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Some Generalizations of Different Types of Quantum Integral Inequalities for Differentiable Convex Functions with Applications

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Abstract: In this paper, we prove a new quantum integral equality involving a parameter, left and right quantum derivatives. Then, we use the newly established equality and prove some new estimates of quantum Ostrowski, quantum midpoint, quantum trapezoidal and quantum Simpson's type inequalities for q-differentiable convex functions. It is also shown that the newly established inequalities are the refinements of the existing inequalities inside the literature. Finally, some examples and applications are given to illustrate the investigated results.

Keywords: midpoint inequalities; trapezoidal inequalities; Ostrowski's inequalities; Simpson's inequalities; quantum calculus; convex functions

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1. Introduction

A. Ostrowski showed an inequality involving a function with bounded derivative in 1938, which became known as the Ostrowski inequality [1]. The following is the inequality:

Theorem 1. *For a differentiable functions $\mathfrak{F}:[\mathfrak{y}_1,\mathfrak{y}_2]\to\mathbb{R}$ over $(\mathfrak{y}_1,\mathfrak{y}_2)$ with $|\mathfrak{F}(\mathfrak{x})|\leq M$, the following inequality holds:*

$$\left|\mathfrak{F}(\mathfrak{x})-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(t)dt\right|\leq M(\mathfrak{y}_2-\mathfrak{y}_1)\left[\frac{(\mathfrak{x}-\mathfrak{y}_1)^2+(\mathfrak{y}_2-\mathfrak{x})^2}{2}\right], \tag{1}$$

where $\mathfrak{x}\in[\mathfrak{y}_1,\mathfrak{y}_2]$.

The following are two possible interpretations of the Ostrowski inequality:

(i) Estimation of the functional value's deviation from its average value.
(ii) A rectangle is used to approximate the area under the curve.

On the other hand, Budak et al. proved the quantum version of the inequality (1) as follows:

Theorem 2 ([2]). *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a q-differentiable function. If ${}_{\mathfrak{y}_1}D_q\mathfrak{F}$ and ${}^{\mathfrak{y}_2}D_q\mathfrak{F}$ are continuous and integrable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with $\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\right|, \left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\right| \leq M$, then the following inequality holds for $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$:*

$$\begin{aligned} &\left|\mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t) \ {}_{\mathfrak{y}_1}d_q t \ + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t) \ {}^{\mathfrak{y}_2}d_q t \ \right]\right| \\ \leq \ & \frac{qM}{(\mathfrak{y}_2 - \mathfrak{y}_1)}\left[\frac{(\mathfrak{r} - \mathfrak{y}_1)^2 + (\mathfrak{y}_2 - \mathfrak{r})^2}{[2]_q}\right]. \end{aligned} \tag{2}$$

It is also well known that $\mathfrak{F}$ is convex if and only if it satisfies the Hermite–Hadamard inequality, stated below:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})d\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{2}, \tag{3}$$

where $\mathfrak{F} : \mathfrak{I} \to \mathbb{R}$ is a convex function and $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathfrak{I}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$. For some recent developments of inequality (3), one can consult [3–6].

In [7], Alp et al. proved the following version of quantum Hermite–Hadamard type for convex functions using the left quantum integrals:

Theorem 3. *For any convex function $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, the following inequality holds:*

$$\mathfrak{F}\left(\frac{q\mathfrak{y}_1 + \mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r}) \ {}_{\mathfrak{y}_1}d_q\mathfrak{r} \leq \frac{q\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{[2]_q}. \tag{4}$$

Recently, Bermudo et al. [8] used the right quantum integrals and proved the following variant of Hermite–Hadamard type inequalities for convex functions:

Theorem 4. *For any convex function $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, the following inequalities holds:*

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1 + q\mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r}) \ {}^{\mathfrak{y}_2}d_q\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1) + q\mathfrak{F}(\mathfrak{y}_2)}{[2]_q} \tag{5}$$

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) \leq \frac{1}{2(\mathfrak{y}_2 - \mathfrak{y}_1)}\left[\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r}) \ {}_{\mathfrak{y}_1}d_q\mathfrak{r} + \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r}) \ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right] \leq \frac{\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{2}. \tag{6}$$

Remark 1. *It is obvious that if we take the limit as $q \to 1^-$ in (4)–(6), then we obtain the inequality (3).*

In the literature there are several papers focused on obtaining some bounds for the left and right estimates of inequalities (4) and (5). For example, Noor et al. [9] and Alp et al. [7] gave some bounds for the right side and right side of the inequality (4), respectively, whereas Budak established similar bounds for the left and right side of the inequality (5) in [10]. In [11], Liu and Zhuang obtained some trapezoid type inequalities by using the twice q-differentiable functions. The authors proved several new estimates by utilizing the generalized convex functions in [12–14]. Brahim et al. established some new version of quantum Hermite–Hadamard inequality in [15]. On the other hand, some papers were devoted to fractional post-quantum inequalities [16,17]. Some authors generalized the quantum Hermite–Hadamard inequalities for coordinated convex functions in [18–20]. In [21–23], the authors used convexity and coordinated convexity to prove some Simpson's and Newton's type inequalities via q-calculus. For the study of Ostrowski's inequalities, one can consult [24,25]. The quantum version of Bernoulli inequality is given by Alomari in [26].

Inspired by the ongoing studies, we prove a new parameterized quantum integral identity involving left and right quantum derivatives to prove different variants of quantum integral inequalities for quantum differentiable convex functions. The main advantage of the newly established inequalities is that these can be turned into quantum Ostrowski's type inequalities for convex functions [2], classical Ostrowski's type inequalities for convex functions [27], several classical integral inequalities for convex functions [28] and several new quantum integral inequalities such as midpoint type, trapezoidal type, Ostrowski's type and Simpson's type without having to prove each one separately.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we establish an identity that plays an essential role in developing the main results of this paper. The different variants of quantum integral inequalities for quantum differentiable convex functions are described in Section 4. The relationship between the findings reported here and similar findings in the literature are also taken into account. In Section 5, we show some examples to illustrate the investigated results. In Section 6, we provide some applications to special means of real numbers by using the newly established results. Section 7 concludes with some recommendations for future research.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well-known inequalities for quantum integrals. Throughout this paper, let $0 < q < 1$ be a constant.

The q-number or q-analog of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{7}$$

Definition 1 ([29])**.** *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then, the left q-derivative of function $\mathfrak{F}$ at $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_1)}{(1-q)(\mathfrak{r} - \mathfrak{y}_1)}, & \text{if } \mathfrak{r} \neq \mathfrak{y}_1; \\ \lim\limits_{\mathfrak{r}\to\mathfrak{y}_1} {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}), & \text{if } \mathfrak{r} = \mathfrak{y}_1. \end{cases} \tag{8}$$

The function $\mathfrak{F}$ is said to be q-differentiable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if $_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})$ exists for all $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that, if $\mathfrak{y}_1 = 0$ and $_0D_q\mathfrak{F}(\mathfrak{r}) = D_q\mathfrak{F}(\mathfrak{r})$, then (8) reduces to

$$D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r})}{(1-q)\mathfrak{r}}, & \text{if } \mathfrak{r} \neq 0; \\ \lim\limits_{\mathfrak{r}\to 0} D_q\mathfrak{F}(\mathfrak{r}), & \text{if } \mathfrak{r} = 0, \end{cases}$$

which is the q-Jackson derivative, see [29–31] for more details.

Theorem 5 ([29])**.** *If $\mathfrak{F}, g : J \to \mathbb{R}$ are q-differentiable functions, then the following identities hold:*

(i) The product $\mathfrak{F}g : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with

$$\begin{aligned} _{\mathfrak{y}_1}D_q(\mathfrak{F}g)(\mathfrak{r}) &= \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r}) + g(q\mathfrak{r} + (1-q)\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) \\ &= g(\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) + \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r}); \end{aligned}$$

(ii) If $g(\mathfrak{r})g(q\mathfrak{r} + (1-q)\mathfrak{r}) \neq 0$, then $\mathfrak{F}/g$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with

$$_{\mathfrak{y}_1}D_q\left(\frac{\mathfrak{F}}{g}\right)(\mathfrak{r}) = \frac{g(\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r})}{g(\mathfrak{r})g(q\mathfrak{r} + (1-q)\mathfrak{r})}.$$

Definition 2 ([29]). *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then, the left q-integral of function $\mathfrak{F}$ at $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$\int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r} = (1-q)(z-\mathfrak{y}_1) \sum_{n=0}^{\infty} q^n \mathfrak{F}(q^n z + (1-q^n)\mathfrak{y}_1). \tag{9}$$

The function $\mathfrak{F}$ is said to be q-integrable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if $\int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r}$ exists for all $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that, if $\mathfrak{y}_1 = 0$, then (9) reduces to

$$\int_0^z \mathfrak{F}(\mathfrak{r})_0 d_q\mathfrak{r} = \int_0^z \mathfrak{F}(\mathfrak{r}) d_q\mathfrak{r} = (1-q)z \sum_{n=0}^{\infty} q^n \mathfrak{F}(q^n z),$$

which is the q-Jackson integral, see [29–31] for more details.

Theorem 6 ([29]). *If $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is a continuous function and $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$, then the following identities hold:*

(i) $\quad {}_{\mathfrak{y}_1}D_q \int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r} = \mathfrak{F}(z)$;

(ii) $\quad \int_c^z {}_{\mathfrak{y}_1}D_q \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r} = \mathfrak{F}(z) - \mathfrak{F}(c)$ *for* $c \in (\mathfrak{y}_1, z)$.

On the other hand, Bermudo et al. [8] defined a new quantum derivative and quantum integral, which are called right q-derivative and right q-integral as follows:

Definition 3 ([8]). *The right q-derivative of mapping $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is defined as:*

$${}^{\mathfrak{y}_2}D_q \mathfrak{F}(\mathfrak{r}) = \frac{\mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_2) - \mathfrak{F}(\mathfrak{r})}{(1-q)(\mathfrak{y}_2 - \mathfrak{r})}, \quad \mathfrak{r} \neq \mathfrak{y}_2.$$

If $\mathfrak{r} = \mathfrak{y}_2$, we define ${}^{\mathfrak{y}_2}D_q \mathfrak{F}(\mathfrak{y}_2) = \lim_{\mathfrak{r} \to \mathfrak{y}_2} {}^{\mathfrak{y}_2}D_q \mathfrak{F}(\mathfrak{r})$ if it exists and it is finite.

Definition 4 ([8]). *The right q-definite integral of mapping $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ on $[\mathfrak{y}_1, \mathfrak{y}_2]$ is defined as:*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})^{\mathfrak{y}_2} d_q\mathfrak{r} = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{k=0}^{\infty} q^k \mathfrak{F}\left(q^k \mathfrak{y}_1 + \left(1-q^k\right)\mathfrak{y}_2\right).$$

Lemma 1 ([32]). *For continuous functions $\mathfrak{F}, g : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, the following equality holds for $c \in [\mathfrak{y}_1, \mathfrak{y}_2]$:*

$$\begin{aligned} &\int_0^c g(t)\, {}_{\mathfrak{y}_1}D_q \mathfrak{F}(t\mathfrak{y}_2 + (1-t)\mathfrak{y}_1) d_q t \\ = \;& \frac{g(t)\mathfrak{F}(t\mathfrak{y}_2 + (1-t)\mathfrak{y}_1)}{\mathfrak{y}_2 - \mathfrak{y}_1}\Bigg|_0^c - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \int_0^c D_q g(t) \mathfrak{F}(qt\mathfrak{y}_2 + (1-qt)\mathfrak{y}_1) d_q t. \end{aligned}$$

Proof. From the fundamental rules of q-integrals and derivatives, we have

$$\begin{aligned}
&\int_0^c g(t)\ {}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{y}_2+(1-t)\mathfrak{y}_1)d_qt \\
=\ &\int_0^c g(t)\frac{\mathfrak{F}(t\mathfrak{y}_2+(1-t)\mathfrak{y}_1)-\mathfrak{F}(qt\mathfrak{y}_2+(1-qt)\mathfrak{y}_1)}{(1-q)(\mathfrak{y}_2-\mathfrak{y}_1)t}d_qt \\
=\ &\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\sum_{k=0}^{\infty} g\left(q^kc\right)\mathfrak{F}\left(q^kc\mathfrak{y}_2+\left(1-q^kc\right)\mathfrak{y}_1\right)-\sum_{k=0}^{\infty} g\left(q^kc\right)\mathfrak{F}\left(q^{k+1}c\mathfrak{y}_2+\left(1-q^{k+1}c\right)\mathfrak{y}_1\right)\right] \\
=\ &\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\sum_{k=0}^{\infty}\left[g\left(q^kc\right)\mathfrak{F}\left(q^kc\mathfrak{y}_2+\left(1-q^kc\right)\mathfrak{y}_1\right)-g\left(q^kc\right)\mathfrak{F}\left(q^{k+1}c\mathfrak{y}_2+\left(1-q^{k+1}c\right)\mathfrak{y}_1\right)\right] \\
&-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\sum_{k=0}^{\infty}\left(g\left(q^kc\right)-g\left(q^{k+1}c\right)\right)\mathfrak{F}\left(q^{k+1}c\mathfrak{y}_2+\left(1-q^{k+1}c\right)\mathfrak{y}_1\right) \\
=\ &\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}[g(c)\mathfrak{F}(c\mathfrak{y}_2+(1-c)\mathfrak{y}_1)-g(0)\mathfrak{F}(\mathfrak{y}_1)] \\
&-\frac{c}{\mathfrak{y}_2-\mathfrak{y}_1}q^kD_qg\left(q^kc\right)\mathfrak{F}\left(q^{k+1}c\mathfrak{y}_2+\left(1-q^{k+1}c\right)\mathfrak{y}_1\right) \\
=\ &\left.\frac{g(t)\mathfrak{F}(t\mathfrak{y}_2+(1-t)\mathfrak{y}_1)}{\mathfrak{y}_2-\mathfrak{y}_1}\right|_0^c-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^c D_qg(t)\mathfrak{F}(qt\mathfrak{y}_2(1-qt)\mathfrak{y}_2)d_qt.
\end{aligned}$$

□

Lemma 2 ([33]). *For continuous functions $\mathfrak{F}, g : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, the following equality holds for $c \in [\mathfrak{y}_1, \mathfrak{y}_2]$:*

$$\begin{aligned}
&\int_0^c g(t)^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{y}_1+(1-t)\mathfrak{y}_2)d_qt \\
=\ &\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^c D_qg(t)\mathfrak{F}(qt\mathfrak{y}_1+(1-qt)\mathfrak{y}_2)d_qt-\left.\frac{g(t)\mathfrak{F}(t\mathfrak{y}_1+(1-t)\mathfrak{y}_2)}{\mathfrak{y}_2-\mathfrak{y}_1}\right|_0^c.
\end{aligned}$$

3. Identities

In this section, we establish a general version of quantum integral equality using the integration by parts method for quantum integrals to obtain the main results.

Lemma 3. *Let $\mathfrak{F} : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a q-differentiable function. If ${}_{\mathfrak{y}_1}D_q\mathfrak{F}$ and ${}^{\mathfrak{y}_2}D_q\mathfrak{F}$ are continuous and integrable on $[\mathfrak{y}_1, \mathfrak{y}_2]$, then for $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ and $\lambda \geq 0$ one has the identity:*

$$\begin{aligned}
&(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)] \\
&-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right] \\
&=\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1(qt-\lambda)\ {}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)d_qt \\
&\quad-\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1(qt-\lambda)\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)d_qt.
\end{aligned} \tag{10}$$

Proof. From Lemma 1 and Definition 2, we have

$$\begin{aligned}
&\int_0^1 (qt-\lambda)\; _{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)d_qt \qquad (11)\\
=& \left.\frac{(qt-\lambda)\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)}{\mathfrak{r}-\mathfrak{y}_1}\right|_0^1 - \frac{q}{\mathfrak{r}-\mathfrak{y}_1}\int_0^1 \mathfrak{F}(qt\mathfrak{r}+(1-qt)\mathfrak{y}_1)d_qt\\
=& \frac{(q-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\mathfrak{F}(\mathfrak{y}_1)}{\mathfrak{r}-\mathfrak{y}_1} - \frac{1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\; _{\mathfrak{y}_1}d_qt + \frac{1-q}{\mathfrak{r}-\mathfrak{y}_1}\mathfrak{F}(\mathfrak{r})\\
=& \frac{1-\lambda}{\mathfrak{r}-\mathfrak{y}_1}\mathfrak{F}(\mathfrak{r}) + \frac{\lambda\mathfrak{F}(\mathfrak{y}_1)}{\mathfrak{r}-\mathfrak{y}_1} - \frac{1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\; _{\mathfrak{y}_1}d_qt.
\end{aligned}$$

Now from Lemma 2 and Definition 3, we have

$$\begin{aligned}
&\int_0^1 (qt-\lambda)\; ^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)d_qt \qquad (12)\\
=& \frac{1}{(\mathfrak{y}_2-\mathfrak{r})^2}\int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\; ^{\mathfrak{y}_2}d_qt - \frac{1-\lambda}{\mathfrak{y}_2-\mathfrak{r}}\mathfrak{F}(\mathfrak{r}) - \frac{\lambda\mathfrak{F}(\mathfrak{y}_2)}{\mathfrak{y}_2-\mathfrak{r}}.
\end{aligned}$$

Thus, we obtain the required equality (10) by subtracting (12) from (11) after multiplying $\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}$ and $\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}$ with (11) and (12), respectively. □

Remark 2. *In Lemma 3, we have*

(i) *If we set $\lambda = 0$, then we obtain the following quantum integral identity:*

$$\begin{aligned}
&\mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\; _{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\; ^{\mathfrak{y}_2}d_qt\right]\\
&= \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 qt\; _{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)d_qt\\
&\quad - \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 qt\; ^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)d_qt,
\end{aligned}$$

which is proved by Budak et al. in [2].

(ii) *If we set $\lambda = 1$ and $q \to 1^-$, then we recapture [28] [Lemma 1].*

(iii) *If we set $\lambda = 0$ and $q \to 1^-$, then we recapture [27] [Lemma 1].*

Corollary 1. *In Lemma 3, we have*

(i) *If we set $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, then we obtain the following equality:*

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\; _{\mathfrak{y}_1}d_qt + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\; ^{\mathfrak{y}_2}d_qt\right]\\
&= \frac{\mathfrak{y}_2-\mathfrak{y}_1}{4}\left[\int_0^1 qt\; _{\mathfrak{y}_1}D_q\mathfrak{F}\left(t\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+(1-t)\mathfrak{y}_1\right)d_qt\right.\\
&\quad \left. - \int_0^1 qt\; ^{\mathfrak{y}_2}D_q\mathfrak{F}\left(t\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+(1-t)\mathfrak{y}_2\right)d_qt\right].
\end{aligned}$$

(ii) *If we set* $\lambda = 1$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following equality:*

$$\begin{aligned}
&\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right] \\
&= \frac{\mathfrak{y}_2-\mathfrak{y}_1}{4}\left[\int_0^1 (qt-1)\ {}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(t\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+(1-t)\mathfrak{y}_1\right)d_qt\right. \\
&\quad \left. - \int_0^1 (qt-1)\ {}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(t\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+(1-t)\mathfrak{y}_2\right)d_qt\right].
\end{aligned}$$

4. Main Results

In this section, we prove different versions of quantum integral inequalities for differentiable convex and bounded functions.

Theorem 7. *Under the assumption of Lemma 3, if* $\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\right|$ *and* $\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\right|$ *are convex mappings over* $[\mathfrak{y}_1,\mathfrak{y}_2]$, *then we have the following inequality:*

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right. \\
&\left. -\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right| \\
&\le \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\Delta_1(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right| + \Delta_2(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|\right) \\
&\quad + \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\Delta_1(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right| + \Delta_2(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|\right),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\Delta_1(\lambda;q) &= \int_0^1 |qt-\lambda| t\ d_qt \\
&= \begin{cases} \frac{\lambda[3]_q - q[2]_q}{[2]_q[3]_q}, & 0<q<\lambda; \\ \frac{2\lambda^3+q[2]_q-\lambda[3]_q}{[2]_q[3]_q}, & \lambda \le q < 1 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2(\lambda;q) &= \int_0^1 |qt-\lambda|(1-t)d_qt \\
&= \begin{cases} \frac{\lambda[2]_q-q}{[2]_q} - \frac{\lambda[3]_q-q[2]_q}{[2]_q[3]_q}, & 0<q<\lambda; \\ \frac{2\lambda^2+q-\lambda[2]_q}{[2]_q} - \frac{2\lambda^3+q[2]_q-\lambda[3]_q}{[2]_q[3]_q}, & \lambda \le q < 1. \end{cases}
\end{aligned}$$

Proof. By taking the modulus in (10) and using the convexity of $\left|{}_{\eta_1}D_q\mathfrak{F}\right|$ and $\left|{}^{\eta_2}D_q\mathfrak{F}\right|$, we have

$$\begin{aligned}
&\Bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\eta_2-\eta_1}[(\mathfrak{r}-\eta_1)\mathfrak{F}(\eta_1)+(\eta_2-\mathfrak{r})\mathfrak{F}(\eta_2)]\\
&-\frac{1}{\eta_2-\eta_1}\left[\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\eta_1}d_qt+\int_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(t)\ {}^{\eta_2}d_qt\right]\Bigg|\\
&\leq\frac{(\mathfrak{r}-\eta_1)^2}{\eta_2-\eta_1}\int_0^1|qt-\lambda|\left|\ {}_{\eta_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\eta_1)\right|d_qt\\
&\quad+\frac{(\eta_2-\mathfrak{r})^2}{\eta_2-\eta_1}\int_0^1|qt-\lambda|\left|\ {}^{\eta_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\eta_2)\right|d_qt\\
&\leq\frac{(\mathfrak{r}-\eta_1)^2}{\eta_2-\eta_1}\int_0^1|qt-\lambda|\left[t\left|{}_{\eta_1}D_q\mathfrak{F}(\mathfrak{r})\right|+(1-t)\left|{}_{\eta_1}D_q\mathfrak{F}(\eta_1)\right|\right]d_qt\\
&+\frac{(\eta_2-\mathfrak{r})^2}{\eta_2-\eta_1}\int_0^1|qt-\lambda|\left[t\left|{}^{\eta_2}D_q\mathfrak{F}(\mathfrak{r})\right|+(1-t)\left|{}^{\eta_2}D_q\mathfrak{F}(\eta_2)\right|\right]d_qt\\
&=\frac{(\mathfrak{r}-\eta_1)^2}{\eta_2-\eta_1}\left(\Delta_1(\lambda;q)\left|{}_{\eta_1}D_q\mathfrak{F}(\mathfrak{r})\right|+\Delta_2(\lambda;q)\left|{}_{\eta_1}D_q\mathfrak{F}(\eta_1)\right|\right)\\
&+\frac{(\eta_2-\mathfrak{r})^2}{\eta_2-\eta_1}\left(\Delta_1(\lambda;q)\left|{}^{\eta_2}D_q\mathfrak{F}(\mathfrak{r})\right|+\Delta_2(\lambda;q)\left|{}^{\eta_2}D_q\mathfrak{F}(\eta_2)\right|\right).
\end{aligned}$$

Thus, the proof is completed. □

Remark 3. *In Theorem 7, we have*

(i) *If we set $\lambda=0$, then we obtain the following quantum Ostrowski type integral inequality:*

$$\begin{aligned}
&\left|\mathfrak{F}(\mathfrak{r})-\frac{1}{\eta_2-\eta_1}\left[\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\eta_1}d_qt+\int_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(t)\ {}^{\eta_2}d_qt\right]\right|\\
&\leq\frac{q}{(\eta_2-\eta_1)[2]_q[3]_q}\left[(\mathfrak{r}-\eta_1)^2\left([2]_q\left|\ {}_{\eta_1}D_q\ \mathfrak{F}(\mathfrak{r})\right|+q^2\left|\ {}_{\eta_1}D_q\ \mathfrak{F}(\eta_1)\right|\right)\right.\\
&\quad\left.+(\eta_2-\mathfrak{r})^2\left([2]_q\left|\ {}^{\eta_2}D_q\ \mathfrak{F}(\mathfrak{r})\right|+q^2\left|\ {}^{\eta_2}D_q\ \mathfrak{F}(\eta_2)\right|\right)\right],
\end{aligned}$$

which is proved by Budak et al. in [2].

(ii) *If we set $\lambda=1$ and $q\to1^-$, then we recapture [28] [Theorem 4].*

Corollary 2. *In Theorem 7, we have*

(i) *If we assume that ${}_{\eta_1}D_q\ \mathfrak{F}(\mathfrak{r}),\ {}^{\eta_2}D_q\ \mathfrak{F}(\mathfrak{r})\leq M$, then we obtain the following new inequality:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\eta_2-\eta_1}[(\mathfrak{r}-\eta_1)\mathfrak{F}(\eta_1)+(\eta_2-\mathfrak{r})\mathfrak{F}(\eta_2)]\\
&-\frac{1}{\eta_2-\eta_1}\left[\int_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\eta_1}d_qt+\int_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(t)\ {}^{\eta_2}d_qt\right]\Bigg|\\
&\leq\frac{M}{\eta_2-\eta_1}\left[(\Delta_1(\lambda;q)+\Delta_2(\lambda;q))\left((\mathfrak{r}-\eta_1)^2+(\eta_2-\mathfrak{r})^2\right)\right].
\end{aligned}\tag{14}$$

(ii) *If we set* $\lambda = 0$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following quantum midpoint type inequality:*

$$\begin{aligned}&\left|\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\ &\leq\frac{q(\mathfrak{y}_2-\mathfrak{y}_1)}{4[2]_q[3]_q}\left[\left([2]_q\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+q^2\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{y}_1)\right|\right)\right.\\ &\left.+\left([2]_q\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+q^2\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{y}_2)\right|\right)\right].\end{aligned}$$

(iii) *If we set* $\lambda = 1$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following quantum trapezoidal type inequality:*

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\ &\leq\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\left[\left(\Delta_1(1;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+\Delta_2(1;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|\right)\right.\\ &\left.+\left(\Delta_1(1;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+\Delta_2(1;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|\right)\right].\end{aligned}$$

(iv) *If we set* $\lambda = \frac{1}{3}$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following quantum Simpson's type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\mathfrak{y}_1)+4\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+\mathfrak{F}(\mathfrak{y}_2)\right]-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\ &\leq\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\left[\left(\Delta_1\left(\frac{1}{3};q\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+\Delta_2\left(\frac{1}{3};q\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|\right)\right.\\ &\left.+\left(\Delta_1\left(\frac{1}{3};q\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+\Delta_2\left(\frac{1}{3};q\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|\right)\right].\end{aligned}$$

Theorem 8. *Under the assumption of Lemma 3, if* $\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\right|^s$, $\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\right|^s$, *where* $s \geq 1$ *are convex mappings over* $[\mathfrak{y}_1, \mathfrak{y}_2]$, *then we have the following quantum integral inequality:*

$$\begin{aligned}&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right.\\ &\left.-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\ &\leq\Delta_5(\lambda;q)^{1-\frac{1}{s}}\left[\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\Delta_1(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s+\Delta_2(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right)^{\frac{1}{s}}\right.\\ &\left.+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\Delta_1(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s+\Delta_2(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right)^{\frac{1}{s}}\right],\end{aligned}\tag{15}$$

where $\Delta_i(\lambda;q), i = 1,2$ *are defined in Theorem 7 and*

$$\Delta_5(\lambda;q)=\int_0^1|qt-\lambda|d_qt=\begin{cases}\frac{\lambda[2]_q-q}{[2]_q}, & 0<q<\lambda;\\ \frac{2\lambda^2+q-\lambda[2]_q}{[2]_q}, & \lambda\leq q<1.\end{cases}$$

Proof. By taking modulus in (10) and using power mean inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right.\\
&\left.-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&\leq\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1|qt-\lambda|\,\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)\right|d_qt\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1|qt-\lambda|\,\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)\right|d_qt\\
&\leq\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1|qt-\lambda|\,d_qt\right)^{1-\frac{1}{s}}\left(\int_0^1|qt-\lambda|\,\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)\right|^sd_qt\right)^{\frac{1}{s}}\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1|qt-\lambda|\,d_qt\right)^{1-\frac{1}{s}}\left(\int_0^1|qt-\lambda|\,\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)\right|^sd_qt\right)^{\frac{1}{s}}.
\end{aligned}$$

Now by applying convexity, we have

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right.\\
&\left.-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&\leq\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1|qt-\lambda|\,d_qt\right)^{1-\frac{1}{s}}\\
&\times\left(\int_0^1|qt-\lambda|\left[t\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s+(1-t)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right]d_qt\right)^{\frac{1}{s}}\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1|qt-\lambda|\,d_qt\right)^{1-\frac{1}{s}}\\
&\times\left(\int_0^1|qt-\lambda|\left[t\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s+(1-t)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right]d_qt\right)^{\frac{1}{s}}\\
&=\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Delta_5(\lambda;q)^{1-\frac{1}{s}}\left(\Delta_1(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s+\Delta_2(\lambda;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right)^{\frac{1}{s}}\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Delta_5(\lambda;q)^{1-\frac{1}{s}}\left(\Delta_1(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s+\Delta_2(\lambda;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right)^{\frac{1}{s}}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 4. *In Theorem 8, we have:*

(i) *If we set $\lambda=0$, then we obtain the following quantum Ostrowski type integral inequality:*

$$\begin{aligned}
&\left|\mathfrak{F}(\mathfrak{r})-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&\leq\frac{q}{(\mathfrak{y}_2-\mathfrak{y}_1)[2]_q}\left[(\mathfrak{r}-\mathfrak{y}_1)^2\left(\frac{[2]_q\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{r})\right|^s+q^2\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{y}_1)\right|^s}{[3]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+(\mathfrak{y}_2-\mathfrak{r})^2\left(\frac{[2]_q\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{r})\right|^s+q^2\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{y}_2)\right|^s}{[3]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}$$

which is proved by Budak et al. in [2].

(ii) *If we set $\lambda = 1$ and $q \to 1^{-}$, then we recapture [28] [Theorem 7].*

Corollary 3. *In Theorem 8, we have*

(i) *If we assume that ${}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{r})$, ${}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{r}) \leq M$, then we obtain the following new inequality:*

$$\begin{aligned}&\bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2 - \mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1) + (\mathfrak{y}_2 - \mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)] \\ &-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\bigg| \\ &\leq \frac{M\Delta_5(\lambda;q)^{1-\frac{1}{s}}}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\Big(\Delta_1(\lambda;q) + \Delta_2(\lambda;q)\Big)^{\frac{1}{s}}\Big((\mathfrak{r}-\mathfrak{y}_1)^2 + (\mathfrak{y}_2-\mathfrak{r})^2\Big)\right].\end{aligned} \tag{16}$$

(ii) *If we set $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, then we obtain the following quantum midpoint type inequality:*

$$\begin{aligned}&\left|\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\ &\leq \frac{q(\mathfrak{y}_2-\mathfrak{y}_1)}{4[2]_q}\left[\left(\frac{[2]_q\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + q^2\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{y}_1)\right|^s}{[3]_q}\right)^{\frac{1}{s}}\right. \\ &\left.+\left(\frac{[2]_q\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + q^2\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{y}_2)\right|^s}{[3]_q}\right)^{\frac{1}{s}}\right].\end{aligned}$$

(iii) *If we set $\lambda = 1$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, then we obtain the following quantum trapezoidal type inequality:*

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\ &\leq \frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\Delta_5(1;q)^{1-\frac{1}{s}}\left[\left(\Delta_1(1;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + \Delta_2(1;q)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right)^{\frac{1}{s}}\right. \\ &\left.+\left(\Delta_1(1;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + \Delta_2(1;q)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right)^{\frac{1}{s}}\right].\end{aligned}$$

(iv) *If we set $\lambda = \frac{1}{3}$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, then we obtain the following quantum Simpson's type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\mathfrak{y}_1) + 4\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) + \mathfrak{F}(\mathfrak{y}_2)\right] - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\ &\leq \frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\Delta_5\left(\frac{1}{3};q\right)^{1-\frac{1}{s}}\left[\left(\Delta_1\left(\frac{1}{3};q\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + \Delta_2\left(\frac{1}{3};q\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right)^{\frac{1}{s}}\right. \\ &\left.+\left(\Delta_1\left(\frac{1}{3};q\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + \Delta_2\left(\frac{1}{3};q\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right)^{\frac{1}{s}}\right].\end{aligned}$$

Theorem 9. *Under the assumption of Lemma 3, if $s > 1$ is a real number and $\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\right|^s$ and $\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\right|^s$ are convex mappings over $[\mathfrak{y}_1, \mathfrak{y}_2]$, then we have the following quantum integral inequality:*

$$\begin{aligned}
&\bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2 - \mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1) + (\mathfrak{y}_2 - \mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)] \\
&-\frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\bigg| \\
&\leq \Delta_7(\lambda; q, r)^{\frac{1}{r}}\left[\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\frac{\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s + q\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right. \\
&\left.+\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\frac{\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s + q\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}$$

where $s^{-1} + r^{-1} = 1$ and

$$\begin{aligned}
\Delta_7(\lambda; q, r) &= \int_0^1 |qt - \lambda|^r\, d_q t \\
&= (1-q)\sum_{n=0}^{\infty} q^n \left|q^{n+1} - \lambda\right|^r.
\end{aligned}$$

Proof. Taking modulus in Lemma 3 and applying the Hölder's inequality, we have

$$\begin{aligned}
&\bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2 - \mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1) + (\mathfrak{y}_2 - \mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)] \\
&-\frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\bigg| \\
&\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_0^1 |qt-\lambda|\,\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_1)\right| d_q t \\
&+\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_0^1 |qt-\lambda|\,\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_2)\right| d_q t \\
&\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\int_0^1 |qt-\lambda|^r\, d_q t\right)^{\frac{1}{r}}\left(\int_0^1 \left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_1)\right|^s d_q t\right)^{\frac{1}{s}} \\
&+\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\int_0^1 |qt-\lambda|^r\, d_q t\right)^{\frac{1}{r}}\left(\int_0^1 \left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_2)\right|^s d_q t\right)^{\frac{1}{s}}.
\end{aligned}$$

Now by applying convexity, we have

$$\begin{aligned}
&\bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2 - \mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1) + (\mathfrak{y}_2 - \mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)] \\
&-\frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\bigg| \\
&\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\int_0^1 |qt-\lambda|^r\, d_q t\right)^{\frac{1}{r}}\left(\int_0^1 \left[t\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s + (1-t)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s\right] d_q t\right)^{\frac{1}{s}} \\
&+\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\int_0^1 |qt-\lambda|^r\, d_q t\right)^{\frac{1}{r}}\left(\int_0^1 \left[t\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s + (1-t)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s\right] d_q t\right)^{\frac{1}{s}} \\
&= \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\Delta_7(\lambda; q, r)^{\frac{1}{r}}\left(\frac{\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s + q\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}} \\
&+\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\Delta_7(\lambda; q, r)^{\frac{1}{r}}\left(\frac{\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s + q\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 5. *In Theorem 9, we have:*

(i) *If we set $\lambda = 0$, then we obtain the following quantum Ostrowski type integral inequality:*

$$\begin{aligned}
&\left|\mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\
&\leq \frac{q}{\mathfrak{y}_2 - \mathfrak{y}_1}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}} \\
&\quad\times\left[(\mathfrak{r} - \mathfrak{y}_1)^2\left(\frac{\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{r})\right|^s + q\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right. \\
&\left.+(\mathfrak{y}_2 - \mathfrak{r})^2\left(\frac{\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{r})\right|^s + q\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}$$

which is proved by Budak et al. in [2].

(ii) *If we set $\lambda = 1$ and $q \to 1^-$, then we recapture [28] [Theorem 5].*

Corollary 4. *In Theorem 8, we have*

(i) *If we assume that ${}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{r}),\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{r}) \leq M$, then we obtain the following new inequality:*

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{\mathfrak{y}_2 - \mathfrak{y}_1}[(\mathfrak{r} - \mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1) + (\mathfrak{y}_2 - \mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right. \\
&\left.-\frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\
&\leq \frac{M\Delta_7(\lambda; q, r)^{\frac{1}{r}}}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[(\mathfrak{r} - \mathfrak{y}_1)^2 + (\mathfrak{y}_2 - \mathfrak{r})^2\right].
\end{aligned} \tag{17}$$

(ii) *If we set $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, then we obtain the following quantum midpoint type inequality:*

$$\begin{aligned}
&\left|\mathfrak{F}\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_q t + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_q t\right]\right| \\
&\leq \frac{q(\mathfrak{y}_2 - \mathfrak{y}_1)}{4}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}} \\
&\quad\times\left[\left(\frac{\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + q\left|\ {}_{\mathfrak{y}_1}D_q\ \mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right. \\
&\left.+\left(\frac{\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + q\left|\ {}^{\mathfrak{y}_2}D_q\ \mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

(iii) *If we set* $\lambda = 1$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following quantum trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&\leq\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\Delta_7(1;q,r)^{\frac{1}{r}}\left[\left(\frac{\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s+q\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s+q\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

(iv) *If we set* $\lambda = \frac{1}{3}$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following quantum Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(\mathfrak{y}_1)+4\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+\mathfrak{F}(\mathfrak{y}_2)\right]\right.\\
&\left.-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&\leq\frac{(\mathfrak{y}_2-\mathfrak{y}_1)}{4}\Delta_7\left(\frac{1}{3};q,r\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\right|^s+q\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\right|^s+q\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\right|^s}{[2]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Remark 6. *We can establish more inequalities of Bullen type and Hermite–Hadamard type with different choices of the parameters and using Young's inequality and we left them for the readers.*

5. Examples

In this section, we provide some examples to support the main theorems.

Example 1. *For a convex mapping* $\mathfrak{F} : [0,2] \to \mathbb{R}$ *is defined by* $\mathfrak{F}(\mathfrak{r}) = \mathfrak{r} + 4$. *From Theorem 7 with* $q = \frac{3}{4}$, $\lambda = 1$ *and* $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *the left side of (13) becomes*

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\right.\\
&\left.-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\\
&=\left|\frac{\mathfrak{F}(0)+\mathfrak{F}(2)}{2}-\frac{1}{2}\left[\int_0^1\mathfrak{F}(t)\ {}_0d_{\frac{3}{4}}t+\int_1^2\mathfrak{F}(t)\ {}^2d_{\frac{3}{4}}t\right]\right|\\
&\approx\left|5-\frac{1}{2}[4.5714+5.4286]\right|=0,
\end{aligned}$$

and the right side of (13) becomes

$$
\begin{aligned}
&\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Big(\Delta_1(\lambda;q)\big|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\big|+\Delta_2(\lambda;q)\big|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\big|\Big)\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Big(\Delta_1(\lambda;q)\big|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\big|+\Delta_2(\lambda;q)\big|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\big|\Big)\\
&=\frac{1}{2}\Bigg[\bigg(\Delta_1\Big(1;\frac{3}{4}\Big)\Big|_0D_{\frac{3}{4}}\mathfrak{F}(1)\Big|+\Delta_2\Big(1;\frac{3}{4}\Big)\Big|_0D_{\frac{3}{4}}\mathfrak{F}(0)\Big|\bigg)\\
&+\bigg(\Delta_1\Big(1;\frac{3}{4}\Big)\Big|^2D_{\frac{3}{4}}\mathfrak{F}(1)\Big|+\Delta_2\Big(1;\frac{3}{4}\Big)\Big|^2D_{\frac{3}{4}}\mathfrak{F}(2)\Big|\bigg)\Bigg]\\
&\approx\frac{1}{2}\Big[\Big(0.2471\cdot|1|+0.3243\cdot|1|\Big)+\Big(0.2471\cdot|1|+0.3243\cdot|1|\Big)\Big]=0.5714.
\end{aligned}
$$

It is clear that

$$0<0.5714,$$

which demonstrates the result described in Theorem 7.

Example 2. *For a convex mapping* $\mathfrak{F}:[0,2]\to\mathbb{R}$ *is defined by* $\mathfrak{F}(\mathfrak{r})=\mathfrak{r}+4$. *From Theorem 8 with* $q=\frac{3}{4},\lambda=1,s=3$ *and* $\mathfrak{r}=\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *the left side of (15) becomes*

$$
\begin{aligned}
&\bigg|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\frac{\lambda}{\mathfrak{y}_2-\mathfrak{y}_1}[(\mathfrak{r}-\mathfrak{y}_1)\mathfrak{F}(\mathfrak{y}_1)+(\mathfrak{y}_2-\mathfrak{r})\mathfrak{F}(\mathfrak{y}_2)]\\
&-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\bigg[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\bigg]\bigg|\\
&=\bigg|\frac{\mathfrak{F}(0)+\mathfrak{F}(2)}{2}-\frac{1}{2}\bigg[\int_0^1\mathfrak{F}(t)\ {}_0d_{\frac{3}{4}}t+\int_1^2\mathfrak{F}(t)\ {}^2d_{\frac{3}{4}}t\bigg]\bigg|\\
&\approx\bigg|5-\frac{1}{2}[4.5714+5.4286]\bigg|=0,
\end{aligned}
$$

and the right side of (15) becomes

$$
\begin{aligned}
&\Delta_5(\lambda;q)^{1-\frac{1}{s}}\Bigg[\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Big(\Delta_1(\lambda;q)\big|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})\big|^s+\Delta_2(\lambda;q)\big|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)\big|^s\Big)^{\frac{1}{s}}\\
&+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\Big(\Delta_1(\lambda;q)\big|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})\big|^s+\Delta_2(\lambda;q)\big|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)\big|^s\Big)^{\frac{1}{s}}\Bigg]\\
&=\Delta_5(\lambda;q)^{1-\frac{1}{3}}\cdot\frac{1}{2}\Bigg[\bigg(\Delta_1\Big(1;\frac{3}{4}\Big)\Big|_0D_{\frac{3}{4}}\mathfrak{F}(1)\Big|^3+\Delta_2\Big(1;\frac{3}{4}\Big)\Big|_0D_{\frac{3}{4}}\mathfrak{F}(0)\Big|^3\bigg)^{\frac{1}{3}}\\
&+\bigg(\Delta_1\Big(1;\frac{3}{4}\Big)\Big|^2D_{\frac{3}{4}}\mathfrak{F}(1)\Big|^3+\Delta_2\Big(1;\frac{3}{4}\Big)\Big|^2D_{\frac{3}{4}}\mathfrak{F}(2)\Big|^3\bigg)^{\frac{1}{3}}\Bigg]\\
&\approx 0.5714\cdot\frac{1}{2}\bigg[\Big(0.2471\cdot|1|^3+0.3243\cdot|1|^3\Big)^{\frac{1}{3}}+\Big(0.2471\cdot|1|^3+0.3243\cdot|1|^3\Big)^{\frac{1}{3}}\bigg]\\
&=0.4742.
\end{aligned}
$$

It is clear that

$$0<0.4742,$$

which demonstrates the result described in Theorem 8.

6. Applications to Special Means of Real Numbers

For any positive number $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$, we consider the following means:

(i) The Arithmetic mean:
$$\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}.$$

(ii) The Harmonic mean:
$$\mathcal{H}(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1 + \mathfrak{y}_2}.$$

(iii) The Geometric mean:
$$\mathcal{G}(\mathfrak{y}_1, \mathfrak{y}_2) = \sqrt{\mathfrak{y}_1\mathfrak{y}_2}.$$

Proposition 1. *For $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*
$$\begin{aligned} &\left|\mathcal{A}^k(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_1, \Theta_2)\right| \\ &\leq M(\mathfrak{y}_2 - \mathfrak{y}_1)\mathcal{A}(\Delta_1(0;q), \Delta_2(0;q)), \end{aligned}$$

where
$$\begin{aligned} \Theta_1 &= (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k, \\ \Theta_2 &= (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k. \end{aligned}$$

Proof. The inequality (14) for function $\mathfrak{F}(t) = t^k$, $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 2. *For $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*
$$\begin{aligned} &\left|\mathcal{A}\left(\mathfrak{y}_1^k, \mathfrak{y}_2^k\right) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_1, \Theta_2)\right| \\ &\leq M(\mathfrak{y}_2 - \mathfrak{y}_1)\mathcal{A}(\Delta_1(1;q), \Delta_2(1;q)), \end{aligned}$$

where
$$\begin{aligned} \Theta_1 &= (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k, \\ \Theta_2 &= (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k. \end{aligned}$$

Proof. The inequality (14) for function $\mathfrak{F}(t) = t^k$, $\lambda = 1$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 3. *For $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*
$$\begin{aligned} &\left|\frac{\mathcal{H}(\mathfrak{y}_1, \mathfrak{y}_2)}{\mathcal{G}(\mathfrak{y}_1, \mathfrak{y}_2)} - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_3, \Theta_4)\right| \\ &\leq M(\mathfrak{y}_2 - \mathfrak{y}_1)\mathcal{A}(\Delta_1(0;q), \Delta_2(0;q)), \end{aligned}$$

where

$$\Theta_3 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)},$$
$$\Theta_4 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (14) for function $\mathfrak{F}(t) = \frac{1}{t}$, $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 4. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\left|\mathcal{H}^{-1}(\mathfrak{y}_1,\mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_3,\Theta_4)\right| \leq M(\mathfrak{y}_2-\mathfrak{y}_1)\mathcal{A}(\Delta_1(1;q),\Delta_2(1;q)),$$

where

$$\Theta_3 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)},$$
$$\Theta_4 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (14) for function $\mathfrak{F}(t) = \frac{1}{t}$, $\lambda = 1$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 5. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\left|\mathcal{A}^k(\mathfrak{y}_1,\mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_1,\Theta_2)\right| \leq 2^{\frac{1-s}{s}}M(\mathfrak{y}_2-\mathfrak{y}_1)\Delta_5(0;q)^{1-\frac{1}{s}}\mathcal{A}^{\frac{1}{s}}(\Delta_1(0;q),\Delta_2(0;q)),$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)^k,$$
$$\Theta_2 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (16) for function $\mathfrak{F}(t) = t^k$, $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 6. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\left|\mathcal{A}\left(\mathfrak{y}_1^k,\mathfrak{y}_2^k\right) - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_1,\Theta_2)\right| \leq 2^{\frac{1-s}{s}}M(\mathfrak{y}_2-\mathfrak{y}_1)\Delta_5(1;q)^{1-\frac{1}{s}}\mathcal{A}^{\frac{1}{s}}(\Delta_1(1;q),\Delta_2(1;q)),$$

where

$$\begin{aligned}\Theta_1 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty} q^n\left(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_1\right)^k,\\ \Theta_2 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty} q^n\left(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_2\right)^k.\end{aligned}$$

Proof. The inequality (16) for function $\mathfrak{F}(t) = t^k$, $\lambda = 1$ and $\mathfrak{r} = \frac{\eta_1+\eta_2}{2}$ leads to the required result. □

Proposition 7. *For $\eta_1, \eta_2 \in \mathbb{R}$ with $\eta_1 < \eta_2$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{\mathcal{H}(\eta_1,\eta_2)}{\mathcal{G}(\eta_1,\eta_2)} - \frac{2}{\eta_2-\eta_1}\mathcal{A}(\Theta_3,\Theta_4)\right|\\ &\leq 2^{\frac{1-s}{s}} M(\eta_2-\eta_1)\Delta_5(0;q)^{1-\frac{1}{s}}\mathcal{A}^{\frac{1}{s}}(\Delta_1(0;q),\Delta_2(0;q)),\end{aligned}$$

where

$$\begin{aligned}\Theta_3 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_1)},\\ \Theta_4 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_2)}.\end{aligned}$$

Proof. The inequality (16) for function $\mathfrak{F}(t) = \frac{1}{t}$, $\lambda = 0$ and $\mathfrak{r} = \frac{\eta_1+\eta_2}{2}$ leads to the required result. □

Proposition 8. *For $\eta_1, \eta_2 \in \mathbb{R}$ with $\eta_1 < \eta_2$, the following inequality holds:*

$$\begin{aligned}&\left|\mathcal{H}^{-1}(\eta_1,\eta_2) - \frac{2}{\eta_2-\eta_1}\mathcal{A}(\Theta_3,\Theta_4)\right|\\ &\leq 2^{\frac{1-s}{s}} M(\eta_2-\eta_1)\Delta_5(1;q)^{1-\frac{1}{s}}\mathcal{A}^{\frac{1}{s}}(\Delta_1(1;q),\Delta_2(1;q)),\end{aligned}$$

where

$$\begin{aligned}\Theta_3 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_1)},\\ \Theta_4 &= (1-q)(\eta_2-\eta_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\eta_1,\eta_2)) + (1-q^n)\eta_2)}.\end{aligned}$$

Proof. The inequality (16) for function $\mathfrak{F}(t) = \frac{1}{t}$, $\lambda = 1$ and $\mathfrak{r} = \frac{\eta_1+\eta_2}{2}$ leads to the required result. □

Proposition 9. *For $\eta_1, \eta_2 \in \mathbb{R}$ with $\eta_1 < \eta_2$, the following inequality holds:*

$$\begin{aligned}&\left|\mathcal{A}^k(\eta_1,\eta_2) - \frac{2}{\eta_2-\eta_1}\mathcal{A}(\Theta_1,\Theta_2)\right|\\ &\leq \frac{M(\eta_2-\eta_1)\Delta_7(\lambda;q,r)^{\frac{1}{r}}}{2},\end{aligned}$$

where

$$\begin{aligned}\Theta_1 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)^k,\\ \Theta_2 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)^k.\end{aligned}$$

Proof. The inequality (17) for function $\mathfrak{F}(t) = t^k$, $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 10. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\begin{aligned}&\left|\mathcal{A}\left(\mathfrak{y}_1^k,\mathfrak{y}_2^k\right) - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_1,\Theta_2)\right|\\ &\le \frac{M(\mathfrak{y}_2-\mathfrak{y}_1)\Delta_7(\lambda;q,r)^{\frac{1}{r}}}{2},\end{aligned}$$

where

$$\begin{aligned}\Theta_1 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)^k,\\ \Theta_2 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)^k.\end{aligned}$$

Proof. The inequality (17) for function $\mathfrak{F}(t) = t^k$, $\lambda = 1$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 11. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{\mathcal{H}(\mathfrak{y}_1,\mathfrak{y}_2)}{\mathcal{G}(\mathfrak{y}_1,\mathfrak{y}_2)} - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_3,\Theta_4)\right|\\ &\le \frac{M(\mathfrak{y}_2-\mathfrak{y}_1)\Delta_7(\lambda;q,r)^{\frac{1}{r}}}{2},\end{aligned}$$

where

$$\begin{aligned}\Theta_3 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)},\\ \Theta_4 &= (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)}.\end{aligned}$$

Proof. The inequality (17) for function $\mathfrak{F}(t) = \frac{1}{t}$, $\lambda = 0$ and $\mathfrak{r} = \frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

Proposition 12. *For $\mathfrak{y}_1,\mathfrak{y}_2 \in \mathbb{R}$ with $\mathfrak{y}_1 < \mathfrak{y}_2$, the following inequality holds:*

$$\begin{aligned}&\left|\mathcal{H}^{-1}(\mathfrak{y}_1,\mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2-\mathfrak{y}_1}\mathcal{A}(\Theta_3,\Theta_4)\right|\\ &\le \frac{M(\mathfrak{y}_2-\mathfrak{y}_1)\Delta_7(\lambda;q,r)^{\frac{1}{r}}}{2},\end{aligned}$$

where

$$\Theta_3 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_1)},$$
$$\Theta_4 = (1-q)(\mathfrak{y}_2-\mathfrak{y}_1)\sum_{n=0}^{\infty}\frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1,\mathfrak{y}_2))+(1-q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (17) for function $\mathfrak{F}(t)=\frac{1}{t}$, $\lambda=1$ and $\mathfrak{x}=\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ leads to the required result. □

7. Conclusions

In this work, we proved a new parameterized quantum integral identity involving left and right quantum derivatives to prove different variants of quantum integral inequalities for quantum differentiable convex functions. We also proved that the newly established inequalities could be turned into quantum Ostrowski's type inequalities for convex functions [2], classical Ostrowski's type inequalities for convex functions [27], several classical integral inequalities for convex functions [28] and several new quantum integral inequalities such as midpoint type, trapezoidal type, Ostrowski's type and Simpson's type without having to prove each one separately. Some examples are presented to illustrate the results. It is a new and interesting problem that the researcher can obtain similar inequalities for other kinds of convexity and coordinated convexity in their future work.

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