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Article

Analysis of the Multi-Dimensional Navier–Stokes Equation by Caputo Fractional Operator

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Abstract: In this article, we investigate the solution of the fractional multidimensional Navier–Stokes equation based on the Caputo fractional derivative operator. The behavior of the solution regarding the Navier–Stokes equation system using the Sumudu transform approach is discussed analytically and further discussed graphically.

Keywords: Navier–Stokes equation; Caputo derivative; existence and uniqueness; Sumudu transform

MSC: Primary 92B05, 92C60; Secondary 26A33

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1. Introduction

Around 1822, a renowned overseeing state of an improvement in viscid liquid stream was found, known as the Navier–Stokes (NS) condition—this condition was feasibly named as Newton's second law of fluid, and is a blend of congruity and strength conditions. These conditions are useful in depicting the actual study of various consistent and planned characteristics of interest. It distinguishes a few things around the wings of the airplane, for example, the fluid stream in pipes, bloodstream and wind current. The Navier–Stokes condition makes the connection between strain and liquid act as outer powers to the liquid stream reaction. These conditions [1,2] and traditional liquid elements have been very successful in acquiring quantitative information on shock waves, disturbances and solutions. Numerous critical peculiarities, such as their thermodynamics, aeronautical sciences, geophysics, the petrol business, plasma physical science, etc., provide a characteristic depiction of the association of a thick liquid with a rigid body. They are considered significant computational devices for a more superior comprehension of various genuine issues [3,4].

A halfway examination is a general improvement of the investigation regarding the total number of solicitations to whimsical solicitation—it was portrayed in advance during correspondence between Leibniz and L'Hospital in 1695. Considering its particular ability to sort out odd ways to deal with acting and memory influences, which are the fundamental elements of tangled idiosyncrasies, an incomplete assessment has been persistently arranged to chip away at current numerical models [5–7]. The numerical legitimization behind fragmentary solicitation assistants was settled by the joint endeavors of specialists such as Caputo, Riemann, Liouville, Ross and Miller, Podlubny and others. Incomplete solicitation math speculation has been related with practical applications and has claimed the speculation of tumult, electrodynamics, signal care, thermodynamics, monetary viewpoints and various sectors [8].

In fragmentary math, we habitually modeled various real characteristics slightly complicatedly, such as differences with regular examinations. The ongoing strategy relies upon the prompt execution of the normal change on the Caputo-portrayed fragmentary solicitation subordinates. Around the completion of the proposed estimation, we obtained a game plan of the fragmentary solicitation of the Navier–Stokes condition regarding the provided fractional solicitation. Hence, we can get different courses of action at other fragmentary solicitations of these conditions. The responsibility of the ongoing systems is that we can separate other components of these conditions by including various fractional solicitation subordinates in a system; we might pick an ideal fragmentary, requesting to find an answer which is as per the specific arrangement of the issue.

In the current paper, we suppose a period fragmentary Navier–Stokes condition for an incompressible liquid [9–11] progression of density ρ and kinematic consistency $v = \frac{\phi}{\rho}$. It is shown as

$$\begin{cases} D_{\mu}{}^{\beta}V + (V.\nabla)V = \rho\nabla^2 V - \frac{1}{\rho}\nabla g, \\ \nabla.V = 0, \\ V = 0, \quad on\,\Omega \times (0, T). \end{cases}$$

Here, $V = (\mu, v, w)$, q, $and\,\mu$ address liquid vector, tension and time, separately. (β, θ, δ) addresses the spatial parts in Ω. The above conditions can likewise be characterized as

$$\begin{aligned} D_{\mu}{}^{\xi}(\phi) + \phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta} &= \rho\left[\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\beta} \\ D_{\mu}{}^{\xi}(\psi) + \phi\frac{\partial\psi}{\partial\beta} + \psi\frac{\partial\psi}{\partial\theta} + \Theta\frac{\partial\psi}{\partial\delta} &= \rho\left[\frac{\partial^2\psi}{\partial\beta^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\delta^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\theta} \\ D_{\mu}{}^{\xi}(\Theta) + \phi\frac{\partial\Theta}{\partial\beta} + \psi\frac{\partial\Theta}{\partial\theta} + \Theta\frac{\partial\Theta}{\partial\delta} &= \rho\left[\frac{\partial^2\Theta}{\partial\beta^2} + \frac{\partial^2\Theta}{\partial\theta^2} + \frac{\partial^2\Theta}{\partial\delta^2}\right] - \frac{1}{\rho}\frac{\partial g}{\partial\delta}. \end{aligned}$$

Here, some essential obligations from the specialists are discussed; assume, for instance, Herrmann and Hilfer outline that fragmentary partial differential circumstances, for example, the time-partial Navier–Stokes conditions, are not altogether firmly established by applications in different areas of science and arranging. El-Shahed and Salem carried out an incomplete appearance of these conditions in 2005. Producers utilized the Laplace change, limited the Hankel change and limited the Fourier Sine change, adding obsolete Navier–Stokes conditions. Kumar et al. logically took care of the nonlinear halfway model of these conditions by joining HPM and LTA. Ganji et al. and Ragab et al. had settled the non-straight time fragmentary NS condition by executing a homotopy assessment system. Odibat, Momani and Birajdar presented ADM for mathematical assessments of the time-partial Navier–Stokes condition. The watchful arrangement of the time-fractional Navier–Stokes condition is accomplished by Kumar et al., using a blend of ADM and the Laplace change, while Chaurasia and Kumar kept an eye on a near-condition by joining the restricted Hankel change and Laplace change.

M. Rawashdeh and S. Maitama introduced NDM in 2014 to handle immediate and nonlinear ODEs and PDEs. A colossal amount of genuine issues had been concentrated by using NDM—for instance, the examination of the partial message condition, partial-order Whitham–Broer–Kaup conditions, partial-order heat and wave conditions, fragmentary actual models, fractional order PDEs with relative deferment and fragmentary-order dissemination conditions [12–15].

The ongoing creation is stressed over the fractional solicitation smart plan of the Navier–Stokes conditions. An arrangement of conventional conditions has been the subject for examiners over a long period. The smart plans of the fractional-ordered condition are the major point of convergence for researchers and mathematicians; this was the moving work to grow or cultivate the ongoing systems for the incomplete solicitation courses of action regarding the Navier–Stokes conditions. An impressive part of them has accomplished and formulated creative techniques to handle the fragmentary-order Navier–Stokes system. In such a way, a stream investigation is a shrewd responsibility to the logical plan of the partial-order Navier–Stokes conditions. In this article, we did not simply execute two logical procedures—specifically, the Sumudu change and predictor–corrector strategy—however, in such a manner, their evaluation stated the importance of the proposed calculations. The continuous evaluation work is driven in an especially major and direct way to accomplish the clever plans of the doled-out issues with a limited measure of mathematical appraisals. The mix of the proposed methods is irrelevant. With everything taken into account, the proposed methodologies are seen as a mind-boggling responsibility towards the logical course of action regarding fragmentary-order fractional differential circumstances, which constantly arise in science and planning.

In recent studies, Chu et al. [16] solved the system by using the Caputo derivative and Laplace transform with the variation iteration transform method. Singh and Kumar [17] found the approximate solution of the system by using the fractional reduced differential transform method, while Kavvas and Ercan [18] found the solution of the Navier–Stokes system with a different approach, using the system of momentum equations.

In this article, first, we use the existence and uniqueness theorem to verify the existence and uniqueness of the system's solution. Further, we will apply the Sumudu transform with an iterative technique to solve the system. This work strives to find an approximate solution to the NS system with a new methodology and approach.

2. Pre-Requisites

2.1. Caputo Fractional Differential Operator

The Caputo derivative [19] is helpful for modeling phenomena that consider past interactions and issues with non-local features. The equation can be regarded as having "memory" in this sense. This contrasts with parabolic equations such as the heat operator, which does not account for the past, and the groundwater flow equations in confined, unconfined and leaky aquifers, as well as other diffusion issues.

The partial-ordered Caputo derivative of $g(\beta)$ is defined below:

$$D^{\xi}g(\beta) = \frac{1}{\Gamma(k-\xi)}\int_{0}^{\beta}(\beta-\tau)^{k-\xi-1}g^{(k)}(\tau)d\tau, \tag{1}$$

for $k-1<\xi<k$, $k\in N, \beta>0$.

2.2. Caputo Fractional Integral Operator

The Caputo fractional integral operator of order α is explained below:

$$^{C}I^{\alpha}g(t) = \frac{1}{\Gamma\alpha}\int_{0}^{t}(t-v)^{\alpha-1}g(v)dv. \tag{2}$$

2.3. Sumudu Transform

Suppose $f(x,t)$ is a function, then the Sumudu transform of the Caputo–Fabrizio [19] fractional differential coefficient of $f(x,t)$ is defined below:

$$ST({}_{0}^{CF}D_{t}^{\xi})(f(x,t)) = M(\xi)\left[\frac{ST(f(x,t))-f(x,0)}{1-\xi+\theta u}\right]. \tag{3}$$

2.4. Sumudu Transform of the Caputo Fractional Derivative

The Sumudu transform of the Caputo fractional derivative is explained below:

$$ST\left[{}^{c}D^{\xi}f(x,t):s\right] = s^{-\xi}ST\{f(x,t)\} - s^{-\xi}\sum_{k=0}^{m-1}s^{k}f^{k}(x,0). \tag{4}$$

The current article is divided into seven pieces; Segment 1 deals with the paper's introduction, while Segment 2 lists prerequisites for the article. The iteration method process is discussed in Section 3. The system's existence is discussed in Section 4 and the system's uniqueness is discussed in Section 5. By using the Sumudu transform approach, we arrive at the model's solution in Section 6, and in Section 7, we explain the results. We have given due credit to the writers and academics whose articles and research contributed to our conclusions. Thus, the references are in the last part.

3. Iteration Method in the Caputo Derivative by Using the Sumudu Transformation

This section explains the iteration transform method [15] of the fractional partial differential system:

$$D_\mu{}^\xi \phi(\beta,\mu) + H_1(\phi,\psi) + M_1(\phi,\psi) - q_1(\beta,\mu) = 0, \tag{5}$$

and

$$D_\mu{}^\xi \psi(\beta,\mu) + H_2(\phi,\psi) + M_2(\phi,\psi) - q_2(\beta,\mu) = 0, \quad m-1 < \xi \le m \tag{6}$$

where starting constraints are,

$$\phi(\beta,0) = g_1(\beta), \quad \psi(\beta,0) = g_2(\beta). \tag{7}$$

where $D_\mu{}^\xi$ is the Caputo fractional derivative of order ξ. H_1, H_2 & M_1, M_2 are linear and nonlinear functions and q_1, q_2 are source operators. Now, we apply the Sumudu transform:

$$ST\left[D_\mu{}^\xi \phi(\beta,\mu) + H_1(\phi,\psi) + M_1(\phi,\psi) - q_1(\beta,\mu)\right] = 0, \tag{8}$$

and

$$ST\left[D_\mu{}^\xi \psi(\beta,\mu) + H_2(\phi,\psi) + M_2(\phi,\psi) - q_2(\beta,\mu)\right] = 0, \tag{9}$$

or

$$ST\left[D_\mu{}^\xi \phi(\beta,\mu)\right] + ST[H_1(\phi,\psi) + M_1(\phi,\psi) - q_1(\beta,\mu)] = 0, \tag{10}$$

and

$$ST\left[D_\mu{}^\xi \psi(\beta,\mu)\right] + ST[H_2(\phi,\psi) + M_2(\phi,\psi) - q_2(\beta,\mu)] = 0, \tag{11}$$

or

$$s^\xi ST\{\phi(\beta,\mu)\} - s^{-\xi} \sum_{k=0}^{m-1} s^k \phi^k(\beta,0) = -ST[H_1(\phi,\psi) + M_1(\phi,\psi) - q_1(\beta,\mu)], \tag{12}$$

and

$$s^\xi ST\{\psi(\beta,\mu)\} - s^{-\xi} \sum_{k=0}^{m-1} s^k \psi^k(\beta,0) = -ST[H_2(\phi,\psi) + M_2(\phi,\psi) - q_2(\beta,\mu)]. \tag{13}$$

Using the repetitive strategy, we find

$$\begin{aligned}
&ST\{\phi_{m+1}(\beta,\mu)\} = ST\{\phi_m(\beta,\mu)\} + \lambda(s)\left[s^\xi \phi_m(\beta,\mu) - \sum_{k=0}^{m-1} s^{\xi-k-1}\phi^k(\beta,0)\right] \\
&-ST[q_1(\beta,\mu)] - ST[H_1(\phi,\psi) + M_1(\phi,\psi)], \\
&and \\
&ST\{\psi_{m+1}(\beta,\mu)\} = ST\{\psi_m(\beta,\mu)\} + \lambda(s)\left[s^\xi \psi_m(\beta,\mu) - \sum_{k=0}^{m-1} s^{\xi-k-1}\psi^k(\beta,0)\right] \\
&-ST[q_2(\beta,\mu)] - ST[H_2(\phi,\psi) + M_2(\phi,\psi)].
\end{aligned} \tag{14}$$

Setting the Lagrange multiplier as $\lambda(s) = -\frac{1}{s^\xi}$ and taking the inverse Sumudu transform on both sides, we have

$$\phi_{m+1}(\beta,\mu)=\phi_m(\beta,\mu)-ST^{-1}\left[\frac{1}{s^{\xi}}\left\{\sum_{k=0}^{m-1}s^{\xi-k-1}\phi^k(\beta,0)\right\}\right]$$
$$-q_1(\beta,\mu)-\{H_1(\phi,\psi)+M_1(\phi,\psi)\},$$
and
$$\psi_{m+1}(\beta,\mu)=\psi_m(\beta,\mu)-ST^{-1}\left[\frac{1}{s^{\xi}}\left\{\sum_{k=0}^{m-1}s^{\xi-k-1}\psi^k(\beta,0)\right\}\right]$$
$$-q_2(\beta,\mu)-\{H_2(\phi,\psi)+M_2(\phi,\psi)\}.$$

4. Existence of Solution in Caputo Case

We have the mechanism described below:

$$\begin{aligned}
D_t^{\xi}(\phi)+\phi\frac{\partial\phi}{\partial\beta}+\psi\frac{\partial\phi}{\partial\theta}+\Theta\frac{\partial\phi}{\partial\delta}&=\rho\left[\frac{\partial^2\phi}{\partial\beta^2}+\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial\delta^2}\right],\\
D_t^{\xi}(\psi)+\phi\frac{\partial\psi}{\partial\beta}+\psi\frac{\partial\psi}{\partial\theta}+\Theta\frac{\partial\psi}{\partial\delta}&=\rho\left[\frac{\partial^2\psi}{\partial\beta^2}+\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\delta^2}\right],\\
D_t^{\xi}(\Theta)+\phi\frac{\partial\Theta}{\partial\beta}+\psi\frac{\partial\Theta}{\partial\theta}+\Theta\frac{\partial\Theta}{\partial\delta}&=\rho\left[\frac{\partial^2\Theta}{\partial\beta^2}+\frac{\partial^2\Theta}{\partial\theta^2}+\frac{\partial^2\Theta}{\partial\delta^2}\right].
\end{aligned}\tag{15}$$

or

$$\begin{aligned}
D_t^{\xi}(\phi)&=\rho\left[\frac{\partial^2\phi}{\partial\beta^2}+\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial\delta^2}\right]-\left[\phi\frac{\partial\phi}{\partial\beta}+\psi\frac{\partial\phi}{\partial\theta}+\Theta\frac{\partial\phi}{\partial\delta}\right],\\
D_t^{\xi}(\psi)&=\rho\left[\frac{\partial^2\psi}{\partial\beta^2}+\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\delta^2}\right]-\left[\phi\frac{\partial\psi}{\partial\beta}+\psi\frac{\partial\psi}{\partial\theta}+\Theta\frac{\partial\psi}{\partial\delta}\right],\\
D_t^{\xi}(\Theta)&=\rho\left[\frac{\partial^2\Theta}{\partial\beta^2}+\frac{\partial^2\Theta}{\partial\theta^2}+\frac{\partial^2\Theta}{\partial\delta^2}\right]-\left[\phi\frac{\partial\Theta}{\partial\beta}+\psi\frac{\partial\Theta}{\partial\theta}+\Theta\frac{\partial\Theta}{\partial\delta}\right].
\end{aligned}\tag{16}$$

The derivatives have now been changed to the Caputo fractional derivative, and now we have

$$\begin{aligned}
{}^{C}D_t^{\xi}(\phi)&=\rho\left[\frac{\partial^2\phi}{\partial\beta^2}+\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial\delta^2}\right]-\left[\phi\frac{\partial\phi}{\partial\beta}+\psi\frac{\partial\phi}{\partial\theta}+\Theta\frac{\partial\phi}{\partial\delta}\right],\\
{}^{C}D_t^{\xi}(\psi)&=\rho\left[\frac{\partial^2\psi}{\partial\beta^2}+\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\delta^2}\right]-\left[\phi\frac{\partial\psi}{\partial\beta}+\psi\frac{\partial\psi}{\partial\theta}+\Theta\frac{\partial\psi}{\partial\delta}\right],\\
{}^{C}D_t^{\xi}(\Theta)&=\rho\left[\frac{\partial^2\Theta}{\partial\beta^2}+\frac{\partial^2\Theta}{\partial\theta^2}+\frac{\partial^2\Theta}{\partial\delta^2}\right]-\left[\phi\frac{\partial\Theta}{\partial\beta}+\psi\frac{\partial\Theta}{\partial\theta}+\Theta\frac{\partial\Theta}{\partial\delta}\right].
\end{aligned}\tag{17}$$

Assume for a moment that the function $f\left(\beta,\theta,\delta,t,\phi,\phi'_{\beta},\phi'_{\theta},\phi'_{\delta},\phi''_{\beta},\phi''_{\theta},\phi''_{\delta}\right)$ meets the Lipschitz condition [20–26],

$$\left\|f\left(\beta,\theta,\delta,t,\phi,\phi'_{\beta},\phi'_{\theta},\phi'_{\delta},\phi''_{\beta},\phi''_{\theta},\phi''_{\delta}\right)-f\left(\beta,\theta,\delta,t,\phi_1,\phi'_{1\beta},\phi'_{1\theta},\phi'_{1\delta},\phi''_{1\beta},\phi''_{1\theta},\phi''_{1\delta}\right)\right\|$$
$$\leq M|\phi-\phi_1|+K_1\left|\phi'_{\beta}-\phi'_{1\beta}\right|+K_2\left|\phi'_{\theta}-\phi'_{1\theta}\right|+K_3\left|\phi'_{\delta}-\phi'_{1\delta}\right|+L_1\left|\phi''_{\beta}-\phi''_{1\beta}\right|$$
$$+L_2\left|\phi''_{\theta}-\phi''_{1\theta}\right|+L_3\left|\phi''_{\delta}-\phi''_{1\delta}\right|.$$

Now, assume that

$$\begin{aligned}
\left\|\phi'_{\beta}-\phi'_{1\beta}\right\|&\leq\vartheta_1\|\phi-\phi_1\|,\ \left\|\phi'_{\theta}-\phi'_{1\theta}\right\|\leq\vartheta_2\|\phi-\phi_1\|,\ \left\|\phi'_{\delta}-\phi'_{1\delta}\right\|\leq\vartheta_3\|\phi-\phi_1\|,\\
\left\|\phi''_{\beta}-\phi''_{1\beta}\right\|&\leq\vartheta_4\|\phi-\phi_1\|,\ \left\|\phi''_{\theta}-\phi''_{1\theta}\right\|\leq\vartheta_5\|\phi-\phi_1\|,\left\|\phi''_{\delta}-\phi''_{1\delta}\right\|\leq\vartheta_6\|\phi-\phi_1\|,
\end{aligned}$$

where $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5\,and\,\vartheta_6\in R^+$ such that $M+K_1\vartheta_1+K_2\vartheta_2+K_3\vartheta_3+L_1\vartheta_4+L_2\vartheta_5+L_3\vartheta_6\leq 1$; then, the system has a solution if we can determine $t_{\max}$, such that:

$$t_{\max}^{\xi}<\Gamma\xi$$

Proof. Using the fundamental hypothesis of fractional math, we get:

$$\phi(\beta,\theta,\delta,t)-\phi(\beta,\theta,\delta,0)=\frac{1}{\Gamma\xi}\int_0^t(t-y)^{\xi-1}f\left(\beta,\theta,\delta,t,\phi,\phi'_{\beta},\phi'_{\theta},\phi'_{\delta},\phi''_{\beta},\phi''_{\theta},\phi''_{\delta}\right)dy,\tag{18}$$

or

$$\phi(\beta,\theta,\delta,t) = \phi_0 + \frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1} f\left(\beta,\theta,\delta,t,\phi,\phi'_\beta,\phi'_\theta,\phi'_\delta,\phi''_\beta,\phi''_\theta,\phi''_\delta\right)dy \tag{19}$$

Now by iteration,

$$\begin{aligned}&\phi_n(\beta,\theta,\delta,t) = \phi_0 + \\ &\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1} f\left(\beta,\theta,\delta,t,\phi_{n-1},\phi'_{\beta,n-1},\phi'_{\theta,n-1},\phi'_{\delta,n-1},\phi''_{\beta,n-1},\phi''_{\theta,n-1},\phi''_{\delta,n-1}\right)dy.\end{aligned} \tag{20}$$

Let $\gamma_n = \phi_n - \phi_{n-1}$.

So,

$$\begin{aligned}\gamma_n = {}&\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1}\Big[f\left(\beta,\theta,\delta,t,\phi_{n-1},\phi'_{\beta,n-1},\phi'_{\theta,n-1},\phi'_{\delta,n-1},\phi''_{\beta,n-1},\phi''_{\theta,n-1},\phi''_{\delta,n-1}\right)\\ &-f\left(\beta,\theta,\delta,t,\phi_{n-2},\phi'_{\beta,n-2},\phi'_{\theta,n-2},\phi'_{\delta,n-2},\phi''_{\beta,n-2},\phi''_{\theta,n-2},\phi''_{\delta,n-2}\right)\Big]dy.\end{aligned} \tag{21}$$

Here, we see that

$$\phi_n = \sum_{i=1}^{n}\gamma_n$$

Now, taking norm both sides, we have:

$$\|\gamma_n\| = \|\phi_n - \phi_{n-1}\|$$

$$\begin{aligned}\leq {}&\frac{1}{\Gamma\xi}\left\|\int_0^t (t-y)^{\xi-1}\Big[f\left(\beta,\theta,\delta,t,\phi_{n-1},\phi'_{\beta,n-1},\phi'_{\theta,n-1},\phi'_{\delta,n-1},\phi''_{\beta,n-1},\phi''_{\theta,n-1},\phi''_{\delta,n-1}\right)\right.\\ &\left.-f\left(\beta,\theta,\delta,t,\phi_{n-2},\phi'_{\beta,n-2},\phi'_{\theta,n-2},\phi'_{\delta,n-2},\phi''_{\beta,n-2},\phi''_{\theta,n-2},\phi''_{\delta,n-2}\right)\Big]dy\right\|,\end{aligned} \tag{22}$$

So,

$$\begin{aligned}\|\gamma_n\| \leq {}&\frac{1}{\Gamma\xi}\Big[M\|\phi_{n-1}-\phi_{n-2}\| + K_1\left\|\phi'_{\beta,n-1}-\phi'_{\beta,n-2}\right\| + K_2\left\|\phi'_{\theta,n-1}-\phi'_{\theta,n-2}\right\|\\ &+K_3\left\|\phi'_{\delta,n-1}-\phi'_{\delta,n-2}\right\| + L_1\left\|\phi''_{\beta,n-1}-\phi''_{\beta,n-2}\right\| + L_2\left\|\phi''_{\theta,n-1}-\phi''_{\theta,n-2}\right\|\\ &+L_3\left\|\phi''_{\delta,n-1}-\phi''_{\delta,n-2}\right\|\Big]\int_0^t (t-y)^{\xi-1}dy\end{aligned} \tag{23}$$

It gives,

$$\begin{aligned}\leq {}&\Big[M\|\phi_{n-1}-\phi_{n-2}\| + K_1\left\|\phi'_{\beta,n-1}-\phi'_{\beta,n-2}\right\| + K_2\left\|\phi'_{\theta,n-1}-\phi'_{\theta,n-2}\right\| + K_3\left\|\phi'_{\delta,n-1}-\phi'_{\delta,n-2}\right\|\\ &+L_1\left\|\phi''_{\beta,n-1}-\phi''_{\beta,n-2}\right\| + L_2\left\|\phi''_{\theta,n-1}-\phi''_{\theta,n-2}\right\| + L_3\left\|\phi''_{\delta,n-1}-\phi''_{\delta,n-2}\right\|\Big]\\ &\left[\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1}dy\right],\end{aligned} \tag{24}$$

Let $\left\|\gamma'_{n-1}\right\| \leq \vartheta_1\|\gamma_{n-1}\|$ and $\left\|\gamma''_{n-1}\right\| \leq \vartheta_2\|\gamma_{n-1}\|$, hence,

$$\begin{aligned}\|\gamma_n\| \leq {}&[M\|\gamma_{n-1}\| + K_1\vartheta_1\|\gamma_{n-1}\| + K_2\vartheta_2\|\gamma_{n-1}\| + K_3\vartheta_3\|\gamma_{n-1}\| + L_1\vartheta_4\|\gamma_{n-1}\| + L_2\vartheta_5\|\gamma_{n-1}\|\\ &+L_3\vartheta_6\|\gamma_{n-1}\|]\left[\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1}dy\right],\end{aligned} \tag{25}$$

or

$$\|\gamma_n\| \leq \|\gamma_{n-1}\|[M + K_1\vartheta_1 + K_2\vartheta_2 + K_3\vartheta_3 + L_1\vartheta_4 + L_2\vartheta_5 + L_3\vartheta_6]\left[\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1}dy\right], \tag{26}$$

or

$$\|\gamma_n\| \leq \|\gamma_0\|[M + K_1\vartheta_1 + K_2\vartheta_2 + K_3\vartheta_3 + L_1\vartheta_4 + L_2\vartheta_5 + L_3\vartheta_6]^n \left[\frac{1}{\Gamma\xi}\int_0^t (t-y)^{\xi-1}dy\right]^n. \tag{27}$$

Now, let $M + K_1\vartheta_1 + K_2\vartheta_2 + K_3\vartheta_3 + L_1\vartheta_4 + L_2\vartheta_5 + L_3\vartheta_6 = \vartheta_7 < 1$, hence, we have

$$\|\gamma_n\| \leq \|\gamma_0\|\vartheta_7^n \left[\frac{t_{\max}^{\xi}}{\Gamma\xi}\right]^n. \tag{28}$$

Thus, the system has a solution if we can find $t_{\max.}$, such that

$$\left(\frac{t_{\max}^{\xi}}{\Gamma\xi}\right)\vartheta_7 < 1, \tag{29}$$

or

$$\left(\frac{t_{\max}^{\xi}}{\Gamma\xi}\right) < 1, \tag{30}$$

$$t_{\max}^{\xi} < \Gamma\xi.$$

Thus, it implies that the solution of our framework exists. □

5. Uniqueness of Solution

In this part, we will prove the uniqueness of the solution of the system.

Suppose that ϕ and ϕ_1 are two solutions of the first equation of the system (15). Suppose that

$$\phi(\beta,\theta,\delta,t) - \phi_1(\beta,\theta,\delta,t) = \frac{1}{\Gamma\alpha}\int_0^x (x-y)^{\alpha-1}\phi(y)dy - \frac{1}{\Gamma\alpha}\int_0^x (x-y)^{\alpha-1}\phi_1(y)dy, \tag{31}$$

Now, applying norm both sides, we get

$$\|\phi(\beta,\theta,\delta,t) - \phi_1(\beta,\theta,\delta,t)\| = \left\|\frac{1}{\Gamma\alpha}\int_0^x (x-y)^{\alpha-1}\phi(y)dy - \frac{1}{\Gamma\alpha}\int_0^x (x-y)^{\alpha-1}\phi_1(y)dy\right\|, \tag{32}$$

or

$$\|\phi(\beta,\theta,\delta,t) - \phi_1(\beta,\theta,\delta,t)\| \leq \frac{1}{\Gamma\alpha}\left\|\int_0^x (x-y)^{\alpha-1}(\phi(y) - \phi_1(y))dy\right\| \tag{33}$$

or

$$\|\phi(\beta,\theta,\delta,t) - \phi_1(\beta,\theta,\delta,t)\| \leq \frac{1}{\Gamma\alpha}\|\phi(y) - \phi_1(y)\|\int_0^x (x-y)^{\alpha-1}dy \tag{34}$$

We know that F is known as a Lipschitz operator if $\|F(f) - F(g)\| \leq \sigma\|f-g\|$, where σ is the smallest number that satisfies the given condition, and where f and g belong to the range set. Thus, using the Lipschitz condition and remembering that the outcome obtained is bounded, we get $\phi = \phi_1$. Similarly, we can also find the other results—$\psi = \psi_1$ and $\Theta = \Theta_1$.

6. Solution of the Model by Sumudu Transform Method with Caputo Fractional Derivative

Here, we will address the time-fractional-ordered (2 + 1)-layered NS condition

$$\begin{aligned}
&D_t{}^{\xi}(\phi) + \phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta} = \rho\left[\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right] + q_1,\\
&D_t{}^{\xi}(\psi) + \phi\frac{\partial\psi}{\partial\beta} + \psi\frac{\partial\psi}{\partial\theta} + \Theta\frac{\partial\psi}{\partial\delta} = \rho\left[\frac{\partial^2\psi}{\partial\beta^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\delta^2}\right] + q_2,\\
&D_t{}^{\xi}(\Theta) + \phi\frac{\partial\Theta}{\partial\beta} + \psi\frac{\partial\Theta}{\partial\theta} + \Theta\frac{\partial\Theta}{\partial\delta} = \rho\left[\frac{\partial^2\Theta}{\partial\beta^2} + \frac{\partial^2\Theta}{\partial\theta^2} + \frac{\partial^2\Theta}{\partial\delta^2}\right] + q_3.
\end{aligned} \tag{35}$$

with initial conditions

$$\begin{cases} \phi(\beta,\theta,\delta,0) = -0.5\beta + \theta + \delta, \\ \psi(\beta,\theta,\delta,0) = \beta - 0.5\theta + \delta, \\ \Theta(\beta,\theta,\delta,0) = \beta + \theta - 0.5\delta. \end{cases} \tag{36}$$

It should be noted that if ρ is known, then $q_1 = -\frac{1}{\rho}\frac{\partial g}{\partial\beta}$, $q_2 = -\frac{1}{\rho}\frac{\partial g}{\partial\theta}$ and $q_3 = -\frac{1}{\rho}\frac{\partial g}{\partial\delta}$ can be found.

Now, taking the Sumudu transformation [27–29] both sides in Equation (35), we get

$$ST\left[D_t{}^{\xi}(\phi)\right] = -ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right) + \rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right], \tag{37}$$

or

$$\begin{aligned}
&s^{-\xi}\left[ST\{\phi(\beta,\theta,\delta,t)\} - \sum_{k=0}^{m-1} s^k\phi^k(\beta,\theta,\delta,0)\right] = -ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\\
&\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right],
\end{aligned} \tag{38}$$

or

$$\begin{aligned}
&ST\{\phi(\beta,\theta,\delta,t)\} - \sum_{k=0}^{m-1} s^k\phi^k(\beta,\theta,\delta,0) = -\frac{1}{s^{-\xi}}ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\\
&\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right].
\end{aligned} \tag{39}$$

Now, by putting m = 1, we get $0 < \xi < 1$, so

$$\begin{aligned}
&ST\{\phi(\beta,\theta,\delta,t)\} = \phi(\beta,\theta,\delta,0) - \frac{1}{s^{-\xi}}ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\\
&\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right],
\end{aligned} \tag{40}$$

or

$$\begin{aligned}
&ST\{\phi(\beta,\theta,\delta,t)\} = -0.5\beta + \theta + \delta - \frac{1}{s^{-\xi}}ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\\
&\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right],
\end{aligned} \tag{41}$$

or

$$\begin{aligned}
&\phi(\beta,\theta,\delta,t) = -0.5\beta + \theta + \delta - ST^{-1}\left[\frac{1}{s^{-\xi}}ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\right.\\
&\left.\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right) + q_1\right]\right],
\end{aligned} \tag{42}$$

put $q_1 = 0$, then

$$\begin{aligned}
&\phi(\beta,\theta,\delta,t) = -0.5\beta + \theta + \delta - ST^{-1}\left[\frac{1}{s^{-\xi}}ST\left[\left(\phi\frac{\partial\phi}{\partial\beta} + \psi\frac{\partial\phi}{\partial\theta} + \Theta\frac{\partial\phi}{\partial\delta}\right)\right.\right.\\
&\left.\left.+\rho\left(\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\delta^2}\right)\right]\right].
\end{aligned} \tag{43}$$

Further, we get,

$$\begin{aligned}
&\phi_{m+1}(\beta,\theta,\delta,t) = \phi_m - ST^{-1}\left[\frac{1}{s^{-\xi}}ST\left[\left(\phi_m\frac{\partial\phi_m}{\partial\beta} + \psi\frac{\partial\phi_m}{\partial\theta} + \Theta\frac{\partial\phi_m}{\partial\delta}\right)\right.\right.\\
&\left.\left.+\rho\left(\frac{\partial^2\phi_m}{\partial\beta^2} + \frac{\partial^2\phi_m}{\partial\theta^2} + \frac{\partial^2\phi_m}{\partial\delta^2}\right)\right]\right],
\end{aligned} \tag{44}$$

Now, put m = 0.

$$\begin{aligned}\phi_1(\beta,\theta,\delta,t) &= \phi_0 - ST^{-1}\left[\tfrac{1}{s^{-\xi}}ST\left[\left(\phi_0\tfrac{\partial\phi_0}{\partial\beta} + \psi\tfrac{\partial\phi_0}{\partial\theta} + \Theta\tfrac{\partial\phi_0}{\partial\delta}\right)\right.\right.\\ &\left.\left.+\rho\left(\tfrac{\partial^2\phi_0}{\partial\beta^2} + \tfrac{\partial^2\phi_0}{\partial\theta^2} + \tfrac{\partial^2\phi_0}{\partial\delta^2}\right)\right]\right],\end{aligned} \tag{45}$$

or

$$\begin{aligned}\phi_1(\beta,\theta,\delta,t) &= -0.5\beta+\theta+\delta - ST^{-1}\left[\tfrac{1}{s^{-\xi}}ST[(-0.5\beta+\theta+\delta)(-0.5\ )\right.\\ &\left.+(\beta-0.5\theta+\delta)+\ (\beta+\theta-0.5\delta)+\rho(0+0+0)]\right],\end{aligned} \tag{46}$$

or

$$\begin{aligned}\phi_1(\beta,\theta,\delta,t) &= -0.5\beta+\theta+\delta - ST^{-1}\left[\tfrac{1}{s^{-\xi}}ST[(.25\beta-0.5\theta-0.5\delta)\right.\\ &\left.+(\beta-0.5\theta+\delta)+\ (\beta+\theta-0.5\delta)]\right],\end{aligned} \tag{47}$$

or

$$\phi_1(\beta,\theta,\delta,t) = -0.5\beta+\theta+\delta - ST^{-1}\left[\frac{1}{s^{-\xi}}ST\{2.25\beta\}\right], \tag{48}$$

or

$$\phi_1(\beta,\theta,\delta,t) = -0.5\beta+\theta+\delta - 2.25\beta.\frac{t^{\xi}}{\Gamma\xi+1}. \tag{49}$$

Similarly, we can also find the other expressions.

$$\begin{aligned}\phi_2(\beta,\theta,\delta,t) &= -0.5\beta+\theta+\delta - 2.25\beta.\tfrac{t^{\xi}}{\Gamma\xi+1}\\ &+2(2.25)\beta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(-0.5\beta+\theta+\delta),\end{aligned} \tag{50}$$

$$\begin{aligned}\phi_3(\beta,\theta,\delta,t) &= -0.5\beta+\theta+\delta - 2.25\beta.\tfrac{t^{\xi}}{\Gamma\xi+1}\\ &+2(2.25)\beta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(-0.5\beta+\theta+\delta) - \tfrac{(2.25)^2\beta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2},\end{aligned} \tag{51}$$

$$\psi_1(\beta,\theta,\delta,t) = \beta-0.5\theta+\delta - 2.25\theta.\frac{t^{\xi}}{\Gamma\xi+1}, \tag{52}$$

$$\begin{aligned}\psi_2(\beta,\theta,\delta,t) &= \beta-0.5\theta+\delta - 2.25\theta.\tfrac{t^{\xi}}{\Gamma\xi+1}\\ &+2(2.25)\theta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(-0.5\beta+\theta+\delta),\end{aligned} \tag{53}$$

$$\begin{aligned}\psi_3(\beta,\theta,\delta,t) &= \beta-0.5\theta+\delta - 2.25\theta.\tfrac{t^{\xi}}{\Gamma\xi+1}\\ &+2(2.25)\theta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(\beta-0.5\theta+\delta) - \tfrac{(2.25)^2\theta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2},\end{aligned} \tag{54}$$

and

$$\Theta_1(\beta,\theta,\delta,t) = \beta+\theta-0.5\delta - 2.25\delta.\frac{t^{\xi}}{\Gamma\xi+1}, \tag{55}$$

$$\Theta_2(\beta,\theta,\delta,t) = \beta+\theta-0.5\delta - 2.25\delta.\frac{t^{\xi}}{\Gamma\xi+1} + 2(2.25)\delta.\frac{t^{2\xi}}{\Gamma2\xi+1}(\beta+\theta-0.5\delta), \tag{56}$$

and

$$\begin{aligned}\Theta_3(\beta,\theta,\delta,t) &= \beta+\theta-0.5\delta - 2.25\delta.\tfrac{t^{\xi}}{\Gamma\xi+1} + 2(2.25)\delta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(\beta+\theta-0.5\delta)\\ &-\tfrac{(2.25)^2\delta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2}.\end{aligned} \tag{57}$$

Now, the solutions are given as

$$\begin{aligned}\phi(\beta,\theta,\delta,t) &= -0.5\beta+\theta+\delta - 2.25\beta.\tfrac{t^{\xi}}{\Gamma\xi+1} + 2(2.25)\beta.\tfrac{t^{2\xi}}{\Gamma2\xi+1}(-0.5\beta+\theta+\delta)\\ &-\tfrac{(2.25)^2\beta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2}+\ldots\end{aligned} \tag{58}$$

and

$$\psi(\beta,\theta,\delta,t) = \beta - 0.5\theta + \delta - 2.25\theta.\frac{t^{\xi}}{\Gamma\xi+1} + 2(2.25)\theta.\frac{t^{2\xi}}{\Gamma 2\xi+1}(\beta - 0.5\theta + \delta) - \frac{(2.25)^2\theta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2} + \dots \tag{59}$$

and

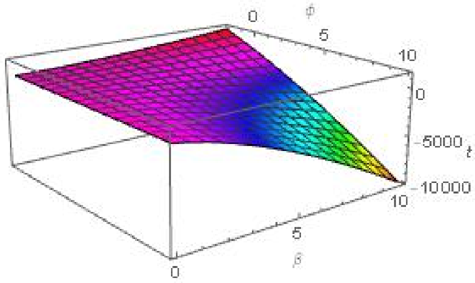
$$\Theta(\beta,\theta,\delta,t) = \beta + \theta - 0.5\delta - 2.25\delta.\frac{t^{\xi}}{\Gamma\xi+1} + 2(2.25)\delta.\frac{t^{2\xi}}{\Gamma 2\xi+1}(\beta + \theta - 0.5\delta) - \frac{(2.25)^2\delta\left[\left\{4(\Gamma\xi+1)^2\right\}+\Gamma(2\xi+1)\right]t^{3\xi}}{\Gamma(2\xi+1)\{(\Gamma\xi+1)\}^2} + \dots \tag{60}$$

which is the required solution of the defined model.

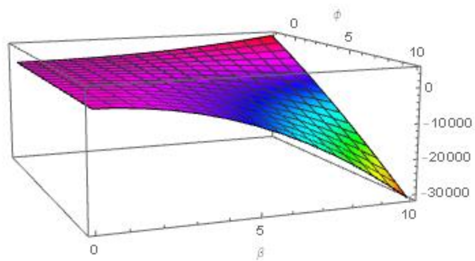
Now, the following are the graphical solutions of the model for various values of ξ.

7. Conclusions

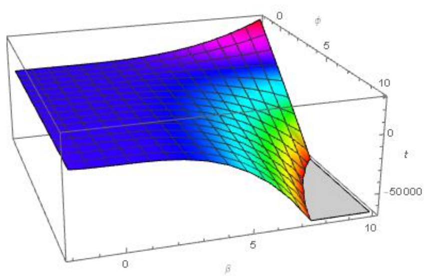
In this paper, we have analyzed the multidimensional Navier–Stokes equation by a fractional derivative of Caputo. We have solved the system by the Sumudu transform for the Caputo derivative and obtained their graphical representations as well. We have also established the validity of the methodology used and found that this new approach also converges and can be used in various problems of fractional calculus. The graphs show the changes in the ϕ component in x-direction (see Figure 1), the ψ component in y-direction (see Figure 2) and the Θ component in z-direction (see Figure 3) for distinct values of ξ. We can apply various different approaches to study the rate of change of flow and we can even find some graphical results to interpret the results and predict the flow for future studies.



(**a**) For ξ = 0.5

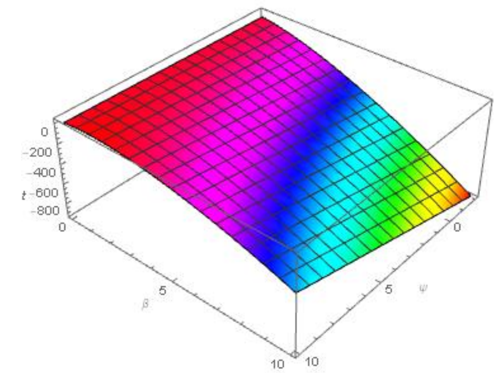


(**b**) For ξ = 0.7

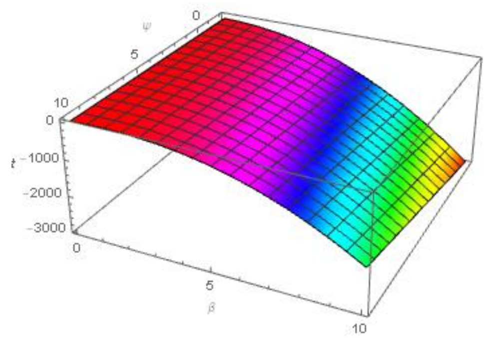


(**c**) For ξ = 1

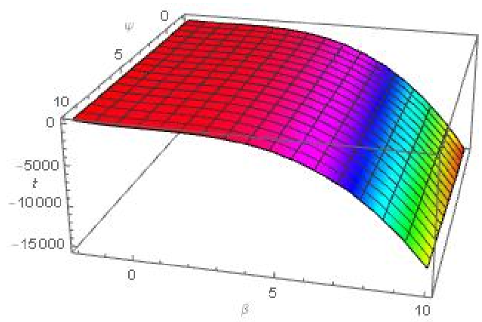
Figure 1. Representation of ϕ for different values of ξ.



(**a**) For $\xi = 0.5$

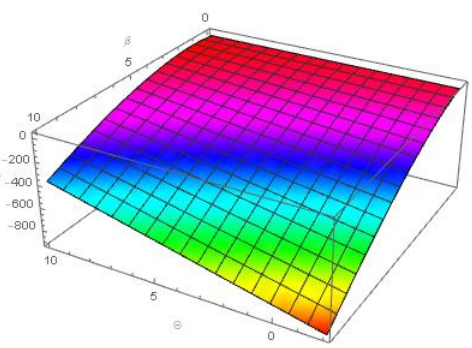


(**b**) For $\xi = 0.7$

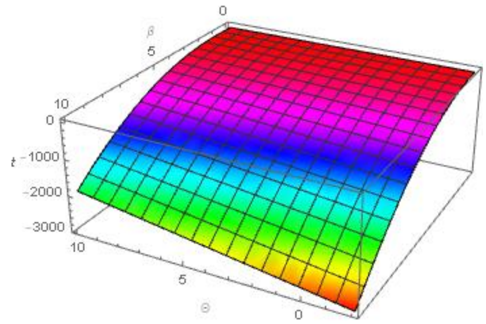


(**c**) For $\xi = 1$

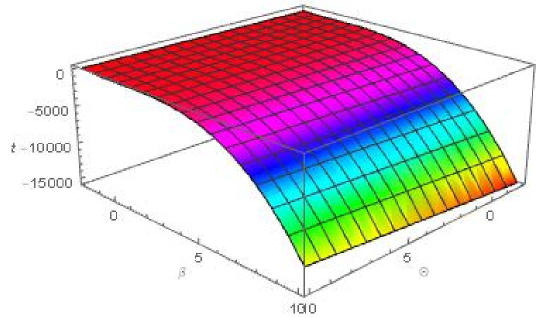
Figure 2. Representation of ψ for different values of ξ.



(**a**) For $\xi = 0.5$



(**b**) For $\xi = 0.7$



(**c**) For $\xi = 1$

Figure 3. Representation of Θ for different values of ξ.

Author Contributions: P.G. led the study, interpreted results and arranged the required literature for study. M.N.M. wrote the manuscript, conducted all the numerical calculations and made the graphs. K.S.A. summarized the data for tables, created the study site map and formatted the final document. All authors have read and agreed to the published version of the manuscript.

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