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Common Fixed Point for Meir–Keeler Type Contraction in Bipolar Metric Space

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Abstract: In mathematical analysis, the Hausdorff derivatives or the fractal derivatives play an important role. Fixed-point theorems and metric fixed-point theory have varied applications in establishing a unique common solution to differential equations and integral equations. In the present work, some fixed-point theorems using the extension of Meir–Keeler contraction in the setting of bipolar metric spaces have been proved. The derived results have been supplemented with non-trivial examples. Our results extend and generalise the results established in the past. We have provided an application to find an analytical solution to an Integral Equation to supplement the derived result.

Keywords: fixed points; bipolar metric space; covariant map; compatible maps; Cauchy bisequence

MSC: 47H10; 54H25

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1. Introduction

Researchers in mathematics and different branches of science and technology studied the Banach Fixed Point Theorem [1] and continued their research to find out if the theorem was applicable to the real world. The Banach Fixed Point Theorem is still popular among computer scientists, physicists, applied mathematicians, as well as people with medical expertise in the 21st century for attempting to apply the theorem to real-life issues. Metric spaces play a significant role in Real Analysis and Functional Analysis due to their most general space that possibly allows one to rethink real-life applications. It is always interesting as well as challenging for mathematicians to understand and apply the concept of topological properties to normed linear spaces as well as metric space in various fields. In the sequel of various generalisations, Meir–Keeler [2] established fixed-point results using weakly uniformly strict contraction in the setting of complete metric spaces.

In addition, metric fixed point theory has a wide range of applications—in dynamic programming, variational inequalities, fractal dynamics, dynamical systems of mathematics, as well as the deployment of satellites in their appropriate orbits in space science, to name a few. It also ensures that patients receive the most appropriate diagnosis, and it examines the intensity of spread of contagious diseases in a variety of cities.

In mathematics, new discoveries of space and their properties are always of interest to researchers. As a result, Gahler [3] introduced the idea of 2-metric spaces in his series of papers, giving us the notion of new dimensions for ordinary metric spaces. The metric adopted here is non-negative real (i.e., $[0,+\infty)$), which has a wide range of applications in this study.

The notion of probabilistic metric spaces, in which the probabilistic distance between two points is examined, has provided a new dimension to the subject and interest in learning more about stars in the cosmos. Similarly, Grabiec [4] and Michalek [5] investigated fuzzy metric spaces, taking into account the degree of agreement and disagreement. It is evident that most of the work was based on real numbers, be it 2-metric, fuzzy metric, modular metric, etc.

Let X be a nonempty set and let $d : X \times X \to \mathbb{R}, \|.\| : X \to \mathbb{R}, d : X \times X \times X \to \mathbb{R}$ and $M(x, y, t) : X \times X \times [0, 1] \to [0, 1]$. "What happens if we replace $\mathbb{R}$ with some other sets that are not totally ordered sets like $\mathbb{R}$?" was a reasonable question. The response of the researchers resulted in various types of metrics, such as the cone metric, the partially ordered metric, the modular metric, and more recently the complex-valued metrics proposed by Huang and Zhang [6], Matthew [7], Azam et al. [8], and Murthy et al. [9]. For more details on the topic, see [10–16]) and the references therein.

The goal of this study is to prove some fixed-point theorems in bipolar metric spaces. In our theorems, the contraction condition is the extension of Meir–Keeler [2] in bipolar metric spaces.

The rest of the paper is organised as follows. In Section 2, we provide some definitions related to bipolar metric spaces, which are used in our main results. In Section 3, we present our main result by establishing fixed-point results using an extension of Meir–Keeler type contraction in the setting of bipolar metric spaces and supplement the derived results with suitable examples. In Section 4, we present an application to find the analytical solution to the integral equation to supplement the derived result.

2. Bipolar Metric Spaces

The following are required in the sequel.

Definition 1 ([10]). *Let A and B be two non-empty sets and $\rho : A \times B \to [0, +\infty)$ be a function. The triplet (A, B, ρ) is called bipolar metric space and ρ is called bipolar metric on (A, B) if the following conditions hold:*

(BP_1) $\rho(a, b) = 0$ *if and only if* $a = b$ *where* $(a, b) \in A \times B$,

(BP_2) *If* $a, b \in A \cap B$ *then* $\rho(a, b) = \rho(b, a)$,

(BP_3) $\rho(a_1, b_2) \leq \rho(a_1, b_1) + \rho(a_2, b_1) + \rho(a_2, b_2)$ *for all* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$.

Definition 2 ([10]). *Let (A, B, ρ) be a bipolar metric space. Elements of A, B and $A \cap B$ are called left, right and central points, respectively. A sequence in A and a sequence in B are called left and right sequences, respectively. By a sequence, we mean either a left or right sequence.*

A sequence $\langle t_n \rangle$ is said to be convergent to a point t if and only if $\langle t_n \rangle$ is a left sequence, t is a right point and $\lim_{n \to +\infty} \rho(t_n, t) = 0$; or $\langle t_n \rangle$ is a right sequence, t is a left point and $\lim_{n \to +\infty} \rho(t, t_n) = 0$.

A sequence $\langle (a_n, b_n) \rangle$ in $A \times B$ is called a bisequence on (A, B). This sequence is simply denoted by (a_n, b_n). If both the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ converge, then the bisequence (a_n, b_n) is said to be convergent. If both the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to a same point $v \in A \cap B$ then (a_n, b_n) is called biconvergent.

If $\lim_{n,m \to +\infty} d(a_n, b_m) = 0$ then the bisequence (a_n, b_n) is called a Cauchy bisequence. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

A bipolar metric space is complete if every Cauchy bisequence is convergent, hence biconvergent.

Definition 3 ([10]). *Let A_1, B_1, A_2 and B_2 be four sets. A function $f : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map if $f(A_1) \subseteq A_2$ and $f(B_1) \subseteq B_2$ and is denoted as $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$. In particular, if (A_1, B_1, ρ_1) and (A_2, B_2, ρ_2) are two bipolar metric spaces, then we use the notaion $f : (A_1, B_1, \rho_1) \rightrightarrows (A_2, B_2, \rho_2)$ for covariant map f.*

Definition 4 ([10]). *Let (A_1, B_1, ρ_1) and (A_2, B_2, ρ_2) be two bipolar metric spaces. A map $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$ is said to be continuous at a point $a_0 \in A_1$, if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $b \in B_1$ and $\rho_1(a_0, b) < \delta$ implies that $\rho_2(f(a_0), f(b)) < \varepsilon$. It is continuous at a point $b_0 \in B_1$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $a \in A_1$ and $\rho_1(a, b_0) < \delta$ implies that $\rho_2(f(a), f(b_0)) < \varepsilon$. If f is continuous at each point $a \in A_1 \cup B_1$, then it is called continuous.*

This definition implies that a covariant map $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$ is continuous if and only if $\{t_n\}$ converges to t on (A_1, B_1, ρ_1) implies $\{f(t_n)\}$ converges to $f(t)$ on (A_2, B_2, ρ_2).

Definition 5 ([15]). *Let (A, B, ρ) be a bipolar metric space and let $S, T : (A, B) \rightrightarrows (A, B)$ be two covariant maps; then the pair (S, T) is said to be compatible if and only if $d(TSa_n, STb_n) \to 0$ and $d(STa_n, TSb_n) \to 0$, whenever (a_n, b_n) is a sequence in $A \times B$ such that $\lim_{n\to+\infty} Sa_n = \lim_{n\to+\infty} Ta_n = \lim_{n\to+\infty} Sb_n = \lim_{n\to+\infty} Tb_n = \xi$ for some $\xi \in A \cap B$.*

Definition 6 ([15]). *If S and T commute at all their coincidence points, then S and T are called weakly compatible.*

Definition 7. *Let (A, B, ρ) be a bipolar metric space and let $F, S, G, T : (A, B) \rightrightarrows (A, B)$ be four covariant maps then the quadruple (F, S, G, T) is said to be compatible if and only if $\rho(TFa_n, FTb_n)$ and $\rho(GSa_n, SGb_n)$ converge to 0, whenever (a_n, b_n) is a sequence in $A \times B$ such that*

$$\lim_{n\to+\infty} Fa_n = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Gb_n = \lim_{n\to\infty} Tb_n = \xi$$

for some $\xi \in A \cap B$.

3. Main Results

We begin this section with some propositions as follows.

Proposition 1. *Let (A, B, ρ) be a bipolar metric space, and let $f, g, S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be four covariant maps satisfying the following conditions:*

For any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \le \rho(Sx, Ty) < \epsilon + \delta \text{ implies } \rho(fx, gy) < \epsilon \tag{1}$$

$$\text{and } Sx = Ty \text{ implies } fx = gy \tag{2}$$

then

$$\rho(fx, gy) < \rho(Sx, Ty) \text{ if } Sx \ne Ty \text{ and} \tag{3}$$

$$\rho(fx, gy) \le \rho(Sx, Ty) \text{ for all } x \in A, y \in B. \tag{4}$$

Proof. Let $Sx \ne Ty$ then $d(Sx, Ty) = \epsilon$ for some $\epsilon > 0$ and from condition (1) we have $d(fx, gy) < \epsilon$ and so (3) holds. From (2) and (3) we get (4). □

Proposition 2. *Let (A, B, ρ) be a bipolar metric space, and let $S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be two covariant maps satisfying the condition*

$$\rho(Tx, Ty) \le \rho(Sx, Sy) \text{ for all } x \in A, y \in B. \tag{5}$$

If S is a continuous function, then T is also a continuous function.

Proof. Let a left sequence $\{a_n\}$ converge to a right point $b \in B$, then $d(Sa_n, Sb)$ tends to zero as S is continuous and so by (5) $d(Ta_n, Tb)$ tends to zero, that is, $\{Ta_n\}$ converges to Tb. Similarly, we can show that if the right sequence $\{b_n\}$ converges to the left point $a \in X$, then $\{Tb_n\}$ converges to Ta. So, T is also continuous. □

Proposition 3. *Let (A, B, ρ) be a bipolar metric space and let $S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be two covariant maps that are compatible. If ξ is a coincidence point of S and T (i.e., $T\xi = S\xi$) then $TS\xi = ST\xi$. That is, a compatible map is weakly compatible.*

Proof. This can be easily proved by taking $a_n = b_n = \xi$ in the Definition 5. □

Proposition 4. *Let (A, B, ρ) be a bipolar metric space and let $f, g, S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be four covariant maps such that the quadruple (f, S, g, T) is compatible. If ξ is a coincidence point of all these four mappings, then $Tf\xi = fT\xi$ and $gS\xi = Sg\xi$.*

We now prove a lemma that will be used in proving our main theorems.

Lemma 1. *Let (A, B, ρ) be a complete bipolar metric space and (u_n, v_n) is a bisequence in $A \times B$ satisfying the condition: For any given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \rho(u_n, v_m) < \epsilon + \delta \text{ implies } \rho(u_{n+1}, v_{m+1}) < \epsilon \tag{6}$$

and

$$u_n = v_m \text{ implies } u_{n+1} = v_{m+1} \tag{7}$$

then the bisequence (u_n, v_n) is the Cauchy bisequence.

Proof. Let $\alpha_n = \rho(u_n, v_n)$ and $\beta_n = \rho(u_n, v_{n+1})$ then from given condition $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ both are monotonic non-increasing bounded below sequences. Hence,

$$\alpha_n \to \epsilon^+ \text{ where } \epsilon \geq 0. \tag{8}$$

If $\epsilon > 0$, then for this ϵ there exists $\delta > 0$ such that (6) holds.
Using (8), we can find $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\begin{aligned} \epsilon &\leq \alpha_n < \epsilon + \delta \\ \epsilon &\leq \rho(u_n, v_n) < \epsilon + \delta. \end{aligned}$$

This implies from (6) that

$$\begin{aligned} \rho(u_{n+1}, v_{n+1}) &< \epsilon \\ \alpha_{n+1} &< \epsilon. \end{aligned}$$

This contradicts (8). So, $\epsilon = 0$ and

$$\alpha_n \to 0^+ \text{ as } n \to +\infty. \tag{9}$$

Similarly

$$\beta_n \to 0^+ \text{ as } n \to +\infty. \tag{10}$$

Claim: (u_n, v_n) is Cauchy. Suppose not. So, there exists $\epsilon > 0$ such that

$$\limsup_{n,m \to +\infty} \rho(u_n, v_m) > 2\epsilon. \tag{11}$$

For this ϵ there exists $\delta > 0$ so that (6) holds.
Let $\delta' = min(\delta, \epsilon)$. So, we have

$$\epsilon \leq \rho(u_n, v_m) < \epsilon + \delta' \text{ implies } \rho(u_{n+1}, v_{m+1}) < \epsilon. \tag{12}$$

Using (9), (10) and (11), we can find integers m, n, M such that,

$$m, n > M, \alpha_M = \rho(u_M, v_M) < \frac{\delta'}{6} \text{ and } \beta_M = \rho(u_M, v_{M+1}) < \frac{\delta'}{6} \tag{13}$$

$$\rho(u_m, v_n) > 2\epsilon \geq \epsilon + \delta'. \tag{14}$$

Now, we consider two cases.
If $n > m$, then for $j \in [m, n] \cap \mathbb{N}$ we have by (B3)

$$\begin{aligned} \rho(u_m, v_j) &\leq \rho(u_m, v_{j+1}) + \rho(u_j, v_{j+1}) + \rho(u_j, v_j) \\ \rho(u_m, v_j) - \rho(u_m, v_{j+1}) &\leq \rho(u_j, v_{j+1}) + \rho(u_j, v_j) = \beta_j + \alpha_j. \end{aligned}$$

Using (13) and (14) above, the inequality implies

$$\rho(u_m, v_j) - \rho(u_m, v_{j+1}) < \frac{\delta'}{3}.$$

Similarly, we can prove that

$$\rho(u_m, v_{j+1}) - \rho(u_m, v_j) < \frac{\delta'}{3}.$$

So that we obtain

$$\left|\rho(u_m, v_j) - \rho(u_m, v_{j+1})\right| < \frac{\delta'}{3}. \tag{15}$$

This implies, since $\rho(u_m, v_m) < \epsilon$ and $\rho(u_m, v_n) > \epsilon + \delta'$, that there exists $j \in [m, n] \cap \mathbb{N}$ such that

$$\epsilon + \frac{2\delta'}{3} \leq \rho(u_m, v_j) < \epsilon + \delta'. \tag{16}$$

This implies (12) that

$$\rho(u_{m+1}, v_{j+1}) < \epsilon.$$

Now

$$\begin{aligned} \rho(u_m, v_j) &\leq \rho(u_m, v_{m+1}) + \rho(u_{m+1}, v_{m+1}) + \rho(u_{m+1}, v_{j+1}) + \rho(u_j, v_{j+1}) \\ &+ \rho(u_j, v_j) \\ &< \frac{\delta'}{6} + \frac{\delta'}{6} + \epsilon + \frac{\delta'}{6} + \frac{\delta'}{6} = \epsilon + \frac{2\delta'}{3}. \end{aligned}$$

This contradicts (16). Similarly, we get the contradiction if $n \leq m$. Therefore, (u_n, v_n) is Cauchy. □

Our first main result is as follows:

Theorem 1. *Let (A, B, ρ) be a complete bipolar metric space and let $S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be two covariant maps satisfying the following conditions*

1. *S and T are compatible mappings.*
2. *S is continuous.*
3. *$T(A \cup B) \subseteq S(A \cup B)$.*
4. *For any given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \rho(Sa, Sb) < \epsilon + \delta \text{ implies } \rho(Ta, Tb) < \epsilon \tag{17}$$

$$\text{and } Sa = Sb \text{ implies } Ta = Tb \tag{18}$$

where $a \in A$ and $b \in B$. Then the functions S and T have a unique common fixed point.

Proof. Let $a_0 \in A$, $b_0 \in B$ and choose $a_1 \in A$ and $b_1 \in B$ such that $Ta_0 = Sa_1 = u_1$ and $Tb_0 = Sb_1 = v_1$. This can be done since $T(A \cup B) \subseteq S(A \cup B)$. In general, we can choose $(a_n, b_n) \in A \times B$ such that $Ta_{n-1} = Sa_n = u_n$ and $Tb_{n-1} = Sb_n = v_n$ for all $n \in \mathbb{N}$.

If $Sa_n = u_n = v_m = Sb_m$ for some $n, m \in \mathbb{N}$ then this implies by a given condition that $u_{n+1} = Ta_n = Ta_m = v_{m+1}$ and if $\epsilon \leq \rho(u_n, v_m) = \rho(Sa_n, Sb_m) < \epsilon + \delta$ then this implies by condition (17) that $\rho(u_{n+1}, v_{m+1}) < \epsilon$. So, by Lemma 1 (α_n, β_n) is a Cauchy bisequence, and as (A, B, ρ) is complete, (u_n, v_n) converges and, thus, biconverges to a point $\xi \in A \cap B$. Hence

$$\lim_{n\to+\infty} Sa_n = \lim_{n\to+\infty} Ta_n = \lim_{n\to+\infty} Sb_n = \lim_{n\to+\infty} Tb_n = \xi.$$

Since S and T are compatible, hence

$$\rho(TSa_n, STb_n) \to 0 \ \text{ and } \ \rho(STa_n, TSb_n) \to 0.$$

Now, by Proposition 2, both the functions S and T are continuous, so we have

$$\begin{aligned} Ta_n \to \xi \quad &\text{implies} \quad STa_n \to S\xi \ \text{ and} \\ Sb_n \to \xi \quad &\text{implies} \quad TSb_n \to T\xi. \end{aligned}$$

By the compatibility of S and T, we have

$$\begin{aligned} &\rho(S\xi, T\xi) = \lim_{n\to\infty} \rho(STa_n, TSb_n) = 0 \\ \text{and this implies} \quad &S\xi = T\xi \\ \text{this implies} \quad &TS\xi = ST\xi. \end{aligned}$$

Let $S\xi = T\xi = u$, then we will show that u is a common fixed point of S and T.
Let $Su \neq u$ then

$$\begin{aligned} \rho(Tu, u) &= \rho(TS\xi, T\xi) < \rho(SS\xi, S\xi) \\ &= \rho(Su, u) = \rho(ST\xi, S\xi) \\ &= \rho(TS\xi, S\xi) = \rho(Tu, u) \end{aligned}$$

which is a contradiction. So $Su = u$

$$\text{i.e.,} \ \ SS\xi = ST\xi = S\xi = T\xi = TS\xi \ \text{ implies } \ Tu = u.$$

So u is a common fixed point of S and T.

Uniqueness: Let us assume that u and v be two distinct common fixed points of S and T. If $Su \neq Sv$, then

$$\begin{aligned} \rho(Tu, Tv) &< \rho(Su, Sv) \\ \Rightarrow \quad \rho(u, v) &< \rho(u, v) \end{aligned}$$

which is a contradiction. So $Su = Sv$, and this implies $u = v$, and the proof is complete. □

In the above theorem, if we take S as an identity mapping, then we get the following corollary.

Corollary 1. *Let (A, B, ρ) be a complete bipolar metric space and let $T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be a covariant map that satisfies the following condition.*

For any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \rho(a, b) < \epsilon + \delta \ \text{ implies } \ \rho(Ta, Tb) < \epsilon$$

then the function T has a unique fixed point.

In the above corollary, if we take $A = B$, then we get the main result of Meir and Keeler [2].

In our next result, we do not require the continuity of S and we have used weakly compatible maps in place of compatible maps.

Theorem 2. *Let (A, B, ρ) be a bipolar metric space, and let $S, T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ be two covariant maps satisfying the following conditions*

1. *S and T are weakly compatible maps.*
2. *$S(A \cup B)$ is complete*
3. *S is injective*
4. *$T(A \cup B) \subseteq S(A \cup B)$.*
5. *For any given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \rho(Sa, Sb) < \epsilon + \delta \text{ implies } \rho(Ta, Tb) < \epsilon \tag{19}$$

$$\text{and } Sa = Sb \text{ implies } Ta = Tb \tag{20}$$

where $a \in A$ and $b \in B$. Then the functions S and T have a unique common fixed point.

Proof. Let the bisequence (u_n, v_n) be as in the proof of Theorem 1, then by the same theorem the bisequence (u_n, v_n) is a Cauchy bisequence and hence biconverges to a point $t \in S(A) \cap S(B) = S(A \cap B)$. Hence $\xi = Su$ for some $u \in A \cap B$. So,

$$\lim_{n\to+\infty} Sa_n = \lim_{n\to+\infty} Ta_n = \lim_{n\to+\infty} Sb_n = \lim_{n\to+\infty} Tb_n = \xi = Su. \tag{21}$$

Now, by using Proposition 1, we have the following.

$$\lim_{n\to+\infty} \rho(Ta_n, Tu) \leq \lim_{n\to+\infty} \rho(Sa_n, Su) = 0.$$

So,

$$\lim_{n\to+\infty} Ta_n = Tu. \tag{22}$$

By (21) and (22), we have

$$Su = Tu = \xi \tag{23}$$

$$\text{implies} \quad STu = TSu \ \text{ (by weakly compatibility of S and T).} \tag{24}$$

Again from (23), we have $STu = S\xi$ and $TSu = T\xi$. So $S\xi = T\xi$. Thus, u and ξ are two coincidence points of S and T. We will prove that $\xi = u$. Suppose not. Then $S\xi \neq Su$ and we get

$$\rho(T\xi, Tu) < \rho(S\xi, Su)$$

$$\rho(T\xi, Tu) < \rho(T\xi, Tu).$$

This is a contradiction. So $\xi = u$ and hence $Su = Tu = u$.

So, u is a common fixed point of S and T. Uniqueness can be proved as in Theorem 1. □

The following example supplements the derived results of Theorem 2.

Example 1. *Let X be the class of singleton subsets of $\mathbb{R}^2$ and Y be the class of nonempty bounded subsets of the metric space $(\mathbb{R}^2, d)$ where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. We define a function $\rho : X \times Y \to [0, \infty)$ by $\rho(\{x\}, A) = sup\{d(x, y) : y \in A\}$. We will show that (X, Y, ρ) is a bipolar metric space.*

(B1) It is clear that $\rho(\{x\}, \{x\}) = 0$, for every $\{x\} \in X = X \cap Y$. Let $\rho(\{x\}, A) = 0$, then $sup\{d(x, y) : y \in A\} = 0$. This implies $A = \{x\}$.

(B2) $\rho(\{x\}, \{y\}) = \rho(\{y\}, \{x\})$ for all $\{x\}, \{y\} \in X \cap Y$.

(B3) *Let* $x = (x_1, x_2), w = (w_1, w_2) \in \mathbb{R}^2$ *and* $A, B \in Y$, *then* $\rho(\{x\}, A) = sup\{d(x,y) : y \in A\} \leq sup\{d(x,z) + d(z,w) + d(w,y) : y \in A, z \in B\} \leq sup\{d(x,z) : z \in B\} + sup\{d(w,z) : z \in B\} + sup\{d(w,y) : y \in A\} = \rho(\{x\}, B) + \rho(\{w\}, B) + \rho(\{w\}, A)$. *Therefore,* (X, Y, ρ) *is a bipolar metric space.*

Let $S, T : X \cap Y \rightrightarrows X \cup Y$ *be two covariant maps defined by*

$$T\{(x_1, x_2)\} = \{(\frac{x_1}{4}, \frac{x_2}{4})\}$$
$$T(A) = \left\{(\frac{x_1}{4}, \frac{x_2}{4}) : (x_1, x_2) \in A\right\}$$
$$S\{(x_1, x_2)\} = \{(\frac{x_1}{2}, \frac{x_2}{2})\}$$
$$S(A) = \left\{(\frac{x_1}{2}, \frac{x_2}{2}) : (x_1, x_2) \in A\right\}$$

for every $\{(x_1, x_2)\} \in X$ *and* $A \in Y$. *Here we observe the following:*

- S *and* T *are weakly compatible maps for let* $Tu = Tu$ *for some* $u \in X \cup Y$, *then* $u = (0,0)$, *so* $STu = TSu$.
- $S(X \cup Y) = X \cup Y$ *is complete.*
- S *is injective.*
- $T(X \cup Y) \subset S(X \cup Y) = X \cup Y$.
- *for any given* $\epsilon > 0$, *if we choose* δ *with* $0 < \delta < 3\epsilon$ *and* $(\{x\}, A) \in X \cup Y$ *with* $\epsilon \leq \rho(S(\{x\}), S(A)) < \epsilon + \delta$ *then this implies that* $\frac{1}{2}\rho(\{x\}, A) < \epsilon + \delta$. *So we get* $\rho(T(\{x\}), T(A)) = \frac{1}{4}\rho(\{x\}, A) < \frac{1}{4}(\epsilon + \delta) < \epsilon$.
- *If* $S(\{x\}) = S(A)$ *then* $\{x\} = A$ *as* S *is injective, so* $T(\{x\}) = T(A)$.

Therefore, all the conditions of Theorem 2 are satisfied, so S *and* T *have a unique common fixed point.*

Remark 1. *In the above example, one can easily see that* (X, Y, ρ) *cannot be a metric space as* $X \neq Y$ *and the triangle inequality is meaningless.*

For a common fixed point of four mappings, we have the following theorem:

Theorem 3. *Let* (A, B, ρ) *be a complete bipolar metric space and let* $S, T, f, g : (A, B, \rho) \rightrightarrows (A, B, \rho)$ *be four covariant maps satisfying the following conditions*

1. *The quadruple (f, S, g, T) is compatible.*
2. *All four mappings are continuous.*
3. $f(A \cup B) \subseteq S(A \cup B)$ *and* $g(A \cup B) \subseteq T(A \cup B)$
4. *For any given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\epsilon \leq \rho(Sa, Tb) < \epsilon + \delta \text{ implies } \rho(fa, gb) < \epsilon \tag{25}$$
$$\text{and } Sa = Tb \text{ implies } fa = gb, \tag{26}$$

where $a \in A$ *and* $b \in B$. *Then the functions* S, T, f *and* g *have a unique common fixed point.*

Proof. Let $a_0 \in A$, $b_0 \in B$ and choose $a_1 \in A$ and $b_1 \in B$ such that $fa_0 = Sa_1 = u_0$ and $gb_0 = Tb_1 = v_0$. This can be done since $f(X \cup B) \subseteq S(A \cup B)$ and $g(A \cup B) \subseteq T(A \cup B)$. In general, we can choose $(a_n, b_n) \in A \times B$ such that $fa_n = Sa_{n+1} = u_n$ and $gb_n = Tb_{n+1} = v_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $\epsilon > 0$ and $\epsilon \leq \rho(u_n, v_m) = \rho(Sa_{n+1}, Tb_{m+1}) \leq \epsilon + \delta$. Then by condition 4 of the theorem we have $\rho(u_{n+1}, v_{m+1}) = \rho(fa_{n+1}, gb_{m+1}) < \epsilon$ and if $\rho(u_n, v_m) = \rho(Sx_{n+1}, Ty_{m+1}) = 0$, then again by condition 4 of the theorem we have $\rho(u_{n+1}, v_{m+1}) = \rho(fx_{n+1}, gy_{m+1}) = 0$.

So, by Lemma 1, the sequence (u_n, v_n) is a Cauchy bisequence. Since (A, B, ρ) is complete, the sequence (u_n, v_n) biconverges to some point $\xi \in A \cap B$. So fa_n, Sa_n, gb_n and Tb_n converge to ξ.

Since the quadruple (f, S, g, T) is compatible, we have $\rho(Tfa_n, fTb_n) \to 0$ and $\rho(gSa_n, Sgb_n) \to 0$. As all four functions f, g, S and T are continuous, this implies $\rho(T\xi, f\xi) = 0$ and $\rho(g\xi, S\xi) = 0$. So $T\xi = f\xi$ and $g\xi = S\xi$. Let $S\xi \neq T\xi$ then $\rho(f\xi, g\xi) < \rho(S\xi, T\xi) = \rho(g\xi, f\xi) = \rho(f\xi, g\xi)$. This is a contradiction. So,

$$T\xi = S\xi = B\xi = A\xi = u \ \ \text{(say)}.$$

By compatibility, this implies $Tf\xi = fT\xi$ and $gS\xi = Sg\xi$ that is, $Tu = fu$ and $gu = Su$. If $Su \neq Tu$ then $\rho(fu, gu) < \rho(Su, Tu) = \rho(gu, fu) = \rho(fu, gu)$. This is a contradiction. So,

$$Su = Tu = fu = gu.$$

Now let $Su \neq u$ that is $ST\xi \neq T\xi$ then $\rho(fu, u) = \rho(fT\xi, g\xi) < \rho(ST\xi, T\xi) = \rho(Su, u) = \rho(gu, u) = \rho(fu, u)$. This is a contradiction. So $Su = u = Tu = fu = gu$. Thus, u is a common fixed point of f, g, S and T.

For uniqueness, assume that u and v are two fixed points of f, g, S and T. If $Su \neq Tv$ such that with $u \neq v$. Then $\rho(fu, gv) < d(Su, Tv)$. This implies $\rho(u, v) < \rho(u, v)$, a contradiction. So $Su = Tv$, that is, $u = v$. □

Remark 2. *In the above theorem, taking $S = T$ and $f = g$, we get Theorem 1 as a corollary.*

Example 2. *Let $A = [0, \frac{1}{3}] \cup \{\frac{5n}{6} : n \in \mathbb{N}\}$ and $B = [0, \frac{1}{3}] \cup \{\frac{5}{18}(3n+1) : n \in \mathbb{N}\}$ and the distance function $\rho : A \times B \to [0, +\infty)$ is defined by $\rho(a, b) = |a - b|$ for all $a \in A$ and $b \in B$.*

Then (A, B, ρ) is a complete bipolar metric space.

Let us consider two covariant maps S and $T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ defined by $Ta = \frac{a}{2}$, $Sa = \frac{5a}{6}$ for all $a \in [0, \frac{1}{3}]$ and $T(\frac{5n}{6}) = \frac{5n}{18(n+1)}$, $S(\frac{5n}{6}) = \frac{5n}{6}$, $T(\frac{5}{18}(3n+1)) = 0$, $S(\frac{5}{18}(3n+1)) = \frac{5}{18}(3n+1)$ for all $n \in \mathbb{N}$. We can see that

$$T(A \cup B) = [0, \frac{1}{6}] \cup \{\frac{5n}{18(n+1)} : n \in \mathbb{N}\}$$

and

$$S(A \cup B) = [0, \frac{5}{18}] \cup \{\frac{5n}{6} : n \in \mathbb{N}\} \cup \{\frac{5}{18}(3n+1) : n \in \mathbb{N}\}.$$

So, $T(A \cup B) \subseteq S(A \cup B)$, S and T are continuous functions.

Now we are going to verify the compatibility of S and T.

For this, let (a_n, b_n) be a sequence in $A \times B$ such that $\lim_{n\to+\infty} Sa_n = \lim_{n\to+\infty} Ta_n = \lim_{n\to+\infty} Sb_n = \lim_{n\to+\infty} Tb_n = \xi$ for some $\xi \in A \cap B = [0, \frac{1}{3}]$.

Without loss of generality, we can assume that $a_n, b_n \in [0, \frac{1}{3}]$.

So, $Sa_n = \frac{5a_n}{6}$ and $Ta_n = \frac{a_n}{2}$. Both Sa_n and Ta_n converge to ξ, so $\xi = 0$.

Now $\lim_{n\to+\infty} \rho(TSa_n, STb_n) = \lim_{n\to+\infty} \rho(T\xi, S\xi) = 0$. Similarly, we have $\lim_{n\to+\infty} \rho(STa_n, STb_n) = 0$. Hence S and T are compatible.

Now we show that S and T satisfy condition 4 of Theorem 1.

For this, let $\epsilon > 0$ be given. Then the maximum value of δ is given by

$$\delta = \begin{cases} \frac{2\epsilon}{3}, & \text{if } \epsilon \in (0, \frac{1}{6}] \cup [\frac{5}{18}, \frac{1}{2}] \cup [\frac{5}{6}, \infty]; \\ \frac{5}{18} - \epsilon, & \text{if } \epsilon \in (\frac{1}{6}, \frac{5}{18}); \\ \frac{5}{6} - \epsilon, & \text{if } \epsilon \in (\frac{1}{2}, \frac{5}{6}). \end{cases}$$

Let us verify the above condition for $\epsilon \in (0, \frac{1}{6}]$. For this ϵ, we take $\delta = \frac{2\epsilon}{3}$. Let $\epsilon \leq d(Sx, Sy) < \epsilon + \delta$. This implies $\epsilon \leq \rho(Sa, Sb) < \frac{5\epsilon}{3}$. This is possible only if $x, y \in [0, \frac{1}{3}]$ so that $\frac{5}{6}|x - y| < \frac{5\epsilon}{3}$.

This gives $\frac{1}{2}|x-y| < \epsilon$ and hence $d(Tx, Ty) < \epsilon$. For other values of ϵ, one can verify in a similar way.

Therefore, all the conditions stipulated in Theorem 1 are satisfied and 0 is the unique common fixed point of S and T.

Example 3. *Let (A, B, ρ) be the bipolar metric space as in the above example. Let us consider four covariant maps f, g, S and $T : (A, B, \rho) \rightrightarrows (A, B, \rho)$ defined by $Ta = \frac{6a}{7}$, $Sa = \frac{5a}{6}$, $fa = \frac{a}{10{,}000}$ and $Ba = \frac{a}{20{,}000}$ for all $a \in [0, \frac{1}{3}]$ and $f(\frac{5n}{6}) = g(\frac{5n}{6}) = \frac{5n}{18(n+1)}$, $T(\frac{5n}{6}) = S(\frac{5n}{6}) = \frac{5n}{6}$, $f(\frac{5}{18}(3n+1)) = 0 = g(\frac{5}{18}(3n+1))$, $T(\frac{5}{18}(3n+1)) = S(\frac{5}{18}(3n+1)) = \frac{5}{18}(3n+1)$ for all $n \in \mathbb{N}$.*

Then we can easily verify that all the conditions given in Theorem 3 are satisfied. Hence, A, B, S and T have zero as a common fixed point.

4. Application

In this section, we study the existence and unique solution of an integral equation as an application of Corollary 1.

Theorem 4. *Let us consider the integral equation*

$$a(\sigma) = \mathfrak{h}(\sigma) + \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} \mathcal{G}(\sigma, \mu, a(\mu))d\mu, \ \sigma \in \mathfrak{Q}_1 \cup \mathfrak{Q}_2,$$

where $\mathfrak{Q}_1 \cup \mathfrak{Q}_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{G} : (\mathfrak{Q}_1^2 \cup \mathfrak{Q}_2^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in L^\infty(\mathfrak{Q}_1) \cup L^\infty(\mathfrak{Q}_2)$,

(T2) *there is a continuous function* $\theta : \mathfrak{Q}_1^2 \cup \mathfrak{Q}_2^2 \to [0, \infty)$ *such that*

$$|\mathcal{G}(\sigma, \mu, a(\mu)) - \mathcal{G}(\sigma, \mu, b(\mu))| \leq \frac{1}{2}|\theta(\sigma, \mu)|(|a(\mu) - b(\mu)|),$$

for $\sigma, \mu \in \mathfrak{Q}_1^2 \cup \mathfrak{Q}_2^2$,

(T3) $|| \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} \theta(\sigma, \mu)d\mu||_\infty \leq 1$ *i.e* $\sup_{\sigma \in \mathfrak{Q}_1 \cup \mathfrak{Q}_2} \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} |\theta(\sigma, \mu)|d\mu \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\mathfrak{Q}_1) \cup L^\infty(\mathfrak{Q}_2)$.

Proof. Let $A = L^\infty(\mathfrak{Q}_1)$ and $B = L^\infty(\mathfrak{Q}_2)$ be two normed linear spaces, where $\mathfrak{Q}_1, \mathfrak{Q}_2$ are Lebesgue measurable sets and $m(\mathfrak{Q}_1 \cup \mathfrak{Q}_2) < \infty$.

Consider $\rho : A \times B \to \mathbb{R}^+$ to be defined by $\rho(a, b) = ||a - b||_\infty$ for all $(a, b) \in A \times B$. Then (A, B, ρ) is a complete bipolar metric space. Define the covariant mapping $T : L^\infty(\mathfrak{Q}_1) \cup L^\infty(\mathfrak{Q}_2) \to L^\infty(\mathfrak{Q}_1) \cup L^\infty(\mathfrak{Q}_2)$ by

$$T(a(\sigma)) = \mathfrak{h}(\sigma) + \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} \mathcal{G}(\sigma, \mu, a(\mu))d\mu, \ \sigma \in \mathfrak{Q}_1 \cup \mathfrak{Q}_2.$$

For any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \rho(a, b) < \epsilon + \delta.$$

Now, we have

$$
\begin{aligned}
\rho(Ta(\sigma), Tb(\sigma)) &= ||Ta(\sigma) - Tb(\sigma)|| \\
&= \left|\mathfrak{h}(\sigma) + \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} \mathcal{G}(\sigma, \mu, a(\mu))d\mu - \left(\mathfrak{h}(\sigma) + \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} \mathcal{G}(\sigma, \mu, a(\mu))d\mu\right)\right| \\
&\leq \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} |\mathcal{G}(\sigma, \mu, a(\mu)) - \mathcal{G}(\sigma, \mu, b(\mu))|d\mu \\
&\leq \frac{1}{2}(||a(\mu) - b(\mu)|) \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} |\theta(\sigma, \mu)|d\mu \\
&\leq \frac{1}{2}(||a(\mu) - b(\mu)|) \sup_{\sigma \in \mathfrak{Q}_1 \cup \mathfrak{Q}_2} \int_{\mathfrak{Q}_1 \cup \mathfrak{Q}_2} |\theta(\sigma, \mu)|d\mu \\
&\leq \frac{1}{2}(||a(\mu) - b(\mu)|| \\
&= \frac{1}{2}\rho(a, b) \\
&< \frac{1}{2}(\epsilon + \delta) \\
&< \epsilon.
\end{aligned}
$$

Hence, all the conditions of a Corollary 1 are satisfied, and consequently, the integral equation has a unique solution. □

5. Conclusions

In this paper, we established some common fixed-point theorems by using the extension of Meir–Keeler type contraction in the setting of bipolar metric spaces. Our results have been validated using nontrivial examples. Our examples illustrate that a bipolar metric space need not be a metric space. We have supplemented the derived results to find an analytical solution to the integral equation in the setting of Bipolar metric space. It will be quite interesting to extend our results in the setting of bipolar p-metric space, neutrosophic metric spaces, orthogonal metric spaces, etc.

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