_p \eta(x) = g(x, \eta(x), \nabla\eta(x)), \text{ in } \Omega, \\ \eta(x) = 0, \text{ on } \partial\Omega, \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. The existence of solutions for the problem (8) was obtained by utilizing Galerkin's approach.

One more reference on convection is Vetro [26], which was devoted to the study of the following $p(x)$-Kirchhoff-type equation:

$$-\Delta^K_{p(x)} \eta(x) = g(x, \eta(x), \nabla\eta(x)), \text{ in } \Omega, \ \eta|_{\partial\Omega} = 0. \tag{9}$$

The existence of weak solutions and generalized solutions for the problem (9) with gradient dependence was obtained via applying a topological method.

The nonlinearity $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, satisfying

H_{g_1}: There exist some constants $c < 1, d > 0$ and a function $\alpha \in [1, p^-)$ such that

$$g(x, \varpi, \nu)\varpi \le c|\nu|^{p(x)} + d(|\varpi|^{\alpha(x)} + 1), \text{ for a.e. } x \in \Omega \text{ and all } (\varpi, \nu) \in \mathbb{R} \times \mathbb{R}^N.$$

H_{g_2}: There exists a positive function $\phi(x) \in L^{p'(x)}(\Omega)$ and some positive constants a and b such that

$$|g(x, \varpi, \nu)| \le h(x) + a|\varpi|^{\phi(x)} + b|\nu|^{\frac{\psi(x)}{p'(x)}}, \text{ for a.e. } x \in \Omega \text{ and all } (\varpi, \nu) \in \mathbb{R} \times \mathbb{R}^N.$$

where $\phi(x) \in C(\overline{\Omega}), \psi(x) \in C(\overline{\Omega})$ such that $0 < \phi^- \le \phi^+ < p^- - 1$, $\left(\frac{\psi}{p'}\right)^+ < p^- - 1$.

The last significant characteristic of the problem (1) is the presence of logarithmic nonlinearity. The interest in studying problems with logarithmic nonlinearity is motivated not only by the purpose of describing mathematical and physical phenomena but also by their application in realistic models. For instance, in the biological population, we use the function $\eta(x)$ to represent the density of the population, and the logarithmic nonlinear term $|\eta|^{r(x)-2}\eta \ln|\eta|$ to denote external influencing factors.

Many scholars make efforts to investigate logarithmic nonlinearity, and, indeed, some important results were obtained; for example, see [27–30]. Peculiarly, Xiang et al. in [31] considered the following equation:

$$\begin{cases} M([\eta]_{s,p}^p)(-\Delta)_p^s \eta = h(x)|\eta|^{\theta p-2}\eta \ln|\eta| + \lambda|\eta|^{q-2}\eta, \ x \in \Omega, \\ \eta(x) = 0, \ x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{10}$$

where $M([\eta]_{s,p}^p) = [\eta]_{s,p}^{p(\theta-1)}$ and $h(x)$ is a sign-changing function. The existence of least energy solutions (10) was obtained by utilizing the Nehari manifold method.

Until now, there have been few papers to handle the equations involving logarithmic nonlinearity with variable exponents. Recently, Boudjeriou in [32] studied the following initial value problem:

$$\begin{cases} \eta_t(x) - \Delta_{p(x)}\eta(x) = |\eta(x)|^{s(x)-2}\eta(x)\log(|\eta(x)|), \ \text{in } \Omega, \ t > 0, \\ \eta(x) = 0, \ \text{in } \partial\Omega, \ t > 0, \\ \eta(x, 0) = \eta_0(x), \ \text{in } \Omega. \end{cases} \tag{11}$$

The weak solutions of Equation (11) were obtained under suitable conditions. Moreover, Zeng et al. in [33] were devoted to the study of equations with logarithmic nonlinearity and variable exponents by applying the logarithmic inequality.

Motivated by the previous and aforementioned cited works, there is no result for the Kirchhoff-type equations, which combine with variable exponents, competing $(p(x), q(x))$-Laplacian, logarithmic nonlinearity, and convection terms; therefore, we will investigate the existence of solutions for these kinds of equations, which are different from the work of [25,26,31,32]. Under weaker conditions on the nonlocal term $M_{v(x)}$ and the nonlinearities g, we prove the existence of finite-dimensional approximate solutions by using the Galerkin method and obtain the existence of weak solutions with the help of the Brezis theorem. One of the main difficulties and innovations of the present article is that we consider competing $(p(x), q(x))$-Laplacian, convective term, and logarithmic nonlinearity with variable exponents; another one is the weaker assumptions on nonlocal term $M_{v(x)}$ and nonlinear term g.

The present article is divided into six sections. Aside from Section 1, we have Section 2 given some preliminary notions and results about Lebesgue spaces and Sobolev spaces, and proved some technical lemmas. The finite-dimensional approximate solutions are obtained in Section 3. Section 4 discusses the existence of weak solutions by applying the Brezis theorem, and we give two examples of application of our theorems in Section 5 and present conclusions in Section 6.

2. Preliminary Results and Some Technical Lemmas

In this section, we briefly review some basic knowledge of generalized Lebesgue spaces and Sobolev spaces with variable exponents, and then give two technical lemmas.

For any real-valued function H defined on a domain Ω, we denote

$$C_+(\overline{\Omega}) := \left\{ H(x) \in C(\overline{\Omega}, \mathbb{R}) : \ 1 < H^- := \inf_{x\in\overline{\Omega}} H(x) \leq H(x) \leq H^+ := \sup_{x\in\overline{\Omega}} H(x) < +\infty \right\}.$$

Letting $\vartheta(x) \in C_+(\overline{\Omega})$, we define the generalized Lebesgue spaces with variable exponents as

$$L^{\vartheta(x)}(\Omega) := \left\{ \eta : \eta \text{ is a measurable function and } \int_\Omega |\eta|^{\vartheta(x)} dx < \infty \right\},$$

provided with the Luxemburg norm

$$\|\eta\|_{\vartheta(x)} = \|\eta\|_{L^{\vartheta(x)}(\Omega)} := \inf\left\{\chi > 0 : \int_\Omega \left|\frac{\eta}{\chi}\right|^{\vartheta(x)} dx \le 1\right\};$$

then, $(L^{\vartheta(x)}(\Omega), \|\cdot\|_{\vartheta(x)})$ is a separable and reflexive Banach spaces; see [34,35].

Lemma 1 (see [35]). *Let $\vartheta(x)$ be the conjugate exponent of $\widetilde{\vartheta}(x) \in C_+(\overline{\Omega})$, that is,*

$$\frac{1}{\vartheta(x)} + \frac{1}{\widetilde{\vartheta}(x)} = 1, \text{ for all } x \in \Omega.$$

Assume that $\eta \in L^{\vartheta(x)}(\Omega)$ and $\xi \in L^{\widetilde{\vartheta}(x)}(\Omega)$; then,

$$\left|\int_\Omega \eta\xi dx\right| \le \left(\frac{1}{\vartheta^-} + \frac{1}{\widetilde{\vartheta}^-}\right)\|\eta\|_{\vartheta(x)}\|\xi\|_{\widetilde{\vartheta}(x)} \le 2\|\eta\|_{\vartheta(x)}\|\xi\|_{\widetilde{\vartheta}(x)}.$$

Proposition 1 (see [36]). *The modular of $L^{\vartheta(x)}(\Omega)$, which is the mapping $\rho_{\vartheta(x)} : L^{\vartheta(x)}(\Omega) \to \mathbb{R}$, is defined by*

$$\rho_{\vartheta(x)}(\eta) := \int_\Omega |\eta|^{\vartheta(x)} dx.$$

Assume that $\eta_n, \eta \in L^{\vartheta(x)}(\Omega)$; then, the following properties hold:

(1) $\|\eta\|_{\vartheta(x)} > 1 \Rightarrow \|\eta\|^{\vartheta^-}_{\vartheta(x)} \le \rho_{\vartheta(x)}(\eta) \le \|\eta\|^{\vartheta^+}_{\vartheta(x)}$,
(2) $\|\eta\|_{\vartheta(x)} < 1 \Rightarrow \|\eta\|^{\vartheta^+}_{\vartheta(x)} \le \rho_{\vartheta(x)}(\eta) \le \|\eta\|^{\vartheta^-}_{\vartheta(x)}$,
(3) $\|\eta\|_{\vartheta(x)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{\vartheta(x)}(\eta) < 1$ (*resp.* $= 1, > 1$),
(4) $\|\eta_n\|_{\vartheta(x)} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho_{\vartheta(x)}(\eta_n) \to 0$ (*resp.* $\to +\infty$),
(5) $\lim_{n\to\infty} |\eta_n - \eta|_{\vartheta(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{\vartheta(x)}(\eta_n - \eta) = 0$.

Now, we consider the following generalized Sobolev spaces with variable exponents

$$W = W^{1,\vartheta(x)}(\Omega) := \left\{\eta \in L^{\vartheta(x)}(\Omega) : |\nabla\eta| \in L^{\vartheta(x)}(\Omega)\right\},$$

endowed with the norm

$$\|\eta\|_W := \|\eta\|_{\vartheta(x)} + \|\nabla\eta\|_{\vartheta(x)};$$

then, $(W, \|\cdot\|_W)$ is a separable and reflexive Banach spaces, see [34].

Lemma 2 (see [34]). *Assume that $\gamma(x) \in C_+(\overline{\Omega})$ fulfills*

$$1 < \gamma^- = \min_{x\in\overline{\Omega}} \gamma(x) \le \gamma(x) < \vartheta^*(x) = \frac{N\vartheta(x)}{N - \vartheta(x)}, \text{ for any } x \in \overline{\Omega}.$$

Then, there exists $C_\gamma = C_\gamma(N, \vartheta, \gamma, \Omega) > 0$ such that

$$\|\eta\|_{\gamma(x)} \le C_\gamma \|\eta\|_W,$$

for any $\eta \in W$. Moreover, the embedding $W \hookrightarrow L^{\gamma(x)}(\Omega)$ is compact.

Let W_0 denote the closure of $C_0^\infty(\Omega)$ in W with respect to the norm $\|\eta\|_{W_0}$, which is the subspace of W. Thus, the spaces $(W_0, \|\cdot\|_{W_0})$ are also separable and reflexive Banach spaces.

Remark 1. *According to the Poincaré inequality, we know that* $\|\nabla\eta\|_{\vartheta(x)}$ *and* $\|\eta\|_{W_0}$ *are equivalent norms in* W_0*. From now on, we work on* W_0 *and replace* $\|\eta\|_{W_0}$ *by* $\|\nabla\eta\|_{\vartheta(x)}$*, that is,*

$$\|\eta\|_{W_0} = \|\nabla\eta\|_{\vartheta(x)}, \textit{ for all } \eta \in W_0.$$

Remark 2. *To simplify the presentation, we will denote the norm of* W_0 *by* $\|\cdot\|$ *instead of* $\|\cdot\|_{W_0}$*.* W_0^* *denotes the dual space of* W_0*.*

Our technique of proof is based on Galerkin methods together with the fixed point theorem, whose proof may be found in Lions [37].

Lemma 3. *Let* W_0 *be a finite dimensional space with the norm* $\|\cdot\|$ *and let* $G : W_0 \to W_0^*$ *be a continuous mapping. Assume that there is a constant* $R > 0$ *such that*

$$\langle G(\eta), \eta\rangle \geq 0, \textit{ for all } \eta \in W_0 \textit{ with } \|\eta\| = R,$$

then $\eta \in W_0$ *exists with* $\|\eta\| \leq R$ *satisfying* $G(\eta) = 0$*.*

The following two Lemmas provide a useful growth estimate, related to logarithmic nonlinear terms, which play an important role during our proof process.

Lemma 4. *Assume that* $h(x) \in C_+(\overline{\Omega})$*; then, we have the following estimate:*

$$\ln t \leq \frac{1}{eh(x)} t^{h(x)} \leq \frac{1}{eh^-} t^{h(x)}, \textit{ for all } t \in [1, +\infty).$$

Proof. Let $h(x) \in C_+(\overline{\Omega})$, and we construct the following function:

$$f(t) = \ln t - \frac{1}{eh(x)} t^{h(x)}, \text{ for all } t \in [1, +\infty).$$

With respect to t, just by taking a simple derivative, we deduce

$$f'(t) = \frac{1}{t} - \frac{1}{e} t^{h(x)-1}, \text{ for all } t \in [1, +\infty),$$

and let $f'(t) = 0$; then, $t^* = e^{h^{-1}(x)}$. It is obvious that t^* is the unique maximum point of the function $f(t)$, so $f(t) \leq f(t^*) = 0$ for all $t \in [1, +\infty)$. Therefore, based on the above discussion, we can obtain the stated conclusion. □

Lemma 5. *Assume that, for all* $\eta \in W_0$ *and* $h(x), r(x) \in C_+(\overline{\Omega})$*, then the following inequality holds:*

$$\int_\Omega |\eta|^{r(x)} \ln|\eta| dx \leq C_{\Omega_1}|\Omega| + \frac{1}{eh^-} \max\left\{C_{h^+ + r^+}\|\eta\|^{h^+ + r^+}, C_{h^- + r^-}\|\eta\|^{h^- + r^-}\right\},$$

where $C_{\Omega_1}, C_{h^+ + r^+}, C_{h^- + r^-}$ *are some positive constants and* $h(x) + r(x) \leq h^+ + r^+ < 2p^- < p^*(x) = \frac{Np(x)}{N - p(x)}$*.*

Proof. Let $\Omega_1 = \{x \in \Omega : |\eta(x)| \leq 1\}$ and $\Omega_2 = \{x \in \Omega : |\eta(x)| \geq 1\}$; then,

$$\int_\Omega |\eta|^{r(x)} \ln|\eta| dx = \int_{\Omega_1} |\eta|^{r(x)} \ln|\eta| dx + \int_{\Omega_2} |\eta|^{r(x)} \ln|\eta| dx.$$

Since $|\eta(x)| \leq 1$, there exist $M_{r_1} > 0$ and $M_{r_2} > 0$ such that $|\eta|^{r(x)} < M_{r_1}$ and $\ln|\eta| < M_{r_2}$. By a simple calculation, we obtain

$$\int_{\Omega_1} |\eta|^{r(x)} \ln |\eta| dx < C_{\Omega_1} |\Omega|, \tag{12}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and $C_{\Omega_1} > 0$. Using Lemma 4 with $h(x) + r(x) \leq h^+ + r^+ < p^*(x)$, we deduce

$$\begin{aligned} \int_{\Omega_2} |\eta|^{r(x)} \ln |\eta| dx \leq & \frac{1}{eh^-} \int_{\Omega_2} |\eta|^{r(x)+h(x)} dx \\ \leq & \frac{1}{eh^-} \max\left\{ \|\eta\|^{h^+ + r^+}_{h(x)+r(x)}, \|\eta\|^{h^- + r^-}_{h(x)+r(x)} \right\}, \end{aligned}$$

in view of Lemma 2, and there exist some constants $C_{h^+ + r^+} > 0$ and $C_{h^- + r^-} > 0$ such that

$$\int_{\Omega_2} |\eta|^{r(x)} \ln |\eta| dx \leq \frac{1}{eh^-} \max\left\{ C_{h^+ + r^+} \|\eta\|^{h^+ + r^+}, C_{h^- + p^-} \|\eta\|^{h^- + r^-} \right\}. \tag{13}$$

It follows from (12) and (13) that

$$\int_{\Omega} |\eta|^{r(x)} \ln |\eta| dx \leq C_{\Omega_1} |\Omega| + \frac{1}{eh^-} \max\left\{ C_{h^+ + r^+} \|\eta\|^{h^+ + r^+}, C_{h^- + r^-} \|\eta\|^{h^- + r^-} \right\}.$$

This yields the stated conclusion. □

3. Finite Dimensional Approximate Solutions

Since W_0 is a reflexive and separable Banach space, see [34], and there exists an orthonormal basis $\{e_1, ..., e_n, ...\}$ in W_0, such that

$$W_0 = \overline{span\{e_1, ..., e_n\}}.$$

Define $X_n = span\{e_1, ..., e_n\}$, which means a sequence of vector X_n subspaces of W_0, satisfying

$$dim(X_n) < \infty \text{ for all } n \geq 1, X_n \subset X_{n+1} \text{ for all } n \geq 1, \text{ and } \overline{\bigcup_{n=1}^{\infty} X_n} = W_0.$$

It is known that X_n and $\mathbb{R}^N$ are isomorphic and, for $\eta \in \mathbb{R}^N$, we have a unique $\xi \in X_n$ by the identification

$$\eta \rightarrow \Sigma_{i=1}^{N} \xi_i e_i = \xi, \ \|\eta\| = |\xi|,$$

where $|\cdot|$ is the Euclidian norm in $\mathbb{R}^N$.

Theorem 1. *Assume that conditions H_m, H_{pq}, and H_{g_1} are satisfied; then,*

- *if $2p^- > p^+$ and $p^- > \alpha^+$, problem (1) admits a approximate solution for all $\mu \geq 0$ and $\lambda \leq 0$,*
- *if $2p^- > p^+$, $p^- > q^+$ and $p^- > \alpha^+$, the problem (1) admits a approximate solution for all $\mu < 0$ and $\lambda \leq 0$,*
- *if $2q^- > p^+$ and $q^- > \alpha^+$, problem (1) admits a approximate solution for all $\mu \geq 0$ and $\lambda \leq 0$,*
- *if $2p^- > r^+ + h^+$ and $p^- > \alpha^+$, problem (1) admits a approximate solution for all $\mu \geq 0$ and $\lambda > 0$,*
- *if $2p^- > r^+ + h^+$, $p^- > q^+$ and $p^- > \alpha^+$, problem (1) admits a approximate solution for all $\mu < 0$ and $\lambda > 0$,*
- *if $2q^- > r^+ + h^+$ and $q^- > \alpha^+$, problem (1) admits a approximate solution for all $\mu \geq 0$ and $\lambda > 0$,*

that is, for all $n \geq 1$ and $\varphi \in X_n$, there exists $\eta_n \in X_n$ such that

$$\begin{aligned} &M_{p(x)}\Big(\delta_{p(x)}(\eta_n)\Big)\langle \eta_n, \varphi\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta_n)\Big)\langle \eta_n, \varphi\rangle_{q(x)} \\ &\quad = \lambda \int_\Omega \Big(|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n|\Big)\varphi dx + \int_\Omega g(x,\eta_n,\nabla\eta_n)\varphi dx. \end{aligned} \tag{14}$$

Proof. For all $\eta \in X_n$, we consider the mapping $G = (G_1, G_2, ..., G_N) : \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned} G_i =& M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle \eta, e_i\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle \eta, e_i\rangle_{q(x)} \\ &- \lambda \int_\Omega \Big(|\eta|^{r(x)-2}\eta \ln|\eta|\Big)e_i dx - \int_\Omega g(x,\eta,\nabla\eta)e_i dx. \end{aligned}$$

The following work shows that, for each $n \geq 1$, problem (1) has an approximate solution η_n in X_n, namely

$$\begin{aligned} &M_{p(x)}\Big(\delta_{p(x)}(\eta_n)\Big)\langle \eta_n, e_i\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta_n)\Big)\langle \eta_n, e_i\rangle_{q(x)} \\ &\quad = \lambda \int_\Omega \Big(|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n|\Big)e_i dx + \int_\Omega g(x,\eta_n,\nabla\eta_n)e_i dx. \end{aligned} \tag{15}$$

For $\eta \in X_n$, we have

$$\begin{aligned} \langle G, \eta\rangle =& M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle \eta, \eta\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle \eta, \eta\rangle_{q(x)} \\ &- \int_\Omega |\eta|^{r(x)} \ln|\eta| dx - \int_\Omega g(x,\eta,\nabla\eta)\eta dx, \\ \geq& \frac{1}{p^+}\left(\int_\Omega |\nabla\eta|^{p(x)}dx\right)^2 + \frac{\mu}{q^+}\left(\int_\Omega |\nabla\eta|^{p(x)}dx\right)^2 \\ &- \lambda \int_\Omega \Big||\eta|^{r(x)} \ln|\eta|\Big| dx - \int_\Omega |g(x,\eta,\nabla\eta)\eta| dx. \end{aligned}$$

From H_{g_1} and Lemma 5, we have the following estimate:

$$\begin{aligned} \langle G, \eta\rangle \geq& \frac{1}{p^+}\left(\int_\Omega |\nabla\eta|^{p(x)}dx\right)^2 + \frac{\mu}{q^+}\left(\int_\Omega |\nabla\eta|^{q(x)}dx\right)^2 \\ &- \frac{\lambda}{eh^-}\max\Big\{C_{h^+ + r^+}\|\eta\|^{h^+ + r^+}, C_{h^- + r^-}\|\eta\|^{h^- + r^-}\Big\} \\ &- \lambda C_{\Omega_1}|\Omega| - c\int_\Omega |\nabla\eta|^{p(x)}dx - d\int_\Omega (|\eta|^{\alpha(x)} + 1)dx. \end{aligned}$$

According to Remark 1 and Lemma 2, there exist some positive constants C_{α^+} and C_{α^-}, such that

$$\begin{aligned} \langle G, \eta\rangle \geq& \frac{1}{p^+}\min\Big\{\|\eta\|^{2p^+}, \|\eta\|^{2p^-}\Big\} + \frac{\mu}{q^+}\min\Big\{\|\eta\|^{2q^+}, \|\eta\|^{2q^-}\Big\} \\ &- \frac{\lambda}{eh^-}\max\Big\{C_{h^+ + r^+}\|\eta\|^{h^+ + r^+}, C_{h^- + r^-}\|\eta\|^{h^- + r^-}\Big\} \\ &- d\max\Big\{C_{\alpha^+}\|\eta\|^{\alpha^+}, C_{\alpha^-}\|\eta\|^{\alpha^-}\Big\} - c\max\Big\{\|\eta\|^{p^+}, \|\eta\|^{p^-}\Big\} \\ &- (\lambda C_{\Omega_1} + d)|\Omega|. \end{aligned}$$

If $\|\eta\| > 1$, then

$$\begin{aligned} \langle G, \eta\rangle \geq& \frac{1}{p^+}\|\eta\|^{2p^-} + \frac{\mu}{q^+}\min\Big\{\|\eta\|^{2q^+}, \|\eta\|^{2q^-}\Big\} - \frac{\lambda C_{h^+ + r^+}}{eh^-}\|\eta\|^{h^+ + r^+} \\ &- dC_\alpha\|\eta\|^{\alpha^+} - c\|\eta\|^{p^+} - (\lambda C_{\Omega_1} + d)|\Omega|. \end{aligned}$$

Combined with the above analysis, we deduce that

Case 1: Utilizing that $2p^- > p^+$ and $p^- > \alpha^+$ with $\mu \geq 0$ and $\lambda \leq 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\langle G, \eta \rangle \geq \frac{1}{p^+} \|\eta\|^{2p^-} - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} - d|\Omega| \geq 0,$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

Case 2: Utilizing that $2p^- > p^+$, $p^- > q^+$ and $p^- > \alpha^+$ with $\mu < 0$ and $\lambda \leq 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\begin{aligned} \langle G, \eta \rangle \geq & \frac{1}{p^+} \|\eta\|^{2p^-} + \frac{\mu}{q^+} \|\eta\|^{2q^+} \\ & - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} - d|\Omega| \geq 0, \end{aligned}$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

Case 3: Utilizing that $2q^- > p^+$ and $q^- > \alpha^+$ with $\mu \geq 0$ and $\lambda \leq 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\langle G, \eta \rangle \geq \frac{\mu}{q^+} \|\eta\|^{2q^-} - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} - d|\Omega| \geq 0,$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

Case 4: Utilizing that $2p^- > r^+ + h^+$ and $p^- > \alpha^+$ with $\mu \geq 0$ and $\lambda > 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\begin{aligned} \langle G, \eta \rangle \geq & \frac{1}{p^+} \|\eta\|^{2p^-} - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} \\ & - \frac{\lambda C_{h^+ + r^+}}{eh^-} \|\eta\|^{h^+ + r^+} - (\lambda C_{\Omega_1} + d)|\Omega| \geq 0, \end{aligned}$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

Case 5: Utilizing that $2p^- > r^+ + h^+$, $p^- > q^+$ and $p^- > \alpha^+$ with $\mu < 0$ and $\lambda > 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\begin{aligned} \langle G, \eta \rangle \geq & \frac{1}{p^+} \|\eta\|^{2p^-} + \frac{\mu}{q^+} \|\eta\|^{2q^+} - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} \\ & - \frac{\lambda C_{h^+ + r^+}}{eh^-} \|\eta\|^{h^+ + r^+} - (\lambda C_{\Omega_1} + d)|\Omega| \geq 0, \end{aligned}$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

Case 6: Utilizing that $2q^- > r^+ + h^+$ and $q^- > \alpha^+$ with $\mu \geq 0$ and $\lambda > 0$, there exists a positive constant R, provided at a sufficiently large size, such that

$$\begin{aligned} \langle G, \eta \rangle \geq & \frac{\mu}{q^+} \|\eta\|^{2q^-} - c\|\eta\|^{p^+} - dC_\alpha \|\eta\|^{\alpha^+} \\ & - \frac{\lambda C_{h^+ + r^+}}{eh^-} \|\eta\|^{h^+ + r^+} - (\lambda C_{\Omega_1} + d)|\Omega| \geq 0, \end{aligned}$$

for all $\eta \in X_n$, with $\|\eta\| = R$.

In the above six cases, G is continuous, so, in view of Lemma 3, problem (1) admits a approximate solution η_n in $X_n \subset W_0$ with $\|\eta_n\| \leq R$. □

Corollary 1. *Assume that the conditions of Theorem 1 are satisfied, then the sequence $\{\eta_n\}_{n \geq 1}$ with $\eta_n \in X_n$ constructed in Theorem 1 is bounded in W_0.*

Proof. If $\|\eta_n\| \leq 1$ for all $n \in \mathbb{N}$, then the sequence $\{\eta_n\}_{n \in N}$ is bounded in W_0.

If $\|\eta_n\| > 1$ for all $n \in \mathbb{N}$, with η_n in place of φ in (14), we have

$$M_{p(x)}\Big(\delta_{p(x)}(\eta_n)\Big)\langle \eta_n, \eta_n\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta_n)\Big)\langle \eta_n, \eta_n\rangle_{q(x)}$$
$$= \lambda \int_\Omega \Big(|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n|\Big)\eta_n dx + \int_\Omega g(x, \eta_n, \nabla\eta_n)\eta_n dx.$$

On the basis of condition H_{g_1} and Lemma 5, it gives

$$\frac{1}{p^+}\Big(\int_\Omega |\nabla\eta_n|^{p(x)}dx\Big)^2 + \frac{\mu}{q^+}\Big(\int_\Omega |\nabla\eta_n|^{q(x)}dx\Big)^2$$
$$\leq c\int_\Omega |\nabla\eta_n|^{p(x)}dx + d\int_\Omega (|\eta_n|^{\alpha(x)} + 1)dx + \lambda C_{\Omega_1}|\Omega|$$
$$+ \frac{\lambda}{eh^-}\max\Big\{C_{h^+ + r^+}\|\eta_n\|^{h^+ + r^+}, C_{h^- + r^-}\|\eta_n\|^{h^- + r^-}\Big\}.$$

Case 1: Recalling that $2p^- > p^+$ and $p^- > \alpha^+$ with $\mu \geq 0$ and $\lambda \leq 0$, and, by Lemmas 1 and 2, we deduce

$$\frac{1}{p^+}\|\eta_n\|^{2p^-} \leq c\|\eta_n\|^{p^+} + dC_{\alpha^+}\|\eta_n\|^{\alpha^+} + d|\Omega|.$$

Case 2: Recalling that $2p^- > p^+$, $p^- > q^+$ and $p^- > \alpha^+$ with $\mu < 0$ and $\lambda \leq 0$, and by Lemmas 1 and 2, we deduce

$$\frac{1}{p^+}\|\eta_n\|^{2p^-} \leq -\frac{\mu}{q^+}\|\eta_n\|^{2q^+} + c\|\eta_n\|^{p^+} + dC_{\alpha^+}\|\eta_n\|^{\alpha^+} + d|\Omega|.$$

Case 3: Recalling that $2q^- > p^+$ and $q^- > \alpha^+$ with $\mu \geq 0$ and $\lambda \leq 0$, and by Lemmas 1 and 2, we deduce

$$\frac{\mu}{q^+}\|\eta_n\|^{2q^-} \leq c\|\eta_n\|^{p^+} + dC_\alpha\|\eta_n\|^{\alpha^+} + d|\Omega|.$$

Case 4: Recalling that $2p^- > r^+ + h^+$ and $p^- > \alpha^+$ with $\mu \geq 0$ and $\lambda > 0$, and by Lemmas 1 and 2, we deduce

$$\frac{1}{p^+}\|\eta_n\|^{2p^-} \leq c\|\eta_n\|^{p^+} + dC_\alpha\|\eta_n\|^{\alpha^+}$$
$$+ \frac{\lambda C_{h^+ + r^+}}{eh^-}\|\eta_n\|^{h^+ + r^+} + (\lambda C_{\Omega_1} + d)|\Omega|.$$

Case 5: Recalling that $2p^- > r^+ + h^+$, $p^- > q^+$ and $p^- > \alpha^+$ with $\mu < 0$ and $\lambda > 0$, and by Lemmas 1 and 2, we deduce

$$\frac{1}{p^+}\|\eta_n\|^{2p^-} \leq -\frac{\mu}{q^+}\|\eta_n\|^{2q^+} + c\|\eta_n\|^{p^+} + dC_\alpha\|\eta_n\|^{\alpha^+}$$
$$+ \frac{\lambda C_{h^+ + r^+}}{eh^-}\|\eta_n\|^{h^+ + r^+} + (\lambda C_{\Omega_1} + d)|\Omega|.$$

Case 6: Recalling that $2p^- > r^+ + h^+$ and $q^- > \alpha^+$ with $\mu \geq 0$ and $\lambda > 0$, and by Lemmas 1 and 2, we deduce

$$\frac{\mu}{q^+}\|\eta_n\|^{2q^-} \leq c\|\eta_n\|^{p^+} - dC_\alpha\|\eta_n\|^{\alpha^+}$$
$$+ \frac{\lambda C_{h^+ + r^+}}{eh^-}\|\eta_n\|^{h^+ + r^+} + (\lambda C_{\Omega_1} + d)|\Omega|.$$

In the above six cases, we conclude that the sequence $\{\eta_n\}_{n\geq 1}$ is bounded in W_0. $\square$

4. Existence of Weak Solutions

In this section, our interest is devoted to the existence of weak solutions for problem (1). The following are the main results of this section.

Theorem 2. *Assume that conditions H_m, H_{pq}, and H_{g_2} are satisfied, then, for all $\mu > 0$,*

- *if $2p^- > \phi^+ + 1$ and $2p^- > (\frac{\psi}{p'})^+$, problem (1) admits at least one weak solution with $\lambda \leq 0$.*
- *if $2p^- > h^+ + r^+$, $2p^- > \phi^+ + 1$ and $2p^- > (\frac{\psi}{p'})^+$, problem (1) admits at least one weak solution with $\lambda > 0$.*
- *if $2q^- > \phi^+ + 1$ and $2q^- > (\frac{\psi}{p'})^+$, problem (1) admits at least one weak solution with $\lambda \leq 0$.*
- *if $2q^- > h^+ + r^+$, $2q^- > \phi^+ + 1$ and $2q^- > (\frac{\psi}{p'})^+$, problem (1) admits at least one weak solution with $\lambda > 0$.*

Corollary 2. *Assume that the conditions of Theorem 2 are satisfied; then, the sequence $\{\eta_n\}_{n\geq 1}$ with $\eta_n \in X_n$ is bounded in W_0.*

Proof. The proof is similar to Corollary 1, which we omit. □

To prove Theorems 2, we use the Brezis theorem for pseudomonotone operators in the separable reflexive space (see (Theorem 27.A [38]). Let us define the operator $T : W_0 \to W_0^*$ as

$$\begin{aligned}\langle T\eta, \varphi\rangle =& M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{q(x)} \\ &- \lambda \int_\Omega \Big(|\eta|^{r(x)-2}\eta \ln|\eta|\Big)\varphi dx - \int_\Omega g(x, \eta, \nabla\eta)\varphi dx,\end{aligned}$$

for all $\eta, \varphi \in W_0$.

Lemma 6. *Assume that the conditions of Theorem 2 are satisfied; then, the operator T is bounded.*

Proof. Let $\eta \in W_0$ be fixed and denote by Φ_η the linear functional on W_0 , defined as

$$\Phi_\eta(\varphi) = \int_\Omega |\nabla\eta|^{v(x)-2}\nabla\eta\nabla\varphi dx,$$

for any $\varphi \in W_0$ and $v(x) \in \{p(x), q(x)\}$. By Hölder inequality,

$$|\Phi_\eta(\varphi)| \leq \|\eta\| \, \|\varphi\|, \text{ for all } \eta, \varphi \in W_0. \tag{16}$$

Obviously, $\Phi_\eta(\varphi)$ is bounded. From the hypothesis H_m and Proposition 1, there exist some constants $C_{v_1}, C_{v_2} > 0$ such that

$$0 < C_{v_1} \leq M_{v(x)}\Big(\delta_{v(x)}(\eta)\Big) \leq C_{v_2},$$

which, together with (16), there exists a constant $C_{v(x)} > 0$ such that

$$|M_{v(x)}\Big(\delta_{v(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{v(x)}| \leq C_{v(x)}. \tag{17}$$

In fact, by a simple calculation for the logarithmic nonlinear term, we deduce

$$\int_\Omega \left| |\eta|^{r(x)-2}\eta \ln|\eta| \right|^{\frac{r^+}{r^+-1}} dx = \int_{\Omega_1} \left| |\eta|^{r(x)-2}\eta \ln|\eta| \right|^{\frac{r^+}{r^+-1}} dx + \int_{\Omega_2} \left| |\eta|^{r(x)-2}\eta \ln|\eta| \right|^{\frac{r^+}{r^+-1}} dx$$
$$\leq C_{\Omega_1}|\Omega| + \int_{\Omega_2} \left| |\eta|^{r(x)-2}\eta \ln|\eta| \right|^{\frac{r^+}{r^+-1}} dx.$$

Since $r^+ < p^*(x)$, then, by using the continuous embedding $L^{p^*(x)}(\Omega) \hookrightarrow L^{r^+}(\Omega)$ and combining Lemma 4, we deduce

$$\int_\Omega \left| |\eta|^{r(x)-2}\eta \ln|\eta| \right|^{\frac{r^+}{r^+-1}} dx \leq C_{\Omega_1}|\Omega| + \int_\Omega |\eta|^{r^+} dx$$
$$\leq C_{\Omega_1}|\Omega| + C_{\Omega_2}\|\eta\|_{p^*(x)}. \tag{18}$$

where $C_{\Omega_2} > 0$. Notice that the relation (18) implies that

$$\left\| |\eta|^{r(x)-1} \ln|\eta| \right\|_{L^{\frac{r^+}{r^+-1}}(\Omega)} \leq C_{\frac{r^+}{r^+-1}},$$

where $C_{\frac{r^+}{r^+-1}} > 0$. Using the Hölder inequality and taking into account the embeddings, for any $\varphi \in W_0$ with $\|\varphi\| \leq 1$,

$$\left| \int_\Omega \varphi |\eta|^{r(x)-2}\eta \ln|\eta| dx \right| \leq \|\varphi\|_{L^{r^+}(\Omega)} \left\| |\eta|^{r(x)-2}\eta \ln|\eta| \right\|_{L^{\frac{r^+}{r^+-1}}(\Omega)} \leq C_{\frac{r^+}{r^+-1}}. \tag{19}$$

From hypothesis $\mathcal{G}_1$ and Jensen's inequality, for all $\eta \in X$, we have

$$\begin{aligned}
&\int_\Omega |g(x,\eta,\nabla\eta)|^{p'(x)} dx \\
\leq &\int_\Omega \left[|h(x)| + |a|\eta|^{\phi(x)}| + |b|\nabla\eta|^{\frac{\psi(x)}{p'(x)}}| \right]^{p'(x)} dx \\
\leq &3^{(q')^+-1} \left[\int_\Omega |h(x)|^{p'(x)} dx + \int_\Omega |a|\eta|^{\phi(x)}|^{p'(x)} dx + \int_\Omega |b|\nabla\eta|^{\frac{\psi(x)}{p'(x)}}|^{p'(x)} dx \right] \\
\leq &C_{p'} \left[\int_\Omega |h(x)|^{p'(x)} dx + \int_\Omega |\eta|^{p'(x)\phi(x)} dx + \int_\Omega |\nabla\eta|^{\psi(x)} dx \right],
\end{aligned} \tag{20}$$

where $C_{p'} = 3^{(q')^+-1} \max\left\{1, a^{(p')^-}, a^{(p')^+}, b^{(p')^-}, b^{(p')^+}\right\}$. It follows from (20) and Proposition 1 that we have

$$\int_\Omega |g(x,\eta,\nabla\eta)|^{p'(x)} dx \leq C_{p'} \left[3 + |h|_{p'}^{(p')^+} + |\eta|_{p'\phi}^{(p'\phi)^+} + \|\eta\|^{\psi^+} \right]$$
$$\leq C_{p'} \left[3 + |h|_{p'}^{(p')^+} + C_{p'\phi}^{(p'\phi)^+} \|\eta\|_{p'\phi}^{(p'\phi)^+} + \|\eta\|^{\psi^+} \right].$$

Hence, invoking Proposition 1, we infer

$$|g(x,\eta,\nabla\eta)|_{p'} \leq \left\{ 1 + C_{p'}[3 + |h|_{p'}^{(p')^+} + C_{p'\phi}^{(p'\phi)^+} \|\eta\|_{p'\phi}^{(p'\phi)^+} + \|\eta\|^{\psi^+}] \right\}^{\frac{1}{(p')^-}}. \tag{21}$$

Utilizing Lemma 1and taking into account the embeddings, for all $\varphi \in W_0$ with $\|\varphi\| \leq 1$,

$$\left| \int_\Omega g(x,\eta,\nabla\eta)\varphi dx \right| \leq 2|g(x,\eta,\nabla\eta)|_{p'}|\varphi|_p \leq 2|g(x,\eta,\nabla\eta)|_{p'}. \tag{22}$$

Thus, it follows from these estimates (17), (19), and (22) that we easily determine the boundedness of T. □

Lemma 7. *Assume that the conditions of Theorem 2 are satisfied; then, the operator T is demicontinuous.*

Proof. Assuming that $\eta_n \to \eta$ in W_0, we show that $T\eta_n \to T\eta$ in W_0^*, that is,

$$
\begin{aligned}
&M_{p(x)}\Big(\delta_{p(x)}(\eta_n)\Big)\langle \eta_n, \varphi\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta_n)\Big)\langle \eta_n, \varphi\rangle_{q(x)} \\
&\quad - \lambda \int_\Omega \Big(|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n|\Big)\varphi dx - \int_\Omega g(x, \eta_n, \nabla\eta_n)\varphi dx \\
&\to M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{q(x)} \\
&\quad - \lambda \int_\Omega \Big(|\eta|^{r(x)-2}\eta \ln|\eta|\Big)\varphi dx - \int_\Omega g(x, \eta, \nabla\eta)\varphi dx.
\end{aligned}
\tag{23}
$$

Since $\eta_n \to \eta$ in W_0, up to a subsequence, we have

$$
\eta_n \to \eta \text{ and } \nabla\eta_n \to \nabla\eta, \text{ a.e. in } \Omega. \tag{24}
$$

Thus, we have

$$
||\nabla\eta_n|^{v(x)-2}\nabla\eta_n|^{p'(x)} \le \|\eta_n\|(1 + \|\eta_n\|_{p'}^{v^+-2}), v \in \{p(x), q(x)\}. \tag{25}
$$

which imply that $\{|\nabla\eta_n|^{v(x)-2}\nabla\eta_n\}$ are bounded in $L^{p'}(\Omega)$.

For $v(x) \in \{p(x), q(x)\}$, we obtain

$$
\begin{cases}
|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n| \to |\eta|^{r(x)-2}\eta \ln|\eta|, \text{ a.e. in } \Omega, \\
|\nabla\eta_n|^{v(x)-2}\nabla\eta_n \to |\nabla\eta|^{v(x)-2}\nabla\eta, \text{ a.e. in } \Omega, \\
g(x, \eta_n, \nabla\eta_n) \to g(x, \eta, \nabla\eta), \text{ a.e. in } \Omega.
\end{cases}
\tag{26}
$$

Moreover, the boundedness of $\{\eta_n\}$ in W_0 and (22) imply that $\{g(x, \eta_n, \nabla\eta_n)\}$ are bounded in $L^{p'}(\Omega)$, and (19) implies that $\{|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n|\}$ are bounded in $L^{r'}(\Omega)$. Thanks to (17), (24), and (26), combined with H_m and Proposition 1, we obtain

$$
M_{v(x)}\Big(\delta_{v(x)}(\eta_n)\Big)\langle \eta_n, \varphi\rangle_{v(x)} \to M_{v(x)}\Big(\delta_{v(x)}(\eta)\Big)\langle \eta, \varphi\rangle_{v(x)}. \tag{27}
$$

Now, we show that the following conclusion holds:

$$
\int_\Omega g(x, \eta_n, \nabla\eta_n)\varphi dx \to \int_\Omega g(x, \eta, \nabla\eta)\varphi dx. \tag{28}
$$

Let $g(x, \eta_n, \nabla\eta_n), g(x, \eta, \nabla\eta) \in L^{p'}(\Omega)$, and

$$
E(N) = \{x \in: |g(x, \eta_n, \nabla\eta_n) - g(x, \eta, \nabla\eta)| \le 1, \text{ for all } n \ge N\}.
$$

Since $\text{meas}(E(N)) \to \text{meas}(\Omega)$ as $N \to \infty$, and setting

$$
\mathcal{F}_N = \Big\{\Psi_N \in L^{p''(x)}(\Omega) : \Psi_N \equiv 0 \text{ a.e. in } \Omega \setminus E(N)\Big\}.
$$

First, we prove that $\mathcal{F}_N$ is dense in $L^{p''(x)}(\Omega)$. Let $f \in L^{p''(x)}(\Omega)$ and

$$
f_N(x) = \begin{cases} f(x) & \text{if} x \in (E(N)), \\ 0 & \text{if} x \in \Omega \setminus (E(N)). \end{cases}
$$

Then,

$$\varrho_{p''(x)}\Big(f_N(x)-f(x)\Big)=\int_{E(N)}|f_N(x)-f(x)|^{p''(x)}dx+\int_{\Omega\setminus E(N)}|f_N(x)-f(x)|^{p''(x)}dx$$
$$=\int_{\Omega\setminus E(N)}|f(x)|^{p''(x)}dx$$
$$=\int_{\Omega}|f(x)|^{p''(x)}\chi_{\Omega\setminus E(N)}dx.$$

Taking $\Phi_N=|f(x)|^{p''(x)}\chi_{\Omega\setminus E(N)}$ for almost every x in Ω, we have

$$\Phi_N\to 0 \text{ a.e. in } \Omega \text{ and } |\Phi_N|\leq |f|^{p''(x)}.$$

Utilizing the dominated convergence theorem, we infer

$$\varrho_{p''(x)}(f_N(x)-f(x))\to 0 \text{ as } N\to\infty,$$

hence $f_N\to f$ in $L^{p''(x)}(\Omega)$. Thus, $\mathcal{F}_{\mathcal{N}}$ is dense in $L^{p''(x)}(\Omega)$.

Next, for all $\varphi\in\mathcal{F}_{\mathcal{N}}$, let us show that

$$\lim_{n\to\infty}\int_\Omega\Big(g(x,\eta_n,\nabla\eta_n)-g(x,\eta,\nabla\eta)\Big)\varphi(x)dx=0. \tag{29}$$

Since $\varphi\equiv 0$ in $\Omega\setminus E(N)$, it suffices to prove that

$$\int_{E(N)}\Big(g(x,\eta_n,\nabla\eta_n)-g(x,\eta,\nabla\eta)\Big)\varphi(x)dx\to 0 \text{ as } n\to\infty.$$

Let $\varphi_n=\varphi(g(x,\eta_n,\nabla\eta_n)-g(x,\eta,\nabla\eta))$. Since $|(g(x,\eta_n,\nabla\eta_n-g(x,\eta,\nabla\eta))\varphi(x)|\leq\varphi(x)$ a.e. in $E(N)$ and $\varphi_n\to 0$ a.e. in Ω, thanks to the dominated convergence theorem, we deduce $\varphi_n\to 0$ in $L^1(\Omega)$, which implies that (29) holds.

It follows from the density of $\mathcal{F}_{\mathcal{N}}$ in $L^{p''(x)}(\Omega)$ that we deduce

$$\lim_{n\to\infty}\int_\Omega g(x,\eta_n,\nabla\eta_n)\varphi(x)dx=\lim_{n\to\infty}\int_\Omega g(x,\eta,\nabla\eta)\varphi(x)dx,$$

for all $\varphi\in\mathcal{F}_{\mathcal{N}}$, which implies that (28) holds.

Using the same discussion as above, one can conclude that

$$\int_\Omega\Big(|\eta_n|^{r(x)-2}\eta_n\ln|\eta_n|\Big)\varphi dx\to\int_\Omega\Big(|\eta|^{r(x)-2}\eta\ln|\eta|\Big)\varphi dx. \tag{30}$$

As a result, it follows from (27), (28), and (30) that (23) holds, that is, the operator T is demicontinuous. □

Lemma 8. *Assume that the conditions of Theorem 2 are satisfied; then, for all $\mu>0$, the operator T is coercive.*

Proof. First, for all $\eta\in W_0$, we note that

$$\langle T\eta,\eta\rangle=M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle\eta,\eta\rangle_{p(x)}+\mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle\eta,\eta\rangle_{q(x)}$$
$$-\lambda\int_\Omega\Big(|\eta|^{r(x)-2}\eta\ln|\eta|\Big)\eta dx-\int_\Omega g(x,\eta,\nabla\eta)\eta dx. \tag{31}$$

To estimate the first and second integral terms, we deduce

$$
\begin{aligned}
&M_{p(x)}\Big(\delta_{p(x)}(\eta)\Big)\langle\eta,\eta\rangle_{p(x)} + \mu M_{q(x)}\Big(\delta_{q(x)}(\eta)\Big)\langle\eta,\eta\rangle_{q(x)} \\
&\quad\geq \frac{1}{p^+}\left(\int_\Omega |\nabla\eta|^{p(x)}dx\right)^2 + \frac{\mu}{q^+}\left(\int_\Omega |\nabla\eta|^{q(x)}dx\right)^2 \\
&\quad\geq \frac{1}{p^+}\min\Big\{\|\eta\|^{2p^+}, \|\eta\|^{2p^-}\Big\} + \frac{\mu}{q^+}\min\Big\{\|\eta\|^{2q^+}, \|\eta\|^{2q^-}\Big\} \\
&\quad\geq \frac{1}{p^+}\Big\{\|\eta\|^{2p^-} - 1\Big\} + \frac{\mu}{q^+}\Big\{\|\eta\|^{2q^-} - 1\Big\}.
\end{aligned} \tag{32}
$$

To estimate the third integral term, let $\Omega_1 = \{x \in \Omega : |\eta(x)| \leq 1\}$ and $\Omega_2 = \{x \in \Omega : |\eta(x)| \geq 1\}$; then,

$$
\int_\Omega |\eta|^{r(x)} \ln|\eta| dx = \int_{\Omega_1} |\eta|^{r(x)} \ln|\eta| dx + \int_{\Omega_2} |\eta|^{r(x)} \ln|\eta| dx.
$$

Using Lemma 4 with $h(x) + r(x) \leq h^+ + r^+ < p^*(x)$, we deduce

$$
\int_{\Omega_2} |\eta|^{r(x)} \ln|\eta| dx \leq \frac{1}{eh^-}\int_{\Omega_2} |\eta|^{r(x)+h(x)} dx \leq \frac{1}{eh^-}\Big(|\eta|^{h^+ + r^+}_{h(x)+r(x)} + 1\Big),
$$

in view of Lemma 2, there exist some constants $C_{h^+ + r^+} > 0$ and $C_{h^- + r^-} > 0$ such that

$$
\int_{\Omega_2} |\eta|^{r(x)} \ln|\eta| dx \leq \frac{1}{eh^-} C_{h^+ + r^+}\Big(\|\eta\|^{h^+ + r^+} + 1\Big).
$$

This implies that

$$
\int_\Omega |\eta|^{r(x)} \ln|\eta| dx \leq C_{\Omega_1}|\Omega| + \frac{1}{eh^-} C_{h^+ + r^+}\Big(\|\eta\|^{h^+ + r^+} + 1\Big), \tag{33}
$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and $C_{\Omega_1} > 0$. This yields the stated conclusion.

To estimate the fourth integral term, we deduce from H_{g_2}, the Hölder-type inequality, and Proposition 1 that

$$
\begin{aligned}
\int_\Omega g(x,\eta,\nabla\eta)\eta dx &\leq \int_\Omega \Big(h(x) + a|\eta|^{\phi(x)} + b|\nabla\eta|^{\frac{\psi(x)}{p'(x)}}\Big)\eta dx \\
&\leq |h|_{p'}|\eta|_p + a\Big(|\eta|^{\phi^+ + 1}_{\phi(x)+1} + 1\Big) + b\Big(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\Big)|\eta|_p \\
&\leq C_p|h|_{p'}\|\eta\| + aC^{\phi^+ + 1}_{\phi(x)+1}\Big(\|\eta\|^{\phi^+ + 1} + 1\Big) + bC_p\|\eta\|\Big(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\Big).
\end{aligned} \tag{34}
$$

It follows (32), (33), and (34) that

$$
\begin{aligned}
\langle T\eta,\eta\rangle \geq &\frac{1}{p^+}\Big\{\|\eta\|^{2p^-} - 1\Big\} + \frac{\mu}{q^+}\Big\{\|\eta\|^{2q^-} - 1\Big\} \\
&- \lambda C_{\Omega_1}|\Omega| - \frac{\lambda}{eh^-} C_{h^+ + r^+}\Big(\|\eta\|^{h^+ + r^+} + 1\Big) \\
&- C_p|h|_{p'}\|\eta\| - aC^{\phi^+ + 1}_{\phi(x)+1}\Big(\|\eta\|^{\phi^+ + 1} + 1\Big) - bC_p\|\eta\|\Big(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\Big).
\end{aligned} \tag{35}
$$

Case 1: Utilizing that $2p^- > \phi^+ + 1$ and $2p^- > (\frac{\psi}{p'})^+$ with $\lambda \leq 0,$ for all $\eta \in W_0$, such that

$$
\begin{aligned}
\langle T\eta,\eta\rangle \geq &\frac{1}{p^+}\Big\{\|\eta\|^{2p^-} - 1\Big\} - C_p|h|_{p'}\|\eta\| \\
&- aC^{\phi^+ + 1}_{\phi(x)+1}\Big(\|\eta\|^{\phi^+ + 1} + 1\Big) - bC_p\|\eta\|\Big(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\Big).
\end{aligned}
$$

Case 2: Utilizing that $2p^- > h^+ + r^+$, $2p^- > \phi^+ + 1$ and $2p^- > (\frac{\psi}{p'})^+$ with $\lambda > 0$, for all $\eta \in W_0$, such that

$$\begin{aligned}\langle T\eta, \eta\rangle \geq &\frac{1}{p^+}\left\{\|\eta\|^{2p^-} - 1\right\} - \lambda C_{\Omega_1}|\Omega| - \frac{\lambda}{eh^-}C_{h^+ + r^+}\left(\|\eta\|^{h^+ + r^+} + 1\right) \\ &- C_p|h|_{p'}\|\eta\| - aC^{\phi^+ + 1}_{\phi(x)+1}\left(\|\eta\|^{\phi^+ + 1} + 1\right) - bC_p\|\eta\|\left(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\right).\end{aligned}$$

Case 3: Utilizing that $2q^- > \phi^+ + 1$ and $2q^- > (\frac{\psi}{p'})^+$ with $\lambda \leq 0$, for all $\eta \in W_0$, such that

$$\begin{aligned}\langle T\eta, \eta\rangle \geq &\frac{\mu}{q^+}\left\{\|\eta\|^{2q^-} - 1\right\} - C_p|h|_{p'}\|\eta\| \\ &- aC^{\phi^+ + 1}_{\phi(x)+1}\left(\|\eta\|^{\phi^+ + 1} + 1\right) - bC_p\|\eta\|\left(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\right).\end{aligned}$$

Case 4: Utilizing that $2q^- > h^+ + r^+$, $2q^- > \phi^+ + 1$ and $2q^- > (\frac{\psi}{p'})^+$ with $\lambda > 0$, for all $\eta \in W_0$, such that

$$\begin{aligned}\langle T\eta, \eta\rangle \geq &\frac{\mu}{q^+}\left\{\|\eta\|^{2q^-} - 1\right\} - \lambda C_{\Omega_1}|\Omega| - \frac{\lambda}{eh^-}C_{h^+ + r^+}\left(\|\eta\|^{h^+ + r^+} + 1\right) \\ &- C_p|h|_{p'}\|\eta\| - aC^{\phi^+ + 1}_{\phi(x)+1}\left(\|\eta\|^{\phi^+ + 1} + 1\right) - bC_p\|\eta\|\left(\|\eta\|^{(\frac{\psi}{p'})^+} + 1\right).\end{aligned}$$

In the above four cases, we deduce the coerciveness of T from (35) as $\|\eta\| \to \infty$. □

Lemma 9. *Assume that the conditions of Theorem 2 are satisfied, then T is an (S_+)-type operator.*

Proof. Let $\{\eta_n\} \in W_0$ be such that $\eta_n \rightharpoonup \eta$ in W_0 as $n \to \infty$ and

$$\limsup_{n\to\infty}\langle T\eta_n - T\eta, \eta_n - \eta\rangle \leq 0.$$

First, note that

$$\begin{aligned}\langle T\eta_n, \eta_n - \eta\rangle = &M_{p(x)}\left(\delta_{p(x)}(\eta)\right)\langle \eta, \eta_n - \eta\rangle_{p(x)} + \mu M_{q(x)}\left(\delta_{q(x)}(\eta)\right)\langle \eta, \eta_n - \eta\rangle_{q(x)} \\ &- \lambda\int_\Omega\left(|\eta|^{r(x)-2}\eta\ln|\eta|\right)(\eta_n - \eta)dx - \int_\Omega g(x, \eta, \nabla\eta)(\eta_n - \eta)dx. \quad (36)\end{aligned}$$

Going if necessary up to a subsequence, we suppose there exists $\eta \in W_0$ such that

$$\begin{aligned}&\eta_n \rightharpoonup \eta, \text{ weakly in } W_0, \\ &\eta_n \to \eta, \text{ strongly in } L^{p(x)}(\Omega), \\ &\eta_n \to \eta, \text{ a.e. in } \Omega.\end{aligned} \quad (37)$$

Indeed, by a simple calculation for the logarithmic nonlinear term, we deduce

$$\begin{aligned}&\int_\Omega\left||\eta_n|^{r(x)-2}\eta_n\ln|\eta_n|\right|^{\frac{r^+}{r^+-1}}dx \\ &= \int_{\Omega_1}\left||\eta_n|^{r(x)-2}\eta_n\ln|\eta_n|\right|^{\frac{r^+}{r^+-1}}dx + \int_{\Omega_2}\left||\eta_n|^{r(x)-2}\eta_n\ln|\eta_n|\right|^{\frac{r^+}{r^+-1}}dx \\ &\leq C_{\Omega_1}|\Omega| + \int_{\Omega_2}\left||\eta_n|^{r(x)-2}\eta_n\ln|\eta_n|\right|^{\frac{r^+}{r^+-1}}dx.\end{aligned}$$

Since $r^+ < p^*(x)$, then, by using the continuous embedding $L^{p^*(x)}(\Omega) \hookrightarrow L^{r^+}(\Omega)$ and combining Lemma 4, we deduce

$$\begin{aligned}\int_\Omega \left| |\eta_n|^{r(x)-2}\eta_n \ln|\eta_n| \right|^{\frac{r^+}{r^+-1}} dx &\le C_{\Omega_1}|\Omega| + \int_\Omega |\eta_n|^{r^+} dx \\ &\le C_{\Omega_1}|\Omega| + C_{\Omega_2}\|\eta_n\|_{p^*(x)}.\end{aligned} \tag{38}$$

where $C_{\Omega_2} > 0$. In conjunction with Hölder's inequality, we obtain

$$\left| \int_\Omega (\eta_n - \eta)|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n| dx \right| \le \|\eta_n - \eta\|_{L^{r^+}(\Omega)} \left\| |\eta_n|^{r(x)-2}\eta_n \ln|\eta_n| \right\|_{L^{\frac{r^+}{r^+-1}}(\Omega)}. \tag{39}$$

Therefore, it follows from (37), (38) and (39) that

$$\left| \int_\Omega (\eta_n - \eta)|\eta_n|^{r(x)-2}\eta_n \ln|\eta_n| dx \right| \to 0, \text{ as } n \to \infty. \tag{40}$$

In the same fashion, utilizing Lemma 1, we have

$$\left| \int_\Omega g(x,\eta,\nabla\eta)(\eta_n - \eta) dx \right| \le 2|g(x,\eta,\nabla\eta)|_{p'}|(\eta_n - \eta)|_p.$$

By the boundedness of $\{\eta_n\} \in W_0$ and (37), we infer from the inequality above and the preceding estimate (21) that

$$\left| \int_\Omega g(x,\eta,\nabla\eta)(\eta_n - \eta) dx \right| \to 0, \text{ as } n \to \infty. \tag{41}$$

If $\eta_n \rightharpoonup \eta$ in W_0 and $\limsup_{n\to\infty}\langle T\eta_n - T\eta, \eta_n - \eta\rangle \le 0$, as a consequence

$$\lim_{n\to\infty}\langle T\eta_n, \eta_n - \eta\rangle = \lim_{n\to\infty}\langle T\eta_n - T\eta, \eta_n - \eta\rangle = 0. \tag{42}$$

By (40), (41), (42), and H_m, for $v \in \{p(x), q(x)\}$, as $n \to \infty$, we deduce

$$\int_\Omega (|\nabla\eta_n|^{v(x)-2}\nabla\eta_n - |\nabla\eta|^{v(x)-2}\nabla\eta)(\nabla\eta_n - \nabla\eta) dx \to 0.$$

Using the following Simon inequalities

$$|u_1 - u_2|^\tau \le \begin{cases} c_\tau[(|u_1|^{\tau-2}u_1 - |u_2|^{\tau-2}u_2)(u_1 - u_2)]^{\frac{\tau}{2}}(|u_1|^\tau + |u_2|^\tau)^{\frac{2-\tau}{\tau}}, \ 1 < \tau < 2, \\ \tilde{c}_\tau(|u_1|^{\tau-2}u_1 - |u_2|^{\tau-2}u_2)(u_1 - u), \ \tau \ge 2, \end{cases} \tag{43}$$

for all $u_1, u_2 \in \mathbb{R}^N$, where c_τ and $\tilde{c}_\tau$ are positive constants depending only on τ, we obtain

$$\int_\Omega |\nabla\eta_n - \nabla\eta|^{v(x)} dx \le \int_\Omega (|\nabla\eta_n|^{v(x)-2}\nabla\eta_n - |\nabla\eta|^{v(x)-2}\nabla\eta)(\nabla\eta_n - \nabla\eta) dx.$$

Hence,

$$\|\eta_n - \eta\| \to 0 \text{ as } n \to \infty,$$

that is, if $\eta_n \rightharpoonup \eta$ in W_0 and $\limsup_{n\to\infty}\langle T\eta_n - T\eta, \eta_n - \eta\rangle \le 0$, then $\eta_n \to \eta$ in W_0. This shows the (S_+)-property of T. □

Proof of Theorem 2. From Section 2, evidently, we know that W_0 is a real, separable, and reflexive Banach spaces. Moreover, it follows from Lemmas 6–9 that the operator T satisfies all conditions of the Brezis theorem. Hence, invoking the Brezis theorem, we obtain that $T\eta = 0$ has at least one solution η in W_0, i.e., problem (1) has at least one weak solution η. □

5. Examples

Now, we give two easy examples of application of our theorems. The first is when $M_{p(x)}(t) = a_p + b_p t$, for all $t \geq 0$ with $a_p > 0$, $b_p \geq 0$ and $M_{q(x)}(t) = a_q + b_q t$, for all $t \geq 0$ with $a_q > 0$, $b_q \geq 0$. In this case, problem (1) reduces to the following form.

Example 1. *Consider the problem*

$$\begin{cases} \Big(a_p + b_p \delta_{p(x)}(\eta)\Big)(-\Delta_{p(x)}\eta) + \mu\Big(a_q + b_q \delta_{q(x)}(\eta)\Big)(-\Delta_{q(x)}\eta) \\ \qquad = \lambda |\eta|^{r(x)-2}\eta \ln|\eta| + g(x,\eta,\nabla\eta), \ in\ \Omega, \\ \eta|_{\partial\Omega} = 0, \end{cases} \tag{44}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with a smooth boundary.

It is clear that $M_{p(x)}(t) \geq a_p > 0$, for all $t \geq 0$ and $M_{q(x)}(t) \geq a_q > 0$, for all $t \geq 0$. That is, the condition H_m is satisfied. Thus, the results obtained in Theorems 1 and 2 stay true for problem (1). The problem and results are all new.

The second is when $p(x), q(x), r(x), \alpha(x)$ are constant, that is, $p(x) = p = constant \in (1,+\infty)$, $q(x) = q = constant \in (1,+\infty)$, $r(x) = r = constant \in (1,+\infty)$, $\alpha(x) = \alpha = constant \in [1,p)$ and $M_{p(x)}(t) = (a_p + p b_p t)^{p-1}$, for all $t \geq 0$ with $a_p > 0$, $b_p \geq 0$ and $M_{q(x)}(t) = (a_q + q b_q t)^{q-1}$, for all $t \geq 0$ with $a_q > 0$, $b_q \geq 0$. In this case, problem (1) becomes the following form.

Example 2. *Consider the problem*

$$\begin{cases} \big(a_p + b_p \int_\Omega |\nabla\eta|^p dx\big)^{p-1}(-\Delta_p\eta) + \mu\big(a_q + b_q \int_\Omega |\nabla\eta|^q dx\big)^{q-1}(-\Delta_q\eta) \\ \qquad = \lambda |\eta|^{r-2}\eta \ln|\eta| + g(x,\eta,\nabla\eta), \ in\ \Omega, \\ \eta|_{\partial\Omega} = 0, \end{cases} \tag{45}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with a smooth boundary.

The function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ given by

$$g(x,\varpi,\nu) = |\varpi|^{\alpha-2}\varpi + \frac{\varpi}{1+\varpi^2}\Big(|\nu|^{p-1} + \gamma(x)\Big), \text{ for all } (x,\varpi,\nu) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

with a constant $\alpha \in [1,p)$, and some $\gamma \in L^\infty(\Omega)$ satisfies conditions H_{g_1} (see [21]). For $p \in (1,+\infty), q \in (1,+\infty)$, the condition H_{pq} is satisfied. It is clear that $M_{p(x)}(t) \geq a_p^{p-1} > 0$, for all $t \geq 0$ and $M_{q(x)}(t) \geq a_q^{q-1} > 0$, for all $t \geq 0$. That is, condition H_m is satisfied. Thus, the results obtained in Theorem 1 stay true for problem (2). The problem and results are also all new.

6. Conclusions

In this article, we study a kind of Kirchhoff-type elliptic problem, which combines with a variable exponent, competing $(p(x), q(x))$-Laplacian, logarithmic nonlinearity, and convection term. Due to the deficit of ellipticity, monotonicity, and variational structure, there are no available techniques to handle problem (1). A fundamental idea of the paper is to seek a solution to (1) as a limit of finite dimensional approximations. With the help of the Galerkin method and Brezis theorem, we obtain the existence of finite-dimensional approximate solutions and weak solutions, respectively. Our study extends previous results, such as from the elliptic problem with logarithmic nonlinearity or the convection term to $(p(x), q(x))$-Kirchhoff-type equations both logarithmic nonlinearity with variable exponents and convection terms. Finally, we consider that it will be a new field to study such problems (1) in fractional Sobolev spaces with variable exponents and in Sobolev spaces with variable exponents and variable fractional order.

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