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Article

Sequential Caputo–Hadamard Fractional Differential Equations with Boundary Conditions in Banach Spaces

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Abstract: We present the existence of solutions for sequential Caputo–Hadamard fractional differential equations (SC-HFDE) with fractional boundary conditions (FBCs). Known fixed-point techniques are used to analyze the existence of the problem. In particular, the contraction mapping principle is used to investigate the uniqueness results. Existence results are obtained via Krasnoselkii's theorem. An example is used to illustrate the results. In this way, our work generalizes several recent interesting results.

Keywords: fractional differential equations; Caputo–Hadamard fractional derivative; fractional boundary conditions; existence and uniqueness

MSC: primary 34A08; secondary 26A33, 34G20

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1. Introduction

The study of fractional-order calculus has been a subject of research for many years. It began as a result of Leibniz and L'Hospital's illustrious discourse, in which the issue of a half derivative was first raised (see, e.g., [1–3]). Nowadays, fractional differential equations (FDEs) have gained more popularity due to the impact of deep applications. Some applications of FDEs are in polymer materials, fractional physics, automatic control theory, abnormal diffusion, and in random processes (see, e.g., [1–4]).

Fractional-order models are quite useful in epidemic models to predict the spread of diseases. In 2017, [5] a fractional order Middle East Respiratory Syndrome Corona Virus (MERSCoV) model used an Adams-type predictor-corrector method for the numerical solution of fractional integral equations.

Over the past 150 years, fixed-point theory (FPT) has made significant progress in mathematical analysis. It has applications in a variety of domains, including optimization theory, mathematical physics, topology, and approximation theory. Poincare launched the investigation of FPT in the nineteenth century. The existence and uniqueness of differential and integral equations solutions were established by Banach's 1922 proof of a classical FPT.

In a Banach space of infinite dimensions, Schauder stated the first FPT called Schauder FPT in 1930 and has several applications in game theory, economics, and engineering (see, e.g., [6,7]).

The field of FDEs is a new branch of mathematics that is a valuable tool in modeling many phenomena in various fields such as cancer treatment, medicine, and signal processing, etc.; we refer to [2,3,8–15]. The most important definitions of fractional derivatives (FD) and fractional integral derivatives are stated as follows:

(i) The derivative of the fractional order $\nu > 0$ of a function $g : (0, \infty) \to R$ is given by

$$\mathfrak{D}_{0+}^{\nu} g(t) = \frac{1}{\Gamma(n-\nu)} \frac{d}{dt}^{n} \int_{0}^{t} \frac{g(s)}{(t-s)^{\nu-n+1}} ds,$$

where $n = [\nu] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.

(ii) The fractional order integral of the function $g \in L^1([0,T], \mathbb{R}_+)$ of order $\nu \in \mathbb{R}_+$ is defined by

$$I^{\nu} g(t) = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-s)^{\nu-1} g(s) ds,$$

where Γ is the Euler's gamma function.

Recent research on the Hadamard equations has focused primarily on the core theoretical areas. In particular, the existence results of the solutions are investigated in [16–18], where the strip conditions and FPT are employed. In [19], the authors investigated the stability of Hadamard fractional systems and provide a new fractional comparison principle. In [20], the asymptotic of higher order Caputo–Hadamard fractional equations is studied.

A few years ago, many authors studied Caputo and Riemann–Liouville FDs. Moreover, Caputo–Hadamard and Hadamard–Caputo FDs are used to prove the existence and uniqueness results. Recently, Hadamard, Caputo–Fabrizio, Atangana–Baleanu FDs are applied in cancer-treatment models, see [21,22].

Jessada Tariboon et al. [23] investigated the existence and uniqueness of solutions for two sequential Caputo–Hadamard and Hadamard–Caputo FDE separated BCs as (with $\delta_i, \kappa_i \in \mathbb{R}, i = 1, 2$)

$$\begin{aligned} &{}^{C}\mathfrak{D}^{\mathfrak{p}}({}^{H}\mathfrak{D}^{\nu} x)(\eta) = f(\eta, x(\eta)), \quad \eta \in (a,b), && {}^{H}\mathfrak{D}^{\nu}({}^{C}\mathfrak{D}^{\mathfrak{p}} x)(\eta) = f(\eta, x(\eta)), \\ &\delta_1 x(a) + \delta_2({}^{H}\mathfrak{D}^{\nu} x)(a) = 0, && \delta_1 x(a) + \delta_2({}^{C}\mathfrak{D}^{\mathfrak{p}} x)(a) = 0, \\ &\kappa_1 x(b) + \kappa_2({}^{H}\mathfrak{D}^{\nu} x)(b) = 0, && \kappa_1 x(b) + \kappa_2({}^{C}\mathfrak{D}^{\mathfrak{p}} x)(b) = 0. \end{aligned}$$

where ${}^{C}\mathfrak{D}^{\mathfrak{p}}$ and ${}^{H}\mathfrak{D}^{\nu}$ are the Caputo and Hadamard FDs of orders $\mathfrak{p}$ and ν, respectively.

In [24], the authors took into account the second-order infinite system of DEs

$$\begin{cases} t\frac{d^2 u_j}{dt^2} + \frac{du_j}{dt} = f_j(t, u(t)), \quad t \in J := [1, q] \\ u_j(1) = u_j(q) = 0, \end{cases}$$

where $u(t) = \{u_j(t)\}_{j=1}^{\infty}$, in Banach sequence space l^p, $p \geq 1$. They used the Darbo-type FPT and the Hausdorff measure of noncompactness to prove the existence of solutions.

It should be remarked that a great amount of research on sequential fractional differential equations has been carried out by Bashir ahmad and his team, as follows. In [25], the existence of solutions for a fully coupled Riemann–Stieltjes, integro-multipoint, boundary value problem of Caputo-type sequential FDEs was studied using a known FPT. In [26], some theoretical existence results on novel combined configurations of a Caputo sequential inclusion problem and the hybrid integro-differential in which the BCs appear were established. In [27], existence and uniqueness results were established for a nonlinear sequential Hadamard FDE with multi-point BCs using known FPT. In [28], the existence and uniqueness of solutions for sequential Caputo FDE equipped with integro multipoint BCs were obtained. In their study, nonlinearity depends on the unknown function as well as its lower order FDs. In [29], the existence of solutions for sequential FD inclusions containing Riemann-–Liouville and Caputo-type derivatives and supplemented with generalized fractional integral BCs were studied using a combination of different tools. The authors in [30] investigated the existence of solutions for boundary value problems of Caputo-type sequential FDEs and inclusions supplemented with nonlocal integro-multipoint BCs using tools from functional analysis. One can see [31] for some nice results on a coupled two-parameter system of sequential fractional integro-differential equations supplemented with nonlocal integro-multipoint BCs; see also [32].

Inspired by the above FPT and cited works, we consider the FBCs for SC-HFDE of the form

$$ {}^{C}\mathfrak{D}^{\nu}({}^{H}\mathfrak{D}^{\nu_1}x)(\eta) = g(\eta, x(\eta)), \quad \eta \in J := [a,b], \quad 1 < \nu, \nu_1 < 2 \tag{1} $$

$$ x(a) = 0, \ \kappa\, {}^{H}\mathfrak{D}^{\delta_1}x(b) + (1-\kappa)\, {}^{H}\mathfrak{D}^{\delta_2}x(b) = \delta_3, \quad \delta_3 \in \mathbb{R} \tag{2} $$

where ${}^{C}\mathfrak{D}^{\nu}$ is the Caputo FD of orders ν, ${}^{H}\mathfrak{D}^{\nu_1}$ is the Hadamard FD of orders ν_1, ${}^{H}\mathfrak{D}^{\delta_1}$ is the Hadamard FD of orders δ_1,the ${}^{H}\mathfrak{D}^{\delta_2}$ is the Hadamard FD of orders δ_2. $0 < \delta_1, \delta_2 < \nu - \nu_1$, $0 \leq \kappa \leq 1$ is some constant and a continuous function $g : J \times \mathbb{R} \to \mathbb{R}$.

We use the following assumptions to prove the results of SC-HFDE involving FDs.

(A_1) The function $g : J = [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous.
(A_2) There exists nondecreasing functions $\phi_g(t) \in C([a,b], \mathbb{R}^+)$:

$$ |g(t,x)| \leq \phi_g(t), \quad \text{for any} \quad x \in \mathbb{R} $$

(A_3) There exists the function $\psi_g(t) \in C([a,b], \mathbb{R}^+)$:

$$ |g(t,x) - g(t,x_1)| \leq \psi_g(t)|x - x_1|, \quad \text{for any} \quad x, x_1 \in \mathbb{R} $$

The most important definitions of the problem (1)–(2) and lemma are stated in [2,3,10,23].

Our contributions are as follows:

1. Generalizing the results obtained in [33], in particular in the BCs.
2. Generalizing the outcomes in [34] in the sense of the BCs and in the used techniques.

The rest of the article is organized as follows. The next section contains some auxiliary results, Section 3 is devoted to the main contribution, Section 4 is for various applications, and in Section 5 we conclude the work.

2. Auxiliary Results

Definition 1 ([1–3])**.** *For at least n-times differentiable function $g : [a, \infty) \to \mathbb{R}$, the Caputo's FD (with order ν) is defined by*

$$ ({}^{C}\mathfrak{D}_0^{\nu})g(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} g^{(n)}(s)ds, \ \textit{for}\ n-1 < \nu < n, $$

where $n = [\nu] + 1$ and $[\nu]$ denotes the integer part of the real number ν.

Definition 2 ([1–3])**.** *The Riemann–Liouville fractional integral (of order ν) for a function $g : [a, \infty) \to \mathbb{R}$ is defined as follows*

$$ ({}^{RL}\mathfrak{J}^{\nu})g(t) = \frac{1}{\Gamma(\nu)} \int_a^t \frac{g(s)}{(t-s)^{1-\nu}} ds, \ \textit{for}\ \nu > 0, $$

provided the integral exists.

Definition 3 ([1–3])**.** *The Hadamard fractional integral of order ν is defined by*

$$ ({}^{H}J^{\nu})g(t) = \frac{1}{\Gamma(\nu)} \int_b^t \left(\log \frac{t}{s}\right)^{\nu-1} \frac{g(s)}{s} ds, \ \ \nu > 0. $$

provided the integral exists.

Definition 4 ([1–3]). *The Caputo-type Hadamard FD is defined as*

$$^{H}\mathfrak{D}^{\nu}g(t) = \frac{1}{\Gamma(n-\nu)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\nu-1}\delta^{n}\frac{g(s)}{s}ds, n-1<\nu<n,\ n=[\nu]+1,$$

where $g:[a,\infty)\to\mathbb{R}$ is an n-times differentiable function and $\delta^{n}=\left(t\frac{d}{dt}\right)^{n}$.

Lemma 1. *The general solution of $^{c}\mathfrak{D}^{\nu}x(\rho)=0$ (with $\nu>0$) is*

$$x(\rho) = c_0 + c_1\rho + \ldots + c_{n-1}(\rho-a)^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ ($n=[\nu]+1$).

In view of Lemma 1, it follows that

$$I^{\nu}\,{}^{C}\mathfrak{D}^{\nu}x(\rho) = x(\rho) + c_0 + c_1(\rho-a) + \ldots + c_{n-1}(\rho-a)^{n-1}, \tag{3}$$

for $i=0,1,2,\ldots,n-1$ ($n=[\nu]+1$) and some $c_i\in\mathbb{R}$.

Lemma 2. *The FBCs*

$$^{C}\mathfrak{D}^{\nu}(^{H}\mathfrak{D}^{\nu_1}x)(\eta) = w(\eta), \quad \eta\in J:=[a,b], \quad 1<\nu,\nu_1\leq 2 \tag{4}$$
$$\kappa\,{}^{H}\mathfrak{D}^{\delta_1}x(b) + (1-\kappa)\,{}^{H}\mathfrak{D}^{\delta_2}x(b) = \delta_3, x(a)=0 \tag{5}$$

is equivalent to

$$\begin{aligned} x(\eta) ={}^{H} I^{\nu_1}(^{RL}I^{\nu}w)(\eta) + \frac{\log(\frac{\eta}{a})^{\nu_1}}{\lambda_1\Gamma(\nu_1+1)}\Big(\delta_3 - \kappa\,{}^{H}I^{\nu_1}(^{RL}I^{\nu-\delta_1}w)(b) \\ -(1-\kappa)\,{}^{H}I^{\nu_1}(^{RL}I^{\nu-\delta_2}w)(b)\Big), \quad \eta\in J:=[a,b], \end{aligned} \tag{6}$$

where

$$\lambda_1 = \frac{\kappa\log(\frac{b}{a})^{1-\delta_1}}{\Gamma(2-\delta_1)} + \frac{(1-\kappa)\log(\frac{b}{a})^{1-\delta_2}}{\Gamma(2-\delta_2)} \neq 0 \tag{7}$$

Proof. Taking the Riemann–Liouville fractional integral (of order q) and Hadamard fractional integral (of order q_1) in Equation (4), we obtain

$$x(\eta) ={}^{H} I^{\nu_1}(^{RL}I^{\nu}w)(\eta) + c_1 + c_2\frac{\log(\frac{\eta}{a})^{\nu_1}}{\Gamma(\nu_1+1)} \tag{8}$$

The first boundary condition of (5) $\Rightarrow c_1=0$ and second boundary condition of (5) in the above, Equation (8), we obtain

$$\delta_3 = \kappa^{H}I^{\nu_1}(^{RL}I^{\nu}w)(b) + c_2\kappa\frac{\log(\frac{b}{a})^{1-\delta_1}}{\Gamma(2-\delta_1)} + (1-\kappa)^{H}I^{\nu_1}(^{RL}I^{\nu}w)(b) + c_2(1-\kappa)\frac{\log(\frac{b}{a})^{1-\delta_2}}{\Gamma(2-\delta_2)} \tag{9}$$

$$c_2 = \frac{1}{\lambda_1}\Big(\delta_3 - \kappa\,{}^{H}I^{\nu_1}(^{RL}I^{\nu-\delta_1}w)(b) - (1-\kappa)\,{}^{H}I^{\nu_1}(^{RL}I^{\nu-\delta_2}w)(b)\Big) \tag{10}$$

Substituting constant c_2 in (8), we obtain the integral Equation (6). The proof is completed. □

Theorem 1 ([35] (Krasnoselskii's FPT)). *Suppose a Banach space $\mathbb{X}$, select a closed, bounded, and convex set $\emptyset\neq B\subset\mathbb{X}$. Let A_1 and A_2 be two operators: (i) $A_1x+A_2y\in B$ whenever*

x,$y \in B$; (ii) A_1 is compact and continuous; (iii) A_2 is a contraction mapping. Therefore, $\exists\, z \in B$: $z = A_1 z + A_2 z$.

3. Main Results

We start by defining $\zeta = C([a,b],\mathbb{R}^+) : [a,b] \to \mathbb{R}$ as the Banach space of all functions (continuous) with the norm $||x|| = \sup\{|x(t)|, t \in [a,b]\}$. Now, define the operator $\Phi : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$\begin{aligned}\Phi x(t) =& {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(g_x))(t) + \frac{\log(\frac{t}{a})^{\nu_1}}{\lambda_1\Gamma(\nu_1+1)}\Big(\delta_3 - \kappa\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(g_x))(b) \\ &- (1-\kappa)\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(g_x))(b)\Big), \quad t \in J := [a,b],\end{aligned} \tag{11}$$

where $g_x(t) = g(t, x(t))$ and set abbreviate notation

$${}^{H}I^{\nu_1}({}^{RL}I^{\nu}(g_x))(t) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu)}\int_a^t\int_a^s (\log\frac{t}{s})^{\nu_1}(s-\sigma)^{\nu-1}g(\sigma, x(\sigma))d\sigma\frac{ds}{s}$$

FPT play an essential role in many interesting recent results, see, e.g., [36–38].

3.1. Uniqueness Via Contraction Mapping Principle

Theorem 2. *Assume that (A_1), (A_3) are holds. If $\lambda_2\psi_g^* < 1$, where*

$\psi_g^* = \sup\{\psi_g(t) : t \in [a,b]\}$

$$\lambda_2 = {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(1))(b) + \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3| - |\kappa|\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(1))(b) - (|1-\kappa|)\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(1))(b)\Big),$$

then the fractional problem (1) and (2) has a unique solution on J.

Proof. Let $B_r = \{x \in C : \|x\| \le r\}$ be a convex and closed bounded subset of C, where the fixed constant r satisfies

$$r \ge \frac{\mathrm{p}\lambda_2}{1-\psi_g^*\lambda_2} \tag{12}$$

where $\mathrm{p} = \sup\{g(t,0) : t \in [a,b]\}$. Next, we prove that $\Phi B_r \subset B_r$ and by using the triangle inequality $|g_x| \le |g_x - g_0| + |g_0|$, we have

$$\begin{aligned}|\Phi x(t)| \le& {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(|g_x|))(t) + \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3| - |\kappa|\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(|g_x|))(b) \\ &- (|1-\kappa|)\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(|g_x|))(b)\Big), \\ |\Phi x(t)| \le& {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(|g_x - g_0| + |g_0|))(t) + \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)} \\ &\Big(|\delta_3| - |\kappa|\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(|g_x - g_0| + |g_0|))(b) - (|1-\kappa|)\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(|g_x - g_0| + |g_0|))(b)\Big), \\ \le& {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(\psi_g^* + p))(t) + \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3| - |\kappa|\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(\psi_g^* + p))(b) \\ &- (|1-\kappa|)\, {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(\psi_g^* + p))(b)\Big), \\ =& \psi_g^* r\lambda_2 + p\lambda_2 \\ \le& r\end{aligned}$$

Therefore, $\Phi B_r \subset B_r$. Let $x_1, x_2 \in B_r$, we have

$$\begin{aligned}|\Phi x_1(t)-\Phi x_2(t)| \leq\ & {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(|g_{x_1}-g_{x_2}|))(t)+\frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3|-|\kappa|\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(|g_{x_1}-g_{x_2}|))(b)\\ &-(|1-\kappa|)\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(|g_{x_1}-g_{x_2}|))(b)\Big),\\ \leq\ & \psi_g^*||x_1-x_2||^{H}I^{\nu_1}({}^{RL}I^{\nu}(1))(t)+\frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3|-|\kappa|\psi_g^*||x_1-x_2||^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(1))(b)\\ &-(|1-\kappa|)\psi_g^*||x_1-x_2||^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(1))(b)\Big),\\ =\ & \psi_g^*\lambda_2||x_1-x_2||,\end{aligned}$$

$\Rightarrow$ $|\Phi x_1(t)-\Phi x_2(t)| \leq \psi_g^*\lambda_2||x_1-x_2||$. Since $\psi_g^*\lambda_4 < 1$, then the operator Φ is a contraction. Now, the operator Φ has unique FP, which implies that problem (1)–(2) has a unique solution on $J=[a,b]$. $\square$

3.2. Existence via Krasnoselkii's Theorem

Theorem 3. *Suppose $(A_1),(A_2)$ are satisfied. If*

$$\psi_g^*\Big[\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(1))(b)\Big]<1, \tag{13}$$

then the BVP's (1) and (2) has at least one solution on $[a,b]$.

Proof. Let $B_\sigma=\{x\in C([a,b],\mathbb{R}):||x||\leq\sigma\}$ where a constant σ satisfies $\sigma\geq\phi_g^*\lambda_2$ and $\phi_g^*=\sup\{\phi_g(t):t\in[a,b]\}$. Divide the operator Φ into the two operators Φ_1 and Φ_2 on B_σ with

$$\Phi_1x(t)=\frac{\log(\frac{t}{a})^{\nu_1}}{\lambda_1\Gamma(\nu_1+1)}\Big(\delta_3-\kappa\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(g_x))(b)-(1-\kappa)\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(g_x))(b)\Big),$$

and

$$\Phi_2x(t)=\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(g_x))(t).$$

The ball B_σ is a bounded, closed and convex subset of the Banach space $C([a,b],\mathbb{R})$. Now, show that $\Phi_1x+\Phi_2y\in B_\sigma$. Let $x,y\in B_\sigma$; then, we have

$$\begin{aligned}|\Phi_1x(t)+\Phi_2y(t)|\leq\ & \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3|-|\kappa|\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(|g_x|))(b)\\ &-(|1-\kappa|)\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(|g_x|))(b)\Big)+\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(|g_y|))(t)\\ \leq\ & \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3|-|\kappa|\ {}^{H}I^{\nu_1}\phi_g^*({}^{RL}I^{\nu-\delta_1}(1))(b)\\ &-(|1-\kappa|)\ {}^{H}I^{\nu_1}\phi_g^*({}^{RL}I^{\nu-\delta_2}(1))(b)\Big)+\ {}^{H}I^{\nu_1}\Psi^*({}^{RL}I^{\nu}(1))(t)\\ =\ & \phi_g^*\lambda_2\\ \leq\ & \sigma,\end{aligned}$$

which implies that $\Phi_1x+\Phi_2y\in B_\sigma$. Next, to prove that Φ_2 is a contraction mapping, for $x,y\in B_\sigma$, we have

$$\begin{aligned}||\Phi_2x-\Phi_2y||\leq\ & {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(|g_x-g_y|))(b)\\ \leq\ & \psi_g^*\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu}(1))(b)||x-y||,\end{aligned}$$

by (A_3), which is a contraction by (13).

Next, we show that the operator Φ_1 is continuous and compact. By using the continuity of g on $[a,b]\times\mathbb{R}$, we can conclude that Φ_1 is continuous. For $x\in B_\sigma$,

$$||\Phi_1x||\leq\phi_g^*\lambda_3,$$

where

$$\lambda_3 = \frac{|\log(\frac{t}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3| - |\kappa|\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(1))(b) - (|1-\kappa|)\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(1))(b)\Big).$$

This implies that $\Phi_1 B_\sigma$ is uniformly bounded. Now, we prove that $\Phi_1 B_\sigma$ is equicontinuous. For $t_1, t_2 \in [a,b]$: $t_1 < t_2$ and for $x \in B_\sigma$, we have

$$\begin{aligned} |\Phi_1 x(t_1) - \Phi_1 x(t_2)| \leq & \frac{|\log(\frac{t_2}{a})^{\nu_1} - \log(\frac{t_1}{a})^{\nu_1}|}{|\lambda_1|\Gamma(\nu_1+1)}\Big(|\delta_3| - |\kappa|\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_1}(|g_x|))(b) \\ & - (|1-\kappa|)\ {}^{H}I^{\nu_1}({}^{RL}I^{\nu-\delta_2}(|g_x|))(b)\Big) \\ \leq & \phi_g^*\lambda_3|\log(\frac{t_2}{a})^{\nu_1} - \log(\frac{t_1}{a})^{\nu_1}|. \end{aligned}$$

It is obvious that the above expression is independent of x and also tends to zero as $t_1 \to t_2$. Therefore $\Phi_1 B_\sigma$ is equicontinuous. Hence $\Phi_1 B_\sigma$ is relatively compact. Now, by applying the Arzela–Ascoli theorem (see, e.g., [39]), the operator Φ_1 is compact on B_σ. Thus, Φ_1 and Φ_2 satisfy the assumptions of Theorem 1. By the conclusion of Theorem 1, we confirm that the problem (1) and (2) has at least one solution on $[a,b]$. □

4. Example

We consider an example to verify the main results as follows.

Example 1. *Suppose the FBCs for SC-HFDEs*

$$ {}^{C}\mathfrak{D}^{\frac{3}{2}}({}^{H}\mathfrak{D}^{\nu_1}x)(\eta) = g(\eta, x(\eta)), \quad \eta \in (\frac{1}{2}, \frac{5}{2}), \tag{14}$$

$$ x(\frac{1}{2}) = 0, \frac{1}{8}\ {}^{H}\mathfrak{D}^{\frac{5}{2}}x(\frac{5}{2}) + \frac{7}{8}\ {}^{H}\mathfrak{D}^{\frac{1}{4}}x(\frac{5}{2}) = \frac{9}{2},. \tag{15}$$

where $\nu = \frac{3}{2}, \nu_1 = \frac{4}{3}, a = \frac{1}{2}, b = \frac{5}{2}, \delta_1 = \frac{1}{2}, \delta_2 = \frac{1}{4}, \delta_3 = \frac{3}{4}$ *and* $\kappa = \frac{1}{8}$, $\lambda_1 = 1.005489449$, ${}^{H}I^{\frac{4}{3}}({}^{RL}I^{\frac{3}{2}}(1))(\frac{5}{2}) = 0.039718$, ${}^{H}I^{\frac{4}{3}}({}^{RL}I^{1}(1))(\frac{5}{2}) = 0.1055989$, ${}^{H}I^{\frac{4}{3}}({}^{RL}I^{\frac{5}{4}}(1))(\frac{5}{2}) = 0.0821249$, $\frac{(\log 5)^{\frac{4}{3}}}{\lambda_1\Gamma(\frac{7}{3})} = 0.519833119$, *and let* $g : (\frac{1}{2}, \frac{5}{2}) \times \mathbb{R} \to \mathbb{R}$ *with*

$$g(\eta, x(\eta)) = \frac{cos^2\eta}{4[(\eta - \frac{1}{2}) + 3]}(\frac{x^2 + |x|}{|x|}) + \frac{1}{7}$$

gives, $|g(\eta, x(\eta)) - g(\eta, y(\eta))| \leq \psi_g^*|x-y|$ *and* $\psi_g^* = \frac{1}{3}$*. Thus,* $\psi_g^*\lambda_4 = 0.782827602 < 1$.

Hence, by Theorem 2, problem (14) and (15) with $g(\eta, x(\eta))$ *has a unique solution on* $(\frac{1}{2}, \frac{5}{2})$*. This illustrates our results.*

5. Conclusions

We investigated the existence and uniqueness results for fractional boundary value problems of SC-HFDE. Potential future works could be to develop new fractional models for Corona Virus, and to find controlled corona-virus conditions using a numerical approach with fractional order. Moreover, we intend to investigate our results based on other FD, such as, e.g., Abu-Shady–Kaabar FD, Katugampola derivative, and conformable derivative.

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