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Generalized p-Convex Fuzzy-Interval-Valued Functions and Inequalities Based upon the Fuzzy-Order Relation

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Abstract: Convexity is crucial in obtaining many forms of inequalities. As a result, there is a significant link between convexity and integral inequality. Due to the significance of these concepts, the purpose of this study is to introduce a new class of generalized convex interval-valued functions called (p,s)-convex fuzzy interval-valued functions ((p,s)-convex *F-I-V-F*s) in the second sense and to establish Hermite–Hadamard (H–H) type inequalities for (p,s)-convex *F-I-V-F*s using fuzzy order relation. In addition, we demonstrate that our results include a large class of new and known inequalities for (p,s)-convex *F-I-V-F*s and their variant forms as special instances. Furthermore, we give useful examples that demonstrate usefulness of the theory produced in this study. These findings and diverse approaches may pave the way for future research in fuzzy optimization, modeling, and interval-valued functions.

Keywords: (p,s)-convex fuzzy-interval-valued function; fuzzy Riemann integral; Jensen type inequality; Schur type inequality; Hermite–Hadamard type inequality; Hermite–Hadamard–Fejér type inequality

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1. Introduction

A convex function has a convex set as its epigraph; therefore, the theory of inequality of convex functions falls under the umbrella of convexity. Nonetheless, it is a significant theory in and of itself, as it affects practically all fields of mathematics. The graphical analysis is most often the initial issue that necessitates the acquaintance with this theory. This is an opportunity to learn about the second derivative test of convexity, which is a useful tool for detecting convexity. The difficulty of identifying the extreme values of functions with many variables, as well as the application of Hessian as a higher dimensional generalization of the second derivative, follows. Holder, Jensen, and Minkowski all made early contributions to convex analysis. The next step is to go on to optimization issues in infinite dimensional spaces; however, despite the technological sophistication required to solve such problems, the fundamental concepts are quite similar to those underlying the one variable situation. Despite numerous applications, many contemporary difficulties in economics and engineering, the relevance of convex analysis is well recognized in optimization theory [1–3], and the idea of convexity no longer suffices.

Over the years, remarkable varieties of convexities, such as harmonic convexity [4], quasi convexity [5], Schur convexity [6], strong convexity [7,8], p-convexity [9], fuzzy convexity [10,11], fuzzy preinvexity [12] and generalized convexity [13], p-convexity [14] and so on, have been introduced to convex sets and convex functions. A fascinating field for research is the definition of convexity with an integral problem. Therefore, several authors have identified a great number of equalities or inequalities as applications of convex functions. The representative results include Gagliardo–Nirenberg-type inequality [15], Hardy-type inequality [16], Ostrowski-type inequality [17], Olsen-type inequality [18], and the most commonly known inequality of, namely, the H–H inequality [19]. Similarly, many authors have devoted themselves to study the fractional integral inequalities for single-valued and interval-valued functions, see [20–28].

In ref. [29], the enormous research work fuzzy set and system has been dedicated on development of different fields, and it plays an important role in the study of a wide class problems arising in pure mathematics and applied sciences including operation research, computer science, managements sciences, artificial intelligence, control engineering and decision sciences. Recently, fuzzy interval analysis and fuzzy interval-valued differential equations have been put forward to deal the ambiguity originate by insufficient data in some mathematical or computer models that determine real-world phenomena [30–40]. There are some integrals to deal with fuzzy-interval-valued functions (in short, *F-I-V-F*s), where the integrands are *F-I-V-F*s. For instance, Osuna-Gomez et al. [41], and Costa et al. [42] constructed Jensen's integral inequality for *F-I-V-F*s through a Kulisch–Miranker order relation, see [43]. By using the same approach, Costa and Roman-Flores also presented Minkowski and Beckenbach's inequalities, where the integrands are *F-I-V-F*s. This paper is motivated by [42–44] and especially by Costa et al. [45] because they established a relation between elements of fuzzy-interval space and interval space, and introduced level-wise fuzzy order relation on fuzzy-interval space through a Kulisch–Miranker order relation defined on interval space. For more information related to fuzzy interval calculus and generalized convex *F-I-V-F*s, see [46–61].

Inspired by the ongoing research work, the new class of generalized convex *F-I-V-F*s is introduced, which is known as (p, s)-convex *F-I-V-F*s. With the help of this class and fuzzy Riemann integral operator, we introduce Jensen, Schur, and fuzzy interval H–H type inequalities via fuzzy order relation. Moreover, we show that our results include a wide class of new and known inequalities for (p, s)-convex *F-I-V-F*s and their variant forms as special cases. Some useful examples are also presented to verify the validity of our main results.

2. Definitions and Basic Results

Let $\mathcal{K}_C$ and $\mathbb{F}_C(\mathbb{R})$ be the collection of all closed and bounded intervals, and fuzzy intervals of $\mathbb{R}$. We use $\mathcal{K}_C^+$ to represent the set of all positive intervals. The collection of all Riemann integrable real-valued functions, Riemann integrable *I-V-F*s and fuzzy Riemann integrable *F-I-V-F*s over $[\mathrm{t},\ \mathrm{s}]$ is denoted by $\mathcal{R}_{[\mathrm{t},\ \mathrm{s}]}$, $\mathcal{IR}_{[\mathrm{t},\ \mathrm{s}]}$, and $\mathcal{FR}_{([\mathrm{t},\ \mathrm{s}])}$, respectively. For more conceptions on interval-valued functions and fuzzy interval-valued functions, see [36,42–44]. Moreover, we have:

The inclusion " $\subseteq$ " means that

$$\xi \subseteq \eta \text{ if and only if, } [\xi_*,\ \xi^*] \ \subseteq\ [\eta_*,\ \eta^*], \text{ if and only if } \eta_* \leq \xi_*,\ \xi^* \leq \eta^*, \tag{1}$$

for all $[\boldsymbol{r}_*,\ \boldsymbol{r}^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$.

Remark 1 ([43]). The relation " $\leq_I$ " defined on $\mathcal{K}_C$ by

$$[\boldsymbol{r}_*,\ \boldsymbol{r}^*]\ \leq_I\ [\eta_*,\ \eta^*] \text{ if and only if } \boldsymbol{r}_* \leq \eta_*,\ \ \boldsymbol{r}^* \leq \eta^*, \tag{2}$$

for all $[\boldsymbol{r}_*,\ \boldsymbol{r}^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$; it is an order relation.

Proposition 1 ([7]). Let $\mathbb{F}_C(\mathbb{R})$ be a set of fuzzy numbers. If $\xi, \varpi \in \mathbb{F}_C(\mathbb{R})$, then relation " $\preccurlyeq$ " defined on $\mathbb{F}_C(\mathbb{R})$ by

$$\xi \preccurlyeq \varpi \text{ if and only if, } [\xi]^\varphi \leq_I [\varpi]^\varphi, \text{ for all } \varphi \in [0,\ 1]; \tag{3}$$

this relation is known as partial order relation.

Theorem 1 ([50]). Let $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a *F-I-V-F*, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi),\ \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t},\ \mathrm{s}]$

and for all $\varphi \in (0,\ 1]$. Then, $\mathfrak{U}$ is fuzzy Riemann integrable over $[\mathrm{t},\ \mathrm{s}]$ if, and only if, $\mathfrak{U}_*(\varkappa,\varphi)$ and $\mathfrak{U}^*(\varkappa,\varphi)$ both are Riemann integrable over $[\mathrm{t},\ \mathrm{s}]$. Moreover, if $\mathfrak{U}$ is fuzzy Riemann integrable over $[\mathrm{t},\ \mathrm{s}]$, then

$$\left((FR)\int_{\mathrm{t}}^{\mathrm{s}}\mathfrak{U}(\varkappa)d\varkappa\right)^{\varphi} = \left((R)\int_{\mathrm{t}}^{\mathrm{s}}\mathfrak{U}_*(\varkappa,\varphi)d\varkappa,\ \ (R)\int_{\mathrm{t}}^{\mathrm{s}}\mathfrak{U}^*(\varkappa,\varphi)d\varkappa\right) = (IR)\int_{t}^{s}\mathfrak{U}_{\varphi}(\varkappa)d\varkappa, \quad (4)$$

for all $\varphi \in (0,\ 1]$.

Definition 1 ([10]). Let K be a convex set. Then, *F-I-V-F* $\mathfrak{U} : K \to \mathbb{F}_C(\mathbb{R})$ is named as a convex *F-I-V-F* on K if the coming inequality

$$\mathfrak{U}(\zeta + (1-\zeta)y) \preccurlyeq \zeta\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)\mathfrak{U}(y) \quad (5)$$

is valid for all , $y \in K$, $\zeta \in [0,\ 1]$, where $\mathfrak{U}(\varkappa) \succcurlyeq \widetilde{0}$. If (5) is reversed, then $\mathfrak{U}$ is named as a concave on $[\mathrm{t},\ \mathrm{s}]$. $\mathfrak{U}$ is affine if and only if it is both a convex and concave function.

Definition 2. Let K_p be a p-convex set and $s \in [0,\ 1]$. Then, *F-I-V-F* $\mathfrak{U} : K_p \to \mathbb{F}_C(\mathbb{R})$ is named as a (p,s)-convex *F-I-V-F* in the second sense on K_p such that

$$\mathfrak{U}\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) \preccurlyeq \zeta^s\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)^s\mathfrak{U}(y), \quad (6)$$

for all $\varkappa,\ y \in K_p,\ \zeta \in [0,\ 1]$, where $\mathfrak{U}(\varkappa) \succcurlyeq \widetilde{0}$. If (6) is reversed, then $\mathfrak{U}$ is named as a (p,s)-concave *F-I-V-F* in the second sense on $[\mathrm{t},\ \mathrm{s}]$. $\mathfrak{U}$ is (p,s)-affine if and only if it is both (p,s)-convex and (p,s)-concave *F-I-V-F* in the second sense.

Remark 2. The (p,s)-convex *F-I-V-F*s in the second sense have some very nice properties similar to convex *F-I-V-F*:

- If we attempt to take $\mathfrak{U}$ as (p,s)-convex *F-I-V-F*, then we can obtain that $\Upsilon\mathfrak{U}$ is also (p,s)-convex *F-I-V-F*, for $\Upsilon \geq 0$;
- If we attempt to take both $\mathcal{F}$ and $\mathfrak{U}$ both as (p,s)-convex *F-I-V-F*s, then we can obtain that $\max(\mathcal{F}(\varkappa),\mathfrak{U}(\varkappa))$ is also a (p,s)-convex *F-I-V-F*.

We now discuss some new and known special cases of (p,s)-convex *F-I-V-F*s in the second sense:

- If we attempt to take $s \equiv 1$, then from (p,s)-convex *F-I-V-F*, we achieve p-convex *F-I-V-F*, that is

$$\mathfrak{U}\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) \preccurlyeq \zeta\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)\mathfrak{U}(y),\ \forall\ \varkappa,\ y \in K,\ \zeta \in [0,\ 1]. \quad (7)$$

- If we attempt to take $p \equiv 1$, then from (p,s)-convex *F-I-V-F*, we achieve s-convex *F-I-V-F*, see [13]; that is,

$$\mathfrak{U}(\zeta\varkappa + (1-\zeta)y) \preccurlyeq \zeta^s\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)^s\mathfrak{U}(y),\ \forall\varkappa,\ y \in K,\ \zeta \in [0,\ 1],\ s \in [0,\ 1]. \quad (8)$$

- If we attempt to take $p \equiv 1$ and $s \equiv 1$, then from (p,s)-convex *F-I-V-F*, we achieve convex *F-I-V-F*, see [13,36], that is

$$\mathfrak{U}(\zeta\varkappa + (1-\zeta)y) \preccurlyeq \zeta\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)\mathfrak{U}(y),\ \forall\ \varkappa,\ y \in K,\ \zeta \in [0,\ 1]. \quad (9)$$

Theorem 2. Let K_p be p-convex set and $\mathfrak{U} : K_p \to \mathbb{F}_C(\mathbb{R})$ be a *F-I-V-F*, whose φ-levels define the family of IVFs $\mathfrak{U}_{\varphi} : K_p \subset \mathbb{R} \to \mathcal{K}_C{}^+ \subset \mathcal{K}_C$ are given by

$$\mathfrak{U}_{\varphi}(\varkappa) = [\mathfrak{U}_*(\varkappa,\varphi),\ \mathfrak{U}^*(\varkappa,\varphi)], \quad (10)$$

for all $\in K_p$ and for all $\varphi \in [0,\ 1]$. Then, $\mathfrak{U}$ is (p,s)-convex *F-I-V-F* in the second sense on K_p, if and only if, for all $\varphi \in [0,\ 1]$, $\mathfrak{U}_*(\varkappa,\ \varphi)$ and $\mathfrak{U}^*(\varkappa,\ \varphi)$ both are (p,s)-convex functions in the second sense.

Proof. Assume that, for each $\varphi \in [0,\ 1]$, $\mathfrak{U}_*(\varkappa,\ \varphi)$ and $\mathfrak{U}^*(\varkappa,\ \varphi)$ are (p,s)-convex function in the second sense on K_p. Then, from Equation (6), we have

$$\mathfrak{U}_*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right) \le \zeta^s\mathfrak{U}_*(\varkappa,\ \varphi) + (1-\zeta)^s\mathfrak{U}_*(y,\ \varphi),\ \forall\ \varkappa, y \in K_p,\ \zeta \in [0,\ 1],$$

and

$$\mathfrak{U}^*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right) \le \zeta^s\mathfrak{U}^*(\varkappa,\ \varphi) + (1-\zeta)^s\mathfrak{U}^*(y,\ \varphi), \forall\ \varkappa, y \in K_p,\ \zeta \in [0,\ 1].$$

Then, by Equation (10), we obtain

$$\begin{aligned}\mathfrak{U}_\varphi\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) &= \left[\mathfrak{U}_*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right),\ \mathfrak{U}^*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right)\right],\\ &\le_I [\zeta^s\mathfrak{U}_*(\varkappa,\ \varphi),\ \zeta^s\mathfrak{U}^*(\varkappa,\ \varphi)] + \left[(1-\zeta)^s\mathfrak{U}_*(y,\ \varphi),\ (1-\zeta)^s\mathfrak{U}^*(y,\ \varphi)\right],\end{aligned}$$

that is

$$\mathfrak{U}\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) \preccurlyeq \zeta^s\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)^s\mathfrak{U}(y), \forall\ \varkappa, y \in K_p,\ \zeta \in [0,\ 1].$$

Hence, $\mathfrak{U}$ is (p,s)-convex *F-I-V-F* in the second sense on K_p.

Conversely, let $\mathfrak{U}$ be (p,s)-convex *F-I-V-F* in the second sense on K_p. Then, for all $\varkappa, y \in K_p$ and $\zeta \in [0,\ 1]$, we have

$$\mathfrak{U}\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) \preccurlyeq \zeta^s\mathfrak{U}(\varkappa)\widetilde{+}(1-\zeta)^s\mathfrak{U}(y).$$

Therefore, from Equation (10), we have

$$\mathfrak{U}_\varphi\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}}\right) = \left[\mathfrak{U}_*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right),\ \mathfrak{U}^*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right)\right].$$

Again, from Equation (10), we obtain

$$\zeta^s\mathfrak{U}_\varphi(\varkappa)\widetilde{+}(1-\zeta)^s\mathfrak{U}_\varphi(\varkappa) = [\zeta^s\mathfrak{U}_*(\varkappa,\ \varphi),\ \zeta^s\mathfrak{U}^*(\varkappa,\ \varphi)] + \left[(1-\zeta)^s\mathfrak{U}_*(y,\ \varphi),\ (1-\zeta)^s\mathfrak{U}^*(y,\ \varphi)\right],$$

Then, by (p,s)-convexity in the second sense of $\mathfrak{U}$, we have

$$\mathfrak{U}_*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right) \le \zeta^s\mathfrak{U}_*(\varkappa,\ \varphi) + (1-\zeta)^s\mathfrak{U}_*(y,\ \varphi),$$

and

$$\mathfrak{U}^*\left([\zeta\varkappa^p + (1-\zeta)y^p]^{\frac{1}{p}},\ \varphi\right) \le \zeta^s\mathfrak{U}^*(\varkappa,\ \varphi) + (1-\zeta)^s\mathfrak{U}^*(y,\ \varphi),$$

for each $\varphi \in [0,\ 1]$. Hence, the result follows. □

Remark 3. On the basis of Theorem 2, we consider the special situation as below:

- If we attempt to take $\mathfrak{U}_*(\varkappa,\ \varphi) = \mathfrak{U}^*(\varkappa,\ \varphi)$ with $\varphi = 1$, then from Definition 2, we obtain the (p,s)-convex function, see [46];
- If we attempt to take $\mathfrak{U}_*(\varkappa,\ \varphi) = \mathfrak{U}^*(\varkappa,\ \varphi)$ with $\varphi = 1$ and $s = 1$, then from Definition 2, we obtain the p-convex function, see [9];

- If we attempt to take $\mathfrak{U}_*(\varkappa,\ \varphi) = \mathfrak{U}^*(\varkappa,\ \varphi)$ with $\varphi = 1$, $p = 1$ and $s = 0$, then from Definition 2, we obtain the P-function, see [47].

Example 1. We consider the *F-I-V-F* $\mathfrak{U} : [0,\ 1] \to \mathbb{F}_C(\mathbb{R})$ defined by

$$\mathfrak{U}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{2\varkappa^p} & \sigma \in [0,\ 2\varkappa^p] \\ \frac{4\varkappa^p - \sigma}{2\varkappa^2} & \sigma \in (2\varkappa^p,\ 4\varkappa^p] \\ 0 & otherwise, \end{cases} \tag{11}$$

Then, for each $\varphi \in [0,\ 1]$, we have $\mathfrak{U}_\varphi(\varkappa) = [2\varphi\varkappa^p, (4 - 2\varphi)\varkappa^p]$. Since end point functions $\mathfrak{U}_*(\varkappa, \varphi)$ and $\mathfrak{U}^*(\varkappa, \varphi)$, both are (p, s)-convex functions in the second sense for each $\varphi \in [0,\ 1]$ and $s \in [0,\ 1]$. Hence, $\mathfrak{U}(\varkappa)$ is (p, s)-convex *F-I-V-F* in the second sense.

3. Discrete Inequalities for (p, s)-Convex *F-I-V-F* in the Second Sense

In the following, we establish the following result:

Theorem 3. (Discrete Jensen type inequality for (p, s)-convex *F-I-V-F*) Let $\omega_j \in \mathbb{R}^+$, $t_j \in [\mathrm{t},\ \mathrm{s}]$, $(j = 1,\ 2,\ 3, \ldots,\ k,\ k \geq 2)$ and $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi),\ \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t},\ \mathrm{s}]$ and for all $\varphi \in [0,\ 1]$, then

$$\mathfrak{U}\left(\left[\frac{1}{W_k}\sum_{j=1}^{k}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right) \preccurlyeq \textstyle\sum_j^k \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}(\mathrm{t}_j), \tag{12}$$

where $W_k = \sum_{j=1}^{k}\omega_j$. If $\mathfrak{U}$ is (p, s)-concave *F-I-V-F*, then inequality Equation (29) is reversed.

Proof. When $k = 2$, then inequality Equation (12) is true. Considering that inequality Equation (29) is true for $k = n - 1$, then

$$\mathfrak{U}\left(\left[\frac{1}{W_{n-1}}\sum_{j=1}^{n-1}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right) \preccurlyeq \textstyle\sum_{j=1}^{n-1} \left(\frac{\omega_j}{W_{n-1}}\right)^s \mathfrak{U}(\mathrm{t}_j)$$

Now, let us prove that inequality (12) holds for $k = n$.

$$\mathfrak{U}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right)$$

$$= \mathfrak{U}\left(\left[\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2}\omega_j \mathrm{t}_j{}^p + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\mathrm{t}_{n-1}{}^p + \frac{\omega_n}{\omega_{n-1}+\omega_n}\mathrm{t}_n{}^p\right)\right]^{\frac{1}{p}}\right).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\mathfrak{U}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}, \varphi\right)$$
$$\mathfrak{U}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}, \varphi\right)$$

$$= \mathfrak{U}_*\left(\left[\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2}\omega_j \mathrm{t}_j{}^p + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\mathrm{t}_{n-1}{}^p + \frac{\omega_n}{\omega_{n-1}+\omega_n}\mathrm{t}_n{}^p\right)\right]^{\frac{1}{p}}, \varphi\right)$$
$$= \mathfrak{U}^*\left(\left[\frac{W_{n-2}}{W_n}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2}\omega_j \mathrm{t}_j{}^p + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\mathrm{t}_{n-1}{}^p + \frac{\omega_n}{\omega_{n-1}+\omega_n}\mathrm{t}_n{}^p\right)\right]^{\frac{1}{p}}, \varphi\right)$$

$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_j, \varphi) + \left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)^s \mathfrak{U}_*\left(\left[\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\mathrm{t}_{n-1}{}^p + \frac{\omega_n}{\omega_{n-1}+\omega_n}\mathrm{t}_n{}^p\right]^{\frac{1}{p}}, \varphi\right)$$
$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_j, \varphi) + \left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)^s \mathfrak{U}^*\left(\left[\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\mathrm{t}_{n-1}{}^p + \frac{\omega_n}{\omega_{n-1}+\omega_n}\mathrm{t}_n{}^p\right]^{\frac{1}{p}}, \varphi\right)$$

$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_j, \varphi) + \left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)^s \left[\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\right)^s \mathfrak{U}_*(\mathrm{t}_{n-1}, \varphi) + \left(\frac{\omega_n}{\omega_{n-1}+\omega_n}\right)^s \mathfrak{U}_*(\mathrm{t}_n, \varphi)\right]$$
$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_j, \varphi) + \left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)^s \left[\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}\right)^s \mathfrak{U}^*(\mathrm{t}_{n-1}, \varphi) + \left(\frac{\omega_n}{\omega_{n-1}+\omega_n}\right)^s \mathfrak{U}^*(\mathrm{t}_n, \varphi)\right]$$

$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_j, \varphi) + \left[\left(\frac{\omega_{n-1}}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_{n-1}, \varphi) + \left(\frac{\omega_n}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_n, \varphi)\right]$$
$$\leq \sum_{j=1}^{n-2}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_j, \varphi) + \left[\left(\frac{\omega_{n-1}}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_{n-1}, \varphi) + \left(\frac{\omega_n}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_n, \varphi)\right]$$

$$= \sum_{j=1}^{n}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_j, \varphi)$$
$$= \sum_{j=1}^{n}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_j, \varphi).$$

From which, we have

$$\left[\mathfrak{U}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}, \varphi\right), \mathfrak{U}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}, \varphi\right)\right]$$
$$\leq_I \left[\sum_{j=1}^{n}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}_*(\mathrm{t}_j, \varphi), \sum_{j=1}^{n}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}^*(\mathrm{t}_j, \varphi)\right],$$

that is,

$$\mathfrak{U}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right) \preccurlyeq \sum_{j=1}^{n}\left(\frac{\omega_j}{W_n}\right)^s \mathfrak{U}(\mathrm{t}_j),$$

and the result follows. □

If $\omega_1 = \omega_2 = \omega_3 = \cdots = \omega_k = 1$, then Theorem 3 reduces to the following result:

Corollary 1. Let $s \in [0, 1]$ $\mathrm{t}_j \in [\mathrm{t}, \mathrm{s}]$,. $(j = 1, 2, 3, \ldots, k, k \geq 2)$ and $\mathfrak{U} : [\mathrm{t}, \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi : [\mathrm{t}, \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ that are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi), \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t}, \mathrm{s}]$ and for all $\varphi \in [0, 1]$; then,

$$\mathfrak{U}\left(\left[\frac{1}{k}\sum_{j=1}^{k}\mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right) \preccurlyeq \sum_{j=1}^{k}\left(\frac{1}{k}\right)^s \mathfrak{U}(\mathrm{t}_j). \tag{13}$$

If $\mathfrak{U}$ is a (p, s)-concave *F-I-V-F*, then inequality Equation (13) is reversed.

The next Theorem 4 gives the Schur-type inequality for (p, s)-convex *F-I-V-F*s.

Theorem 4. (Discrete Schur-type inequality for (p, s)-convex *F-I-V-F*) Let $s \in [0, 1]$ and $\mathfrak{U} : [\mathrm{t}, \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of IVFs $\mathfrak{U}_\varphi : [\mathrm{t}, \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi), \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t}, \mathrm{s}]$ and for all $\varphi \in [0, 1]$. If $\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in [\mathrm{t}, \mathrm{s}]$, such that $\mathrm{t}_1 < \mathrm{t}_2 < \mathrm{t}_3$ and $\mathrm{t}_3{}^p - \mathrm{t}_1{}^p$, $\mathrm{t}_3{}^p - \mathrm{t}_2{}^p$, $\mathrm{t}_2{}^p - \mathrm{t}_1{}^p \in [0, 1]$, we have

$$(\mathrm{t}_3{}^p - \mathrm{t}_1{}^p)^s \mathfrak{U}(\mathrm{t}_2) \preccurlyeq (\mathrm{t}_3{}^p - \mathrm{t}_2{}^p)^s \mathfrak{U}(\mathrm{t}_1) + (\mathrm{t}_2{}^p - \mathrm{t}_1{}^p)^s \mathfrak{U}(\mathrm{t}_3). \tag{14}$$

If $\mathfrak{U}$ is a (p, s)-concave *F-I-V-F*, then inequality Equation (14) is reversed.

Proof. Let t_j such that $L < \mathrm{t}_j \left\langle U \ (j = 1,\ 2,\ 3, \ldots,\ k),\ ({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \right\rangle 0$. Then, by hypothesis, we have

$$\left(\frac{{\mathrm{t}_3}^p - {\mathrm{t}_2}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s = \frac{({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s} \text{ and } \left(\frac{{\mathrm{t}_2}^p - {\mathrm{t}_1}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s = \frac{({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s}.$$

Consider $\zeta = \frac{{\mathrm{t}_3}^p - {\mathrm{t}_2}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}$, then ${\mathrm{t}_2}^p = \zeta {\mathrm{t}_1}^p + (1-\zeta){\mathrm{t}_3}^p$. Since $\mathfrak{U}$ is a (p, s)-convex *F-I-V-F* then, by hypothesis, we have

$$\mathfrak{U}(\mathrm{t}_2) \preccurlyeq \left(\frac{{\mathrm{t}_3}^p - {\mathrm{t}_2}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}(\mathrm{t}_1) + \left(\frac{{\mathrm{t}_2}^p - {\mathrm{t}_1}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}(\mathrm{t}_3).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\begin{aligned} \mathfrak{U}_*(\mathrm{t}_2, \varphi) &\le \left(\tfrac{{\mathrm{t}_3}^p - {\mathrm{t}_2}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}_*(\mathrm{t}_1, \varphi) + \left(\tfrac{{\mathrm{t}_2}^p - {\mathrm{t}_1}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}_*(\mathrm{t}_3, \varphi), \\ \mathfrak{U}^*(\mathrm{t}_2, \varphi) &\le \left(\tfrac{{\mathrm{t}_3}^p - {\mathrm{t}_2}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}^*(\mathrm{t}_1, \varphi) + \left(\tfrac{{\mathrm{t}_2}^p - {\mathrm{t}_1}^p}{{\mathrm{t}_3}^p - {\mathrm{t}_1}^p}\right)^s \mathfrak{U}^*(\mathrm{t}_3, \varphi) \end{aligned} \tag{15}$$

$$\begin{aligned} &= \tfrac{({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s} \mathfrak{U}_*(\mathrm{t}_1, \varphi) + \tfrac{({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s} \mathfrak{U}_*(\mathrm{t}_3, \varphi) \\ &= \tfrac{({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s} \mathfrak{U}^*(\mathrm{t}_1, \varphi) + \tfrac{({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s}{({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s} \mathfrak{U}^*(\mathrm{t}_3, \varphi). \end{aligned} \tag{16}$$

From Equation (16), we have

$$\begin{aligned} ({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}_*(\mathrm{t}_2, \varphi) \le ({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s \mathfrak{U}_*(\mathrm{t}_1, \varphi) + ({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}_*(\mathrm{t}_3, \varphi), \\ ({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}^*(\mathrm{t}_2, \varphi) \le ({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s \mathfrak{U}^*(\mathrm{t}_1, \varphi) + ({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}^*(\mathrm{t}_3, \varphi), \end{aligned}$$

that is

$$\begin{gathered} \left[({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}_*(\mathrm{t}_2, \varphi),\ ({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}^*(\mathrm{t}_2, \varphi)\right] \\ \le_I \left[({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s \mathfrak{U}_*(\mathrm{t}_1, \varphi) + ({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}_*(\mathrm{t}_3, \varphi),\ ({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s \mathfrak{U}^*(\mathrm{t}_1, \varphi) + ({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}^*(\mathrm{t}_3, \varphi)\right]. \end{gathered}$$

Hence,

$$({\mathrm{t}_3}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}(\mathrm{t}_2) \preccurlyeq ({\mathrm{t}_3}^p - {\mathrm{t}_2}^p)^s \mathfrak{U}(\mathrm{t}_1) + ({\mathrm{t}_2}^p - {\mathrm{t}_1}^p)^s \mathfrak{U}(\mathrm{t}_3).$$

□

A refinement of Jensen type inequality for (p, s)-convex *F-I-V-F* is given in the following theorem.

Theorem 5. Let $s \in [0,\ 1]$, $\omega_j \in \mathbb{R}^+$, $\mathrm{t}_j \in [\mathrm{t},\ \mathrm{s}]$, $(j = 1,\ 2,\ 3, \ldots, ,k,\ k \ge 2)$ and $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi),\ \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t},\ \mathrm{s}]$ and for all $\varphi \in [0,\ 1]$. If $(L,\ U) \subseteq [\mathrm{t}, \mathrm{s}]$, then

$$\textstyle\sum_{j=1}^k \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}\left(\mathrm{t}_j\right) \preccurlyeq \sum_{j=1}^k \left(\left(\frac{U^p - {\mathrm{t}_j}^p}{U^p - L^p}\right)^s \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}(L, \varphi) + \left(\frac{{\mathrm{t}_j}^p - L^p}{U^p - L^p}\right)^s \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}(U, \varphi) \right), \tag{17}$$

where $W_k = \sum_{j=1}^k \omega_j$. If $\mathfrak{U}$ is (p, s)-concave *F-I-V-F*, then inequality Equation (17) is reversed.

Proof. Consider t_j such that $L < \mathrm{t}_j < U \ (j = 1,\ 2,\ 3, \ldots,\ k)$. Then, by hypothesis and inequality Equation (15), we have

$$\mathfrak{U}(\mathrm{t}_j) \le \left(\frac{U^p - {\mathrm{t}_j}^p}{U^p - L^p}\right)^s \mathfrak{U}(L, \varphi) + \left(\frac{{\mathrm{t}_j}^p - L^p}{U^p - L^p}\right)^s \mathfrak{U}(U, \varphi).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\begin{aligned}\mathfrak{U}_*(\mathrm{t}_j,\varphi) &\le \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \mathfrak{U}_*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \mathfrak{U}_*(U,\varphi),\\ \mathfrak{U}^*(\mathrm{t}_j,\varphi) &\le \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \mathfrak{U}^*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \mathfrak{U}^*(U,\varphi).\end{aligned}$$

The above inequality can be written as

$$\begin{aligned}\left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(\mathrm{t}_j,\varphi) &\le \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(U,\varphi),\\ \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(\mathrm{t}_j,\varphi) &\le \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(U,\varphi)\end{aligned} \tag{18}$$

Taking the sum of all inequalities (18) for $j = 1,\ 2,\ 3, \ldots,\ k$, we have

$$\begin{aligned}\textstyle\sum_{j=1}^k \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(\mathrm{t}_j,\varphi) &\le \textstyle\sum_{j=1}^k \left(\left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(U,\varphi)\right),\\ \textstyle\sum_{j=1}^k \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(\mathrm{t}_j,\varphi) &\le \textstyle\sum_{j=1}^k \left(\left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(L,\varphi) + \left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(U,\varphi)\right),\end{aligned}$$

that is

$$\textstyle\sum_{j=1}^k \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_\varphi(\mathrm{t}_j) = \left[\sum_{j=1}^k \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(\mathrm{t}_j,\varphi),\ \sum_{j=1}^k \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(\mathrm{t}_j,\varphi)\right]$$

$$\le_I \left[\textstyle\sum_{j=1}^k \begin{pmatrix} \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(L,\varphi) \\ +\left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_*(U,\varphi)\end{pmatrix},\ \sum_{j=1}^k \begin{pmatrix} \left(\tfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(L,\varphi) \\ +\left(\tfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\tfrac{\omega_j}{W_k}\right)^s \mathfrak{U}^*(U,\varphi)\end{pmatrix}\right]$$

$$= \textstyle\sum_{j=1}^k \left(\dfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s [\mathfrak{U}_*(L,\varphi),\ \mathfrak{U}^*(L,\varphi)] + \sum_{j=1}^k \left(\dfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s [\mathfrak{U}_*(U,\varphi),\ \mathfrak{U}^*(U,\varphi)]$$

$$= \textstyle\sum_{j=1}^k \left(\dfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_\varphi(L) + \sum_{j=1}^k \left(\dfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}_\varphi(U).$$

Thus,

$$\textstyle\sum_{j=1}^k \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}(\mathrm{t}_j) \preccurlyeq \sum_{j=1}^k \left(\left(\dfrac{U^p-\mathrm{t}_j{}^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}(L) + \left(\dfrac{\mathrm{t}_j{}^p-L^p}{U^p-L^p}\right)^s \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}(U)\right),$$

and this completes the proof. □

We now consider some special cases of Theorems 3 and 5.
If $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}_*(\varkappa,\varphi)$, then Theorems 3 and 5 reduce to the following results:

Corollary 2 ([21]). (Jensen inequality for (p,s)-convex function) Let $s \in [0,\ 1]$, $\omega_j \in \mathbb{R}^+$, $\mathrm{t}_j \in [\mathrm{t},\ \mathrm{s}]$, $(j = 1,\ 2,\ 3, \ldots,\ k,\ k \ge 2)$ and let $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathfrak{U}$ is a (p,s)-convex function, then

$$\mathfrak{U}\left(\left[\frac{1}{W_k}\sum_{j=1}^k \omega_j \mathrm{t}_j{}^p\right]^{\frac{1}{p}}\right) \le \textstyle\sum_{j=1}^k \left(\dfrac{\omega_j}{W_k}\right)^s \mathfrak{U}(\mathrm{t}_j), \tag{19}$$

where $W_k = \sum_{j=1}^k \omega_j$. If $\mathfrak{U}$ is (p,s)-concave function, then inequality (19) is reversed.

Corollary 3. Let $s \in [0,\ 1], \omega_j \in \mathbb{R}^+, \mathrm{t}_j \in [\mathrm{t},\ \mathrm{s}], (j = 1,\ 2,\ 3, \ldots,\ k,\ k \geq 2)$, and $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathfrak{U}$ is a (p, s)-convex function and $\mathrm{t}_1,\ \mathrm{t}_2, \ldots, \mathrm{t}_j \in (L,\ U) \subseteq [\mathrm{t}, \mathrm{s}]$, then

$$\sum_{j=1}^{k} \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}\left(\mathrm{t}_j\right) \leq \sum_{j=1}^{k} \left(\left(\frac{U^p - \mathrm{t}_j{}^p}{U^p - L^p}\right)^s \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}(L) + \left(\frac{\mathrm{t}_j{}^p - L^p}{U^p - L^p}\right)^s \left(\frac{\omega_j}{W_k}\right)^s \mathfrak{U}(U) \right), \quad (20)$$

where $W_k = \sum_{j=1}^{k} \omega_j$. If $\mathfrak{U}$ is a (p, s)-concave function, then inequality (20) is reversed.

4. Hermite–Hadamard Type Inequalities for (p, s)-Convex *F-I-V-F* in the Second Sense

In this section, we will continue with the H–H inequality for (p, s)-convex fuzzy-*I-V-F*s as well as the fuzzy-interval H–H Fejér inequality for (p, s)-convex fuzzy-*I-V-F*s using the fuzzy order relation. Firstly, we start with the following H–H inequality for (p, s)-convex fuzzy-*I-V-F*s:

Theorem 6. Let $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of *I-V-F*s. $\mathfrak{U}_\varphi : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi),\ \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t},\ \mathrm{s}]$ and for all $\varphi \in [0,\ 1]$. If $\mathfrak{U} \in \mathcal{FR}_{([\mathrm{t},\ \mathrm{s}])}$, then

$$2^{s-1}\, \mathfrak{U}\left(\left[\frac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}} \right) \preccurlyeq \frac{p}{\mathrm{s}^p - \mathrm{t}^p}\ (FR) \int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1} \mathfrak{U}(\varkappa) d\varkappa \leq_p \frac{\mathfrak{U}(\mathrm{t}) \widetilde{+} \mathfrak{U}(\mathrm{s})}{s+1}. \quad (21)$$

If $\mathfrak{U}$ is a (p, s)-concave *F-I-V-F*, then

$$2^{s-1}\, \mathfrak{U}\left(\left[\frac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}} \right) \succcurlyeq \frac{p}{\mathrm{s}^p - \mathrm{t}^p}\ (FR) \int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1} \mathfrak{U}(\varkappa) d\varkappa \succcurlyeq \frac{\mathfrak{U}(\mathrm{t}) \widetilde{+} \mathfrak{U}(\mathrm{s})}{s+1}. \quad (22)$$

Proof. Let $\mathfrak{U}$ be a $(p,\ s)$-convex *F-I-V-F*. Then, by hypothesis, we have

$$2^s \mathfrak{U}\left(\left[\frac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}} \right) \preccurlyeq \mathfrak{U}\left([\zeta \mathrm{t}^p + (1-\zeta)\mathrm{s}^p]^{\frac{1}{p}} \right) \widetilde{+} \mathfrak{U}\left([(1-\zeta)\mathrm{t}^p + \zeta \mathrm{s}^p]^{\frac{1}{p}} \right).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\begin{aligned} 2^s \mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) &\leq \mathfrak{U}_*\left([\zeta \mathrm{t}^p + (1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}, \varphi \right) + \mathfrak{U}_*((1-\zeta)\mathrm{t}^p + \zeta \mathrm{s}^p, \varphi), \\ 2^s \mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) &\leq \mathfrak{U}^*\left([\zeta \mathrm{t}^p + (1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}, \varphi \right) + \mathfrak{U}^*((1-\zeta)\mathrm{t}^p + \zeta \mathrm{s}^p, \varphi). \end{aligned}$$

Then,

$$\begin{aligned} 2^s \int_0^1 \mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) d\zeta &\leq \int_0^1 \mathfrak{U}_*\left([\zeta \mathrm{t}^p + (1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}, \varphi \right) d\zeta + \int_0^1 \mathfrak{U}_*((1-\zeta)\mathrm{t}^p + \zeta \mathrm{s}^p, \varphi) d\zeta, \\ 2^s \int_0^1 \mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) d\zeta &\leq \int_0^1 \mathfrak{U}^*\left([\zeta \mathrm{t}^p + (1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}, \varphi \right) d\zeta + \int_0^1 \mathfrak{U}^*((1-\zeta)\mathrm{t}^p + \zeta \mathrm{s}^p, \varphi) d\zeta. \end{aligned}$$

It follows that

$$\begin{aligned} 2^{s-1} \mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) &\leq \tfrac{p}{\mathrm{s}^p - \mathrm{t}^p}\ \int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1} \mathfrak{U}_*(\varkappa, \varphi) d\varkappa, \\ 2^{s-1} \mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p + \mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi \right) &\leq \tfrac{p}{\mathrm{s}^p - \mathrm{t}^p}\ \int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1} \mathfrak{U}^*(\varkappa, \varphi) d\varkappa. \end{aligned}$$

That is,

$$2^{s-1}\left[\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right),\ \mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\right] \leq_I \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\left[\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)d\varkappa,\ \int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)d\varkappa\right].$$

Thus,

$$2^{s-1}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa. \tag{23}$$

In a similar way as above, we have

$$\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \frac{1}{s+1}\left[\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})\right]. \tag{24}$$

Combining Equations (23) and (24), we have

$$2^{s-1}\ \mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \frac{1}{s+1}\left[\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})\right].$$

Hence, we obtain the required result. □

Remark 4. On the basis of Theorem 6, we consider the certain the special situation as below:

- If we attempt to take $\mathfrak{U}_*(\varkappa,\ \varphi) = \mathfrak{U}^*(\varkappa,\ \varphi)$ with $\varphi = 1$, then we achieve the (p,s)-convex function, see [9];
- If we attempt to take $s = 1$, then we achieve the result for p-convex *F-I-V-F-*:

$$\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{2}; \tag{25}$$

- If we attempt to take $p = 1$, then we achieve the result for s-convex *F-I-V-F*, see [13]:

$$\mathfrak{U}\left(\frac{\mathrm{t}+\mathrm{s}}{2}\right) \preccurlyeq \frac{1}{\mathrm{s}-\mathrm{t}}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{s+1}; \tag{26}$$

- If we attempt to take $s = 1$ and $p = 1$, then we achieve the result for p-convex *F-I-V-F*, see [13]:

$$\mathfrak{U}\left(\frac{\mathrm{t}+\mathrm{s}}{2}\right) \preccurlyeq \frac{1}{\mathrm{s}-\mathrm{t}}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{2}; \tag{27}$$

- If we attempt to take $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}^*(\varkappa,\varphi)$ with $\varphi = 1$, then we acquire the result for classical (p,s)-convex function, see [21]:

$$2^{s-1}\ \mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (R)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \leq \frac{1}{s+1}\left[\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})\right]; \tag{28}$$

- If we attempt to take $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}^*(\varkappa,\varphi)$ with $\varphi = 1$ and $s = 1$, then we acquire the result for classical p-convex function:

$$\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (R)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \leq \frac{\mathfrak{U}(\mathrm{t})+\mathfrak{U}(\mathrm{s})}{2}; \tag{29}$$

- If we attempt to take $\mathfrak{U}_*(\varkappa, \varphi) = \mathfrak{U}^*(\varkappa, \varphi)$ with, $\varphi = 1$, $p = 1$ and $s = 1$, then we acquire the result for classical convex function:

$$\mathfrak{U}\left(\frac{\mathrm{t}+\mathrm{s}}{2}\right) \leq \frac{1}{\mathrm{s}-\mathrm{t}}\ (R)\int_{\mathrm{t}}^{\mathrm{s}} \mathfrak{U}(\varkappa)d\varkappa \leq \frac{\mathfrak{U}(\mathrm{t})+\mathfrak{U}(\mathrm{s})}{2}. \tag{30}$$

Example 2. Let p be an odd number and $s \in [0,\ 1]$, and the *F-I-V-F* $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] = [2,\ 3] \to \mathbb{F}_C(\mathbb{R})$ defined by

$$\mathfrak{U}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma}{\left(2-\varkappa^{\frac{p}{2}}\right)}, & \sigma \in \left[0,\ 2-\varkappa^{\frac{p}{2}}\right] \\ \frac{2\left(2-\varkappa^{\frac{p}{2}}\right)-\sigma}{\left(2-\varkappa^{\frac{p}{2}}\right)}, & \sigma \in \left(2-\varkappa^{\frac{p}{2}},\ 2\left(2-\varkappa^{\frac{p}{2}}\right)\right] \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Then, for each $\varphi \in [0,\ 1]$, we have $\mathfrak{U}_\varphi(\varkappa) = \left[\varphi\left(2-\varkappa^{\frac{p}{2}}\right),\ (2-\varphi)\left(2-\varkappa^{\frac{p}{2}}\right)\right]$. Since end point functions $\mathfrak{U}_*(\varkappa, \varphi) = \varphi\left(2-\varkappa^{\frac{p}{2}}\right)$, $\mathfrak{U}^*(\varkappa,\ \varphi) = (2-\varphi)\left(2-\varkappa^{\frac{p}{2}}\right)$ are (p, s)-convex functions for each $\varphi \in [0,\ 1]$. Then, $\mathfrak{U}(\varkappa)$ is (p, s)-convex *F-I-V-F*. We now compute the following:

$$\begin{aligned} 2^{s-1}\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi\right) &= \tfrac{4-\sqrt{10}}{2}\varphi, \\ 2^{s-1}\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}, \varphi\right) &= \tfrac{4-\sqrt{10}}{2}(2-\varphi), \\ \tfrac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{s} \varkappa^{p-1}\mathfrak{U}_*(\varkappa,\ \varphi)d\varkappa = \varphi\int_2^3\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa &= \tfrac{21}{50}\varphi, \\ \tfrac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{s} \varkappa^{p-1}\mathfrak{U}^*(\varkappa,\ \varphi)d\varkappa = (2-\varphi)\int_2^3\left(2-\varkappa^{\frac{p}{2}}\right)d\varkappa &= \tfrac{21}{50}(2-\varphi), \\ \tfrac{\mathfrak{U}_*(\mathrm{t},\ \varphi)+\mathfrak{U}_*(\mathrm{s},\ \varphi)}{s+1} &= \tfrac{4-\sqrt{2}-\sqrt{3}}{2}\varphi, \\ \tfrac{\mathfrak{U}^*(\mathrm{t},\ \varphi)+\mathfrak{U}^*(\mathrm{s},\ \varphi)}{s+1} &= \tfrac{4-\sqrt{2}-\sqrt{3}}{2}(2-\varphi), \end{aligned}$$

for all $\varphi \in [0,\ 1]$. That means

$$\left[\frac{4-\sqrt{10}}{2}\varphi,\ \frac{4-\sqrt{10}}{2}(2-\varphi)\right] \leq_I \left[\frac{21}{50}\varphi,\ \frac{21}{50}(2-\varphi)\right] \leq_I \left[\frac{4-\sqrt{2}-\sqrt{3}}{2}\varphi,\ \frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\varphi)\right], \text{for all } \varphi \in [0,\ 1],$$

and the Theorem 6 has been verified.

Theorem 7. Let $\mathfrak{U} : [\mathrm{t},\ \mathrm{s}] \to \mathbb{F}_C(\mathbb{R})$ be a (p, s)-convex *F-I-V-F*, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi : [\mathrm{t},\ \mathrm{s}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa) = [\mathfrak{U}_*(\varkappa, \varphi),\ \mathfrak{U}^*(\varkappa, \varphi)]$ for all $\in [\mathrm{t},\ \mathrm{s}]$ and for all $\varphi \in [0,\ 1]$. If $\mathfrak{U} \in\ \mathcal{FR}_{([\mathrm{t},\ \mathrm{s}])}$, then

$$4^{s-1}\ \mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \rhd_1 \preccurlyeq \frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{s+1}\left[\frac{1}{2}+\frac{1}{2^s}\right], \tag{32}$$

where

$$\rhd_1 = \frac{\frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{2}\widetilde{+}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right)}{s+1}, \rhd_2 = 2^{s-2}\left[\mathfrak{U}\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\widetilde{+}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\right],$$

and $\rhd_1 = [\rhd_{1_*},\ \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2_*},\ \rhd_2{}^*]$.

Proof. Take $\left[\mathrm{t}^p,\ \frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]$, and we have

$$\begin{aligned}&2^s\mathfrak{U}\left(\left[\frac{\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}+\frac{(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}\right]^{\frac{1}{p}}\right)\\ &\preccurlyeq \mathfrak{U}\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right)\widetilde{+}\mathfrak{U}\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right).\end{aligned}$$

Therefore, for each $\varphi\in[0,\ 1]$, we have

$$\begin{aligned}&2^s\mathfrak{U}_*\left(\left[\frac{\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}+\frac{(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}\right]^{\frac{1}{p}},\varphi\right)\\ &\quad\leq \mathfrak{U}_*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}_*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right),\\ &2^s\mathfrak{U}^*\left(\left[\frac{\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}+\frac{(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}{2}\right]^{\frac{1}{p}},\varphi\right)\\ &\quad\leq \mathfrak{U}^*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}^*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right).\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}2^{s-2}\mathfrak{U}_*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)&\leq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)d\varkappa,\\ 2^{s-2}\mathfrak{U}^*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)&\leq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)d\varkappa.\end{aligned}$$

That is,

$$\begin{aligned}&2^{s-2}\left[\mathfrak{U}_*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right),\ \mathfrak{U}^*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)\right]\\ &\leq_I \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\left[\int_{\mathrm{t}}^{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)d\varkappa,\ \int_{\mathrm{t}}^{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)d\varkappa\right].\end{aligned}$$

It follows that

$$2^{s-2}\mathfrak{U}\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}\varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa. \tag{33}$$

In a similar way as above, we have

$$2^{s-2}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\frac{\mathrm{t}^p+\mathrm{s}^p}{2}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa. \tag{34}$$

Combining Equations (33) and (34), we have

$$2^{s-2}\left[\mathfrak{U}\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\widetilde{+}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right)\right]\preccurlyeq\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa.$$

By using Theorem 6, we have

$$4^{s-1}\,\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right)=4^{s-1}\,\mathfrak{U}\left(\left[\frac{1}{2}.\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}+\frac{1}{2}.\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}}\right).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\begin{aligned}
4^{s-1}\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right) &= 4^{s-1}\,\mathfrak{U}_*\left(\left[\frac{1}{2}.\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}+\frac{1}{2}.\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right),\\
4^{s-1}\mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right) &= 4^{s-1}\mathfrak{U}^*\left(\left[\frac{1}{2}.\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}+\frac{1}{2}.\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)\\
&\le 2^{s-2}\left[\mathfrak{U}_*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)\right]\\
&\le 2^{s-2}\left[\mathfrak{U}^*\left(\left[\frac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\varphi\right)\right]\\
&= \rhd_{2*}\\
&= \rhd_2{}^*\\
&\le \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)d\varkappa\\
&\le \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)d\varkappa\\
&\le \frac{1}{s+1}\left[\frac{\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi)}{2}+\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\right]\\
&\le \frac{1}{s+1}\left[\frac{\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)}{2}+\mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\right]\\
&= \rhd_{1*}\\
&= \rhd_1{}^*\\
&\le \frac{1}{s+1}\left[\frac{\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi)}{2}+\frac{1}{2^s}(\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi))\right]\\
&\le \frac{1}{s+1}\left[\frac{\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)}{2}+\frac{1}{2^s}(\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi))\right]\\
&= \frac{1}{s+1}[\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi)]\left[\frac{1}{2}+\frac{1}{2^s}\right]\\
&= \frac{1}{s+1}[\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)]\left[\frac{1}{2}+\frac{1}{2^s}\right],
\end{aligned}$$

that is

$$4^{s-1}\,\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\,(FR)\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)d\varkappa \preccurlyeq \rhd_1 \preccurlyeq \frac{\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})}{s+1}\left[\frac{1}{2}+\frac{1}{2^s}\right],$$

hence, the result follows. □

Example 3. Let p be an odd number and the *F-I-V-F* $\mathfrak{U}:[\mathrm{t},\ \mathrm{s}]=[2,\ 3]\to\mathbb{F}_C(\mathbb{R})$ defined by, $\mathfrak{U}_\varphi(\varkappa)=\left[\varphi\left(2-\varkappa^{\frac{p}{2}}\right),(2-\varphi)\left(2-\varkappa^{\frac{p}{2}}\right)\right]$, as in Example 2, then $\mathfrak{U}(\varkappa)$ is (p,s)-convex *F-I-V-F* and satisfies Equation (21). We have

$$\mathfrak{U}_*(\varkappa,\ \varphi)=\varphi\left(2-\varkappa^{\frac{p}{2}}\right)\text{ and }\mathfrak{U}^*(\varkappa,\ \varphi)=(2-\varphi)\left(2-\varkappa^{\frac{p}{2}}\right)$$

We now compute the following:

$$\begin{aligned}
&\tfrac{\mathfrak{U}_*(\mathrm{t},\,\varphi)+\mathfrak{U}_*(\mathrm{s},\,\varphi)}{s+1}\left[\tfrac{1}{2}+\tfrac{1}{2^s}\right]=\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\varphi,\\
&\tfrac{\mathfrak{U}^*(\mathrm{t},\,\varphi)+\mathfrak{U}^*(\mathrm{s},\,\varphi)}{s+1}\left[\tfrac{1}{2}+\tfrac{1}{2^s}\right]=\tfrac{4-\sqrt{2}-\sqrt{3}}{2}(2-\varphi),\\
&\rhd_{1*}=\frac{\frac{\mathfrak{U}_*(\mathrm{t},\,\varphi)+\mathfrak{U}_*(\mathrm{s},\,\varphi)}{2}+\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\,\varphi\right)}{s+1}=\tfrac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\varphi,\\
&\rhd_1{}^*=\frac{\frac{\mathfrak{U}^*(\mathrm{t},\,\varphi)+\mathfrak{U}^*(\mathrm{s},\,\varphi)}{2}+\mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\,\varphi\right)}{s+1}=\tfrac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\varphi),\\
&\rhd_{2*}=2^{s-2}\left[\mathfrak{U}_*\left(\left[\tfrac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\,\varphi\right)+\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\,\varphi\right)\right]=\tfrac{5-\sqrt{11}}{4}\varphi,\\
&\rhd_2{}^*=2^{s-2}\left[\mathfrak{U}^*\left(\left[\tfrac{3\mathrm{t}^p+\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\,\varphi\right)+\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+3\mathrm{s}^p}{4}\right]^{\frac{1}{p}},\,\varphi\right)\right]=\tfrac{5-\sqrt{11}}{4}(2-\varphi),\\
&4^{s-1}\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)=\tfrac{4-\sqrt{10}}{2}\varphi,\\
&4^{s-1}\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)=\tfrac{4-\sqrt{10}}{2}(2-\varphi).
\end{aligned}$$

Then, we obtain that

$$\begin{gathered}
\tfrac{4-\sqrt{10}}{2}\varphi\le\tfrac{5-\sqrt{11}}{4}\varphi\le\tfrac{21}{50}\varphi\le\tfrac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\varphi\le\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\varphi,\\
\tfrac{4-\sqrt{10}}{2}(2-\varphi)\le\tfrac{5-\sqrt{11}}{4}(2-\varphi)\le\tfrac{21}{50}(2-\varphi)\le\tfrac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\varphi)\le\tfrac{4-\sqrt{2}-\sqrt{3}}{2}(2-\varphi).
\end{gathered}$$

Hence, Theorem 7 is verified.

The next Theorems 8 and 9 give the second H–H Fejér inequality and the first H–H Fejér inequality for (p,s)-convex *F-I-V-F*, respectively.

Theorem 8. (Second H–H Fejér inequality for (p,s)-convex *F-I-V-F*) Let $\mathfrak{U}:[\mathrm{t},\ \mathrm{s}]\to\mathbb{F}_C(\mathbb{R})$ be a (p,s)-convex *F-I-V-F* with $\mathrm{t}<\mathrm{s}$, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi:[\mathrm{t},\ \mathrm{s}]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa)=[\mathfrak{U}_*(\varkappa,\varphi),\ \mathfrak{U}^*(\varkappa,\varphi)]$ for all $\varkappa\in[\mathrm{t},\ \mathrm{s}]$ and for all $\varphi\in[0,\ 1]$. If $\mathfrak{U}\in\ \mathcal{FR}_{([\mathrm{t},\ \mathrm{s}])}$ and $\Psi:[\mathrm{t},\ \mathrm{s}]\to\mathbb{R},\ \Psi(\varkappa)\ge 0,$ p-symmetric with respect to $\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}$, then

$$\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)\Psi(\varkappa)d\preccurlyeq\left[\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})\right]\int_0^1\zeta^s\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta. \tag{35}$$

If $\mathfrak{U}$ is (p,s)-concave *F-I-V-F*, then Equation (35) is reversed.

Proof. Let $\mathfrak{U}$ be a $(p,\ s)$-convex *F-I-V-F*. Then, for each $\varphi\in[0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{U}_*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)\\
&\qquad\le\left(\zeta^s\mathfrak{U}_*(\mathrm{t},\varphi)+(1-\zeta)^s\mathfrak{U}_*(\mathrm{s},\varphi)\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right),\\
&\mathfrak{U}^*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)\\
&\qquad\le\left(\zeta^s\mathfrak{U}^*(\mathrm{t},\varphi)+(1-\zeta)^s\mathfrak{U}^*(\mathrm{s},\varphi)\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right).
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\mathfrak{U}_*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\\
&\qquad\le\left((1-\zeta)^s\mathfrak{U}_*(\mathrm{t},\varphi)+\zeta^s\mathfrak{U}_*(\mathrm{s},\varphi)\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right),\\
&\mathfrak{U}^*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\\
&\qquad\le\left((1-\zeta)^s\mathfrak{U}^*(\mathrm{t},\varphi)+\zeta^s\mathfrak{U}^*(\mathrm{s},\varphi)\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right).
\end{aligned}\tag{37}$$

After adding Equations (36) and (37), and integrating over $[0,\ 1]$, we get

$$\begin{aligned}
&\int_0^1\mathfrak{U}_*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\qquad+\int_0^1\mathfrak{U}_*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\le\int_0^1\left[\begin{array}{c}\mathfrak{U}_*(\mathrm{t},\varphi)\left\{\zeta^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)+(1-\zeta)^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\right\}\\+\mathfrak{U}_*(\mathrm{s},\varphi)\left\{(1-\zeta)^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)+\zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\right\}\end{array}\right]d\zeta,\\
&\int_0^1\mathfrak{U}^*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\qquad+\int_0^1\mathfrak{U}^*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\le\int_0^1\left[\begin{array}{c}\mathfrak{U}^*(\mathrm{t},\varphi)\left\{\zeta^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)+(1-\zeta)^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\right\}\\+\mathfrak{U}^*(\mathrm{s},\varphi)\left\{(1-\zeta)^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)+\zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\right\}\end{array}\right]d\zeta.\\
&=2\mathfrak{U}_*(\mathrm{t},\varphi)\int_0^1\zeta^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta+2\mathfrak{U}_*(\mathrm{s},\varphi)\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta\\
&=2\mathfrak{U}^*(\mathrm{t},\varphi)\int_0^1\zeta^s\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta+2\mathfrak{U}^*(\mathrm{s},\varphi)\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta.
\end{aligned}$$

Since Ψ is symmetric, then

$$\begin{aligned}
&=2[\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi)]\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta\\
&=2[\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)]\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta.
\end{aligned}\tag{38}$$

Since

$$\begin{aligned}
&\int_0^1\mathfrak{U}_*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\qquad=\int_0^1\mathfrak{U}_*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&\qquad=\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ \int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\\
&\int_0^1\mathfrak{U}^*\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&=\int_0^1\mathfrak{U}^*\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}},\varphi\right)\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta\\
&=\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ \int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,
\end{aligned}\tag{39}$$

From Equation (39) and integrating with respect to ζ over $[0,\ 1]$, we have

$$\begin{aligned}
&\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ \int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\le[\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi)]\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta,\\
&\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ \int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\ \le[\mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)]\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\left[\ \int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\qquad\int_\mathrm{t}^\mathrm{s}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\right]\\
&\qquad\le_I\ [\mathfrak{U}_*(\mathrm{t},\varphi)+\mathfrak{U}_*(\mathrm{s},\varphi),\ \mathfrak{U}^*(\mathrm{t},\varphi)+\mathfrak{U}^*(\mathrm{s},\varphi)]\int_0^1\ \zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\zeta,
\end{aligned}$$

hence

$$\frac{p}{\mathrm{s}^p-\mathrm{t}^p}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)\Psi(\varkappa)d\varkappa \preccurlyeq \left[\mathfrak{U}(\mathrm{t})\widetilde{+}\mathfrak{U}(\mathrm{s})\right]\int_0^1\zeta^s\Psi\left(\left[(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p\right]^{\frac{1}{p}}\right)d\zeta.$$

□

Theorem 9. (First H–H Fejér inequality for $(p,\ s)$-convex *F-I-V-F*) Let $\mathfrak{U}:[\mathrm{t},\ \mathrm{s}]\to\mathbb{F}_C(\mathbb{R})$ be a $(p,\ s)$-convex *F-I-V-F* with $\mathrm{t}<\mathrm{s}$, whose φ-levels define the family of *I-V-F*s $\mathfrak{U}_\varphi:[\mathrm{t},\ \mathrm{s}]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ are given by $\mathfrak{U}_\varphi(\varkappa)=[\mathfrak{U}_*(\varkappa,\varphi),\ \mathfrak{U}^*(\varkappa,\varphi)]$ for all $\in[\mathrm{t},\ \mathrm{s}]$ and for all $\varphi\in[0,\ 1]$. If $\mathfrak{U}\in\mathcal{FR}_{([\mathrm{t},\ \mathrm{s}])}$ and $\Psi:[\mathrm{t},\ \mathrm{s}]\to\mathbb{R}$, $\Psi(\varkappa)\geq 0$, p-symmetric with respect to $\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}$, and $\int_{\mathrm{t}}^{\mathrm{s}}\Psi(\varkappa)d\varkappa>0$, then

$$2^{s-1}\,\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\Psi(\varkappa)d\varkappa}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa)\Psi(\varkappa)d\varkappa. \tag{40}$$

If $\mathfrak{U}$ is $(p,\ s)$-concave *F-I-V-F*, then inequality (40) is reversed.

Proof. Since $\mathfrak{U}$ is a $(p,\ s)$-convex *F-I-V-F*, then, for each $\varphi\in[0,\ 1]$, we have

$$\begin{aligned}2^s\,\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)&\leq\mathfrak{U}_*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}_*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right),\\ 2^s\,\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)&\leq\mathfrak{U}^*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)+\mathfrak{U}^*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right).\end{aligned} \tag{41}$$

By multiplying Equation (41) by $\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)=\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)$ and integrating it by ζ over $[0,\ 1]$, we obtain

$$\begin{aligned}&2^s\,\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\int_0^1\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)d\zeta\\ &\quad\leq\left(\begin{array}{c}\int_0^1\mathfrak{U}_*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ +\int_0^1\mathfrak{U}_*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\end{array}\right),\\ &2^s\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\int_0^1\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ &\quad\leq\left(\begin{array}{c}\int_0^1\mathfrak{U}^*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ +\int_0^1\mathfrak{U}^*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\end{array}\right).\end{aligned} \tag{42}$$

Since

$$\begin{aligned}&\int_0^1\mathfrak{U}_*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ &\quad=\int_0^1\mathfrak{U}_*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ &\quad=\tfrac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\\ &\int_0^1\mathfrak{U}^*\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([\zeta\mathrm{t}^p+(1-\zeta)\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ &\quad=\int_0^1\mathfrak{U}^*\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}},\varphi\right)\Psi\left([(1-\zeta)\mathrm{t}^p+\zeta\mathrm{s}^p]^{\frac{1}{p}}\right)d\zeta\\ &\quad=\tfrac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\end{aligned} \tag{43}$$

From Equation (43), we have

$$\begin{aligned} 2^{s-1}\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right) &\leq \frac{p}{\int_{\mathrm{t}}^{\mathrm{s}}\Psi(\varkappa)d\varkappa}\ \int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\\ 2^{s-1}\mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right) &\leq \frac{p}{\int_{\mathrm{t}}^{\mathrm{s}}\Psi(\varkappa)d\varkappa}\ \int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa. \end{aligned}$$

From this, we have

$$\begin{aligned} &2^{s-1}\left[\mathfrak{U}_*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right),\ \mathfrak{U}^*\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\varphi\right)\right]\\ &\leq_I \frac{p}{\int_{\mathrm{t}}^{\mathrm{s}}\Psi(\varkappa)d\varkappa}\left[\ \int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,\ \int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\right], \end{aligned}$$

that is

$$2^{s-1}\mathfrak{U}\left(\left[\frac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\Psi(\varkappa)d\varkappa}\ (FR)\int_{\mathrm{t}}^{\mathrm{s}}\varkappa^{p-1}\mathfrak{U}(\varkappa,\varphi)\Psi(\varkappa)d\varkappa,$$

and this completes the proof. □

Remark 5. If we attempt to take $s = 1$ in Theorems 8 and 9, then we achieve the appropriate theorems for p-convex *F-I-V-F*s, see [13]:

- If we attempt to take $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}^*(\varkappa,\varphi)$ with $\varphi = 1$, then, from Theorems 8 and 9, we achieve classical first and second H–H Fejér inequality for (p,s)-convex function, [21];
- If in Theorems 8 and 9, we attempt to take $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}^*(\varkappa,\varphi)$ with $\varphi = 1$ and $s = 1$, then we acquire the classical appropriate theorems for p-convex function, see [49];
- If, in Theorems 8 and 9, we attempt to take $\mathfrak{U}_*(\varkappa,\varphi) = \mathfrak{U}^*(\varkappa,\varphi)$ with $\varphi = 1$, $s = 1$ and $p = 1$, then we acquire the appropriate theorems for a convex function [48];
- If we attempt to take $\Psi(\varkappa) = 1$, then combining Theorem 8 and Theorem 9, we acquire Theorem 4.1.

Example 4. We consider the *F-I-V-F* $\mathfrak{U} : [1,\ 4] \to \mathbb{F}_C(\mathbb{R})$ defined by

$$\mathfrak{U}(\varkappa)(\sigma) = \begin{cases} \frac{\sigma-e^{\varkappa p}}{e^{\varkappa p}}, & \sigma\in[e^{\varkappa p},\ 2e^{\varkappa p}],\\ \frac{4e^{\varkappa p}-\sigma}{2e^{\varkappa p}}, & \sigma\in(2e^{\varkappa p},\ 4e^{\varkappa p}],\\ 0, & otherwise, \end{cases} \tag{44}$$

Then, for each $\varphi \in [0,\ 1]$, we have $\mathfrak{U}_\varphi(\varkappa) = [(1+\varphi)e^{\varkappa p}, 2(2-\varphi)e^{\varkappa p}]$. Since end point functions $\mathfrak{U}_*(\varkappa,\varphi)$, $\mathfrak{U}^*(\varkappa,\varphi)$ are $(p,\ s)$-convex functions, for each $s,\varphi \in [0,\ 1]$, then $\mathfrak{U}(\varkappa)$ is (p,s)-convex *F-I-V-F*. If

$$\Psi(\varkappa) = \begin{cases} \varkappa^p - 1, & \sigma \in \left[1, \frac{5}{2}\right],\\ 4-\varkappa^p, & \sigma\in\left(\frac{5}{2},\ 4\right], \end{cases} \tag{45}$$

where $p = 1$. Then, we have

$$\begin{aligned} \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_1^4\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa &= \frac{1}{3}\int_1^4\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\\ &= \frac{1}{3}\int_1^{\frac{5}{2}}\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa + \frac{1}{3}\int_{\frac{5}{2}}^4\varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\\ &= \frac{1}{3}(1+\varphi)\int_1^{\frac{5}{2}}e(-1)d\varkappa + \frac{1}{3}(1+\varphi)\int_{\frac{5}{2}}^4 e(4-)d\varkappa \approx 11(1+\varphi),\\ \frac{p}{\mathrm{s}^p-\mathrm{t}^p}\int_1^4\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa &= \frac{1}{3}\int_1^4\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\\ &= \frac{1}{3}\int_1^{\frac{5}{2}}\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa + \frac{1}{3}\int_{\frac{5}{2}}^4\varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa\\ &= \frac{2}{3}(2-\varphi)\int_1^{\frac{5}{2}}e(-1)d\varkappa + \frac{2}{3}(2-\varphi)\int_{\frac{5}{2}}^4 e(4-)d\varkappa \approx 22(2-\varphi), \end{aligned} \tag{46}$$

and

$$\begin{aligned}
&[\mathfrak{U}^*(\mathrm{t},\ \varphi) \quad +\mathfrak{U}^*(\mathrm{s},\ \varphi)]\int_0^1 \ \zeta^s\Psi\left(\left[(1-\tau)\mathrm{t}^p+\tau\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\tau \\
&[\mathfrak{U}_*(\mathrm{t},\ \varphi) \quad +\mathfrak{U}_*(\mathrm{s},\varphi)]\int_0^1 \ \zeta^s\Psi\left(\left[(1-\tau)\mathrm{t}^p+\tau\mathrm{s}^p\right]^{\frac{1}{p}}\right)\ d\tau \\
&\qquad = (1+\varphi)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}} 3\tau^2 d+\int_{\frac{1}{2}}^1 \tau(3-3\tau)d\tau\right] \approx \tfrac{43}{2}(1+\varphi) \\
&\qquad = 2(2-\varphi)\left[e+e^4\right]\left[\int_0^{\frac{1}{2}} 3\tau^2 d+\int_{\frac{1}{2}}^1 \tau(3-3\tau)d\tau\right] \approx 43(2-\varphi).
\end{aligned} \tag{47}$$

From Equations (46) and (47), we have

$$[11(1+\varphi),\ 22(2-\varphi)] \quad \leq \quad _I\left[\frac{43}{2}(1+\varphi),\ 43(2-\varphi)\right], \text{for each } \varphi\in[0,\ 1].$$

Hence, Theorem 8 is verified.
For Theorem 9, we have

$$\begin{aligned}
2^{s-1}\mathfrak{U}_*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\ \varphi\right) &\approx \tfrac{61}{5}(1+\varphi), \\
2^{s-1}\mathfrak{U}^*\left(\left[\tfrac{\mathrm{t}^p+\mathrm{s}^p}{2}\right]^{\frac{1}{p}},\ \varphi\right) &\approx \tfrac{122}{5}(2-\varphi),
\end{aligned} \tag{48}$$

$$\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\Psi(\varkappa)d\varkappa = \int_1^{\frac{5}{2}}(\varkappa-1)d\varkappa\int_{\frac{5}{2}}^4(4-)d\varkappa = \frac{9}{4},$$

$$\begin{aligned}
\tfrac{p}{\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\Psi(\varkappa)d\varkappa}\ \int_1^4 \varkappa^{p-1}\mathfrak{U}_*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa &\approx \tfrac{73}{5}(1+\varphi), \\
\tfrac{p}{\int_{\mathrm{t}}^{\mathrm{s}} \varkappa^{p-1}\Psi(\varkappa)d\varkappa}\ \int_1^4 \varkappa^{p-1}\mathfrak{U}^*(\varkappa,\varphi)\Psi(\varkappa)d\varkappa &\approx \tfrac{293}{10}(2-\varphi).
\end{aligned} \tag{49}$$

From Equations (48) and (49), we have

$$\left[\frac{61}{5}(1+\varphi),\ \frac{122}{5}(2-\varphi)\right] \quad \leq \quad _I\left[\frac{73}{5}(1+\varphi),\ \frac{293}{10}(2-\varphi)\right].$$

Hence, Theorem 9 has been demonstrated.

5. Conclusions and Future Developments

Through this study, we have provided a reformative version of the different inequalities in the frame of fuzzy interval space, which offers a better approximation than the interval integral inequalities.

Then, for mappings satisfying the property "the product of two $(p,\ s)$-convex *F-I-V-F*s is a $(p,\ s)$-convex *F-I-V-F*", we created certain fuzzy interval integral inequalities in terms of the fuzzy interval H–H type inequalities. It is a fascinating topic to apply these fuzzy interval inequalities to φ-type special means, numerical integration, and probability density functions. With the methods and ideas provided in this article, the interested readers are encouraged to further excavation on fuzzy interval inequalities. In the future, we will try to explore this concept and its generalizations with the help of fuzzy fractional integral operators.

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Abbreviations

$\mathcal{K}_C$	Collection of all closed and bounded intervals
$\mathcal{K}_C^+$	Collection of all closed and bounded positive intervals
$\mathbb{F}_C(\mathbb{R})$	Collection of all closed and bounded fuzzy intervals
*F-I-V-F*s	Fuzzy-interval-valued functions
*I-V-F*s	Interval-valued functions
$\leq_I$	order relation
$\preccurlyeq$	fuzzy order relation
(p,s)-convex *F-I-V-Fs*	(p,s)-Convex fuzzy-interval-valued functions
H–H inequality	Hermite–Hadamard inequality
H–H Fejér inequality	Hermite–Hadamard–Fejér inequality
(FR)-integrable	Fuzzy Riemann integrable
$\mathcal{R}_{[t,\ s]}$	Riemann integrable real-valued functions
$\mathcal{IR}_{[t,\ s]}$	Riemann integrable *I-V-F*s
$\mathcal{FR}_{([t,\ s])}$	Riemann integrable *F-I-V-F*s

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