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Article

Fractional Biswas–Milovic Equation in Random Case Study

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Abstract: We apply two mathematical techniques, specifically, the unified solver approach and the $\exp(-\varphi(\xi))$-expansion method, for constructing many new solitary waves, such as bright, dark, and singular soliton solutions via the fractional Biswas–Milovic (FBM) model in the sense of conformable fractional derivative. These solutions are so important for the explanation of some practical physical problems. Additionally, we study the stochastic modeling for the fractional Biswas–Milovic, where the parameter and the fraction parameters are random variables. We consider these parameters via beta distribution, so the mathematical methods that were used in this paper may be called random methods, and the exact solutions derived using these methods may be called stochastic process solutions. We also determined some statistical properties of the stochastic solutions such as the first and second moments. The proposed techniques are robust and sturdy for solving wide classes of nonlinear fractional order equations. Finally, some selected solutions are illustrated for some special values of parameters.

Keywords: conformable fractional derivative; unified solver method; $\exp(-\varphi(\xi))$-expansion method; traveling wave solutions; FBM equation; stochastic solutions

MSC: 34A08; 35A20; 35C07; 26A33; 60H15; 35R11

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1. Introduction

Nonlinear fractional differential equations (NFDEs) play vital roles in many interesting applications in chemical engineering, fluid mechanics, biology, electromagnetic theory, physics, and others [1–4]. These equations are powerful instruments for depicting real-world problems more accurately than the classical integer-order equations. Thus, the investigation of solitary wave solutions for NFDEs becomes very useful in scientific research. Recently, many researches have proposed and developed various numerical and analytical methods for solving NFDEs. Shah et al. applied the time-fractional Caputo and Caputo–Fabrizio fractional derivatives to the Chua type nonlinear chaotic system [5]. Alshehry et al. presented the Laplace residual-power-series method (LRPSM), a powerful new technique for solving fractional partial differential equations [6].

For the above reasons, recently, several efficient mathematical approaches have been proposed to obtain solutions of NFDEs, such as the $(\frac{G'}{G})-$ expansion method [7], the fractional sub-equation method [8], the first integral method [9], the tanh-sech method [10], the unified solver method [11], the exponential function method [12] and others [13–16]. Recently, the dynamical behavior of the exact traveling wave solutions and their phase portrait analysis were extensively studied using the dynamical system theory. Zhu et al. [17] studied the exact traveling wave solutions and bifurcations of the fractional Klein–Gordon equation and the fractional generalized Hirota–Satsuma coupled KdV system. Specifically, they applied this technique for the first time to NFDEs. Based on these powerful results, some other authors followed the same technique, for example, see [18–20]. Li et al. [18] studied the dynamical behavior of a time–space fractional Phi-4 equation using the bifurcation method of a planar dynamical system via conformable fractional derivative. Liu et al. [19] investigated the dynamical behavior and bifurcation of solutions of the traveling wave system for a generalized (3 + 1)-dimensional time-fractional gCH-KP equation. Li and Han [20] investigated the bifurcation and new exact solutions for the (2 + 1)-dimensional conformable time-fractional Zoomeron equation.

The Biswas–Milovic equation studied in this paper is given by [21]

$$i(q^m)_t + a(q^m)_{xx} + bF(|\, q\,|^2)q^m = 0, \quad i = \sqrt{-1}, \tag{1}$$

where x and t are two independent variables, and q is a complex valued function. The coefficients a and b are constants with $ab > 0$, and $m \geq 1$ is a parameter. Moreover, F is a real-valued algebraic function, which is necessary to have smoothness of the complex function $F(|\, q\,|^2) : \mathbb{C} \to \mathbb{C}$. Equation (1) is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in it. This equation emerges in the study of long-distance optical communications and all-optical ultra fast switching devices. Furthermore, this equation has been indicated to manage the evolution of a wave packet in a weakly nonlinear and dispersive medium and has eventuated diverse fields, such as plasma, nonlinear optics, and water waves.

To highlight numerous complex phenomena in various fields of nature, such as biology, economy, engineering, chemical engineering, signal processing, solid state physics, and electromagnetic theory, it is crucial to take into account random problem effects. Practical considerations require that the stochastic type perturbations be considered. More focus has been placed in recent years on the impact of noise on the spread of these solton solutions. Therefore, we must take into consideration the stochastic effect.

The novelties of this paper are mainly exhibited in four aspects. First, we present a general form of a new fractional Biswas–Milovic equation (FBM). Second, we use a new method, the so-called unified solver method [22], in order to solve the FBM. Many other nonlinear models developing in applied science and new physics can be solved using the solver as a box solver. In comparison to previous methods, this solver has some advantages, such as avoiding difficult and time-consuming calculations and presenting important solutions explicitly. Moreover, we use the exp-function method [23,24] in order to solve the proposed equation. Indeed, this method presents powerful solutions in vital applications in natural science [25,26]. Third, we obtain new types of exact analytical solutions. Comparing our results with other results, one can see that our results are new and more extensive. Fourth, we study the stochastic modeling for the fractional Biswas–Milovic through the parameter where the fraction parameters are random variables. Thus, we must discuss our fractional problem under beta distribution [27].

The remainder of this paper is arranged as follows. In Section 2, some preliminaries, notions of local fractional calculus, and the beta random distribution method are introduced. Section 3 presents the description of the two methods applied in this work, namely, the unified solver method and the exp-function method. In Section 4, some exact solutions of the fractional BM equation are presented, using the two methods. Stochastic study with beta distribution is studied in Section 5. Conclusions are in Section 6.

2. Preliminaries

Here, we give some fundamental notions of fractional calculus theory, which turn out to be very useful to complete this article in a unified way. There are various types of fractional derivatives [2], such as *Grünwald*–Letnikov, Caputo local fractional derivative [28], Abel–Riemann fractional derivative [29] and He's fractional derivative [15,16]. Khalil et al. [30] presented a new vital definition of fractional derivatives, the so-called conformable fractional derivative. This definition preserves various advantages that cannot be satisfied by the known fractional derivatives, such as Rolle's theorem, mean value theorem, the product of two functions, and the chain rule [31]. The conformable fractional derivative has attracted important attention due to its simplicity. Thus, many studies has been done on it by many scientists. First, we introduce some properties of the conformable fractional derivative. Second, we give the beta random distribution.

Definition 1 ([30]). *Let a function $\phi : (0, \infty) \to \mathbb{R}$, then the conformable fraction derivative of ϕ of order α is*

$$D_t^\alpha(\phi)(t) = \lim_{\iota \to 0} \frac{\phi(t + \iota t^{1-\alpha}) - \phi(t)}{\iota}, \quad t > 0, \ \ 0 < \alpha \leq 1.$$

The conformable fractional derivative satisfies:

(i) $D_t^\alpha(a\,\chi + b\,\phi) = aD_t^\alpha(\chi) + bD_t^\alpha(\phi), \ \ a, b \in \mathbb{R}$,
(ii) $D_t^\alpha(t^n) = nt^{n-\alpha}, \ \ n \in \mathbb{R}$,
(iii) $D_t^\alpha(\chi\,\phi) = \chi\,D_t^\alpha(\phi) + \phi\,D_t^\alpha(\chi)$,
(vi) $D_t^\alpha(\frac{\chi}{\phi}) = \frac{\phi\,D_t^\alpha(\chi) - \chi\,D_t^\alpha(\phi)}{\phi^2}$,
(v) If ϕ is differentiable, thus $D_t^\alpha(\phi)(t) = t^{1-\alpha}\frac{d\phi}{dt}$.

Theorem 1 ([30]). *Let $\chi, \phi : (0, \infty) \to \mathbb{R}$ be differentiable and also α-differentiable, then:*

$$D_t^\alpha(\chi \circ \phi)(t) = t^{1-\alpha}\phi'(t)\psi'(\phi(t)). \tag{2}$$

Beta Distribution

If we have a real random variable X defined on the probability space (Ω, F, P) that has the probability density function $f(x)$, we can defined the statistical moments as follows:

Remark 1. *1. The first moment:*

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx \tag{3}$$

2. *The second moment:*

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f(x)dx. \tag{4}$$

Therefore, if X has a beta distribution, then the probability density function defined as

$$f(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}} \qquad a \leq x \leq b; \alpha, \beta > 0$$

is called a beta random variable. Here, α and β are the shape parameters, a and b are the lower and upper bounds, respectively, of the distribution. In addition, $B(\alpha, \beta)$ is the beta function, and if we have $a = 0$ and $b = 1$, it is called the standard beta distribution.

Additionally, $\mathrm{E}[\mathrm{X}] = \frac{\alpha}{\alpha+\beta}$, where, E[ ] denotes the expectation value operator. In this work, we will deal with the standard beta distribution.

Any NFDEs in two independent variables x and t can be expressed as follows:

$$\Lambda(\phi, D_t^\alpha\phi, D_x^\alpha\phi, D_t^\alpha D_x^\alpha\phi, D_x^\alpha D_x^\alpha\phi, \ldots) = 0, \tag{5}$$

$1 \geq \alpha > 0$.

Utilizing the traveling wave transformation

$$\phi(x,t) = \Phi(\xi), \qquad \xi = \frac{x^\alpha}{\alpha} + w\,\frac{t^\alpha}{\alpha} \tag{6}$$

converts Equation (5) to the following ODE:

$$\Lambda(\Phi, \Phi', \Phi'', \Phi''', \ldots) = 0. \tag{7}$$

If we have any NFDEs in two independent variables x and t, then we develop it only in the beta random distribution because the range of the beta random variable is $1 \geq \alpha > 0$. Therefore, the random case of (5) is given when α has a beta random distribution. We use the beta random distribution traveling wave transformation (6), where k, λ are non zero constants and α has beta random distribution. Therefore, we can complete the beta random distribution methods as in the deterministic α where $0 < \alpha \leq 1$. In this paper, the beta random distribution method is considered through two methods. When we use the beta random variable as in the random problem itself or when we use beta random distribution traveling wave transformation, the proposed methods, namely the unified solver and the $\exp(-\varphi(\xi))$-expansion methods, are called random methods. Additionally, the solutions produced by these random methods are stochastic process solutions.

3. Description of the Methods

We briefly give the unified solver [22] and the $\exp[-\varphi(\xi)]$-expansion techniques [23,24]. Various NFDEs (5) reduce to the following ODE:

$$\Lambda_1 \Phi'' + \Lambda_2 \Phi^3 + \Lambda_3 \Phi = 0, \tag{8}$$

where Λ_1, Λ_2, and Λ_3 are constants that depend on the proposed equation's constants and the wave transformations' speed.

3.1. Unified Solver Method

In light of the unified solver [22], the solutions of Equation (8) are:

(i) Rational solutions: (when $\Lambda_3 = 0$)

$$\Phi_{1,2}(x,t) = \left(\mp \sqrt{\frac{-\Lambda_2}{2\Lambda_1}} (\xi + \varepsilon) \right)^{-1}. \tag{9}$$

(ii) Trigonometric solutions: (when $\frac{\Lambda_3}{\Lambda_1} < 0$)

$$\Phi_{3,4}(x,t) = \pm \sqrt{\frac{\Lambda_3}{\Lambda_2}}\, tan \left(\sqrt{\frac{-\Lambda_3}{2\Lambda_1}} (\eta + \varepsilon) \right) \tag{10}$$

and

$$\Phi_{5,6}(x,t) = \pm \sqrt{\frac{\Lambda_3}{\Lambda_2}}\, cot \left(\sqrt{\frac{-\Lambda_3}{2\Lambda_1}} (\eta + \varepsilon) \right). \tag{11}$$

(iii) Hyperbolic solutions: (when $\frac{\Lambda_3}{\Lambda_1} > 0$)

$$\Phi_{7,8}(x,t) = \pm \sqrt{\frac{-\Lambda_3}{\Lambda_2}}\, tanh \left(\sqrt{\frac{\Lambda_3}{2\Lambda_1}} (\eta + \varepsilon) \right) \tag{12}$$

and

$$\Phi_{9,10}(x,t) = \pm \sqrt{\frac{-\Lambda_3}{\Lambda_2}}\, coth \left(\sqrt{\frac{\Lambda_3}{2\Lambda_1}} (\eta + \varepsilon) \right). \tag{13}$$

Here, ε is an arbitrary constant.

3.2. The Exp$[-\varphi(\xi)]$-Expansion Method

In view of the $\exp[-\varphi(\xi)]$-expansion approach [23,24], the solution of Equation (8) is rewritten in the following polynomial form of $\exp[-\varphi(\xi)]$:

$$\Phi(\xi) = B_0 + B_1 \exp[-\varphi(\xi)] \tag{14}$$

B_0 and $B_1 \neq 0$ are constants and $\varphi(\xi)$ satisfies

$$\varphi'(\xi) = \exp[-\varphi(\xi)] + \nu \exp[\varphi(\xi)] + \lambda\,. \tag{15}$$

The solutions of Equation (15) are:

1. At $\lambda^2 - 4\nu > 0 \,\&\, \nu \neq 0$,

$$\varphi(\xi) = ln\left(\frac{-\sqrt{\lambda^2 - 4\nu}\; tanh\left(\frac{\sqrt{\lambda^2-4\nu}}{2}(\xi + C)\right) - \lambda}{2\nu}\right), \tag{16}$$

2. At $\lambda^2 - 4\nu < 0 \,\&\, \nu \neq 0$,

$$\varphi(\xi) = ln\left(\frac{\sqrt{4\nu - \lambda^2}\; tan\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\,(\xi + C)\right) - \lambda}{2\nu}\right), \tag{17}$$

3. At $\lambda^2 - 4\nu > 0 \,\&\, \lambda \neq 0 \,\&\, \nu = 0$,

$$\varphi(\xi) = -ln\left(\frac{\lambda}{\exp[\lambda(\xi + C)] - 1}\right), \tag{18}$$

4. At $\lambda^2 - 4\nu = 0 \,\&\, \lambda \neq 0 \,\&\, \nu \neq 0$,

$$\varphi(\xi) = ln\left(-\frac{2(\lambda(\xi + C) + 2)}{\lambda^2(\xi + C)}\right), \tag{19}$$

5. At $\lambda^2 - 4\nu = 0 \,\&\, \lambda = 0 \& \nu = 0$,

$$\varphi(\xi) = ln(\xi + C)\,. \tag{20}$$

Here, C is an arbitrary constant.

Setting Equation (14) and Equation (15) into Equation (8) and adding all phrases with the same power $\exp[-m\varphi(\xi)]$, $m = 0, 1, 2, 3$. Then, putting them with zero yields algebraic equations. Solving these equations provides the values of B_0, B_1. Then, we obtain the solutions (14) that produce the exact solutions of Equation (7).

4. Application

In this section, we are concerned with the fractional type of BM equation [32]

$$iD_t^{\alpha} q^m + \gamma D_x^{2\alpha} q^m + \delta \mid q \mid^2 q^m = 0\,, \tag{21}$$

where $1 \geq \alpha > 0$, $m \geq 1$. There are many approaches that have been applied to provide solutions for Equation (21), such as the four trigonometric analytical methods [33], Adomian decomposition method [34], the extended fractional sinh–Gordon equation expansion approach [32], etc. Indeed, most standard papers considered the same methods to study the fractional differential equations in a deterministic sense. In contrast to these papers, we considered the stochastic modeling for the fractional Biswas–Milovic through the parameter, and the fraction parameters are random variables. In probability theory, this means that the only random distribution for these parameters is the beta distribution, so the mathematical methods used in this paper may be called random methods, and the exact solutions found using these methods may be called stochastic process solutions. In addition, we determined some of statistical properties of the stochastic solutions as the first and the second moments. To perform this procedure, we apply the unified solver method and the exp-function method. We consider the traveling wave transformation:

$$q(x,t) = u(\xi)e^{i\psi},\; \xi = \frac{x^{\alpha}}{\alpha} - \nu\frac{t^{\alpha}}{\alpha},\; \psi = -k\frac{x^{\alpha}}{\alpha} + w\frac{t^{\alpha}}{\alpha} + \theta, \tag{22}$$

where ν, k, θ, and w denote the speed of the wave, frequency, phase constant, and wave number, respectively. Substituting (22) into (21) gives [32]

$$a\,u'' + bu^3 - (w + ak^2)\,u = 0 \tag{23}$$

and

$$\nu = -2\,a\,k. \tag{24}$$

Now we apply the unified solver method and exp-function methods for Equation (24).

4.1. On Solving Equation (21) Using the Unified Solver Method

In view of the unified solver approach introduced in [22], the solutions for Equation (24) are:

4.1.1. Rational Solutions

The rational solution of Equation (24) is

$$u_1(x,t) = \left(\mp\sqrt{\frac{-b}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right)^{-1}. \tag{25}$$

Using Equations (22) and (25), the solutions of Equation (21) take the form:

$$q_1(x,t) = e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\left(\mp\sqrt{\frac{-b}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right)^{-1}. \tag{26}$$

4.1.2. Trigonometric Solutions

The trigonometric solutions of Equation (24) are

$$u_{2,3}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\,tan\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right) \tag{27}$$

and

$$u_{4,5}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\,cot\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right). \tag{28}$$

Using Equations (22), (28) and (43), the solutions of Equation (21) take the forms:

$$q_{2,3}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\,e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\,tan\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right) \tag{29}$$

and

$$q_{4,5}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\,e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\,cot\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right). \tag{30}$$

4.1.3. Hyperbolic Solutions

The hyperbolic solutions of Equation (24) are

$$u_{6,7}(x,t) = \pm\sqrt{\frac{w+ak^2}{b}}\,tanh\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right) \tag{31}$$

and

$$u_{8,9}(x,t) = \pm\sqrt{\frac{w+ak^2}{b}}\, coth\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right). \tag{32}$$

Using Equations (22), (31) and (32), the solutions of Equation (21) take the forms:

$$q_{6,7}(x,t) = \pm\sqrt{\frac{w+ak^2}{b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\, tanh\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right) \tag{33}$$

and

$$q_{8,9}(x,t) = \pm\sqrt{\frac{w+ak^2}{b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\, coth\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right). \tag{34}$$

4.2. On Solving Equation (21) Using the Exp$(-\varphi(\xi))$-Expansion Method

In view of the $\exp(-\varphi(\xi))$-expansion technique [23,24], the solutions of Equation (24) have the following solution:

$$u = A_0 + A_1\exp(-\varphi), \tag{35}$$

where A_0 and A_1 are constants to be determined, such that $A_1 \neq 0$. It is easy to see that

$$u'' = A_1\Big(2\exp(-3\varphi) + 3\lambda\exp(-2\varphi) + \nu\lambda + 2\nu\exp(-\varphi) + \lambda^2\exp(-\varphi)\Big), \tag{36}$$

$$u^3 = A_1^3\exp(-3\varphi) + 3A_0A_1^2\exp(-2\varphi) + 3A_0^2A_1\exp(-\varphi) + A_0^3 \tag{37}$$

Substituting u, u'', u^3 into Equation (24) and then equating the coefficients of $\exp(-\varphi)$ to zero gives a system of algebraic equations. Solving this system yields

$$u(\xi) = \pm\sqrt{\frac{-c}{2b}}(\lambda + 2\exp(-\varphi(\xi)))\,. \tag{38}$$

Hence, the solutions of Equation (24) are:
Case 1. At $\lambda^2 - 4\nu > 0, \nu \neq 0,$

$$\tilde{u}_{1,2}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 - \frac{4\nu}{\sqrt{\lambda^2-4\nu}\tanh\Big(\frac{\sqrt{\lambda^2-4\nu}}{2}\big(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\big)\Big) + \lambda}\right), \tag{39}$$

$$\tilde{u}_{3,4}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 - \frac{4\nu}{\sqrt{\lambda^2-4\nu}\coth\Big(\frac{\sqrt{\lambda^2-4\nu}}{2}\big(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\big)\Big) + \lambda}\right). \tag{40}$$

Using Equations (22), (39) and (40), the solutions of Equation (21) take the forms:

$$\tilde{q}_{1,2}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 - \frac{4\nu}{\sqrt{\lambda^2-4\nu}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\,\tanh\Big(\frac{\sqrt{\lambda^2-4\nu}}{2}\big(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\big)\Big) + \lambda}\right), \tag{41}$$

$$\tilde{q}_{3,4}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 - \frac{4\nu}{\sqrt{\lambda^2-4\nu}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\,\coth\Big(\frac{\sqrt{\lambda^2-4\nu}}{2}\big(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\big)\Big) + \lambda}\right). \tag{42}$$

Case 2. At $\lambda^2 - 4\nu < 0, \nu \neq 0$,

$$\tilde{u}_{5,6}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu-\lambda^2}\tan\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right), \tag{43}$$

$$\tilde{u}_{7,8}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu-\lambda^2}\cot\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right). \tag{44}$$

Using Equations (22), (43) and (44), the solutions of Equation (21) take the forms:

$$\tilde{q}_{5,6}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu-\lambda^2}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ \tan\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right), \tag{45}$$

$$\tilde{q}_{7,8}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu-\lambda^2}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ \cot\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right). \tag{46}$$

Case 3. At $\lambda^2 - 4\nu > 0, \nu = 0, \lambda \neq 0$

$$\tilde{u}_{9,10}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{2\lambda}{\exp\left(\lambda\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - 1}\right). \tag{47}$$

Using Equations (22) and (47), the solutions of Equation (21) take the forms:

$$\tilde{q}_{9,10}(\xi) = \pm\sqrt{\frac{-c}{2b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\left(1 + \frac{2\lambda}{\exp\left(\lambda\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - 1}\right). \tag{48}$$

5. Discussion in Some Stochastic Cases under Beta Random Distribution

In this section, we consider Equation (21) in a stochastic sense, where α is a beta random variable. Namely, we study the statistical properties for the stochastic processes solutions as follows:

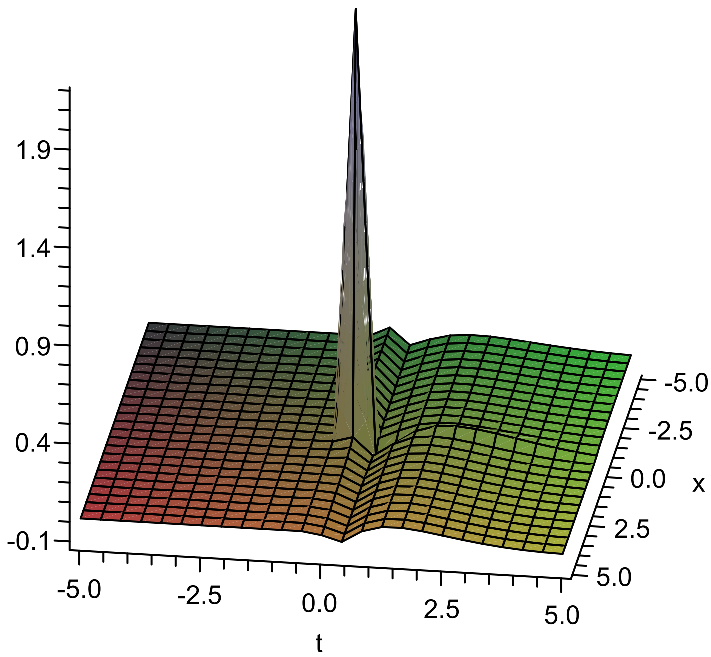
5.1. Stochastic Solutions of (21) via the Unified Solver Method

Here, we can write the stochastic solutions of problem (21) by using the unified solver method to show the statistical properties of the stochastic processes solutions. As in the deterministic case, we find the random relations as follows:

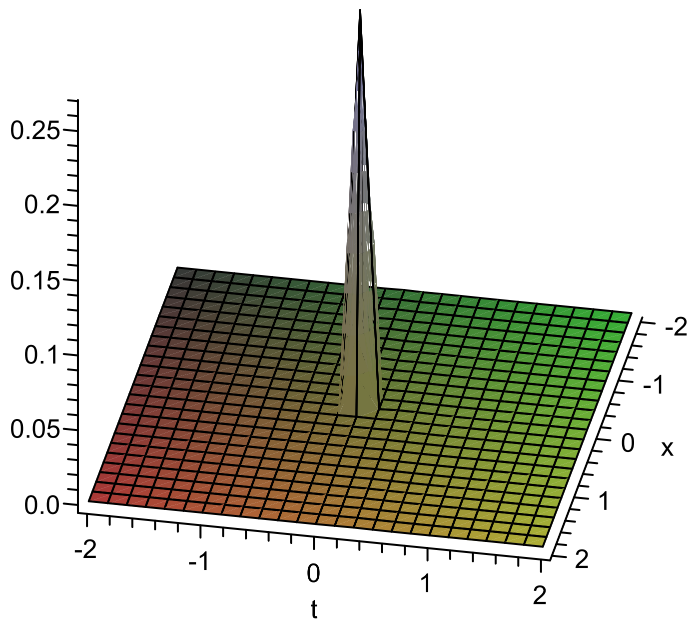
5.1.1. Rational Stochastic Solutions

$$sq_1(x,t) = e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\left(\mp\sqrt{\frac{-b}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right)^{-1}. \tag{49}$$

The mean value $E[sq_1]$ and the second moment $E[sq_1^2]$ are depicted in Figure 1a,b, respectively.



(**a**)



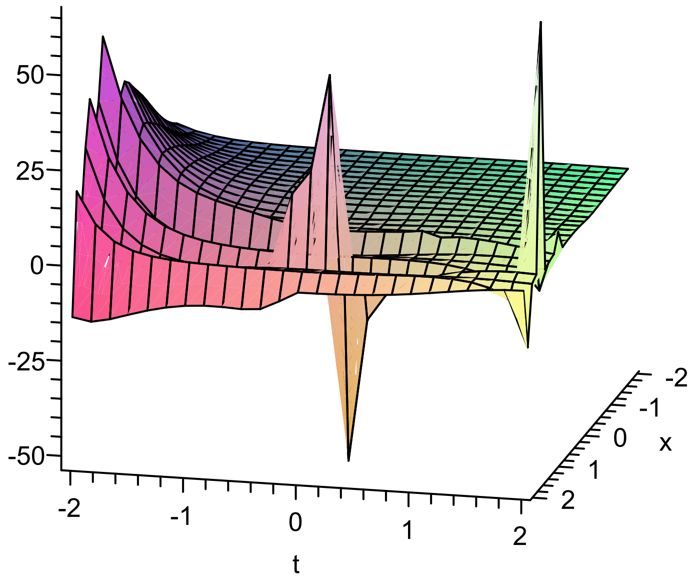
(**b**)

Figure 1. The first moment for the random solution sq_1 in (49); α is a beta random variable and $-2 \leq t, x \leq 2$.

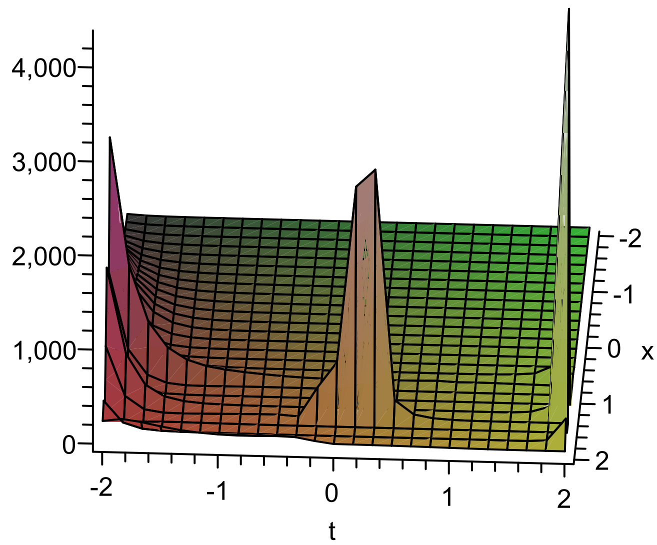
5.1.2. Trigonometric Stochastic Solutions

$$sq_{2,3}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha}+w\frac{t^\alpha}{\alpha}+\theta\right)}\, tan\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}+\mu\right)\right) \quad (50)$$

The mean value $E[sq_3]$ and the second moment $E[sq_3^2]$ are depicted in Figure 2a,b, respectively.



(**a**)

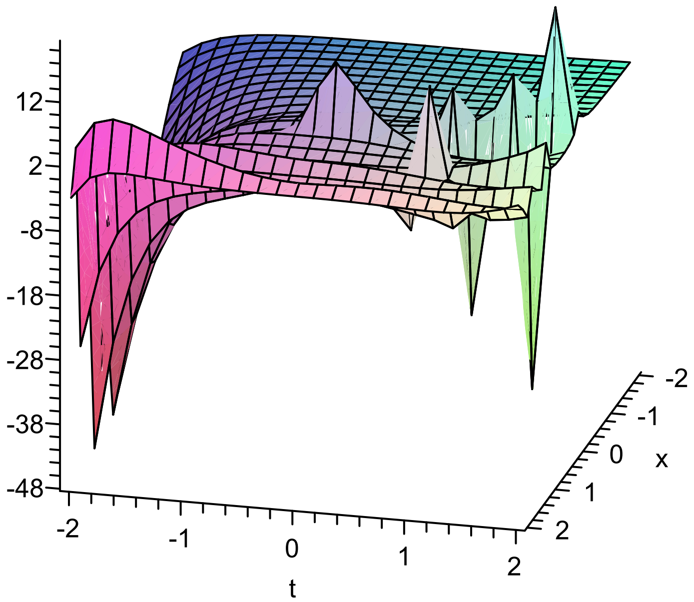


(**b**)

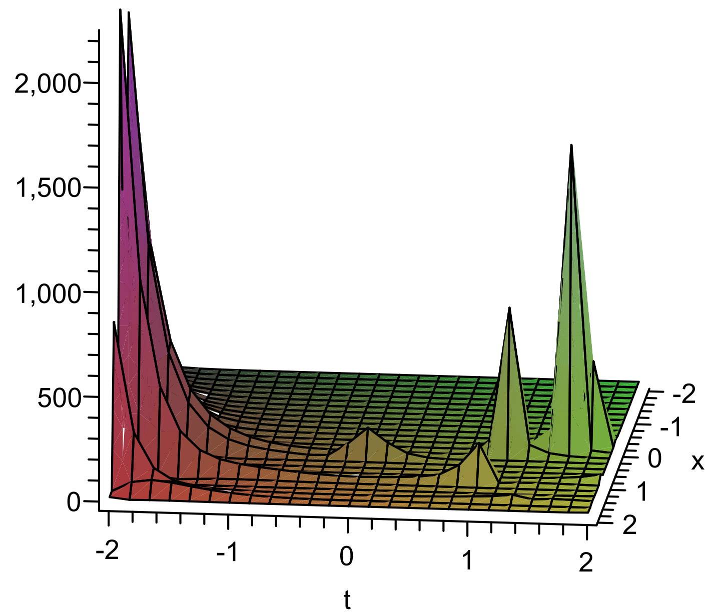
Figure 2. The first and second moments for the random solution sq_3 in (50); α is a beta distribution random variable and $-2 \leq t, x \leq 2$.

$$sq_{4,5}(x,t) = \pm\sqrt{\frac{-(w+ak^2)}{b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha}+w\frac{t^\alpha}{\alpha}+\theta\right)}\, cot\left(\sqrt{\frac{(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}+\mu\right)\right). \quad (51)$$

The mean value $E[sq_5]$ and the second moment $E[sq_5^2]$ are depicted in Figure 3a,b, respectively.



(**a**)



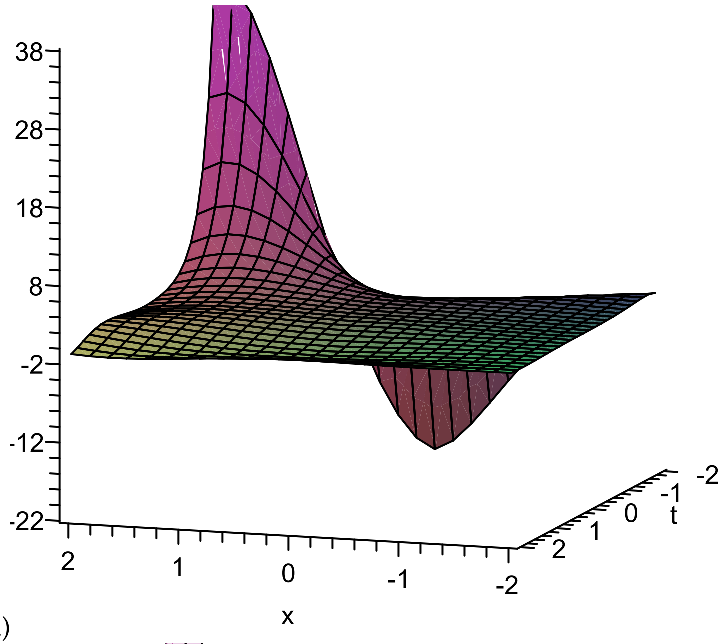
(**b**)

Figure 3. The first and second moments for the random solution sq_5 in (51); α is a beta random variable and $-2 \leq t, x \leq 2$.

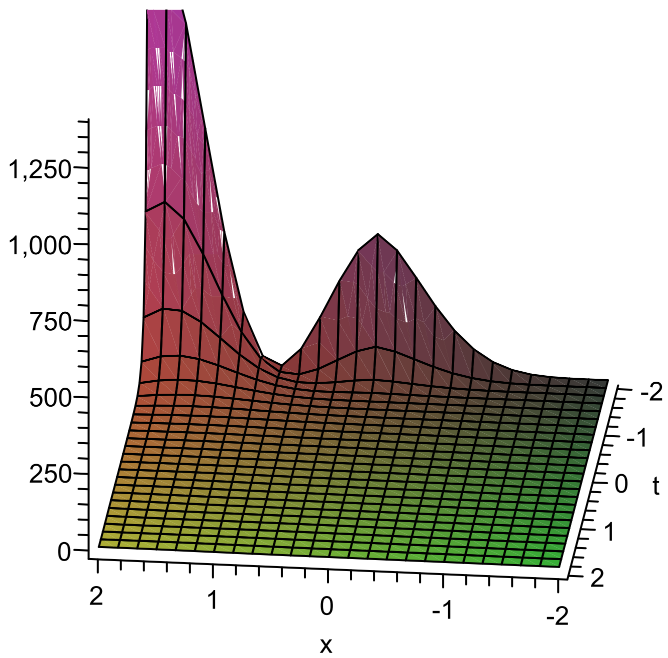
5.1.3. Hyperbolic Stochastic Solutions

$$sq_{6,7}(x,t) = \pm\sqrt{\frac{w + ak^2}{b}}\ e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ tanh\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + \mu\right)\right) \quad (52)$$

The mean value $E[sq_7]$ and the second moment $E[sq_7^2]$ are depicted in Figure 4a,b, respectively.



(**a**)



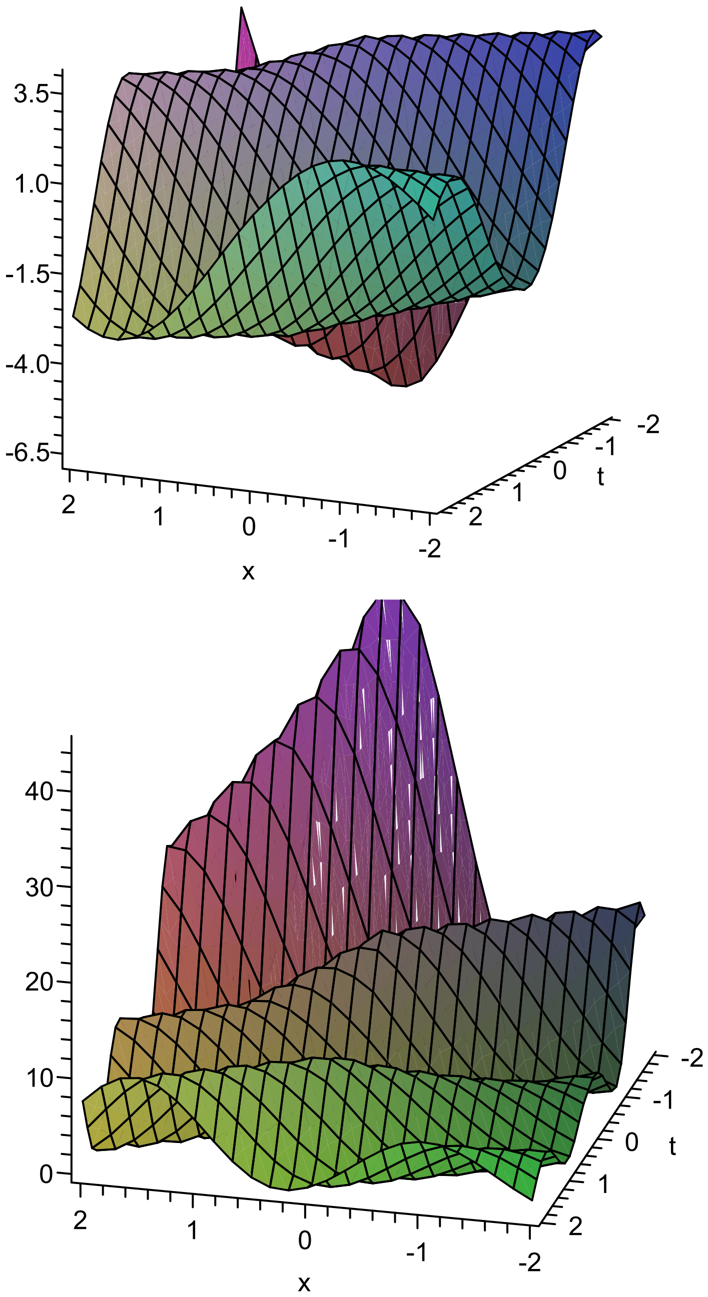
(**b**)

Figure 4. The first and second moments for the random solution sq_7 in (52); α is a beta random variable and $-2 \le t, x \le 2$.

And

$$sq_{8,9}(x,t) = \pm\sqrt{\frac{w+ak^2}{b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha}+w\frac{t^\alpha}{\alpha}+\theta\right)}\, coth\left(\sqrt{\frac{-(w+ak^2)}{2a}}\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}+\mu\right)\right). \quad (53)$$

The mean value $E[sq_9]$ and the second moment $E[sq_9^2]$ are depicted in Figure 5a,b, respectively.



(**a**)

(**b**)

Figure 5. The first and second moments for the random solution sq_9 in (53); α is a beta random variable and $-2 \leq t, x \leq 2$.

5.2. Stochastic Solutions of (21) via the Exp$(-\varphi(\xi))$-Expansion Method

Here, we can write the stochastic solutions of problem (21) using the $\exp(-\varphi(\xi))$-expansion method and try to find the statistical properties of the stochastic processes solutions. As in the deterministic case, we can find the random relations as follows:

Case 1. At $\lambda^2 - 4\nu > 0, \nu \neq 0$,

$$\tilde{sq}_{1,2}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 - \frac{4\nu}{\sqrt{\lambda^2 - 4\nu}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ \tanh\left(\frac{\sqrt{\lambda^2-4\nu}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) + \lambda}\right), \tag{54}$$

where α is a beta random variable.

The mean value $\mathbb{E}[\tilde{sq}_1]$ and the second moment $\mathbb{E}[\tilde{sq}_1^2]$ are depicted in Figure 6a,b, respectively.

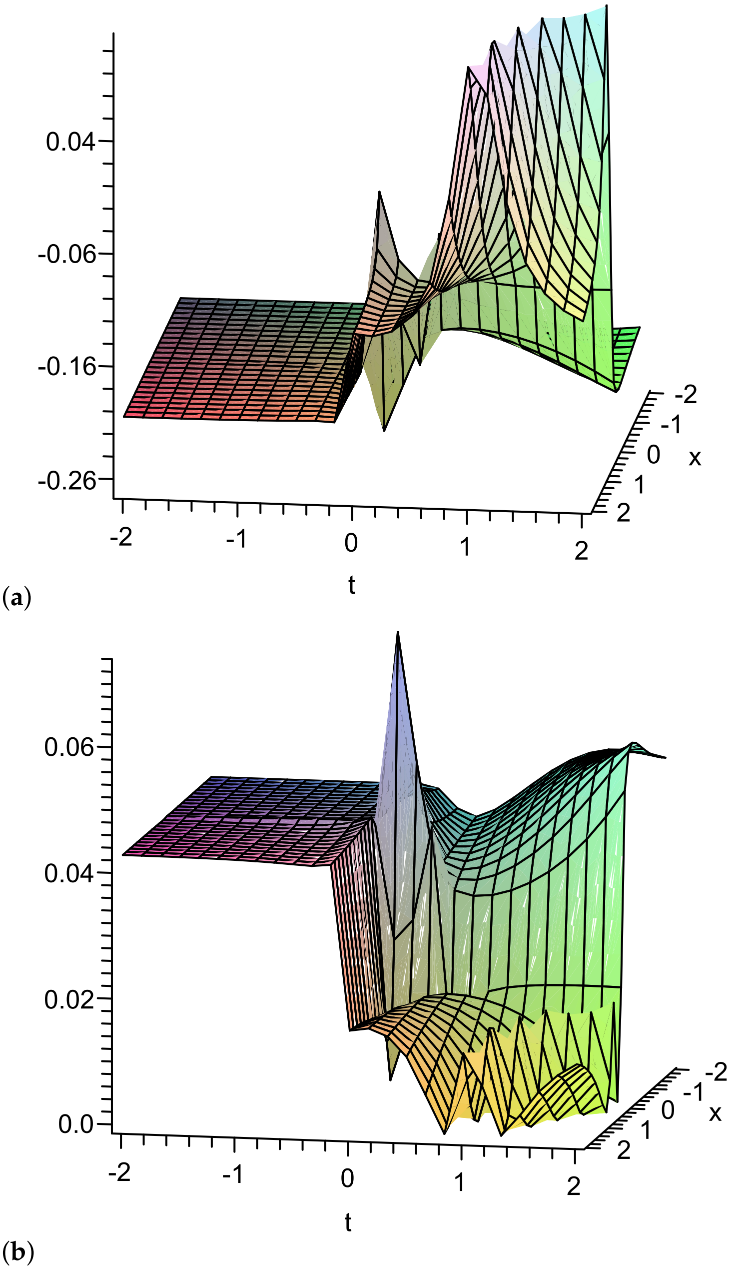


Figure 6. The first and second moments for the random solution $\tilde{sq}_1$ in (54); α is a beta random variable and $-2 \leq t, x \leq 2$, the first moment is on the upper part and the second moment is on the lower part.

Case 2. At $\lambda^2 - 4\nu < 0, \nu \neq 0$,

$$\tilde{sq}_{3,4}(\xi) = \pm\sqrt{\frac{-c}{2b}} \left(1 - \frac{4\nu}{\sqrt{\lambda^2 - 4\nu}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)} \coth\left(\frac{\sqrt{\lambda^2 - 4\nu}}{2} \left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K \right) \right) + \lambda} \right), \tag{55}$$

where α is a beta random distribution.

The mean value $\mathbb{E}[\tilde{sq}_3]$ and the second moment $\mathbb{E}[\tilde{sq}_3^2]$ are depicted in Figure 7a,b, respectively.

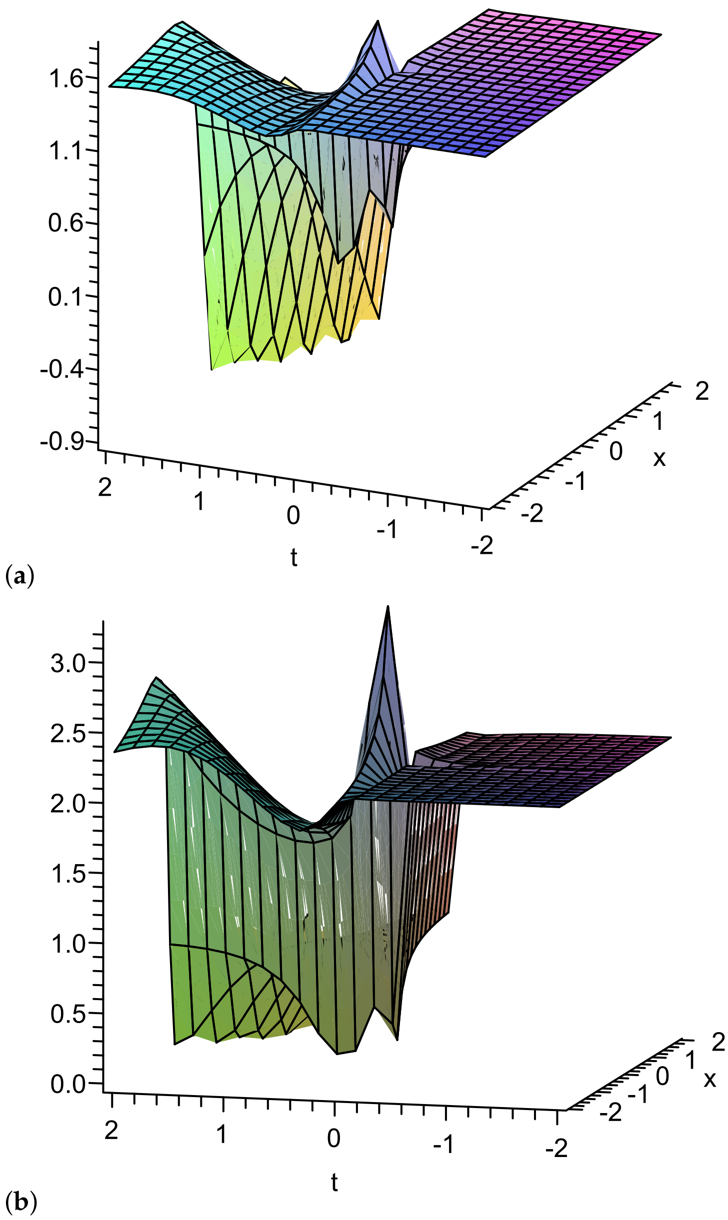


Figure 7. The first and second moments for the random solution $\tilde{sq}_3$ in (55); α is a beta distribution random variable and $-2 \le t, x \le 2$.

Case 3. At $\lambda^2 - 4\nu > 0, \nu = 0, \lambda \neq 0$

$$\tilde{sq}_{5,6}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu - \lambda^2}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ \tan\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right), \tag{56}$$

where α is a beta random variable.

The mean value $\mathbb{E}[\tilde{sq}_5]$ and the second moment $\mathbb{E}[\tilde{sq}_5^2]$ are depicted in Figure 8a,b, respectively.

And

$$\tilde{sq}_{7,8}(\xi) = \pm\sqrt{\frac{-c}{2b}}\left(1 + \frac{4\nu}{\sqrt{4\nu - \lambda^2}\, e^{i\left(-k\frac{x^\alpha}{\alpha} + w\frac{t^\alpha}{\alpha} + \theta\right)}\ \tan\left(\frac{\sqrt{4\nu-\lambda^2}}{2}\left(\frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha} + K\right)\right) - \lambda}\right). \tag{57}$$

where α is a beta random variable.

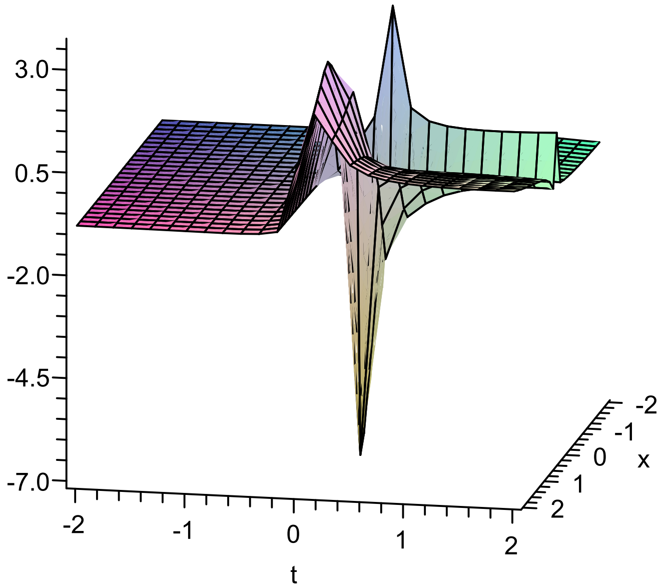
The mean value $\mathbb{E}[\tilde{sq}_7]$ and the second moment $\mathbb{E}\left[\tilde{sq}_7^2\right]$ are depicted in Figure 9a,b, respectively.

$$\tilde{sq}_{9,10}(\xi) = \pm\sqrt{\frac{-c}{2b}}\, e^{i\left(-k\frac{x^\alpha}{\alpha}+w\frac{t^\alpha}{\alpha}+\theta\right)}\left(1+\frac{2\lambda}{\exp\left(\lambda\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}+K\right)\right)-1}\right), \tag{58}$$

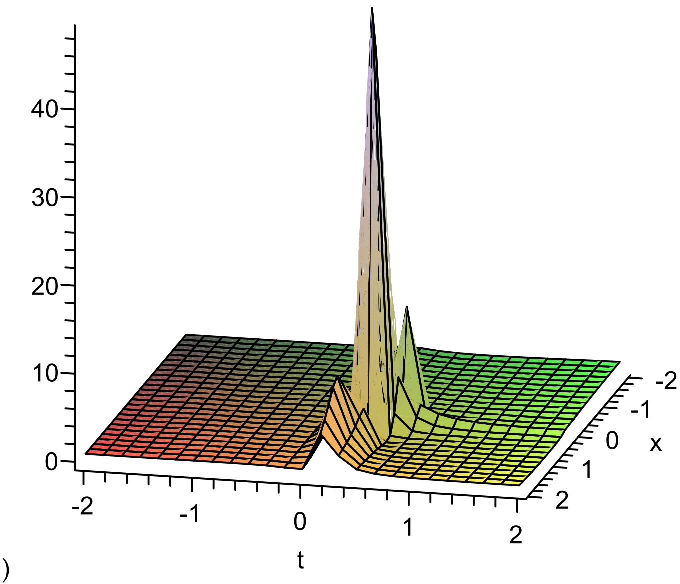
where α is a beta random variable. The mean value $\mathbb{E}[\tilde{sq}_9]$ and the second moment $\mathbb{E}\left[\tilde{sq}_9^2\right]$ are depicted in Figure 10a,b, respectively.

5.3. The Influence of Randomness

Here, we show the effect of the beta random variable on the behavior of solutions. We present a number of graphs for various values of beta distribution parameters. As illustrated in Figures 1, 5 and 7, some are more dispersive than the others. Additionally, Figures 8 and 9 show a near stable solution, whereas in Figures 4, 6 and 10, the surface becomes more planar after minor adjustments.

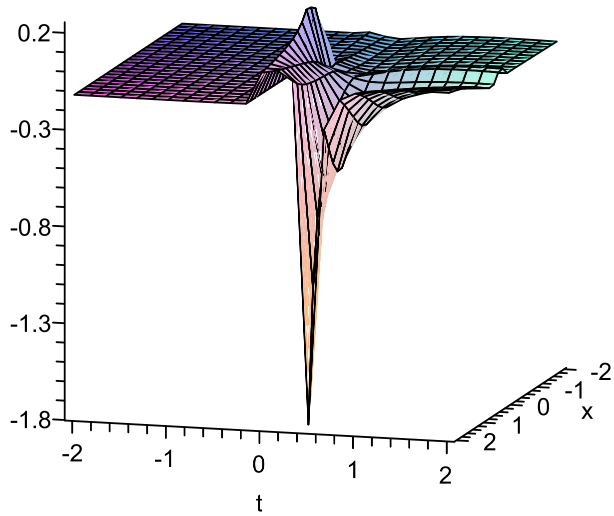


(**a**)

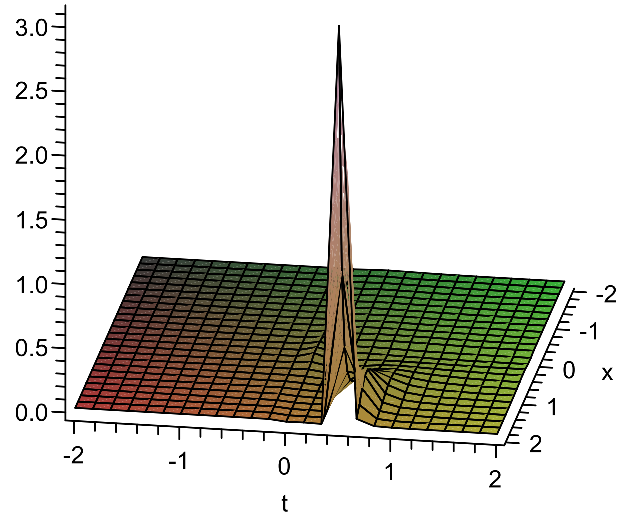


(**b**)

Figure 8. The first and second moments for the random solution $\tilde{sq}_5$ in (56); α is a beta random variable and $-2 \leq t, x \leq 2$.

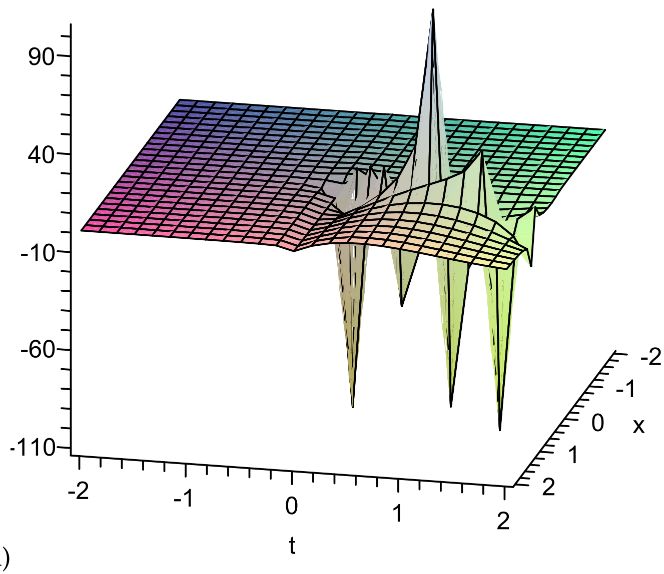


(**a**)



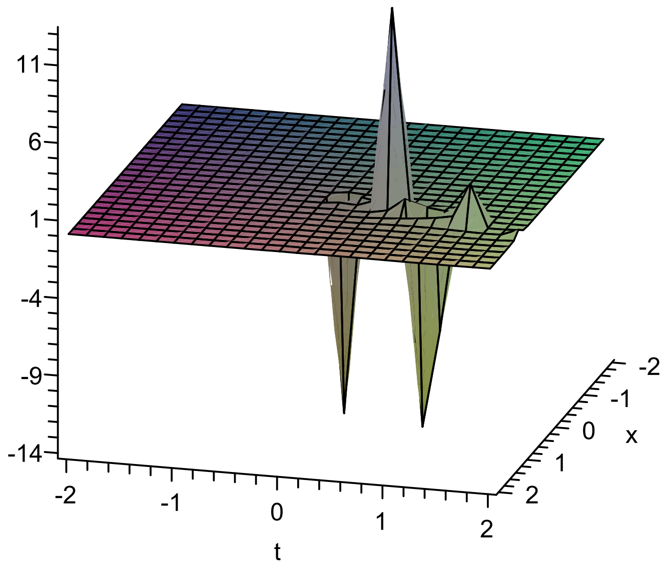
(**b**)

Figure 9. The first and second moments for the random solution $\tilde{sq}_7$ in (57); α is a beta distribution random variable and $-2 \leq t, x \leq 2$.



(**a**)

Figure 10. *Cont.*



(**b**)

Figure 10. The first and second moments for the random solution $\widetilde{sq}_9$ in (58); α is a beta distribution random variable and $-2 \leq t, x \leq 2$.

6. Conclusions

We have successfully applied the unified solver and the $\exp(-\varphi(\xi))$-expansion approaches to extract some solitary waves through deterministic and beta distribution cases for the (stochastic) fractional Biswas–Milovic (FBM) equation in the sense of conformable fractional derivatives. Specifically, some new random solutions for the FBM equation have successfully been obtained. These solutions may have important significance for the explanation of some practical physical phenomena. The graphs of some solutions are illustrated for suitable coefficients. Additionally, the first and the second moments are obtained with graphical representation for the beta distribution case. This method can be applied for other stochastic NFDEs with complex valued solutions appearing in applied sciences.

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