fractal and fractional

Article

The Existence, Uniqueness, and Carathéodory's Successive Approximation of Fractional Neutral Stochastic Differential Equation

Xiaolin Yuan, Yongguang Yu, Guojian Ren *, Wei Chen and Weiyi Xu

School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China
* Correspondence: gjren@bjtu.edu.cn

Abstract: The existence, uniqueness, and Carathéodory's successive approximation of the fractional neutral stochastic differential equation (FNSDE) in Hilbert space are considered in this paper. First, we give the Carathéodory's approximation solution for the FNSDE with variable time delays. We then establish the boundedness and continuity of the mild solution and Carathéodory's approximation solution, respectively. We prove that the mean-square error between the exact solution and the approximation solution depends on the supremum of time delay. Next, we give the Carathéodory's approximation solution for the general FNSDE without delay. Under uniform Lipschitz condition and linear growth condition, we show that the proof of the convergence of the Carathéodory approximation represents an alternative to the procedure for establishing the existence and uniqueness of the solution. Furthermore, under the non-Lipschitz condition, which is weaker than Lipschitz one, we establish the existence and uniqueness theorem of the solution for the FNSDE based on the Carathéodory's successive approximation. Finally, a simulation is given to demonstrate the effectiveness of the proposed methods.

Keywords: Carathéodory's approximation solution; fractional calculus; neutral stochastic differential equation; variable delays; existence and uniqueness theorem

check for updates

Citation: Yuan, X.; Yu, Y.; Ren, G.; Chen, W.; Xu, W. The Existence, Uniqueness, and Carathéodory's Successive Approximation of Fractional Neutral Stochastic Differential Equation. *Fractal Fract.* **2022**, *6*, 700. https://doi.org/10.3390/fractalfract6120700

Academic Editor: Natália Martins

Received: 3 September 2022
Accepted: 1 November 2022
Published: 26 November 2022

1. Introduction

Nowadays, stochastic modeling is playing an important role in many fields of science and industry such that more and more stochastic differential equations (SDEs) are established. In general, the solution for the SDEs does not have an explicit expression, except in the linear case. Therefore, it is necessary and meaningful to seek the approximation solution rather than the accurate solution. Usually, the existence and uniqueness theorem of the solution for SDEs are proved by taking the method of Picard successive approximation [1]. During the production of the Picard iteration, to compute the approximation solution $x_n(t)$ at the nth step, all past information $x_0(t), x_1(t), \dots, x_{n-1}(t)$ is needed, which involves lots of calculations on stochastic integrals. Therefore, to reduce the calculation, the Carathéodory successive approximation was first introduced by Constantine Carathéodory in the early part of the 20th century for ordinary differential equations (ODEs) [2], in which $x_n(t)$ is computed directly. The Carathéodory's approximation solutions for some general SDEs were given in the monograph [1]. Moreover, the Carathéodory approximation solution for the SDEs with pathwise uniqueness was given in [3]. The Carathéodory's approximation solution for a class of perturbed SDEs with reflecting boundary was given in [4]. Considering that the future state of the system may be determined by the present state and some of the past states in some applications, then the functional SDEs are established. Furthermore, some results were obtained on the Carathéodory approximation solutions for functional SDEs with variable delays; for examples, see Refs. [5–9]. In particular, the neutral SDEs are a class of SDEs depending on past and present values but that involve derivatives with delays as well as the function itself. Examples are the problem of lossless transmission, the equation of vibrating masses attached to an elastic bar [10], the collision problem in electrodynamics [11], and so on.

Fractional calculus is a generalization of integral calculus and has properties of memory and heredity. In the 1970s, B.B. Mandelbrot first pointed out that there are a large number of fractional dimensions in nature and many technical fields, as well as self-similarity between the whole and the part. Since then, fractional calculus has been applied to many fields, such as chemistry, viscoelasticity, anomalous diffusion process, complex networks, neural networks, etc. [12–17]. With this background, fractional SDEs are established. The existence and uniqueness theorems of a solution for a class of fractional SDEs were obtained by using the Picard approximation sequence [18,19] or by using the theorem of the Banach fixed point [20–24]. Then, the Carathéodory approximations and stability of solutions to non-Lipschitz fractional SDEs of the Itô–Doob type were investigated in [25]. The Carathéodory's approximation for a type of Caputo fractional SDEs was obtained in [26]. A class of fractional SDEs driven by Lévy noise was studied by using Carathéodory approximation in [27]. The approximations for solutions of Lévy-type SDEs were given in [28], and so on.

Inspired by the above discussion, some results on the existence, uniqueness and Carathéodory's successive approximation of FNSDE are given in this paper. The contributions of this paper are listed: (1) The Carathéodory's approximation for the FNSDE with and without time delay is established, respectively. (2) The boundedness and continuity of the mild solution and Carathéodory's approximation solution are given. (3) The mean-square error between the mild solution and Carathéodory's approximation solution is obtained. (4) Under the non-Lipschitz condition, the existence and uniqueness theorem of the solution for the FNSDE without delay is established based on the method of Carathéodory's successive approximation.

The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. The Carathéodory's approximation solution for the FNSDE with variable time delays is given in Section 3. The Carathéodory's approximation solution for the general FNSDE without delay is given in Section 4. The existence and uniqueness theorem of the solution for the FNSDE under the non-Lipschitz condition is given in Section 5. A numerical example is given in Section 6. Finally, the conclusion is given in Section 7.

Notations: Denote $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ as the set of natural, real and complex numbers, respectively. Let $\mathbb{H}$, $\mathbb{V}$ be two separable Hilbert spaces, $\mathcal{L}(\mathbb{V},\mathbb{H})$ be the space of bounded linear operators from $\mathbb{V}$ into $\mathbb{H}$, $\mathcal{L}(\mathbb{H}) := \mathcal{L}(\mathbb{H},\mathbb{H})$. $\|\cdot\|$ denotes the norms in $\mathbb{H}$, $\mathbb{V}$, $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}(\mathbb{V},\mathbb{H})$. Let $(\cdot,\cdot)$ denote the inner product, where $\mathbb{E}(\cdot)$ represents the mathematical expectation. $\mathcal{C}^n([a,b],\mathbb{R}^n)$ represents the family of continuously n-times differentiable $\mathbb{R}^n$-valued functions defined on $[a,b]$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

2. Preliminaries

Assume that there exists a complete orthonormal basis $\{e_n\}_{n\geq 1}$ in $\mathbb{V}$, and $\{W(t)\}_{t\geq 0}$ is a cylindrical $\mathbb{V}$-valued Wiener process [29] defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ with a finite trace nuclear covariance operator $Q \geq 0$. Denote $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < +\infty$, with $Qe_n = \lambda_n e_n$, $n \in \mathbb{N}$. Let $\{\beta_n(t)\}_{n\geq 1}$ be a sequence of the one-dimensional standard Wiener process mutually independent of $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ such that

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \geq 0.$$

For $\Sigma,\Theta \in \mathcal{L}(\mathbb{V},\mathbb{H})$, define $(\Sigma,\Theta) = \mathrm{Tr}[\Sigma Q\Theta^*]$, and Θ^* is the adjoint of the operator Θ. For any bounded operator $\Theta \in \mathcal{L}(\mathbb{V},\mathbb{H})$, then $\|\Theta\|_Q^2 = \mathrm{Tr}[\Theta Q\Theta^*] = \sum\limits_{n=1}^{\infty} \|\sqrt{\lambda_n}\Theta e_n\|^2$. If $\|\Theta\|_Q^2 < +\infty$, then Θ is called a Q-Hilbert–Schmidt operator. Denote $\mathcal{L}^2(\Omega;\mathbb{H})$ as the set of all $\mathcal{F}_t$-measurable, square-integral $\mathbb{H}$-valued random variables ζ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$, which is a Banach space equipped with the norm $\mathbb{E}\|\zeta\|^2 < +\infty$. Denote $\mathcal{C}([a,b];\mathcal{L}^2(\Omega;\mathbb{H}))$ as the space of all continuous $\mathbb{H}$-valued functions Θ defined on $[a,b]$, which is a Banach space equipped with the norm $\mathbb{E}\left(\sup_{t\in[a,b]}\|\Theta(t)\|^2\right)^{1/2} < +\infty$. Denote $\mathcal{L}^p([a,b];\mathbb{H})$ as the family of $\mathbb{H}$-valued $\mathcal{F}_t$-adapted process $\{h(t)\}_{a\le t\le b}$ such that $\int_a^b \|h(s)\|^p ds < +\infty$ almost surely.

Lemma 1 ([29]). *If Θ is an $\mathcal{L}(\mathbb{V},\mathbb{H})$-valued stochastic process such that $\Theta(t)$ is measurable relative to $\mathcal{F}_t$, and $\int_0^T \mathbb{E}\|\Theta(s)\|^2 ds < \infty$ for some $0 \le T < +\infty$, then*

$$\mathbb{E}\left\|\int_0^t \Theta(s)dW(s)\right\|^2 \le \mathrm{Tr}(Q)\int_0^t \mathbb{E}\|\Theta(s)\|^2 ds, \quad 0 \le t \le T.$$

Definition 1 ([30]). *The α-order Caputo fractional derivative for a function $f(t) \in \mathcal{C}^n([t_0,t],\mathbb{R})$ is defined by*

$${}^{C}_{t_0}\mathcal{D}^{\alpha}_t f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t f^{(n)}(s)k(t-s)ds, \quad t \ge t_0,$$

where $k(t) = t^{n-\alpha-1}$, $n \in \mathbb{Z}$ satisfies $n-1 < \alpha < n$.

Definition 2 ([30]). *The α-order Riemann–Liouville (R-L) fractional integral for a function $f(t)$ is defined by*

$${}_{t_0}\mathcal{I}^{\alpha}_t f(t) = \int_{t_0}^t f(s)k(t-s)ds, \quad t \ge t_0,$$

where $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

Definition 3 ([30]). *The α-order R-L fractional derivative for a function $f(t)$ is defined by*

$${}^{R}_{t_0}\mathcal{D}^{\alpha}_t f(t) = \frac{d^n}{dt^n}\left[{}_{t_0}\mathcal{I}^{n-\alpha}_t f(t)\right], \quad t \ge t_0,$$

where $n \in \mathbb{Z}$ satisfies $n-1 < \alpha < n$.

Lemma 2 ([30]). *Let $\alpha \in \mathbb{R}$, $n = [\alpha]+1$, for $f(t) \in \mathcal{C}^n([t_0,t],\mathbb{R})$, then*

$${}_{t_0}\mathcal{I}^{\alpha}_t\left[{}^{C}_{t_0}\mathcal{D}^{\alpha}_t f(t)\right] = a(t) - \sum_{m=0}^{n-1}\frac{f^{(m)}(t_0)}{m!}t^m.$$

In particular, when $0 < \alpha \le 1$ and $f(t) \in \mathcal{C}^1([t_0,t],\mathbb{R})$, then

$${}_{t_0}\mathcal{I}^{\alpha}_t\left[{}^{C}_{t_0}\mathcal{D}^{\alpha}_t f(t)\right] = f(t) - f(t_0).$$

Definition 4 ([30]). *A two-parameter Mittag–Leffler function is defined by*

$$E_{\gamma,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\gamma+\beta)},$$

where $z,\gamma,\beta \in \mathbb{C}$, $\Re(\gamma) > 0$. Specially, $E_\gamma(z) = E_{\gamma,1}(z)$, $E_1(z) = e^z$.

Lemma 3 ([31]). *For any $p,q \ge 0$ and $x \in (0,1)$, then $(p+q)^2 \le \frac{p^2}{x} + \frac{q^2}{1-x}$.*

Lemma 4 (Hölder's inequality [31]). *Suppose that $x > 1$, $\frac{1}{x} + \frac{1}{y} = 1$. If $p(t) \in \mathcal{L}^x(\Omega)$ and $q(t) \in \mathcal{L}^y(\Omega)$, then*

$$\int_\Omega p(s)q(s)ds \le \left(\int_\Omega |p(s)|^x ds\right)^{\frac{1}{x}} \left(\int_\Omega |q(s)|^y ds\right)^{\frac{1}{y}}.$$

Lemma 5 (Generalized Grönwall inequality [32]). *For $J := [0, T]$ with $0 \le T \le +\infty$, suppose that $\alpha > 0$, $c(t)$ is a nonnegative, nondecreasing, and locally integrable function on J, $b(t)$ is a nonnegative, nondecreasing continuous function defined on J, with $b(t) \le c$, and c is a constant. For $t \in J$, if $a(t)$ is non-negative and locally integrable with*

$$a(t) \le c(t) + b(t) \int_0^t (t-s)^{\alpha-1} a(s)ds,$$

then

$$a(t) \le c(t) E_\alpha[b(t)\Gamma(\alpha)t^\alpha].$$

Lemma 6 (Bihari's inequality [1]). *For $J := [0, T]$ with $0 \le T \le +\infty$, let $c > 0$ is a positive constant and $K : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $K(t) > 0$ for all $t > 0$. Let $a(t)$ be a Borel-measurable bounded non-negative function on J, and $b(t)$ be a non-negative integrable function on J. For $t \in J$, if*

$$a(t) \le c + \int_0^t b(s)K(a(s))ds,$$

then

$$a(t) \le H^{-1}\left(H(c) + \int_0^t b(s)ds\right),$$

holds with $H(c) + \int_0^t b(s)ds \in \mathrm{Dom}(H^{-1})$, $H(t) = \int_1^t \frac{ds}{K(s)}$ on t > 0, and $H^{-1}(\cdot)$ is the inverse function of $H(\cdot)$.

Remark 1. *Lemmas 5 and 6 are both generalizations of the classical Grönwall's inequality, which will be used in the following analysis. In addition, there are many generalizations of Grönwall's inequality, for example, the fractional version of the stochastic Grönwall inequalities [33,34], and so on.*

3. Carathéodory's Approximation Solution for the FNSDE with Variable Time Delays

In this section, the Carathéodory's approximation solution for the FNSDE with variable time delays is given. For $0 \le T < +\infty$, let $\vartheta(t)$ be a continuous nonnegative function on $\mathbb{R}_+$ with $\vartheta = \sup\{\vartheta(t) : t \ge 0\}$. Denote $\mathcal{C}_{\mathcal{F}} := \mathcal{C}([-\vartheta, T]; \mathcal{L}^2(\Omega; \mathbb{H})) \subseteq \mathbb{H}$. Consider the following FNSDE with variable time delays:

$$\begin{cases} d\left[\int_0^t k(t-s)(y(t) - h(y(t)) - \xi + h(\xi))ds\right] \\ = P(y(t), y(t-\vartheta(t)), t)dt + Q(y(t), y(t-\vartheta(t)), t)dW(t), \quad 0 \le t \le T, \\ y(t) = \xi \in \mathcal{L}^2(\Omega; \mathbb{H}), \quad -\vartheta \le t \le 0, \end{cases} \tag{1}$$

where $y(t) \in \mathbb{H}$, $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $\frac{1}{2} < \alpha < 1$, $\{h(y(t))\} \in \mathcal{L}^1(\mathbb{H}; \mathbb{H})$, $\{P(y(t), y(t-\vartheta(t)), t)\} \in \mathcal{L}^1(\mathbb{H} \times \mathbb{H} \times [0, T]; \mathbb{H})$, and $\{Q(y(t), y(t-\vartheta(t)), t)\} \in \mathcal{L}^2(\mathbb{H} \times \mathbb{H} \times [0, T]; \mathcal{L}(\mathbb{K}, \mathbb{H}))$ are continuous nonlinear mapping functions.

Divide both sides of Equation (1) by dt, then Equation (1) is equivalent to

$$\begin{cases} {}_0^R\mathcal{D}_t^\alpha[y(t)-h(y(t))-\xi+h(\xi)] \\ = P(y(t),y(t-\vartheta(t)),t)+Q(y(t),y(t-\vartheta(t)),t)\frac{dW(t)}{dt}, \quad 0\le t\le T, \\ y(t)=\xi, \quad -\vartheta\le t\le 0, \end{cases} \tag{2}$$

which is the α-order R-L derivative of $y(t)-h(y(t))-\xi+h(\xi)$. Furthermore, Equation (2) is equivalent to

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha[y(t)-h(y(t))] = P(y(t),y(t-\vartheta(t)),t)+Q(y(t),y(t-\vartheta(t)),t)\frac{dW(t)}{dt}, \quad 0\le t\le T, \\ y(t)=\xi, \quad -\vartheta\le t\le 0, \end{cases} \tag{3}$$

which is the α-order Caputo derivative of $y(t)-h(y(t))$. Therefore, it could also said that the FNSDE (3) is considered in this paper. It should be noted that $\frac{dW(t)}{dt}$ is only seen as a kind of notation in form, which usually be used in the studies of SDEs [18,20–26]. Taking the α-order R-L fractional integral on both sides of Equation (3), Equation (3) is equivalent to the following stochastic integral equation:

$$\begin{aligned} y(t) = \;& \xi-h(\xi)+h(y(t))+\frac{1}{\Gamma(\alpha)}\int_0^t\big[(t-s)^{\alpha-1}P(y(s),y(s-\vartheta(s)),s)\big]ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\big[(t-s)^{\alpha-1}Q(y(s),y(s-\vartheta(s)),s)\big]dW(s), \quad 0\le t\le T. \end{aligned} \tag{4}$$

Definition 5. An $\mathbb{H}$-valued stochastic process $\{y(t)\}_{0\le t\le T}$ is called a mild solution of Equation (3) if it has the following properties:

(i) $\{y(t)\}$ is t-continuous and $\mathcal{F}_t$-adapted.
(ii) $\{h(\varsigma_1(t))\} \in \mathcal{L}^1(\mathbb{H};\mathbb{H})$, $\{P(\varsigma_1(t),\varsigma_2(t),t)\} \in \mathcal{L}^1(\mathbb{H}\times\mathbb{H}\times[0,T];\mathbb{H})$, and $\{Q(\varsigma_1(t),\varsigma_2(t),t)\}\in\mathcal{L}^2(\mathbb{H}\times\mathbb{H}\times[0,T];\mathcal{L}(\mathbb{K},\mathbb{H}))$.
(iii) Equation (4) holds for every $t\in[0,T]$ with probability 1.

To continue, the following assumptions are necessary:

Assumption 1. *(Linear growth condition) There exists a positive constant $K_1>0$ such that for all $(\varsigma_1,\varsigma_2,t)\in\mathbb{H}\times\mathbb{H}\times[0,T]$, then $\|P(\varsigma_1,\varsigma_2,t)\|^2\vee\|Q(\varsigma_1,\varsigma_2,t)\|^2\le K_1(1+\|\varsigma_1\|^2+\|\varsigma_2\|^2)$.*

Assumption 2. *(Lipschitz condition) There exists a positive constant $K_2>0$ such that for all $(\varsigma_1,\varsigma_2,t)\in\mathbb{H}\times\mathbb{H}\times[0,T]$ and $(\bar\varsigma_1,\bar\varsigma_2,t)\in\mathbb{H}\times\mathbb{H}\times[0,T]$, then $\|P(\varsigma_1,\varsigma_2,t)-P(\bar\varsigma_1,\bar\varsigma_2,t)\|^2\vee\|Q(\varsigma_1,\varsigma_2,t)-Q(\bar\varsigma_1,\bar\varsigma_2,t)\|^2\le K_2(\|\varsigma_1-\bar\varsigma_1\|^2+\|\varsigma_2-\bar\varsigma_2\|^2)$.*

Assumption 3. *There exists a positive constant $K_3\in(0,1)$ such that for all $\varsigma_1,\varsigma_2\in\mathbb{H}$, then $\|h(\varsigma_1)-h(\varsigma_2)\|\le K_3\|\varsigma_1-\varsigma_2\|$.*

Remark 2. *Assumption 3 is a common hypothesis for neutral SDEs, which means that $h(\cdot)$ is uniformly Lipschitz continuous with the Lipschitz coefficient less than 1. It is known from [1] that the Assumption 3 is obtained from a series of experimental data.*

For $n\ge\max\{1,2/\vartheta\}$, define $D_n=\left\{t\in[0,T]:\vartheta(t)<\frac{1}{n}\right\}$, $D_n^c=[0,T]-D_n$. The Carathéodory's approximation solution for the FNSDE (3) is defined by

$$\begin{cases} Y_n(t) = & \xi - h(\xi) + h\left(Y_n\left(t - \frac{1}{n}\right)\right) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \left[I_{D_n^c}(s)(t-s)^{\alpha-1} P\left(Y_n\left(s - \frac{1}{n}\right), Y_n(s - \vartheta(s)), s\right)\right] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \left[I_{D_n}(s)(t-s)^{\alpha-1} P\left(Y_n\left(s - \frac{1}{n}\right), Y_n\left(s - \vartheta(s) - \frac{1}{n}\right), s\right)\right] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \left[I_{D_n^c}(s)(t-s)^{\alpha-1} Q\left(Y_n\left(s - \frac{1}{n}\right), Y_n(s - \vartheta(s)), s\right)\right] dW(s) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \left[I_{D_n}(s)(t-s)^{\alpha-1} Q\left(Y_n\left(s - \frac{1}{n}\right), Y_n\left(s - \vartheta(s) - \frac{1}{n}\right), s\right)\right] dW(s),\ 0 \le t \le T, \\ Y_n(t) = & \xi, \quad -\vartheta \le t \le 0, \end{cases} \tag{5}$$

where I_{D_n} and $I_{D_n^c}$ represent indicator functions of D_n and D_n^c, respectively. Then, $Y_n(\cdot)$ can be determined explicitly by the stepwise iterated Itô integrals over the intervals $\left[0, \frac{1}{n}\right]$, $\left(\frac{1}{n}, \frac{2}{n}\right]$, $\left(\frac{2}{n}, \frac{3}{n}\right]$, etc.

Remark 3. *The main idea of the Carathéodory's approximation solution is to replace the present state $y(t)$ with the past state $y\left(t - \frac{1}{n}\right)$, replace the state $y(t - \vartheta(t))$ with $y\left(t - \vartheta(t) - \frac{1}{n}\right)$ when $0 < \vartheta(t) < \frac{1}{n}$, and keep the state $y(t - \vartheta(t))$ unchanged when $\vartheta(t) \ge \frac{1}{n}$.*

Remark 4. *Usually, the Picard approximation is defined as*

$$\begin{cases} Y_n(t) = & \xi - h(\xi) + h(Y_{n-1}(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t-s)^{\alpha-1} P(Y_{n-1}(s), Y_{n-1}(s - \vartheta(s)), s)\right] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t-s)^{\alpha-1} Q(Y_{n-1}(s), Y_{n-1}(s - \vartheta(s)), s)\right] dW(s), \quad 0 \le t \le T, \\ Y_n(t) = & \xi, \quad -\vartheta \le t \le 0. \end{cases}$$

During this produce, the past states $Y_0(t)$, $Y_1(t)$, ..., $Y_{n-1}(t)$ need to be computed in order to compute $Y_n(t)$, which involve lots of calculations on stochastic integrals. Better than the Picard approximation, $Y_n(t)$ can be calculated directly during the Carathéodory's approximation.

Theorem 1. *Assume that Assumptions 1–3 hold. Let $y(t)$ be the unique mild solution of Equation (1) on $[0, T]$. Then, for $n \ge 1$,*

$$\mathbb{E}\left(\sup_{0 \le t \le T} \|Y_n(t) - y(t)\|^2\right) \le H(T) E_{2\alpha-1}\left[2W_7 \Gamma(2\alpha - 1) T^{2\alpha-1}\right], \tag{6}$$

where $W_6 = \frac{4K_2[T + \mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$, $W_7 = \frac{2W_6}{1-\sqrt{K_3}}$, and

$$\begin{aligned} H(T) \;&=\; W_4 W_6 \left(\frac{1}{n}\right)^{2\alpha-1} + \frac{8W_3W_7}{2\alpha-1}\left[T^{2\alpha-1} - \left(T - \frac{1}{n}\right)^{2\alpha-1}\right] + \frac{3W_4W_7}{2\alpha-1}\left(\frac{1}{n}\right)^{2\alpha-1}\left(T - \frac{1}{n}\right)^{2\alpha-1} \\ &\quad + \frac{4W_3W_7}{2\alpha-1}\left[T^{2\alpha-1} - \left(T - \vartheta - \frac{1}{n}\right)^{2\alpha-1}\right]. \end{aligned}$$

Next, four lemmas are given, which is helpful to prove Theorem 1.

Lemma 7. *Under Assumptions 1 and 3, for all $n \ge \max\{1, 2/\vartheta\}$, then $Y_n(t) \in \mathcal{C}_{\mathcal{F}}$, that is*

$$\mathbb{E}\left(\sup_{-\vartheta \le t \le T} \|Y_n(t)\|^2\right) \le W_1 E_{2\alpha-1}[W_2 \Gamma(2\alpha - 1) T^{2\alpha-1}] := W_3, \tag{7}$$

where $W_1 = \frac{1}{2} + \frac{6+K_3\sqrt{K_3}}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, *and* $W_2 = \frac{10K_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$.

Proof. From Equation (5), Lemmas 1–4, Assumptions 1 and 3, then

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)\|^2\right)\\
&\le \frac{1}{K_3}\mathbb{E}\left\|h\left(Y_n\left(t-\frac{1}{n}\right)\right)-h(\xi)\right\|^2+\frac{5}{1-K_3}\mathbb{E}\|\xi\|^2\\
&\quad+\frac{5t}{(1-K_3)\Gamma(\alpha)^2}\left\{\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|P\left(Y_n\left(s-\frac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right\|^2\right]ds\right.\\
&\quad\left.+\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|P\left(Y_n\left(s-\frac{1}{n}\right),Y_n\left(s-\vartheta(s)-\frac{1}{n}\right),s\right)\right\|^2\right]ds\right\}\\
&\quad+\frac{5\mathrm{Tr}(Q)}{(1-K_3)\Gamma(\alpha)^2}\left\{\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|Q\left(Y_n\left(s-\frac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right\|^2\right]ds\right.\\
&\quad\left.+\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|Q\left(Y_n\left(s-\frac{1}{n}\right),Y_n\left(s-\vartheta(s)-\frac{1}{n}\right),s\right)\right\|^2\right]ds\right\}\\
&\le K_3\mathbb{E}\left\|Y_n\left(t-\frac{1}{n}\right)-\xi\right\|^2+\frac{5}{1-K_3}\mathbb{E}\|\xi\|^2+\frac{5K_1[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\\
&\quad\times\left\{\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2+\mathbb{E}\|Y_n(s-\vartheta(s))\|^2\right)\right]ds\right.\\
&\quad\left.+\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2+\mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\frac{1}{n}\right)\right\|^2\right)\right]ds\right\}\\
&\le \sqrt{K_3}\mathbb{E}\left(\sup_{-\vartheta\le r\le t}\|Y_n(r)\|^2\right)+\left(\frac{K_3}{1-\sqrt{K_3}}+\frac{5}{1-K_3}\right)\mathbb{E}\|\xi\|^2\\
&\quad+\frac{5K_1[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\left(1+2\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left(\sup_{-\vartheta\le r\le t}\|Y_n(r)\|^2\right)&\le \mathbb{E}\|\xi\|^2+\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)\|^2\right)\\
&\le \sqrt{K_3}\mathbb{E}\left(\sup_{-\vartheta\le r\le t}\|Y_n(r)\|^2\right)+\left(1+\frac{K_3}{1-\sqrt{K_3}}+\frac{5}{1-K_3}\right)\mathbb{E}\|\xi\|^2\\
&\quad+\frac{10K_1[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\left(\frac{1}{2}+\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{1}{2}+\mathbb{E}\left(\sup_{-\vartheta\le r\le t}\|Y_n(r)\|^2\right)&\le \frac{1}{2}+\frac{6+K_3\sqrt{K_3}}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2\\
&+\frac{10K_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\left(\frac{1}{2}+\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds\\
&:= W_1+W_2\int_0^t\left[(t-s)^{2\alpha-2}\left(\frac{1}{2}+\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds,
\end{aligned}
$$

where $W_1 = \frac{1}{2} + \frac{6+K_3\sqrt{K_3}}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, and $W_2 = \frac{10K_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$. From Lemma 5, then

$$\frac{1}{2} + \mathbb{E}\left(\sup_{-\vartheta\le r\le t}\|Y_n(r)\|^2\right) \le W_1 E_{2\alpha-1}\left[W_2\Gamma(2\alpha-1)t^{2\alpha-1}\right], \quad \forall 0\le t\le T.$$

In particular, take $t = T$, then

$$\mathbb{E}\left(\sup_{-\vartheta\le r\le T}\|Y_n(r)\|^2\right) \le W_1 E_{2\alpha-1}\left[W_2\Gamma(2\alpha-1)T^{2\alpha-1}\right].$$

The proof is completed. □

Lemma 8. *Under Assumptions 1 and 3, then $y(t)\in\mathcal{C}_{\mathcal{F}}$, that is*

$$\mathbb{E}\left(\sup_{-\vartheta\le r\le T}\|y(r)\|^2\right) \le \overline{W}_1 E_{2\alpha-1}\left[\overline{W}_2\Gamma(2\alpha-1)T^{2\alpha-1}\right] := \overline{W}_3, \tag{8}$$

where $\overline{W}_1 = \frac{1}{2} + \frac{4+K_3\sqrt{K_3}}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, *and* $\overline{W}_2 = \frac{6K_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$.

Proof. This lemma can be proved in the same way as Lemma 7. □

Lemma 9. *Under Assumptions 1 and 3, for all $n\ge\max\{1,2/\vartheta\}$, and any $0\le t_2<t_1\le T$ with $t_1-t_2\le 1$, then*

$$\mathbb{E}\|Y_n(t_2)-Y_n(t_1)\|^2 \le W_4(t_1-t_2)^{2\alpha-1}, \tag{9}$$

where $W_4 = \frac{W_5}{1-K_3}$, *and* $W_5 = \frac{16K_1(1+2W_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}$.

Proof. For any $0\le t_2<t_1\le T$ with $t_1-t_2\le 1$, then

$$\begin{aligned}
&Y_n(t_2)-Y_n(t_1)\\
&= h\left(Y_n\left(t_2-\tfrac{1}{n}\right)\right) - h\left(Y_n\left(t_1-\tfrac{1}{n}\right)\right)\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[I_{D_n^c}(s)\Pi(t_1,t_2)P\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right]ds\\
&\quad-\tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[I_{D_n^c}(s)(t_1-s)^{\alpha-1}P\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right]ds\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[I_{D_n}(s)\Pi(t_1,t_2)P\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right),s\right)\right]ds\\
&\quad-\tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[I_{D_n}(s)(t_1-s)^{\alpha-1}P\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right),s\right)\right]ds\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[I_{D_n^c}(s)\Pi(t_1,t_2)Q\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right]dW(s)\\
&\quad-\tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[I_{D_n^c}(s)(t_1-s)^{\alpha-1}Q\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n(s-\vartheta(s)),s\right)\right]dW(s)\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[I_{D_n}(s)\Pi(t_1,t_2)Q\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right),s\right)\right]dW(s)\\
&\quad-\tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[I_{D_n}(s)(t_1-s)^{\alpha-1}Q\left(Y_n\left(s-\tfrac{1}{n}\right),Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right),s\right)\right]dW(s),
\end{aligned}$$

where $\Pi(t_1, t_2) = (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}$. Furthermore,

$$\begin{aligned} \mathbb{E}\|Y_n(t_2) - Y_n(t_1)\|^2 &\le \frac{1}{K_3}\mathbb{E}\left\|h\left(Y_n\left(t_2 - \frac{1}{n}\right)\right) - h\left(Y_n\left(t_1 - \frac{1}{n}\right)\right)\right\|^2 + I(t) \\ &\le K_3\mathbb{E}\left\|Y_n\left(t_2 - \frac{1}{n}\right) - Y_n\left(t_1 - \frac{1}{n}\right)\right\|^2 + I(t), \end{aligned}$$

with

$$\begin{aligned} & I(t) \\ \le\ & \frac{8K_1[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\left[I_{D_n^c}(s)\Pi(t_1,t_2)^2\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2 + \mathbb{E}\|Y_n(s-\vartheta(s))\|^2\right)\right]ds \\ & + \frac{8K_1[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}\left[I_{D_n^c}(s)(t_1-s)^{2\alpha-2}\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2 + \mathbb{E}\|Y_n(s-\vartheta(s))\|^2\right)\right]ds \\ & + \frac{8K_1[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\left[I_{D_n}(s)\Pi(t_1,t_2)^2\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2 + \mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\frac{1}{n}\right)\right\|^2\right)\right]ds \\ & + \frac{8K_1[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}\left[I_{D_n}(s)(t_1-s)^{2\alpha-2}\left(1+\mathbb{E}\left\|Y_n\left(s-\frac{1}{n}\right)\right\|^2 + \mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\frac{1}{n}\right)\right\|^2\right)\right]ds \\ \le\ & \frac{8K_1[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}\left[(t_1-s)^{2\alpha-2}\left(1+2\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds \\ & + \frac{8K_1[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\left[\Pi(t_1,t_2)^2\left(1+2\mathbb{E}\left(\sup_{-\vartheta\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds \\ \le\ & \frac{8K_1(1+2W_3)[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\Pi(t_1,t_2)^2 ds + \frac{8K_1(1+2W_3)[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}(t_1-s)^{2\alpha-2}ds. \end{aligned}$$

Noted that $2\alpha - 2 \in (-1, 0)$, then

$$\begin{aligned} \int_0^{t_2}\Pi(t_1,t_2)^2 ds &= \int_0^{t_2}\left[(t_2-s)^{2\alpha-2} + (t_1-s)^{2\alpha-2} - 2(t_2-s)^{\alpha-1}(t_1-s)^{\alpha-1}\right]ds \\ &\le \int_0^{t_2}\left[(t_2-s)^{2\alpha-2} + (t_1-s)^{2\alpha-2} - 2(t_1-s)^{2\alpha-2}\right]ds \\ &= \int_0^{t_2}\left[(t_2-s)^{2\alpha-2} - (t_1-s)^{2\alpha-2}\right]ds \\ &= -\frac{1}{2\alpha-1}(t_2-s)^{2\alpha-1}\Big|_0^{t_2} + \frac{1}{2\alpha-1}(t_1-s)^{2\alpha-1}\Big|_0^{t_2} \\ &= \frac{1}{2\alpha-1}(t_1-t_2)^{2\alpha-1} + \frac{1}{2\alpha-1}t_2^{2\alpha-1} - \frac{1}{2\alpha-1}t_1^{2\alpha-1} \\ &< \frac{1}{2\alpha-1}(t_1-t_2)^{2\alpha-1}, \end{aligned}$$

and

$$\int_{t_2}^{t_1}(t_1-s)^{2\alpha-2}ds = -\frac{1}{2\alpha-1}(t_1-s)^{2\alpha-1}\Big|_{t_2}^{t_1} = \frac{1}{2\alpha-1}(t_1-t_2)^{2\alpha-1}.$$

Furthermore,

$$I(t) \le \frac{16K_1(1+2W_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}(t_1-t_2)^{2\alpha-1} := W_5(t_1-t_2)^{2\alpha-1},$$

where $W_5 = \frac{16K_1(1+2W_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}$. Next,

$$\begin{aligned}
&\mathbb{E}\|Y_n(t_2) - Y_n(t_1)\|^2 \\
&\leq K_3\mathbb{E}\left\|Y_n\left(t_2 - \tfrac{1}{n}\right) - Y_n\left(t_1 - \tfrac{1}{n}\right)\right\|^2 + W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq K_3^2\mathbb{E}\|Y_n(t_2 - \tfrac{2}{n}) - Y_n(t_1 - \tfrac{2}{n})\|^2 + K_3W_5(t_1 - t_2)^{2\alpha-1} + W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq K_3^3\mathbb{E}\|Y_n(t_2 - \tfrac{3}{n}) - Y_n(t_1 - \tfrac{3}{n})\|^2 + \sum_{i=0}^{2} K_3^i W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq K_3^{t_2 n}\mathbb{E}\|Y_n(0) - Y_n(t_1 - t_2)\|^2 + \sum_{i=0}^{t_2 n-1} K_3^i W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq K_3^{t_2 n+1}\mathbb{E}\left\|Y_n(0) - Y_n(t_1 - t_2 - \tfrac{1}{n})\right\|^2 + \sum_{i=0}^{t_2 n} K_3^i W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq K_3^{t_1 n}\mathbb{E}\|Y_n(0) - Y_n(0)\|^2 + \sum_{i=0}^{t_1 n-1} K_3^i W_5(t_1 - t_2)^{2\alpha-1} \\
&\leq \sum_{i=0}^{Tn-1} K_3^i W_5(t_1 - t_2)^{2\alpha-1} = \tfrac{1-K_3^{Tn}}{1-K_3} W_5(t_1 - t_2)^{2\alpha-1}.
\end{aligned}$$

Since $K_3 \in (0,1)$, then $\mathbb{E}\|Y_n(t_2) - Y_n(t_1)\|^2 \leq W_4(t_1 - t_2)^{2\alpha-1}$. The proof is completed. □

Lemma 10. *Under Assumptions 1 and 3, for any $0 \leq t_2 < t_1 \leq T$ with $t_1 - t_2 \leq 1$, then*

$$\mathbb{E}\|y(t_2) - y(t_1)\|^2 \leq \overline{W}_4(t_1 - t_2)^{2\alpha-1}, \tag{10}$$

where $\overline{W}_4 = \frac{2K_1(1+2\overline{W}_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}$.

Proof. From Equation (4), Lemmas 1–4, Assumptions 1 and 3,

$$\begin{aligned}
&\mathbb{E}\|y(t_2) - y(t_1)\|^2 \\
&\leq \tfrac{1}{K_3}\mathbb{E}\|h(y(t_2)) - h(y(t_1))\|^2 \\
&\quad + \tfrac{2K_1[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2} \int_{t_2}^{t_1} \left[(t_1 - s)^{2\alpha-2}\left(1 + \mathbb{E}\|y(s)\|^2 + \mathbb{E}\|y(s - \vartheta(s))\|^2\right)\right] ds \\
&\quad + \tfrac{2K_1[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2} \int_0^{t_2} \left[\Pi(t_1, t_2)^2\left(1 + \mathbb{E}\|y(s)\|^2 + \mathbb{E}\|y(s - \vartheta(s))\|^2\right)\right] ds \\
&\leq K_3\mathbb{E}\|y(t_2) - y(t_1)\|^2 + \tfrac{2K_1(1+2\overline{W}_3)[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2} \int_{t_2}^{t_1} (t_1 - s)^{2\alpha-2} ds \\
&\quad + \tfrac{2K_1(1+2\overline{W}_3)[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2} \int_0^{t_2} \Pi(t_1, t_2)^2 ds,
\end{aligned}$$

where $\Pi(t_1, t_2) = (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}$. Since

$$\int_{t_2}^{t_1} (t_1 - s)^{2\alpha-2} ds < \tfrac{1}{2\alpha-1}(t_1 - t_2)^{2\alpha-1}, \quad \int_0^{t_2} \Pi(t_1, t_2)^2 ds < \tfrac{1}{2\alpha-1}(t_1 - t_2)^{2\alpha-1},$$

then

$$\mathbb{E}\|y(t_2) - y(t_1)\|^2 \leq \frac{2K_1(1 + 2\overline{W}_3)[T + \mathrm{Tr}(Q)]}{(2\alpha - 1)(1 - K_3)^2\Gamma(\alpha)^2}(t_1 - t_2)^{2\alpha-1} = \overline{W}_4(t_1 - t_2)^{2\alpha-1},$$

where $\overline{W}_4 = \frac{2K_1(1+2\overline{W}_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}$. The proof is completed. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. From Equations (4) and (5) and Assumptions 2 and 3,

$$
\begin{aligned}
&\mathbb{E}\|Y_n(t)-y(t)\|^2 \\
&\le K_3\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-y(t)\right\|^2+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\left(\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-y(s)\right\|^2\right.\right. \\
&\quad\left.\left.+\mathbb{E}\|Y_n(s-\vartheta(s))-y(s-\vartheta(s))\|^2\right)\right]ds+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\right. \\
&\quad\left.\times\left(\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-y(s)\right\|^2+\mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right)-y(s-\vartheta(s))\right\|^2\right)\right]ds \\
&=K_3\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)+Y_n(t)-y(t)\right\|^2 \\
&\quad+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)+Y_n(s)-y(s)\right\|^2\right]ds \\
&\quad+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\mathbb{E}\|Y_n(s-\vartheta(s))-y(s-\vartheta(s))\|^2\right]ds \\
&\quad+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right)-y(s-\vartheta(s))\right\|^2\right]ds \\
&\le\sqrt{K_3}\mathbb{E}\|Y_n(t)-y(t)\|^2+\tfrac{K_3}{1-\sqrt{K_3}}\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)\right\|^2 \\
&\quad+\tfrac{8K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds \\
&\quad+\tfrac{8K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\|Y_n(s)-y(s)\|^2\right]ds \\
&\quad+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n^c}(s)(t-s)^{2\alpha-2}\mathbb{E}\|Y_n(s-\vartheta(s))-y(s-\vartheta(s))\|^2\right]ds \\
&\quad+\tfrac{4K_2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right)-y(s-\vartheta(s))\right\|^2\right]ds.
\end{aligned}
$$

Denote $W_6=\frac{K_3}{(1-\sqrt{K_3})^2}$ and $W_7=\frac{8K_2[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$, then

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)-y(r)\|^2\right) \\
&\le 2W_7\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left(\sup_{0\le r\le s}\|Y_n(r)-y(r)\|^2\right)\right]ds+\sum_{i=1}^{3}H_i(t),
\end{aligned}
$$

with

$$
\begin{aligned}
H_1(t)&=W_6\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)\right\|^2, \\
H_2(t)&=W_7\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds, \\
H_3(t)&=W_7\int_0^t\left[I_{D_n}(s)(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right)-Y_n(s-\vartheta(s))\right\|^2\right]ds.
\end{aligned}
$$

From Lemma 5, then

$$
\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)-y(r)\|^2\right)\le[H_1(t)+H_2(t)+H_3(t)]E_{2\alpha-1}[2W_7\Gamma(2\alpha-1)t^{2\alpha-1}].
$$

In particular, take $t = T$. Then from Lemma 9,

$$\begin{aligned}
H_1(T) &\leq W_4 W_6 \left(\tfrac{1}{n}\right)^{2\alpha-1}, \\
H_2(T) &= W_7 \int_0^{\frac{1}{n}} \left[(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(s-\tfrac{1}{n}\right) - Y_n(s) \right\|^2 \right] ds \\
&\quad + W_7 \int_{\frac{1}{n}}^{T} \left[(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(s-\tfrac{1}{n}\right) - Y_n(s) \right\|^2 \right] ds \\
&\leq 2W_7 \int_0^{\frac{1}{n}} \left[(T-s)^{2\alpha-2} \left(\mathbb{E} \left\| Y_n\left(s-\tfrac{1}{n}\right) \right\|^2 + \mathbb{E}\|Y_n(s)\|^2 \right) \right] ds \\
&\quad + W_7 \int_{\frac{1}{n}}^{T} \left[(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(T-\tfrac{1}{n}\right) - Y_n(s) \right\|^2 \right] ds \\
&\leq 4W_3 W_7 \int_0^{\frac{1}{n}} (T-s)^{2\alpha-2} ds + W_4 W_7 \left(\tfrac{1}{n}\right)^{2\alpha-1} \int_{\frac{1}{n}}^{T} (T-s)^{2\alpha-2} ds \\
&= \tfrac{4W_3 W_7}{2\alpha-1} \left[T^{2\alpha-1} - \left(T-\tfrac{1}{n}\right)^{2\alpha-1} \right] + \tfrac{W_4 W_7}{2\alpha-1} \left(\tfrac{1}{n}\right)^{2\alpha-1} \left(T-\tfrac{1}{n}\right)^{2\alpha-1}.
\end{aligned}$$

Denote $D_0(t) = \{t \in [0,T] : \vartheta(t) = 0\}$, and $\overline{D}_n(t) = D_n(t) - D_0(t)$, then

$$\begin{aligned}
H_3(T) &= W_7 \int_0^T \left[I_{D_0}(s)(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(s-\tfrac{1}{n}\right) - Y_n(s) \right\|^2 \right] ds \\
&\quad + W_7 \int_0^T \left[I_{\overline{D}_n}(s)(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right) - Y_n(s-\vartheta(s)) \right\|^2 \right] ds \\
&:= H_{31}(T) + H_{32}(T).
\end{aligned}$$

Similar to the analysis of $H_2(T)$, then

$$H_{31}(T) \leq \tfrac{4W_3 W_7}{2\alpha-1} \left[T^{2\alpha-1} - \left(T-\tfrac{1}{n}\right)^{2\alpha-1} \right] + \tfrac{W_4 W_7}{2\alpha-1} \left(\tfrac{1}{n}\right)^{2\alpha-1} \left(T-\tfrac{1}{n}\right)^{2\alpha-1},$$

and

$$\begin{aligned}
&H_{32}(T) \\
&\leq 2W_7 \int_0^{\vartheta+\frac{1}{n}} \left[I_{\overline{D}_n}(s)(t-s)^{2\alpha-2} \left(\mathbb{E} \left\| Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right) \right\|^2 + \mathbb{E}\|Y_n(s-\vartheta(s))\|^2 \right) \right] ds \\
&\quad + W_7 \int_{\vartheta+\frac{1}{n}}^{T} \left[I_{\overline{D}_n}(s)(T-s)^{2\alpha-2} \mathbb{E} \left\| Y_n\left(s-\vartheta(s)-\tfrac{1}{n}\right) - Y_n(s-\vartheta(s)) \right\|^2 \right] ds \\
&\leq 4W_3 W_7 \int_0^{\vartheta+\frac{1}{n}} \left[I_{\overline{D}_n}(s)(T-s)^{2\alpha-2} \right] ds + W_4 W_7 \left(\tfrac{1}{n}\right)^{2\alpha-1} \int_{\vartheta+\frac{1}{n}}^{T} \left[I_{\overline{D}_n}(s)(T-s)^{2\alpha-2} \right] ds.
\end{aligned}$$

Noted that $(T-s)^{2\alpha-2} > 0$ on $0 \leq s \leq T$, then

$$\begin{aligned}
H_{32}(T) &\leq 4W_3 W_7 \int_0^{\vartheta+\frac{1}{n}} (T-s)^{2\alpha-2} ds + W_4 W_7 \left(\tfrac{1}{n}\right)^{2\alpha-1} \int_{\vartheta+\frac{1}{n}}^{T} (T-s)^{2\alpha-2} ds \\
&= \tfrac{4W_3 W_7}{2\alpha-1} \left[T^{2\alpha-1} - \left(T-\vartheta-\tfrac{1}{n}\right)^{2\alpha-1} \right] + \tfrac{W_4 W_7}{2\alpha-1} \left(\tfrac{1}{n}\right)^{2\alpha-1} \left(T-\vartheta-\tfrac{1}{n}\right)^{2\alpha-1} \\
&\leq \tfrac{4W_3 W_7}{2\alpha-1} \left[T^{2\alpha-1} - \left(T-\vartheta-\tfrac{1}{n}\right)^{2\alpha-1} \right] + \tfrac{W_4 W_7}{2\alpha-1} \left(\tfrac{1}{n}\right)^{2\alpha-1} \left(T-\tfrac{1}{n}\right)^{2\alpha-1}.
\end{aligned}$$

From the above analysis, then

$$\mathbb{E}\left(\sup_{0\le r\le T}\|Y_n(r)-y(r)\|^2\right)\le H(T)E_{2\alpha-1}\left[2W_7\Gamma(2\alpha-1)T^{2\alpha-1}\right].$$

The proof is completed. □

4. Carathéodory's Approximation Solution for the General FNSDE without Delay

In this section, the Carathéodory's approximation solution for the general FNSDE without delay is given. Denote $\mathcal{C}:=\mathcal{C}([0,T];\mathcal{L}^2(\Omega;\mathbb{H}))\subseteq\mathbb{H}$. Consider the following FNSDE:

$$\begin{cases} d\left[\int_0^t k(t-s)(y(t)-h(y(t))-\xi+h(\xi))ds\right]=P(y(t),t)dt+Q(y(t),t)dW(t),\ 0\le t\le T,\\ y(t)=\xi\in\mathcal{L}^2(\Omega;\mathbb{H}),\quad -1\le t\le 0,\end{cases}\tag{11}$$

where $y(t)\in\mathbb{H}$, $\frac{1}{2}<\alpha<1$, $\{h(y(t))\}\in\mathcal{L}^1(\mathbb{H};\mathbb{H})$, $\{P(y(t),t)\}\in\mathcal{L}^1(\mathbb{H}\times[0,T];\mathbb{H})$, and $\{Q(y(t),t)\}\in\mathcal{L}^2(\mathbb{H}\times[0,T];\mathcal{L}(\mathbb{K},\mathbb{H}))$ are continuous nonlinear mapping functions.

Divide both sides of Equation (11) by dt, then Equation (11) is equivalent to

$$\begin{cases} {}_0^R\mathcal{D}_t^\alpha[y(t)-h(y(t))-\xi+h(\xi)]=P(y(t),t)+Q(y(t),t)\frac{dW(t)}{dt},\ \ 0\le t\le T,\\ y(t)=\xi,\quad -1\le t\le 0,\end{cases}\tag{12}$$

which is the α-order R-L derivative of $y(t)-h(y(t))-\xi+h(\xi)$. Furthermore, Equation (12) is equivalent to

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha[y(t)-h(y(t))]=P(y(t),t)+Q(y(t),t)\frac{dW(t)}{dt},\quad 0\le t\le T,\\ y(t)=\xi,\quad -1\le t\le 0,\end{cases}\tag{13}$$

which is the α-order Caputo derivative of $y(t)-h(y(t))$.

Taking the α-order R-L fractional integral on both sides of Equation (13), then this equation is equivalent to the following stochastic integral equation:

$$\begin{aligned} y(t)=\ &\xi-h(\xi)+h(y(t))+\frac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}P(y(s),s)\right]ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}Q(y(s),s)\right]dW(s),\ 0\le t\le T.\end{aligned}\tag{14}$$

Definition 6. *An $\mathbb{H}$-valued stochastic process $\{y(t)\}_{0\le t\le T}$ is called a mild solution of Equation (13) if it has the following properties:*

(i) *$\{y(t)\}$ is t-continuous, and $\mathcal{F}_t$-adapted.*
(ii) *$\{h(\varsigma(t))\}\in\mathcal{L}^1(\mathbb{H};\mathbb{H})$, $\{P(\varsigma(t),t)\}\in\mathcal{L}^1(\mathbb{H}\times[0,T];\mathbb{H})$, and $\{Q(\varsigma(t),t)\}\in\mathcal{L}^2(\mathbb{H}\times[0,T];\mathcal{L}(\mathbb{K},\mathbb{H}))$.*
(iii) *Equation (14) holds for every $t\in[0,T]$ with probability 1.*

To continue, the following assumptions are necessary:

Assumption 4. *(Linear growth condition) There exists a positive constant $\overline{K}_1>0$ such that for all $(\varsigma,t)\in\mathbb{H}\times[0,T]$, $\|P(\varsigma,t)\|^2\vee\|Q(\varsigma,t)\|^2\le\overline{K}_1(1+\|\varsigma\|^2)$.*

Assumption 5. *(Lipschitz condition) There exists a positive constant $\overline{K}_2>0$ such that for all $(\varsigma_1,t)\in\mathbb{H}\times[0,T]$ and $(\varsigma_2,t)\in\mathbb{H}\times[0,T]$, $\|P(\varsigma_1,t)-P(\varsigma_2,t)\|^2\vee\|Q(\varsigma_1,t)-Q(\varsigma_2,t)\|^2\le\overline{K}_2\|\varsigma_1-\varsigma_2\|^2$.*

The Carathéodory's approximation solution of the FNSDE (11) is defined as follows:

$$\begin{cases} Y_n(t) = & \xi - h(\xi) + h\Big(Y_n\Big(t-\frac{1}{n}\Big)\Big) + \frac{1}{\Gamma(\alpha)}\int_0^t \Big[(t-s)^{\alpha-1}P\Big(Y_n\Big(s-\frac{1}{n}\Big),s\Big)\Big]ds \\ & +\frac{1}{\Gamma(\alpha)}\int_0^t \Big[(t-s)^{\alpha-1}Q\Big(Y_n\Big(s-\frac{1}{n}\Big),s\Big)\Big]dW(s), \quad 0 \le t \le T, \\ Y_n(t) = & \xi, \quad -1 \le t \le 0. \end{cases} \tag{15}$$

Theorem 2. *Assume that Assumptions 3–5 hold. Let $y(t)$ be the unique mild solution of Equation (11) on $[0,T]$. Then, for $n \ge 1$,*

$$\lim_{n\to+\infty} \mathbb{E}\left(\sup_{0\le t\le T} \|Y_n(t) - y(t)\|^2\right) = 0. \tag{16}$$

Next, four lemmas are given, which is helpful to prove Theorem 2.

Lemma 11. *Under Assumptions 3 and 4, for all $n \ge \max\{1, 1/\vartheta\}$, $Y_n(t) \in \mathcal{C}$, that is*

$$\mathbb{E}\left(\sup_{0\le r\le T} \|Y_n(r)\|^2\right) \le Q_1 E_{2\alpha-1}\Big[Q_2\Gamma(2\alpha-1)T^{2\alpha-1}\Big] := Q_3, \tag{17}$$

where $Q_1 = 1 + \frac{K_3(1+\sqrt{K_3})+3}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, and $Q_2 = \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$.

Proof. From Equation (15), Lemmas 1–4, Assumptions 3 and 4, then

$$\begin{aligned}
&\mathbb{E}\|Y_n(t)\|^2 \\
&\le \frac{1}{K_3}\mathbb{E}\Big\|h\Big(Y_n\Big(t-\frac{1}{n}\Big)\Big) - h(\xi)\Big\|^2 + \frac{3}{1-K_3}\mathbb{E}\|\xi\|^2 + \frac{3\overline{K}_1 T}{(1-K_3)\Gamma(\alpha)^2}\int_0^t \Big[(t-s)^{2\alpha-2} \\
&\quad \times \Big(1+\mathbb{E}\Big\|Y_n\Big(s-\frac{1}{n}\Big)\Big\|^2\Big)\Big]ds + \frac{3\overline{K}_1\mathrm{Tr}(Q)}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\Big[(t-s)^{2\alpha-2}\Big(1+\mathbb{E}\Big\|Y_n\Big(s-\frac{1}{n}\Big)\Big\|^2\Big)\Big]ds \\
&\le K_3\mathbb{E}\Big\|Y_n\Big(t-\frac{1}{n}\Big)-\xi\Big\|^2 + \frac{3}{1-K_3}\mathbb{E}\|\xi\|^2 \\
&\quad + \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\Big[(t-s)^{2\alpha-2}\Big(1+\mathbb{E}\Big\|Y_n\Big(s-\frac{1}{n}\Big)\Big\|^2\Big)\Big]ds \\
&\le \sqrt{K_3}\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(s)\|^2\right) + \Big(\frac{K_3}{1-\sqrt{K_3}} + \frac{3}{1-K_3}\Big)\mathbb{E}\|\xi\|^2 \\
&\quad + \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\Big[(t-s)^{2\alpha-2}\Big(1+\mathbb{E}\Big\|Y_n\Big(s-\frac{1}{n}\Big)\Big\|^2\Big)\Big]ds.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&1+\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)\|^2\right) \\
&\le 1 + \Big[\frac{K_3}{(1-\sqrt{K_3})^2} + \frac{3}{(1-\sqrt{K_3})(1-K_3)}\Big]\mathbb{E}\|\xi\|^2 \\
&\quad + \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\left(1+\mathbb{E}\left(\sup_{0\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds \\
&:= Q_1 + Q_2\int_0^t\left[(t-s)^{2\alpha-2}\left(1+\mathbb{E}\left(\sup_{0\le r\le s}\|Y_n(r)\|^2\right)\right)\right]ds,
\end{aligned}$$

where $Q_1 = 1 + \frac{K_3(1+\sqrt{K_3})+3}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, and $Q_2 = \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$. Then, from Lemma 5,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)\|^2\right) \le 1 + \mathbb{E}\left(\sup_{0\le r\le t}\|Y_n(r)\|^2\right) \le Q_1 E_{2\alpha-1}\left[Q_2\Gamma(2\alpha-1)t^{2\alpha-1}\right], \quad \forall 0 \le t \le T.$$

In particular, take $t = T$, then

$$\mathbb{E}\left(\sup_{0\le r\le T}\|Y_n(r)\|^2\right) \le Q_1 E_{2\alpha-1}\left[Q_2\Gamma(2\alpha-1)T^{2\alpha-1}\right].$$

The proof is completed. □

Lemma 12. *Under Assumptions 3 and 4, then $y(t) \in \mathcal{C}$, that is*

$$\mathbb{E}\left(\sup_{0\le r\le T}\|y(r)\|^2\right) \le \overline{Q}_1 E_{2\alpha-1}\left[\overline{Q}_2\Gamma(2\alpha-1)T^{2\alpha-1}\right] := \overline{Q}_3, \tag{18}$$

where $\overline{Q}_1 = 1 + \frac{K_3(1+\sqrt{K_3})+3}{(1-\sqrt{K_3})(1-K_3)}\mathbb{E}\|\xi\|^2$, and $\overline{Q}_2 = \frac{3\overline{K}_1[T+\mathrm{Tr}(Q)]}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$.

Proof. This lemma can be proved in the same way as Lemma 11. □

Lemma 13. *Under Assumptions 3 and 4, for all $n \ge \max\{1, 1/\vartheta\}$, and any $0 \le t_2 < t_1 \le T$ with $t_1 - t_2 \le 1$, then*

$$\mathbb{E}\|Y_n(t_2) - Y_n(t_1)\|^2 \le Q_4(t_1 - t_2)^{2\alpha-1}, \tag{19}$$

where $Q_4 = \frac{Q_5}{1-K_3}$, and $Q_5 = \frac{8(1+Q_3)[T+\mathrm{Tr}(Q)]\overline{K}_1}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}$.

Proof. From Equation (15), then

$$\begin{aligned}
Y_n(t_2) - Y_n(t_1) &= h\left(Y_n\left(t_2 - \tfrac{1}{n}\right)\right) - h\left(Y_n\left(t_1 - \tfrac{1}{n}\right)\right) \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[\left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right)P\left(Y_n\left(s - \tfrac{1}{n}\right), s\right)\right]ds \\
&\quad - \tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[(t_1 - s)^{\alpha-1}P\left(Y_n\left(s - \tfrac{1}{n}\right), s\right)\right]ds \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[\left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right)Q\left(Y_n\left(s - \tfrac{1}{n}\right), s\right)\right]dW(s) \\
&\quad - \tfrac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[(t_1 - s)^{\alpha-1}Q\left(Y_n\left(s - \tfrac{1}{n}\right), s\right)\right]dW(s).
\end{aligned}$$

Furthermore, from Lemmas 1–4, Assumptions 3 and 4, then

$$\begin{aligned}
&\mathbb{E}\|Y_n(t_2) - Y_n(t_1)\|^2 \\
&\le K_3\mathbb{E}\left\|Y_n\left(t_2 - \tfrac{1}{n}\right) - Y_n\left(t_1 - \tfrac{1}{n}\right)\right\|^2 \\
&\quad + \tfrac{4(t_1-t_2)\overline{K}_1}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}\left[(t_1 - s)^{2\alpha-2}\left(1 + \mathbb{E}\left\|Y_n\left(s - \tfrac{1}{n}\right)\right\|^2\right)\right]ds \\
&\quad + \tfrac{4t_2\overline{K}_1}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\left[\Pi(t_1, t_2)^2\left(1 + \mathbb{E}\left\|Y_n\left(s - \tfrac{1}{n}\right)\right\|^2\right)\right]ds
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{4\mathrm{Tr}(Q)\overline{K}_1}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}\left[(t_1-s)^{2\alpha-2}\left(1+\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)\right\|^2\right)\right]ds\\
&+\tfrac{4\mathrm{Tr}(Q)\overline{K}_1}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\left[\Pi(t_1,t_2)^2\left(1+\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)\right\|^2\right)\right]ds\\
\le{}& K_3\mathbb{E}\left\|Y_n\left(t_2-\tfrac{1}{n}\right)-Y_n\left(t_1-\tfrac{1}{n}\right)\right\|^2+\tfrac{4\overline{K}_1(1+Q_3)[t_1-t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_{t_2}^{t_1}(t_1-s)^{2\alpha-2}ds\\
&+\tfrac{4\overline{K}_1(1+Q_3)[t_2+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^{t_2}\Pi(t_1,t_2)^2ds\\
\le{}& K_3\mathbb{E}\left\|Y_n\left(t_2-\tfrac{1}{n}\right)-Y_n\left(t_1-\tfrac{1}{n}\right)\right\|^2+\tfrac{8\overline{K}_1(1+Q_3)[t_1+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}(t_1-t_2)^{2\alpha-1}\\
:={}& K_3\mathbb{E}\left\|Y_n\left(t_2-\tfrac{1}{n}\right)-Y_n\left(t_1-\tfrac{1}{n}\right)\right\|^2+Q_5(t_1-t_2)^{2\alpha-1}\\
\le{}& Q_4(t_1-t_2)^{2\alpha-1},
\end{aligned}
$$

where $\Pi(t_1,t_2)=(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}$, $Q_4=\frac{Q_5}{1-K_3}$, and $Q_5=\frac{8\overline{K}_1(1+Q_3)[t_1+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)\Gamma(\alpha)^2}$.

The proof is completed. □

Lemma 14. Under Assumptions 3 and 4, for any $0\le t_2<t_1\le T$ with $t_1-t_2\le 1$, then

$$\mathbb{E}\|y(t_2)-y(t_1)\|^2\le\overline{Q}_4(t_1-t_2)^{2\alpha-1}, \tag{20}$$

where $\overline{Q}_4=\frac{8\overline{K}_1(1+Q_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}$.

Proof. From Equations (14), Lemmas 1–5, Assumptions 3 and 4, then

$$\mathbb{E}\|y(t_2)-y(t_1)\|^2\le K_3\mathbb{E}\|y(t_2)-y(t_1)\|^2+\tfrac{8\overline{K}_1(1+Q_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}(t_1-t_2)^{2\alpha-1}.$$

Furthermore, it could be obtained that

$$\mathbb{E}\|y(t_2)-y(t_1)\|^2\le\overline{Q}_4(t_1-t_2)^{2\alpha-1},$$

where $\overline{Q}_4=\frac{8\overline{K}_1(1+Q_3)[T+\mathrm{Tr}(Q)]}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}$. The proof is completed. □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. From Equations (14) and (15), then

$$
\begin{aligned}
Y_n(t)-y(t)={}& h\left(Y_n\left(t-\tfrac{1}{n}\right)\right)-h(y(t))\\
&+\tfrac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}\left(P\left(Y_n\left(s-\tfrac{1}{n}\right),s\right)-P(y(s),s)\right)\right]ds\\
&+\tfrac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}\left(Q\left(Y_n\left(s-\tfrac{1}{n}\right),s\right)-Q(y(s),s)\right)\right]dW(s).
\end{aligned}
$$

Furthermore, from Lemmas 1–4, Assumptions 3 and 5, then

$$
\begin{aligned}
&\mathbb{E}\|Y_n(t)-y(t)\|^2\\
\le{}& K_3\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-y(t)\right\|^2+\tfrac{2t\overline{K}_2}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-y(s)\right\|^2\right]ds\\
&+\tfrac{2\mathrm{Tr}(Q)\overline{K}_2}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-y(s)\right\|^2\right]ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq K_3\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)+Y_n(t)-y(t)\right\|^2 \\
&\quad+\tfrac{2[t+\mathrm{Tr}(Q)]\overline{K}_2}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)+Y_n(s)-y(s)\right\|^2\right]ds \\
&\quad+\tfrac{4[T+\mathrm{Tr}(Q)]\overline{K}_2}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\|Y_n(s)-y(s)\|^2\right]ds \\
&\quad+\tfrac{4[T+\mathrm{Tr}(Q)]\overline{K}_2}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds.
\end{aligned}
$$

Denote $Q_6=\frac{K_3}{(1-\sqrt{K_3})^2}$ and $Q_7=\frac{4[T+\mathrm{Tr}(Q)]\overline{K}_2}{(1-\sqrt{K_3})(1-K_3)\Gamma(\alpha)^2}$, then

$$
\begin{aligned}
&\mathbb{E}\|Y_n(t)-y(t)\|^2 \\
&\leq Q_6\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)\right\|^2+Q_7\int_0^t[(t-s)^{2\alpha-2}\mathbb{E}\|Y_n(s)-y(s)\|^2]ds \\
&\quad+Q_7\int_0^t(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq r\leq t}\|Y_n(r)-y(r)\|^2\right) \\
&\leq Q_7\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left(\sup_{0\leq r\leq s}\|Y_n(r)-y(r)\|^2\right)\right]ds+J_1(t)+J_2(t),
\end{aligned}
$$

with

$$
\begin{aligned}
J_1(t)&=Q_6\mathbb{E}\left\|Y_n\left(t-\tfrac{1}{n}\right)-Y_n(t)\right\|^2, \\
J_2(t)&=Q_7\int_0^t\left[(t-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds.
\end{aligned}
$$

From Lemma 5, then

$$
\mathbb{E}\left(\sup_{0\leq r\leq t}\|Y_n(r)-y(r)\|^2\right)\leq[J_1(t)+J_2(t)]E_{2\alpha-1}[Q_7\Gamma(2\alpha-1)t^{2\alpha-1}].
$$

In particular, take $t=T$, then

$$
\begin{aligned}
J_1(T)&=Q_6\mathbb{E}\left\|Y_n\left(T-\tfrac{1}{n}\right)-Y_n(T)\right\|^2\leq Q_4\left(\tfrac{1}{n}\right)^{2\alpha-1}, \\
J_2(T)&=Q_7\int_0^T\left[(T-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds \\
&\leq 2Q_7\int_0^{\frac{1}{n}}\left[(T-s)^{2\alpha-2}\left(\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)\right\|^2+\mathbb{E}\|Y_n(s)\|^2\right)\right]ds \\
&\quad+Q_7\int_{\frac{1}{n}}^T\left[(T-s)^{2\alpha-2}\mathbb{E}\left\|Y_n\left(s-\tfrac{1}{n}\right)-Y_n(s)\right\|^2\right]ds \\
&\leq 4Q_3Q_7\int_0^{\frac{1}{n}}(T-s)^{2\alpha-2}ds+Q_4Q_7\left(\tfrac{1}{n}\right)^{2\alpha-1}\int_{\frac{1}{n}}^T(T-s)^{2\alpha-2}ds \\
&=\tfrac{4Q_3Q_7}{2\alpha-1}\left[T^{2\alpha-1}-\left(T-\tfrac{1}{n}\right)^{2\alpha-1}\right]+\tfrac{Q_4Q_7}{2\alpha-1}\left(\tfrac{1}{n}\right)^{2\alpha-1}\left(T-\tfrac{1}{n}\right)^{2\alpha-1}.
\end{aligned}
$$

Obviously, $\lim_{n\to+\infty} J_1(T) = 0$, and $\lim_{n\to+\infty} J_2(T) = 0$, then

$$\mathbb{E}\left(\sup_{0\le r\le T} \|Y_n(r) - y(r)\|^2\right) \to 0, \text{ as n} \to +\infty.$$

The proof is completed. □

5. Existence and Uniqueness Theorem under Non-Lipschitz Condition

In this section, by using the method of Carathéodory's successive approximation, the existence and uniqueness theorem of the solution for the FNSDE (13) is established under the non-Lipschitz condition, which is weaker than the Lipschitz one.

Assumption 6. *Let $P(x,t)$ and $Q(x,t)$ be continuous functions. Assume that there exists a continuous increasing concave function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(0) = 0$ such that $\int_{0^+} \frac{ds}{\kappa(s)} = +\infty$ and for all $(\varsigma_1, t) \in \mathbb{H} \times [0,T]$ and $(\varsigma_2, t) \in \mathbb{H} \times [0,T]$, then $\|P(\varsigma_1,t) - P(\varsigma_2,t)\|^2 \vee \|Q(\varsigma_1,t) - Q(\varsigma_2,t)\|^2 \le \kappa(\|\varsigma_1 - \varsigma_2\|^2)$.*

Remark 5 ([25,35])**.** *The concrete form of the concave function $\kappa(\cdot)$ can be selected as*

$$\kappa_1(\varsigma) = K\varsigma, \quad 0 \le \varsigma,$$

$$\kappa_2(\varsigma) = \begin{cases} \varsigma \log\left(\frac{1}{\varsigma}\right), & 0 \le \varsigma \le \delta, \\ \delta \log\left(\frac{1}{\delta}\right) + \kappa_2'(\delta-)(\varsigma - \delta), & \delta < \varsigma, \end{cases}$$

$$\kappa_3(\varsigma) = \begin{cases} \varsigma \log\left(\frac{1}{\varsigma}\right) \log\left(\log\left(\frac{1}{\varsigma}\right)\right), & 0 \le \varsigma \le \delta, \\ \delta \log\left(\frac{1}{\delta}\right) \log\left(\log\left(\frac{1}{\delta}\right)\right) + \kappa_3'(\delta-)(\varsigma - \delta), & \delta < \varsigma, \end{cases}$$

where $\delta \in (0,1)$ is sufficiently small. Note that if $\kappa(\cdot) = \kappa_1(\cdot)$, then Assumption 6 yields to Assumption 5 (Lipschitz condition).

Lemma 15 ([1])**.** *Assumption 6 implies the linear growth condition (Assumption 4).*

Proof. Since $\kappa(\cdot)$ is a concave and non-negative function, there exists a positive constant $c_1 > 0$ such that $\kappa(y) \le c_1(1+y)$ for $y \ge 0$. Then

$$\begin{aligned} &\|P(y,t)\|^2 \vee \|Q(y,t)\|^2 \\ &\le 2\big(\|P(y,t) - P(0,t)\|^2 \vee \|Q(y,t) - Q(0,t)\|^2\big) + 2\big(\|P(0,t)\|^2 \vee \|Q(0,t)\|^2\big) \\ &\le 2c_2 + 2\kappa(\|y\|^2) \le 2(c_1 + c_2)(1 + \|y\|^2), \end{aligned}$$

where $c_2 = \sup_{0\le t\le T} \big(\|P(0,t)\|^2 \vee \|Q(0,t)\|^2\big)$. The proof is completed. □

Define $H(t) = \int_1^t \frac{ds}{\kappa(s)}$ for $t > 0$. Denote $H^{-1}(\cdot)$ as the inverse function of $H(\cdot)$. From Assumption 6, then $\lim_{\epsilon\to 0} H(\epsilon) = -\infty$ and $\mathrm{Dom}(H^{-1}) = (-\infty, H(\infty))$.

Theorem 3. *Under Assumptions 3 and 6, the Equation (13) has a unique mild solution on $[0,T]$.*

Proof. *Proof of uniqueness.* Let $y(t)$ and $z(t)$ be two mild solutions of the FNSDE (13) with initial value y_0 and z_0, respectively. From Lemmas 1–4 and Assumptions 3 and 6,

$$\begin{aligned}
&\mathbb{E}\|y(t)-z(t)\|^2\\
&\le \mathbb{E}\Big\|h(y(t))-h(z(t))+\tfrac{1}{\Gamma(\alpha)}\int_0^t[(t-s)^{\alpha-1}(P(y(s),s)-P(z(s),s))]ds\\
&+\tfrac{1}{\Gamma(\alpha)}\int_0^t[(t-s)^{\alpha-1}(Q(y(s),s)-Q(z(s),s))]dW(s)\Big\|^2\\
&\le K_3\mathbb{E}\|y(t)-z(t)\|^2+\tfrac{2[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\Big[(t-s)^{2\alpha-2}\mathbb{E}\Big(\kappa\Big(\|y(s)-z(s)\|^2\Big)\Big)\Big]ds.
\end{aligned}$$

Since $\kappa(\cdot)$ is concave, then from the Jensen inequality,

$$\mathbb{E}\Big(\kappa\Big(\|y(s)-z(s)\|^2\Big)\Big)\le\kappa\Big(\mathbb{E}\Big(\|y(s)-z(s)\|^2\Big)\Big)\le\kappa\left(\mathbb{E}\left(\sup_{0\le r\le s}\|y(r)-z(r)\|^2\right)\right).$$

Furthermore, for any $\epsilon>0$, then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le t}\|y(r)-z(r)\|^2\right)\\
&\le\epsilon+\tfrac{2[T+\mathrm{Tr}(Q)]}{(1-K_3)^2\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\kappa\left(\mathbb{E}\left(\sup_{0\le r\le s}\|y(r)-z(r)\|^2\right)\right)\right]ds.
\end{aligned}$$

In view of Lemma 6, then

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le r\le T}\|y(r)-z(r)\|^2\right)&\le H^{-1}\Big[H(\epsilon)+\tfrac{3[T+\mathrm{Tr}(Q)]}{(1-3K_3)\Gamma(\alpha)^2}\int_0^T(T-s)^{2\alpha-2}ds\Big]\\
&=H^{-1}\Big[H(\epsilon)+\tfrac{3[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-3K_3)\Gamma(\alpha)^2}\Big].
\end{aligned}$$

Letting $\epsilon\to0$ gives $\mathbb{E}\left(\sup_{0\le r\le T}\|y(r)-z(r)\|^2\right)=0$, which implies that $y(t)=z(t)$ for all $0\le t\le T$ almost surely. Therefore, the pathwise uniqueness of the solution for Equation (13) holds. The proof of the uniqueness is completed.

Proof of existence. Consider the Carathéodory's successive approximation defined by (15). From Lemma 15, then Assumption 6 is satisfied. Furthermore, according to Lemma 11, then $Y_n(t)\in\mathcal{C}$, $\forall t\in[0,T]$. Next, it will prove that $\{Y_n(t)\}$ is a Cauchy sequence in $\mathcal{C}$ for each $t\in[0,T]$. Let $m>n\ge1$, then

$$\begin{aligned}
&Y_m(t)-Y_n(t)\\
&=h\Big(Y_m\Big(t-\tfrac{1}{m}\Big)\Big)-h\Big(Y_n\Big(t-\tfrac{1}{n}\Big)\Big)\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^t\Big[(t-s)^{\alpha-1}\Big(P\Big(Y_m\Big(s-\tfrac{1}{m}\Big),s\Big)-P\Big(Y_n\Big(s-\tfrac{1}{n}\Big),s\Big)\Big)\Big]ds\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^t\Big[(t-s)^{\alpha-1}\Big(Q\Big(Y_m\Big(s-\tfrac{1}{m}\Big),s\Big)-Q\Big(Y_n\Big(s-\tfrac{1}{n}\Big),s\Big)\Big)\Big]dW(s)\\
&=\Big[h\Big(Y_m\Big(t-\tfrac{1}{m}\Big)\Big)-h\Big(Y_n\Big(t-\tfrac{1}{m}\Big)\Big)\Big]+\Big[h\Big(Y_n\Big(t-\tfrac{1}{m}\Big)\Big)-h\Big(Y_n\Big(t-\tfrac{1}{n}\Big)\Big)\Big]\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^t\Big[(t-s)^{\alpha-1}\Big(P\Big(Y_m\Big(s-\tfrac{1}{m}\Big),s\Big)-P\Big(Y_n\Big(s-\tfrac{1}{m}\Big),s\Big)\Big)\Big]ds\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_0^t\Big[(t-s)^{\alpha-1}\Big(P\Big(Y_n\Big(s-\tfrac{1}{m}\Big),s\Big)-P\Big(Y_n\Big(s-\tfrac{1}{n}\Big),s\Big)\Big)\Big]ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}\left(Q\left(Y_m\left(s-\frac{1}{m}\right),s\right)-Q\left(Y_n\left(s-\frac{1}{m}\right),s\right)\right)\right]dW(s)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}\left(Q\left(Y_n\left(s-\frac{1}{m}\right),s\right)-Q\left(Y_n\left(s-\frac{1}{n}\right),s\right)\right)\right]dW(s).$$

From Assumptions 3, 6, Lemmas 1–4, and Jensen inequality, then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le T}\|Y_m(r)-Y_n(r)\|^2\right)\\
&\le K_3\mathbb{E}\left(\sup_{0\le r\le T}\|Y_m(r)-Y_n(r)\|^2\right)+\frac{5K_3^2}{1-K_3}\mathbb{E}\left(\sup_{0\le r\le T}\left\|Y_n\left(r-\frac{1}{m}\right)-Y_n\left(r-\frac{1}{n}\right)\right\|^2\right)\\
&\quad+\frac{5[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\kappa\left(\mathbb{E}\left(\sup_{0\le r\le s}\|Y_m(r)-Y_n(r)\|^2\right)\right)\right]ds\\
&\quad+\frac{5[T+\mathrm{Tr}(Q)]}{(1-K_3)\Gamma(\alpha)^2}\int_0^t\left[(t-s)^{2\alpha-2}\kappa\left(\mathbb{E}\left\|Y_n\left(s-\frac{1}{m}\right)-Y_n\left(s-\frac{1}{n}\right)\right\|^2\right)\right]ds.
\end{aligned}$$

Furthermore, then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le T}\|Y_m(r)-Y_n(r)\|^2\right)\\
&\le\frac{5[T+\mathrm{Tr}(Q)]}{(1-K_3)^2\Gamma(\alpha)^2}\int_0^T\left[(T-s)^{2\alpha-2}\kappa\left(\mathbb{E}\left(\sup_{0\le r\le s}\|Y_m(r)-Y_n(r)\|^2\right)\right)\right]ds+P_1(T)+P_2(T),
\end{aligned}$$

with

$$\begin{aligned}
P_1(T)&=\frac{5K_3^2}{(1-K_3)^2}\mathbb{E}\left(\sup_{0\le r\le T}\left\|Y_n\left(r-\frac{1}{m}\right)-Y_n\left(r-\frac{1}{n}\right)\right\|^2\right),\\
P_2(T)&=\frac{5[T+\mathrm{Tr}(Q)]}{(1-K_3)^2\Gamma(\alpha)^2}\int_0^T\left[(T-s)^{2\alpha-2}\kappa\left(\mathbb{E}\left\|Y_n\left(s-\frac{1}{m}\right)-Y_n\left(s-\frac{1}{n}\right)\right\|^2\right)\right]ds.
\end{aligned}$$

From Lemma 13, then

$$\begin{aligned}
P_1(T)&\le\frac{5K_3^2Q_4}{(1-K_3)^2}\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1},\\
P_2(T)&\le\frac{5[T+\mathrm{Tr}(Q)]}{(1-K_3)^2\Gamma(\alpha)^2}\int_0^T\left[(T-s)^{2\alpha-2}\kappa\left(Q_4\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}\right)\right]ds\\
&\le\frac{5[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}\kappa\left(Q_4\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}\right).
\end{aligned}$$

In view of Lemma 6, then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le r\le T}\|Y_m(r)-Y_n(r)\|^2\right)\\
&\le H^{-1}\left[H\left(\frac{5K_3^2Q_4}{(1-K_3)^2}\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}+\frac{5[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}\kappa\left(Q_4\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}\right)\right)\right.\\
&\quad\left.+\frac{5[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}\right],
\end{aligned}$$

Note that

$$\frac{5K_3^2Q_4}{(1-K_3)^2}\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}+\frac{5[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}\kappa\left(Q_4\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}\right)\to 0,$$
$$\text{as}\ \ m,n\to+\infty,$$

such that

$$H\left(\frac{5K_3^2Q_4}{(1-K_3)^2}\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}+\frac{5[T+\mathrm{Tr}(Q)]T^{2\alpha-1}}{(2\alpha-1)(1-K_3)^2\Gamma(\alpha)^2}\kappa\left(Q_4\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha-1}\right)\right)\to-\infty,$$
$$\text{as } m,n\to+\infty.$$

Then,

$$\mathbb{E}\left(\sup_{0\le r\le T}\|Y_m(r)-Y_n(r)\|^2\right)\to 0,\ \text{ as } m,n\to+\infty,$$

which implies that $\{Y_n(t)\}$ is a uniformly Cauchy sequence in $\mathcal{C}$. Therefore, there exists a continuous function $y(t)$ in $\mathcal{C}$ such that

$$\lim_{n\to+\infty}\mathbb{E}\left(\sup_{0\le t\le T}\|Y_n(t)-y(t)\|^2\right)=0.$$

According to Lemma 15, the linear growth condition (Assumption 4) holds under Assumption 6. From Theorem 2, it could be proven that the limit $y(t)$ of the sequence $\{Y_n(t)\}_{0\le t\le T}$ is a solution of Equation (13). The proof of existence is completed.

Therefore, the proof of Theorem 3 is completed. □

Remark 6. *In this section, only the Lipschitz condition and the linear growth condition that the functions $P(\cdot)$ and $Q(\cdot)$ satisfied are weakened to the non-Lipschitz condition, the assumption condition of the function $h(\cdot)$ is not changed, that is, the function $h(\cdot)$ still satisfies the Lipschitz condition. This is because the FNSDE is a model summarized from the actual systems. It turns out that $h(\cdot)$ should be Lipschitz continuous with the Lipschitz coefficient less than 1 [1].*

Remark 7. *When $\alpha=1$, Equations (3) and (13) yield the integer-order SDEs considered in [1]. Therefore, the results of this paper can be regarded as a generalization of the results in [1].*

Remark 8. *Compared with [1–9], where the Carathéodory's approximation solutions of various of SDEs were given, the FNSDE with memory and heredity is considered herein. Different from [18,19], in which the existence and uniqueness of the solution of the fractional SDE were proved by defining Picard's successive approximation, the existence and uniqueness of the solution of the FNSDE are established by using Carathéodory's successive approximation in this paper.*

6. Some Examples

In this section, two explicit examples are given to show that the obtained results can be used in real-life models. A numerical example is given to demonstrate the effectiveness of the proposed methods.

Example 1. *Consider the fractional neutral stochastic complex networks*

$$\begin{cases} d\left[\int_0^t k(t-s)(y_i(t)-h(y_i(t))-\xi+h(\xi))ds\right] \\ =\left[f(t,y_i(t))+k\sum_{j=1}^{M}m_{ij}\Gamma y_j(t)\right]dt+g(t,y_i(t))dW(t),\ \ i=1,2,\dots,M,\ 0\le t\le T, \\ y_i(t)=\xi,\quad -1\le t\le 0, \end{cases}\tag{21}$$

where $y_i(t)\in\mathbb{R}^n$ represents the state of the ith node at time t, $k(t)=t^{-\alpha}/\Gamma(1-\alpha)$ with $\alpha\in(0,1)$, $h(\cdot)\in\mathcal{L}^1(\mathbb{R}^n;\mathbb{R}^n)$, $f(\cdot,\cdot)\in\mathcal{L}^1([0,\infty)\times\mathbb{R}^n;\mathbb{R}^n)$, and $g(\cdot,\cdot)\in\mathcal{L}^2([0,\infty)\times\mathbb{R}^n;\mathcal{L}(\mathbb{U},\mathbb{R}^n))$ are continuous differentiable nonlinear mapping functions, $k>0$ represents the coupling strength, $\Gamma=\mathrm{diag}\{\gamma_1,\gamma_2,\dots,\gamma_n\}$ represents the inner linking matrix with $\gamma_i>0$, $\mathcal{M}=(m_{ij})_{M\times M}\in\mathbb{R}^{M\times M}$ is the coupling configuration matrix which reflect the topological structure of the network.

Example 2. *Consider the fractional neutral stochastic neural networks*

$$\begin{cases} d\Big[\int_0^t k(t-s)(y_i(t)-h(y_i(t))-\xi+h(\xi))ds\Big] \\ = \big[-c_iy_i(t)+\sum\limits_{j=1}^{n}a_{ij}f_j(y_j(t))+I_i(t)\big]dt+g(t,y_i(t))dW(t),\ \ i=1,2,\ldots,n,\ 0\le t\le T, \\ y_i(t)=\xi, \quad -1\le t\le 0, \end{cases} \tag{22}$$

where $y_i(t)\in\mathbb{R}$ represents the state of the ith neuron at time t, $c_i>0$ represents the rate at which the ith neuron returns to its resting state without any connection, a_{ij} represents the connection weight between the jth neuron and the ith neuron, $f_j(\cdot)$ represents the activation function of the jth neuron, and $I_i(t)$ represents the external input.

Example 3. *Consider the following system:*

$$\begin{cases} d\Big[\int_0^t k(t-s)(y(t)+\tanh(y(t))-10-\tanh(10))ds\Big] \\ =[-0.5y(t)+\cos(y(t))]dt+0.2\sin(y(t))dW(t), \quad 0\le t\le T, \\ y(t)=10, \quad -\vartheta\le t\le 0, \end{cases} \tag{23}$$

with $\alpha=0.9$. By using the predictor–corrector scheme proposed in [36], the trajectory of the mild solution of the system (23) is depicted in Figure 1. At the same time, by using Carathéodory's successive approximation, the trajectory of $Y_n(t)$ is also depicted in Figure 1. It is shown in Figure 1 that $Y_n(t)$ converges to $y(t)$ as time passes, which is consistent with the conclusion of Theorem 2.

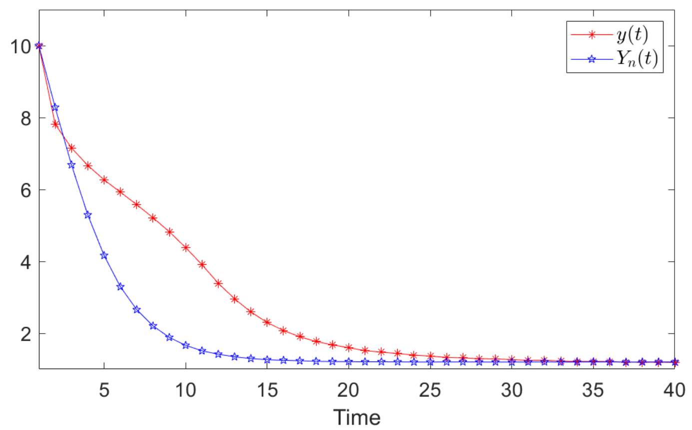

Figure 1. Trajectories of $Y_n(t)$ and $y(t)$.

Remark 9. *Examples 1 and 2 are given to show that the dynamics of nodes in complex networks and the dynamics of neurons in neural networks can be modeled by the FNSDEs. Since any property of the solution is based on the existence of the solution, it is very important to study the existence and uniqueness of the solution of the SDEs. The results obtained in this paper can be used to prove the existence and uniqueness theorem of the solution for the fractional neutral stochastic complex networks and fractional neutral stochastic neural networks. Furthermore, based on the existence and uniqueness of the solution of the system, a series of problems, such as the mean-square synchronization control problem of the fractional-order stochastic complex networks and the stability problem of the fractional-order neural networks, can be studied [13–16]. Due to the complexity of the system, the explicit solutions of some complex systems are difficult to be obtained. Therefore, based on Caratheodory's successive approximation, the numerical solution of the FNSDEs is given. It can be proved theoretically that the convergence order between the approximation and exact solution is $O(1/n)$.*

Remark 10. *For the case of FNSDEs with variable time delays, Theorem 1 shows that the mean-square error between $Y_n(t)$ and $y(t)$ depends on the supremum of time delay, that is $\mathbb{E}(\sup_{0\le t\le T}\|Y_n(t)-y(t)\|^2)\neq 0$. Considering the poor convergence effect of the numerical example, the numerical result for this case is not given in the paper.*

7. Conclusions

In this paper, the existence, uniqueness, and Carathéodory's successive approximation of FNSDE in Hilbert space were investigated. The Carathéodory's approximation solution for the FNSDE with and without delay was established, respectively. Next, the mean-square error between the mild solution and Carathéodory's approximation solution was obtained. Furthermore, by using the defined Carathéodory's successive approximation, the existence and uniqueness theorem of the solution for the FNSDE was established under the non-Lipschitz condition. Finally, some examples were given to demonstrate the effectiveness of the proposed methods.

Author Contributions: Conceptualization, X.Y. and G.R.; methodology, X.Y., Y.Y. and G.R.; software, W.C. and W.X.; validation, X.Y., Y.Y. and G.R.; formal analysis, X.Y.; investigation, X.Y. and G.R.; writing—original draft preparation, X.Y.; writing—review and editing, X.Y., Y.Y. and G.R.; visualization, W.C. and W.X.; funding acquisition, X.Y., G.R. and Y.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported in part by the Fundamental Research Funds for the Central Universities (No. YA22YJS00020), the National Natural Science Foundation of China (No. 62003026, No. 62173027 and No. 61973329), and the Natural Science Foundation of Beijing Municipality (No. Z180005).

Data Availability Statement: Not applicable

Acknowledgments: All authors contributed to the study, conception, and design. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Sample Availability: The datasets generated during the current study are available from the corresponding author on reasonable request.

Abbreviations

The following abbreviations are used in this manuscript:

SDEs	Stochastic differential equations
FNSDE	Fractional neutral stochastic differential equation
ODEs	Ordinary differential equations

References

1. Mao, X. *Stochastic Differential Equations and Their Applications*; Horwood Publishing: Chichester, UK, 2007.
2. Coddington, E.A.; Levinson, N. *Theory of Ordinary Differential Equations*; McGraw-Hill: New York, NY, USA; Toronto, ON, Canada; London, UK, 1995.
3. Mao, X. Approximate solutions for stochastic differential equations with pathwise uniqueness. *Stoch. Anal. Appl.* **1994**, *12*, 355–367. [CrossRef]
4. Benabdallah, M.; Bourza, M. Carathéodory approximate solutions for a class of perturbed stochastic differential equations with reflecting boundary. *Stoch. Anal. Appl.* **2019**, *37*, 936–954. [CrossRef]
5. Mao, X. Approximate solutions for a class of stochastic evolution equations with variable delays . *Numer. Func. Anal. Opt.* **1991**, *12*, 525–533. [CrossRef]
6. Mao, X. Approximate solutions for a class of delay stochastic differential equations. *Stochastics Stoch. Rep.* **1991**, *35*, 111–123. [CrossRef]
7. Mao, X. Approximate solutions for a class of stochastic evolution equations with variable delays. II. *Numer. Func. Anal. Opt.* **1994**, *15*, 65–76. [CrossRef]
8. Liu, K. Carathéodory approximate solutions for a class of semilinear stochastic evolution equations with time delays. *J. Math. Anal. Appl.* **1998**, *220*, 349–364. [CrossRef]

9. Turo, J. Carathéodory approximation solutions to a class of stochastic functional differential equations. *Appl. Anal.* **2007**, *61*, 121–128. [CrossRef]
10. Rubanik, V.P. *Oscillations of Quasilinear Systems with Retardation*; Nauk: Moscow, Russia, 1969.
11. Driver, R.D. A functional differential system of neutral type arising in a two-body problem of classical electrodynamics. In *Nonlinear Differential Equations and Nonlinear Mechanics*; Academic Press: Cambridge, MA, USA, 1963; pp. 474–484.
12. Caputo, M.; Mainardi, F. Linear models of dissipation in anelastic solids. *Riv. Clel Nuovo C* **1971**, *1*, 161–198. [CrossRef]
13. Yuan, X.; Ren, G.; Wang, H.; Yu, Y. Mean-square synchronization of fractional-order stochastic complex network via pinning control. *Neurocomputing* **2022**, *513*, 153–164. [CrossRef]
14. Wang, H.; Yu, Y.; Wen, G.; Zhang, S.; Yu, J. Global stability analysis of fractional-order Hopfield neural networks with time delay. *Neurocomputing* **2015**, *154*, 15–23. [CrossRef]
15. Zhang, S.; Yu, Y.; Wang, H. Mittag-Leffler stability of fractional-order Hopfield neural networks. *Nonlinear Anal. Hybrid* **2015**, *16*, 104–121. [CrossRef]
16. Zhang, S.; Yu, Y.; Yu, J. LMI conditions for global stability of fractional-order neural networks. *IEEE Trans. Neural Netw. Learn Syst.* **2017**, *28*, 2423–2433. [CrossRef]
17. Gemant, A. A method of analyzing experimental results obtained from elasto-viscous bodies. *Physics* **1936**, *7*, l–317. [CrossRef]
18. Li, Y.; Wang, Y. The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay. *J. Differ. Equations* **2019**, *266*, 3514–3558. [CrossRef]
19. Ramkumar, K.; Ravikumar, K.; Varshini, S. Fractional neutral stochastic differential equations with Caputo fractional derivative: Fractional Brownian motion, Poisson jumps, and optimal control. *Stochastic Anal. Appl.* **2021**, *39*, 157–176. [CrossRef]
20. El-Borai, M.M.; El-Nadi, K.E.S.; Fouad, H.A. On some fractional stochastic delay differential equations. *Comput. Math. Appl.* **2010**, *59*, 1165–1170. [CrossRef]
21. Cui, J.; Yan, L. Existence result for fractional neutral stochastic integro-differential equations with infinite delay. *J. Phys A. Math. Theor.* **2011**, *44*, 335201. [CrossRef]
22. Sakthivel, R.; Revathi, P.; Ren, Y. Existence of solutions for nonlinear fractional stochastic differential equations. *Nonlinear Anal.* **2013**, *81*, 70–86. [CrossRef]
23. Chen, P.; Zhang, X.; Li, Y. Nonlocal problem for fractional stochastic evolution equations with solution operators. *Fract. Calc. Appl. Anal.* **2016**, *19*, 1507–1526. [CrossRef]
24. Benchaabane, A.; Sakthivel, R. Sobolev-type fractional stochastic differential equations with non-Lipschitz coefficients. *J. Comput. Appl. Math.* **2017**, *312*, 65–73. [CrossRef]
25. Abouagwa, M.; Liu, J.; Li, J. Carathéodory approximations and stability of solutions to non-Lipschitz stochastic fractional differential equations of Itô-Doob type. *Appl Math Comp.* **2018**, *329*, 143–153. [CrossRef]
26. Guo, Z.; Hu, J.; Wang, W. Caratheodory's approximation for a type of Caputo fractional stochastic differential equations. *Adv. Differ. Equ.* **2020**, 636. [CrossRef]
27. Abouagwa, M.; Li, J. Stochastic fractional differential equations driven by Lévy noise under Carathéodory conditions. *J. Math Phys.* **2019**, *60*, 022701. [CrossRef]
28. Baran, M. Approximations for solutions of Lévy-type stochastic differential equations. *Stochastic Anal. Appl.* **2009**, *27*, 924–961. [CrossRef]
29. Curtain, R.F.; Falb, P.L. Ito's Lemma in infinite dimensions. *J. Math. Anal. Appl.* **1970**, *31*, 434–448. [CrossRef]
30. Podlubny, I. *Fractional Differential Equations of Nathematics in Science and Engineering*; Academic Press: San Diego, CA, USA, 1999.
31. Bhatia, R. *Matrix Analysis*; Springer: Berlin/Heidelberg, Germany, 1996.
32. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
33. Hendy, A.S.; Zaky, M.A.; Doha, E.H. On a discrete fractional stochastic Grönwall inequality and its application in the numerical analysis of stochastic FDEs involving a martingale. *Int. J. Nonlin. Sci. Num.* **2021**. [CrossRef]
34. Hendy, A.S.; Zaky, M.A.; Suragan, D. Discrete fractional stochastic Grönwall inequalities arising in the numerical analysis of multi-term fractional order stochastic differential equations. *Math. Comput. Simulat.* **2022**, 193, 269–279. [CrossRef]
35. Dai, X.; Bu, W.; Xiao, A. Well-posedness and EM approximations for non-Lipschitz stochastic fractional integro-differential equations. *J. Comput. Appl. Math.* **2019**, *356*, 377–390. [CrossRef]
36. Bhalekar, S.; Daftardar-Gejji, V. A predictor-corrector scheme for solving nonlinaer delay differential equations of fractional order. *J. Fract. Calc. Appl.* **2011**, *5*, 1–9.