fractal and fractional

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Well-Posedness and Regularity Results for Fractional Wave Equations with Time-Dependent Coefficients

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Abstract: Fractional wave equations with time-dependent coefficients are natural generations of classical wave equations which can be used to characterize propagation of wave in inhomogeneous media with frequency-dependent power-law behavior. This paper discusses the well-posedness and regularity results of the weak solution for a fractional wave equation allowing that the coefficients may have low regularity. Our analysis relies on mollification arguments, Galerkin methods, and energy arguments.

Keywords: fractional wave equations; energy estimate; well-posedness; regularity

MSC: 35R11; 49K40; 35B65

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1. Introduction

Fractional calculus has become an important topic thanks to its effective characterization of the ubiquitous power-law phenomena as well as its widespread applications in many areas of science and engineering such as porous media, turbulence, bioscience, geoscience, viscoelastic material, and so on. The most important mathematical equations among such models are fractional partial differential equations, which can be more relevant for describing the underlying anomalous features, non-local interactions, manifesting in memory effects, sharp peaks, power law distributions, and self-similar structures. For such kinds of equations, there is a large and rapidly growing number of publications. See the monographs of Herrmann [1], Hilfer [2], Jin [3], Kilbas et al. [4], and Zhou [5], and the references therein.

In this paper we consider the following fractional wave equation in a bound domain $\Omega \subset \mathbb{R}^N (N > 2)$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_t^\alpha u(t,x) - \mathcal{A}u(t,x) = f(t,x), & (t,x) \in (0,T] \times \Omega, \\ u(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0,\ \partial_t u(0,x) = u_1, & x \in \Omega, \end{cases} \tag{1}$$

where ∂_t^α is a fractional derivative of order $\alpha \in (1,2)$, which will be defined in the following contexts, and

$$\mathcal{A}u(t,x) = \sum_{i,j=1}^{N} \partial_i\big(a_{i,j}(t,x)\partial_j u(t,x)\big) + \sum_{j=1}^{N} b_j(t,x)\partial_j u(t,x) + c(t,x)u(t,x),$$

$\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, ..., N$ and $b_j \in L^\infty((0,T) \times \Omega)$, $c \in L^\infty(0,T, L^{\frac{2q}{q-2}}(\Omega))$ with $q \in [2, \frac{2N}{N-2})$, $a_{i,j} \in W^{1,\infty}(0,T; L^\infty(\Omega))$, and $a_{i,j} = a_{j,i}$. What is more, we assume that $\mathcal{A}$ is uniformly elliptic, i.e., there exist positive constants μ, ν such that

$$\mu|\zeta|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(t,x)\zeta_i\zeta_j \leq \nu|\zeta|^2 \tag{2}$$

for a.a. $(t,x) \in [0,T] \times \Omega$, $\forall\, \zeta \in \mathbb{R}^N$.

Recall that the initial boundary value problem (1) would resolve itself into fractional diffusion equations when $\alpha \in (0,1)$. It has attracted a growing interest due to its widespread applications in sub-diffusive processes. The authors in [6] constructed fundamental solutions to the problem using Fox's H-functions and the Levi method, then the parametrix estimates were established. Zacher [7] studied the well-posedness of weak solutions of abstract evolutionary integro-differential equations based on the Galerkin method and energy estimates. Later, Kubica and Yamamoto [8] used the same method to obtain well-posedness of weak solutions of fractional diffusion equations with time-dependent coefficients. In [9], the authors considered the problem with Caputo derivative on $\mathbb{R}^N$ in L^q- framework and then the uniqueness, existence, and $L^q(L^p)$-estimates of solutions are obtained. In [10], the authors investigated the well-posedness for this problem with time independent elliptic operators but general non-homogenous boundary conditions by mean of an eigenfunction representation involving the Mittag-Leffter functions. For other results for fractional diffusion equations, we refer to [11–15] and the references therein.

Recently, problem (1) has been the focus of many studies due to its significant application in super-diffusive models of anomalous diffusion such as diffusion in heterogeneous media and viscoelastic problems such as the propagation of stress waves in viscoelastic solids. More specifically, significant development has been made in well-posedness as well as regularity results of the weak solution to fractional wave equations. For example, in [16], the authors used Laplace transform to define weak solutions and used the Strichartz estimate to derive its well-posedness. Later, Otárola and Salgado [17] also gave a definition of weak solutions similar to that of inter-order cases and established the well-posedness together with regularity estimates. In [18,19], the authors obtained results on the existence and regularity of local and global weak solutions of semi-linear cases. In [20] the authors used integrated cosine family to give the representation of solutions and then provided the existence and regularity results of mild solutions. For other results for fractional wave equations, we refer to [21] for existence and regularity, [22] for the subordination principle, [23] for the global existence of small data solutions, [24] for approximate controllability, [25] for asymptotic behavior, [26] for well-posedness and regularity, and the references therein.

In the literature mentioned on fractional wave equations, the main technique to construct solutions for deriving such existence and regularity results is based on Fourier series, cosine family, or resolvent operators, and solutions are expressed by the Mittag-Leffler functions. In fact, as it is well known, the smoothness of solutions is followed by the properties of Mittag-Leffler functions. The main novelties of the present paper lie in two aspects. Compared with the existing research on fractional wave equations, our analysis is rather general and relies on Galerkin methods and energy arguments, which can be applied to the general problem that Fourier expansive of solutions cannot be used and cannot be converted to ordinary differential equations. Very recently, Huang and Yamamoto [27] discussed the well-posedness of initial-boundary value problems for time fractional diffusion-wave equations with time-dependent coefficients using the Galerkin method. On the other hand, in contrast to the classical integer-order case, the main technical difficulty in the rigorous analysis of the well-posedness and regularity of fractional wave equations stems from establishing the energy estimates of the problem. This is mainly due to the fact that integration by parts formula for integer-order derivatives cannot be generalized directly to a fractional-order case and properties of composition and conjugation of the fractional Caputo derivative ∂_t^α ($\alpha \in (1,2)$) do not exist. Therefore, we found it more challenging in dealing with the well-posedness and regularity of fractional wave equations.

The paper is organized as follows. In Section 2 we recall some notations, definitions, and preliminary facts used throughout this work. In Section 3 we discuss approximation equations and show the existence of their solutions by means of mollification arguments and the Galerkin methods, which reduce the regularity of coefficients $a_{i,j}, b_j, c, f$. The energy estimates of approximation solutions are established in Section 4. Finally, we derive the well-posedness and regularity results of fractional wave equations using the weak compactness arguments.

2. Preliminaries

Here we recall some notations, definitions, and preliminary facts which are used throughout this paper.

Let X be a Banach space and $v : [0,\infty) \to X$. The left Riemann–Liouville fractional integral of order $\alpha > 0$ for the function v is defined as

$$ {}_0I_t^\alpha v(t) = (g_\alpha * v)(t), \ \ t > 0, $$

where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $*$ denotes the convolution.

Further, ${}^L\partial_t^\alpha v$ and $\partial_t^\alpha v$ represent the left Riemann–Liouville fractional derivative and Caputo fractional derivative of order $\alpha > 0$ for the function v, respectively, which are defined by

$$ {}^L\partial_t^\alpha v(t) = \frac{d^n}{dt^n}[{}_0I_t^{n-\alpha} v(t)] \ \text{ and } \ \partial_t^\alpha v(t) = {}^L\partial_t^\alpha \left[v(t) - \sum_{k=0}^{n-1} \frac{v^{(k)}(0)}{k!} t^k \right], \ \ t > 0, $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Here we denote by $AC^n([0,T],X)$ the space of functions v that $v \in C^{n-1}([0,T],X)$ and $v^{(n-1)} \in AC([0,T],X)$. In particular, $AC^1([0,T],X) = AC([0,T],X)$. It is worth mentioning that if $v \in AC^n([0,T],X)$, then the Caputo fractional derivative $\partial_t^\alpha v(t)$ exists almost everywhere on $[0,T]$, which is represented by

$$ \partial_t^\alpha v(t) = [g_{n-\alpha} * v^{(n)}](t) \ \text{ for } t \in [0,T]. $$

For more insight into the topic, see Kilbas et al. [4] and Zhou [5].

Lemma 1 ([4])**.** *If $v \in AC^2([0,T],X)$ and $\alpha \in (1,2]$, then ${}_0I_t^\alpha \partial_t^\alpha v(t) = v(t) - v(0) - v'(0)t$ and $\partial_t^\alpha {}_0I_t^\alpha v(t) = v(t)$.*

Lemma 2. *Let $\alpha \in (1,2)$. If $v \in AC^2([0,T],X)$, then we have*

$$ \partial_t^\alpha v(t) = \frac{d}{dt}\partial_t^{\alpha-1} v(t) - v'(0) g_{2-\alpha}(t) = \partial_t^{\alpha-1} v'(t) $$

for a.e. $t \in (0,T)$.

Proof. If $v \in AC^2([0,T],X)$, then $v'(t)$ exists for a.e. $t \in (0,T)$. From the definition of ∂_t^α we know that

$$\begin{aligned}\partial_t^\alpha v(t) =& \frac{d^2}{dt^2}\int_0^t g_{2-\alpha}(s)[v(t-s)-v(0)-v'(0)(t-s)]ds\\ =& \frac{d}{dt}\int_0^t g_{2-\alpha}(s)[v'(t-s)-v'(0)]ds\\ =& \frac{d}{dt}\int_0^t g_{2-\alpha}(t-s)[v'(s)-v'(0)]ds\\ =& \frac{d}{dt}\partial_t^{\alpha-1}v(t) - v'(0)g_{2-\alpha}(t).\end{aligned}$$

On the other hand, since $\alpha - 1 \in (0,1)$, we see that $\frac{d}{dt}\int_0^t g_{2-\alpha}(t-s)[v'(s)-v'(0)]ds = {}^L\partial_t^{\alpha-1}[v'(t)-v'(0)] = \partial_t^{\alpha-1}v'(t)$. Thus, the proof is complete. □

Before proceeding further, we state an important lemma, which is a direct consequence of an estimate borrowed from [7].

Lemma 3. *Let $T > 0$ and H be a real Hilbert space with a scalar product $(\cdot,\cdot)$. Assume $k \in L^1(0,T)$, $k' \in L^{1,loc}(0,T)$, $k \geq 0$, $k' \leq 0$. Then for any $v \in H^1(0,T,H)$, there holds*

$$\int_0^t \left(\frac{d}{ds}(k*v)(s), v(s)\right)ds \geq \frac{1}{2}(k*\|v\|^2)(t) + \frac{1}{2}\int_0^t k(s)\|v(s)\|^2 ds$$

for any $t \in [0,T]$.

Next, a very significant example is provided which will play a crucial role in the proof of energy estimates.

Example 1. *For $\alpha \in (1,2)$, we choose $k(t) = g_{2-\alpha}(t)$. Then for any $v \in H^2(0,T,H)$ and $t \in [0,T]$, there holds*

$$\int_0^t \left(\frac{d}{ds}[\partial_s^{\alpha-1}v(s)], v'(s)\right)ds \geq \frac{1}{2}(g_{2-\alpha}*\|v'\|^2)(t) + \int_0^t \frac{g_{2-\alpha}(s)}{2}\|v'(s)\|^2 ds.$$

The following property presents the lower bound of the uniformly elliptic operator if the function has enough regularity, which was proved by [28] (see also [8]).

Lemma 4. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of C^2 class and (2) holds. If $u \in H^3(\Omega)$ and $u|_{\partial\Omega} = 0$ and $\Delta u|_{\partial\Omega} = 0$, then*

$$\frac{\mu}{4}\|\nabla^2 u\|^2 - C\|\nabla u\|^2 \leq \sum_{i,j=1}^N \int_\Omega \partial_i(a_{i,j}(t,x)\partial_j u)\Delta u dx,$$

where C depends continuously on $\max_{i,j}\|\nabla a_{i,j}(t,x)\|_{L^\infty}$ and the C^2-norm of $\partial\Omega$, and $\nabla^2 u = \{u_{x_i x_j}\}_{i,j=1}^N$.

We consider the space

$${}_0H^2(0,T) = \{v \in H^2(0,T):\ v(0) = 0, v'(0) = 0\}.$$

Next, we introduce the definition of the weak solution of Equation (1).

Definition 1. *Let $T \in (0,\infty)$ and $f \in L^2(0,T,L^2(\Omega))$. For given functions u_0 and u_1, we say a function*

$$u \in L^2(0,T,H_0^1(\Omega)) \ \text{ with } \ {}_0I_t^{2-\alpha}(u-u_0-u_1t) \in {}_0H^2(0,T,H^{-1}(\Omega))$$

is a weak solution of Equation (1) provided

$$\frac{\partial^2}{\partial t^2}\int_\Omega {}_0I_t^{2-\alpha}(u(t,x)-u_0-u_1t)\omega(x)dx + \int_\Omega \sum_{i,j=1}^N a_{i,j}(t,x)\partial_j u(t,x)\cdot\partial_i\omega(x)dx$$
$$= \int_\Omega \sum_{j=1}^N b_j(t,x)\partial_j u(t,x)\omega(x)dx + \int_\Omega c(t,x)u(t,x)\omega(x)dx + \int_\Omega f(t)\omega(x)dx$$

for each $\omega \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$.

The vectors u_0 and u_1 can be regarded as initial data for $u(t)$ and $u'(t)$ at least in a weak sense, respectively. If, for example, $u \in AC^2([0,T],H^{-1}(\Omega))$, then the condition ${}_0I_t^{2-\alpha}(u-u_0-u_1t) \in {}_0H^2(0,T,H^{-1}(\Omega))$ implies $u(0)=u_0$ and $\partial_t u(0) = u_1$.

Remark 1. *In view of Definition 1, we know $u' \in C([0,T],H^{-1}(\Omega))$ for $\alpha > \frac{3}{2}$.*

Proof. Indeed, for $t_1,t_2 \in [0,T]$ with $t_1 < t_2$, it follows from Lemma 2 and Hölder's inequality that

$$\begin{aligned}
&\|u'(t_2)-u'(t_1)\|_{H^{-1}}\\
=&\left\| \int_0^{t_2} g_{\alpha-1}(t_2-s)\partial_s^\alpha u(s)ds - \int_0^{t_1} g_{\alpha-1}(t_1-s)\partial_s^\alpha u(s)ds \right\|_{H^{-1}}\\
\le& \int_{t_1}^{t_2} g_{\alpha-1}(t_2-s)\|\partial_s^\alpha u(s)\|_{H^{-1}}ds + \int_0^{t_1}[g_{\alpha-1}(t_1-s)-g_{\alpha-1}(t_2-s)]\|\partial_s^\alpha u(s)\|_{H^{-1}}ds\\
\le& \frac{(t_2-t_1)^{\alpha-\frac{3}{2}}}{(2\alpha-3)^{\frac{1}{2}}\Gamma(\alpha-1)}\|\partial_t^\alpha u\|_{L^2(0,T,H^{-1})}\\
&+ \|\partial_t^\alpha u\|_{L^2(0,T,H^{-1})}\left(\int_0^{t_1}[g_{\alpha-1}(t_1-s)-g_{\alpha-1}(t_2-s)]^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

In view of the inequality $\xi_1^\sigma - \xi_2^\sigma \le (\xi_1-\xi_2)^\sigma$ for $\xi_1,\xi_2 > 0$ and $0 \le \sigma \le 1$, we calculate the integral

$$\begin{aligned}
\int_0^{t_1}[g_{\alpha-1}(t_1-s)-g_{\alpha-1}(t_2-s)]^2ds \le& \frac{1}{\Gamma^2(\alpha-1)}\int_0^{t_1}(t_1-s)^{2(\alpha-2)}-(t_2-s)^{2(\alpha-2)}ds\\
\le& \frac{(t_2-t_1)^{2\alpha-3}}{(2\alpha-3)\Gamma^2(\alpha-1)}.
\end{aligned}$$

The second term is bounded by $\|\partial_t^\alpha u\|_{L^2(0,T,H^{-1})}\frac{(t_2-t_1)^{\alpha-\frac{3}{2}}}{(2\alpha-3)^{\frac{1}{2}}\Gamma(\alpha-1)}$. This ensures

$$\|u'(t_2)-u'(t_1)\|_{H^{-1}} \to 0 \ \text{ as } t_1 \to t_2.$$

The proof is complete. □

3. Approximation Solution

In this section we provide the Galerkin approximate scheme and derive the corresponding existence results. We will suppose initially that

$$
\begin{aligned}
&A \in (W^{1,\infty}(0,T;L^\infty(\Omega)))^{N\times N},\ b_j \in L^\infty((0,T)\times\Omega),\\
&c \in L^\infty(0,T,L^{\frac{2q}{q-2}}(\Omega)),\ \ f \in L^2(0,T,L^2(\Omega))
\end{aligned}
\tag{3}
$$

for $q \in [2, \frac{2N}{N-2})$, where $A(t,x) = \{a_{i,j}(t,x)\}_{i,j=1}^N$ and $\mathbf{b} = (b_1, b_2, ..., b_N)$.

Let ϱ_ε be the standard mollifier satisfying

$$
\varrho_\varepsilon \in C^\infty(\mathbb{R}),\ \ \mathrm{supp}\ \varrho_\varepsilon = \{t : |t| < \frac{\varepsilon}{T}\},\ \ \int_{\mathbb{R}} \varrho_\varepsilon(t)dt = 1.
$$

Then we introduce the mollification v_ε of the function $v \in L^{1,loc}(\mathbb{R})$ as

$$
v_\varepsilon(t) = (\varrho_\varepsilon * v)(t).
$$

We note first that $v_\varepsilon \in C^\infty(\mathbb{R})$ and if $v \in L^p(\mathbb{R})$ for $p \geq 1$, then $v_\varepsilon \to v$ in $L^p(\mathbb{R})$.

Moreover, we denote $a^n_{i,j}, b^n_j, c^n, f_{\frac{1}{n}}$ by the mollification of $a_{i,j},\ b_j, c, f$, which are defined by

$$
\begin{aligned}
&a^n_{i,j}(t,x) = \big(\varrho_{\frac{1}{n}} * a_{i,j}(\cdot,x)\big)(t), \quad b^n_j(t,x) = \big(\varrho_{\frac{1}{n}} * b_j(\cdot,x)\big)(t),\\
&c^n(t,x) = \big(\varrho_{\frac{1}{n}} * c(\cdot,x)\big)(t), \quad f_{\frac{1}{n}}(t) = \big(\varrho_{\frac{1}{n}} * f(\cdot,x)\big)(t),
\end{aligned}
$$

where $a_{i,j}$ is the continuation by even reflection to $(-T,T)$ and zero elsewhere, b_j and c are the continuation by zero for $t \notin (0,T)$, and f is the continuation by odd reflection to $(-T,T)$ and zero elsewhere. Then $\lim_{n\to\infty} a^n_{i,j}(t) = a_{i,j}$ in $L^2((0,T)\times\Omega)$ for $a_{i,j} \in L^\infty((0,T)\times\Omega)$ (due to (2)).

Next, we seek approximate solutions $u_n(t,x)$ for Equation (1) in the form:

$$
u_n(t,x) = \sum_{k=1}^{n} d_{n,k}(t)e_k(x) \ \text{ for } n \in \mathbb{N},
\tag{4}
$$

where $\{e_k\}$ denotes the complete orthonormal system of eigenfunctions which forms an orthogonal basis of $L^2(\Omega) \cap H^1_0(\Omega)$ such that

$$
-\Delta e_k = \lambda_k e_k \ \text{ in } \Omega, \quad e_k|_{\partial\Omega} = 0,\ k = 1,2,...
$$

For the sake of selecting $d_{n,k}(t)$, one considers the following approximate equation:

$$
\begin{cases}
\partial^\alpha_t u_n(t,x) - \mathcal{A}^n u_n(t,x) = f^n(t), \quad (t,x) \in (0,T]\times\Omega,\\
u_n(0,x) = u_{n0},\ \partial_t u_n(0,x) = u_{n1},
\end{cases}
\tag{5}
$$

where

$$
\begin{aligned}
&\mathcal{A}^n u_n(t,x) = \sum_{i,j=1}^{N} \partial_i\big(a^n_{i,j}(t,x)\partial_j u_n(t,x)\big) + \sum_{j=1}^{N} b^n_j(t,x)\partial_j u_n(t,x) + c^n(t,x)u_n(t,x),\\
&f^n(t,x) = \sum_{k=1}^{n} \big(f_{\frac{1}{n}}(t,\cdot), e_k(\cdot)\big)e_k(x),\\
&u_{n0}(t,x) = \sum_{k=1}^{n} \big(u_0(\cdot), e_k(\cdot)\big)e_k(x),\ \ u_{n1}(t,x) = \sum_{k=1}^{n} \big(u_1(\cdot), e_k(\cdot)\big)e_k(x).
\end{aligned}
$$

Let us introduce the time-dependent bilinear form

$$\mathcal{B}^n[u,v;t] := \int_\Omega \sum_{i,j=1}^N a_{i,j}^n(t,x)\partial_j u \cdot \partial_i v - \sum_{j=1}^N b_j^n(t,x)\partial_j uv - c^n(t,x)uvdx.$$

Taking the scalar product of (5) with e_l for $l = 1, ..., n$, we obtain

$$\begin{cases} (\partial_t^\alpha u_n(t,\cdot), e_l) + \mathcal{B}^n[u_n, e_l; t] = (f_{\frac{1}{n}}(t), e_l), \\ (u_n(0,\cdot), e_l) = (u_{n0}, e_l),\ (\partial_t u_n(0,\cdot), e_l) = (u_{n1}, e_l). \end{cases} \tag{6}$$

More precisely, we write

$$\begin{aligned} &d_n(t) = \big(d_{n,1}(t), \cdots, d_{n,n}(t)\big), \\ &\mathcal{L}^n(t) = \{L_{k,l}^n(t)\}_{k,l=1}^n, \quad L_{k,l}^n(t) = \mathcal{B}^n[e_l, e_k; t], \\ &F^n(t) = \big(f_{\frac{1}{n}}(t), e_l\big)_{l=1}^n, \\ &d_{n0} = \big(u_0, e_l\big)_{l=1}^n, \quad d_{n1} = \big(u_1, e_l\big)_{l=1}^n. \end{aligned}$$

Then (6) can be reduced to the following linear differential system for the functions d_n:

$$\begin{cases} \partial_t^\alpha d_n(t) + \mathcal{L}^n(t) d_n(t) = F^n(t) \ \text{ for } t \in (0,T], \\ d_n(0) = d_{n0},\ d_n'(0) = d_{n1}. \end{cases} \tag{7}$$

Now we consider the nonlinear integral system for the functions

$$d_n(t) = d_{n0} + d_{n1}t + \big[g_\alpha * \big(\mathcal{L}^n(\cdot)d_n(\cdot)\big)\big](t) + \big[g_\alpha * F^n\big](t) \ \text{ for } t \in [0,T]. \tag{8}$$

We shall show that system (8) has a unique solution d_n which belongs to $AC^2[0,T]$. By Lemma 1, then the solution d_n of Equation (8) is also the solution of Equation (7). To accomplish this, we introduce the space

$$E_T = \big\{d \in C^1([0,T],\mathbb{R}^n) : \ d(0) = d_{n0},\ d'(0) = d_{n1},\ t^{2-\alpha}d''(t) \in C([0,T],\mathbb{R}^n)\big\},$$

and define a metric on E_T as

$$\|d\|_{E_T} = \|d\|_{C[0,T]} + \|d'\|_{C[0,T]} + \|t^{2-\alpha}d''\|_{C[0,T]}.$$

It is easy to show that $(E_T, \|\cdot\|_{E_T})$ is a complete metric space. We notice that $E_T \subset AC^2([0,T],\mathbb{R}^n)$.

Theorem 1. *Let $T \in (0,\infty)$ and (3) hold. For every $n \in \mathbb{N}$, Equation (8) has a unique solution in E_T.*

Proof. Consider the operator $\mathcal{T} : E_T \to E_T$ given by

$$\mathcal{T}d(t) = d_{n0} + d_{n1}t + \big[g_\alpha * \big(\mathcal{L}^n(\cdot)d(\cdot)\big)\big](t) + \big(g_\alpha * F^n\big)(t), \ \text{ for } t \in [0,T].$$

Then it is well-defined. Indeed, let $d \in E_T$, then $\mathcal{T}d(0) = d_{n0}$. Further, we immediately take the first and second derivatives of $\mathcal{T}d$ with respect to t to obtain

$$(\mathcal{T}d)'(t) = d_{n1} + g_\alpha(t)\big[\mathcal{L}^n(0)d(0) + F^n(0)\big] + \big[g_\alpha * \big(\mathcal{L}^n(\cdot)d(\cdot) + F^n(\cdot)\big)'\big](t) \ \text{ for } t \in [0,T],$$

and

$$(\mathcal{T}d)''(t) = g_{\alpha-1}(t)\big[\mathcal{L}^n(0)d(0) + F^n(0)\big] + g_\alpha(t)\big[(\mathcal{L}^n)'(0)d(0) + \mathcal{L}^n(0)d'(0) + (F^n)'(0)\big]$$
$$+ g_\alpha * \big[\mathcal{L}^n(t)d(t) + F^n(t)\big]'' \quad \text{for } t \in (0,T].$$

For convenience we let $G_d(t) = \mathcal{L}^n(t)d(t) + F^n(t)$. Then $G_d'(t) = (\mathcal{L}^n)'(t)d(t) + \mathcal{L}^n(t)d'(t) + (F^n)'(t)$ and $G_d, G_d' \in C([0,T], \mathbb{R}^n)$. We can easily check that $\mathcal{T}d$ and $(\mathcal{T}d)'$ are continuous on $C([0,T], \mathbb{R}^n)$, which also ensures that $(\mathcal{T}d)'(0) = d_{n1}$. Therefore it remains to consider the continuity of $t^{2-\alpha}(\mathcal{T}d)''(t)$. It is easy to verify the continuity of the first two components. To deal with the third one we estimate for $0 \le t_1 < t_2 \le T$

$$\begin{aligned}
&\left| t_2^{2-\alpha} \int_0^{t_2} g_\alpha(s) G_d''(t_2 - s)ds - t_1^{2-\alpha} \int_0^{t_1} g_\alpha(s) G_d''(t_1 - s)ds \right| \\
\le& |t_2^{2-\alpha} - t_1^{2-\alpha}| \int_0^{t_2} g_\alpha(s)|G_d''(t_2 - s)|ds + t_1^{2-\alpha} \int_{t_1}^{t_2} g_\alpha(s)|G_d''(t_2 - s)|ds \\
&+ t_1^{2-\alpha} \int_0^{t_1} g_\alpha(s)|G_d''(t_2 - s) - G_d''(t_1 - s)|ds \\
=:& I_1(t_1,t_2) + I_2(t_1,t_2) + I_3(t_1,t_2).
\end{aligned}$$

On the other hand, from the definition of G_d and $d \in E_T$, it follows that

$$G_d''(t) = (\mathcal{L}^n)''(t)d(t) + 2(\mathcal{L}^n)'(t)d'(t) + \mathcal{L}^n(t)d''(t) + (F^n)''(t).$$

From the representation of $\mathcal{L}^n(t)$ and F^n, we know that $\mathcal{L}^n$ and F^n belong to the space E_T, which yields that $G_d \in E_T$ and

$$|G_d''(t)| \le \|G_d\|_{E_T} t^{\alpha-2}. \tag{9}$$

Thus one can immediately calculate $I_1(t_1,t_2)$ and $I_2(t_1,t_2)$ as follows

$$\begin{aligned}
I_1(t_1,t_2) \le& \|G_d\|_{E_T} |t_2^{2-\alpha} - t_1^{2-\alpha}| \int_0^{t_2} g_\alpha(s)(t_2 - s)^{\alpha-2}ds \\
=& \frac{\|G_d\|_{E_T}}{\Gamma(\alpha)} B(\alpha, \alpha-1) t_2^\alpha \Big[1 - \Big(\frac{t_1}{t_2}\Big)^{2-\alpha}\Big] \to 0, \text{ as } t_2 \to t_1,
\end{aligned}$$

and

$$\begin{aligned}
I_2(t_1,t_2) \le& \|G_d\|_{E_T} t_1^{2-\alpha} \int_{t_1}^{t_2} g_\alpha(s)(t_2 - s)^{\alpha-2}ds \\
\le& \frac{\|G_d\|_{E_T}}{\Gamma(\alpha)} t_2^\alpha \Big(\frac{t_1}{t_2}\Big)^{2-\alpha} \int_{t_1/t_2}^1 s^{\alpha-1}(1-s)^{\alpha-2}ds \to 0, \text{ as } t_2 \to t_1.
\end{aligned}$$

Finally, for $I_3(t_1,t_2)$, choosing a $\delta \in (0, t_1)$ sufficient small for $t_1 > 0$, one can derive from the increasing property of g_α and (9) that

$$\begin{aligned}
&I_3(t_1,t_2) \\
=& t_1^{2-\alpha} \int_0^{t_1-\delta} g_\alpha(s)|G_d''(t_2 - s) - G_d''(t_1 - s)|ds + t_1^{2-\alpha} \int_{t_1-\delta}^{t_1} g_\alpha(s)|G_d''(t_2 - s) - G_d''(t_1 - s)|ds \\
\le& t_1^{2-\alpha} g_\alpha(t_1 - \delta) \int_0^{t_1-\delta} |G_d''(t_2 - s) - G_d''(t_1 - s)|ds + 2\|G_d\|_{E_T} t_1^{2-\alpha} \int_{t_1-\delta}^{t_1} g_\alpha(s)(t_1 - s)^{\alpha-2}ds \\
\le& t_1^{2-\alpha} g_\alpha(t_1) \int_\delta^{t_1} |G_d''(t_2 - t_1 + s) - G_d''(s)|ds + \frac{2\|G_d\|_{E_T}}{\Gamma(\alpha)} t_1^\alpha \int_{1-\delta/t_1}^1 s^{\alpha-1}(1-s)^{\alpha-2}ds.
\end{aligned}$$

It is clear that the second term tends to zero for some sufficient small δ. Then we choose one of such δ, it follows from the uniform continuity of G'' (due to the continuity of G''_d on $[\delta, T]$) that for any $\varepsilon > 0$, there exists $\delta' < \delta$ with $|t_2 - t_1| < \delta'$ such that $|G''_d(t_2 - t_1 + s) - G''_d(s)| < \varepsilon$. Thus, this yields that the first term can be bounded by $\varepsilon t_1^{2-\alpha} g_\alpha(t_1)(t_1 - \delta)$, which together with $I_3(0, t_2) = 0$ shows that $I_3(t_1, t_2) \to 0$ as $t_2 \to t_1$ for $0 \le t_1 < t_2 \le T$.

Therefore, we have $\mathcal{T}d \in E_T$ for $d \in E_T$.

Moreover, for $d_1, d_2 \in E_T$, we have

$$\begin{aligned}
|\mathcal{T}d_1(t) - \mathcal{T}d_2(t)| \le& \Big[g_\alpha * \big|G_{d_1}(\cdot) - G_{d_2}(\cdot)\big|\Big](t) \\
\le& \frac{\|\mathcal{L}^n\|}{\Gamma(\alpha)} \|d_1 - d_2\|_{E_T} \int_0^t (t-s)^{\alpha-1} ds \\
\le& \frac{\|\mathcal{L}^n\| t^\alpha}{\Gamma(1+\alpha)} \|d_1 - d_2\|_{E_T},
\end{aligned} \tag{10}$$

where we have used

$$\begin{aligned}
\big|G_{d_1}(s) - G_{d_2}(s)\big| \le& |\mathcal{L}^n(s)||d_1(s) - d_2(s)| \\
\le& \|\mathcal{L}^n\| \|d_1 - d_2\| \\
\le& \|\mathcal{L}^n\| \|d_1 - d_2\|_{E_T} \ \text{ for } s \in [0, t].
\end{aligned} \tag{11}$$

Similarly, in view of

$$\begin{aligned}
\big|G'_{d_1}(s) - G'_{d_2}(s)\big| \le& |(\mathcal{L}^n)'(s)||d_1(s) - d_2(s)| + |\mathcal{L}^n(s)||d'_1(s) - d'_2(s)| \\
\le& \|(\mathcal{L}^n)'\| \|d_1 - d_2\| + \|\mathcal{L}^n\| \|d'_1 - d'_2\| \\
\le& 2\|\mathcal{L}^n\|_{C^1[0,T]} \|d_1 - d_2\|_{E_T} \ \text{ for } s \in [0, t],
\end{aligned} \tag{12}$$

we proceed to estimate $(\mathcal{T}d_1)' - (\mathcal{T}d_2)'$ as follows:

$$\begin{aligned}
|(\mathcal{T}d_1)'(t) - (\mathcal{T}d_2)'(t)| \le& g_\alpha(t)\big|G_{d_1}(0) - G_{d_2}(0)\big| + \big[g_\alpha * \big|G'_{d_1}(\cdot) - G'_{d_2}(\cdot)\big|\big](t) \\
\le& \frac{2\|\mathcal{L}^n\|_{C^1[0,T]}}{\Gamma(\alpha)} \|d_1 - d_2\|_{E_T} \int_0^t (t-s)^{\alpha-1} ds \\
\le& \frac{2\|\mathcal{L}^n\|_{C^1[0,T]}}{\Gamma(1+\alpha)} t^\alpha \|d_1 - d_2\|_{E_T},
\end{aligned} \tag{13}$$

where it is easy to show that $G_{d_1}(0) - G_{d_2}(0) = 0$ due to $d_1(0) - d_2(0) = 0$ and (11).

Finally, we will estimate $t^{2-\alpha}(\mathcal{T}d_1)'' - t^{2-\alpha}(\mathcal{T}d_2)''$. Taking account of the following inequality

$$\begin{aligned}
\big|G''_{d_1}(s) - G''_{d_2}(s)\big| \le& |(\mathcal{L}^n)''(s)||d_1(s) - d_2(s)| + 2|(\mathcal{L}^n)'(s)||d'_1(s) - d'_2(s)| \\
&+ |\mathcal{L}^n(s)||d''_1(s) - d''_2(s)| \\
\le& \|(\mathcal{L}^n)''\| \|d_1 - d_2\| + 2\|(\mathcal{L}^n)'\| \|d'_1 - d'_2\| \\
&+ s^{\alpha-2} \|\mathcal{L}^n\| \|s^{2-\alpha} d''_1 - s^{2-\alpha} d''_2\| \\
\le& 3\|\mathcal{L}^n\|_{C^2[0,T]} (1 + s^{\alpha-2}) \|d_1 - d_2\|_{E_T} \ \text{ for } s \in (0, t],
\end{aligned}$$

it holds that

$$\begin{aligned}
&|t^{2-\alpha}(\mathcal{T}d_1)''(t) - t^{2-\alpha}(\mathcal{T}d_2)''(t)| \\
\le& t^{2-\alpha}g_{\alpha-1}(t)\big|G_{d_1}(0) - G_{d_2}(0)\big| + t^{2-\alpha}g_\alpha(t)\big|G'_{d_1}(0) - G'_{d_2}(0)\big| \\
&+ t^{2-\alpha}\big[g_\alpha * \big|G''_{d_1}(\cdot) - G''_{d_2}(\cdot)\big|\big](t) \\
\le& \frac{3\|\mathcal{L}^n\|_{C^2[0,T]}}{\Gamma(\alpha)}\|d_1 - d_2\|_{E_T} t^{2-\alpha}\int_0^t (t-s)^{\alpha-1}(1+s^{\alpha-2})ds \\
\le& 3\|\mathcal{L}^n\|_{C^2[0,T]}\left(\frac{t^2}{\Gamma(1+\alpha)} + \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}t^\alpha\right)\|d_1 - d_2\|_{E_T},
\end{aligned} \tag{14}$$

where we know from (12) that $G'_{d_1}(0) - G'_{d_2}(0) = 0$.

For the sake of convenience, we let

$$M(t) = 3t^\alpha\|\mathcal{L}^n\|_{C^2[0,T]}\left(\frac{1}{\Gamma(1+\alpha)} + \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\right) + \|\mathcal{L}^n\|_{C^2[0,T]}\frac{3t^2}{\Gamma(1+\alpha)}.$$

Then one can choose a $T_1 \in (0,T)$ small enough which ensures that $M(T_1) < 1$. Therefore, combining (10), (13) with (14), we deduce that

$$\|\mathcal{T}d_1 - \mathcal{T}d_2\|_{E_{T_1}} \le M(T_1)\|d_1 - d_2\|_{E_T}.$$

This also shows that the operator $\mathcal{T}$ is a strict contraction on $E(T_1)$. It follows that $\mathcal{T}$ has a fixed point, thus Equation (8) has a unique solution in E_{T_1}.

Now, we will deal with the continuation of the solution to the interval $[0,T]$. Let us make the assumption that we have obtained the solution $\bar{d}$ of Equation (8) on the interval $[0,T_l]$ for $T_l > 0$. We shall define the solution for $t \in [T_l, T_{l+1}]$ with $T_{l+1} > T_l$. To accomplish this, we introduce the complete space

$$\bar{E}_{T_{l+1}} = \big\{d \in C^2((0,T_{l+1}],\mathbb{R}^n) : \ d(t) = \bar{d}(t) \ \text{ for } t \in [0,T_l]\big\},$$

with the distance $\|d\|_{\bar{E}_{T_{l+1}}} = \|d\|_{C^2[T_l,T_{l+1}]}$. Let $d \in \bar{E}_{T_{l+1}}$, then $d \in E_{T_{l+1}}$. According to the previous proof, we know that $\mathcal{T}d \in E_{T_{l+1}}$, which implies that $\mathcal{T}d \in C^1([0,T_{l+1}],\mathbb{R}^n)$ and $t^{2-\alpha}(\mathcal{T}d)'' \in C([0,T_{l+1}],\mathbb{R}^n)$. It holds that $(\mathcal{T}d)'' \in C((0,T_{l+1}],\mathbb{R}^n)$ and then $\mathcal{T}d \in \bar{E}_{T_{l+1}}$.

Next, we will show that the operator $\mathcal{T}$ is also a strict contraction on $\bar{E}_{T_{l+1}}$ when $T_{l+1} - T_l$ is sufficiently small. We shall rewrite $\mathcal{T}$ in the following form:

$$\mathcal{T}d(t) = d_{n0} + d_{n1}t + \frac{1}{\Gamma(\alpha)}\int_0^{T_l} g_\alpha(t-s)G_d(s)ds + \frac{1}{\Gamma(\alpha)}\int_{T_l}^t g_\alpha(t-s)G_d(s)ds.$$

For $d_1, d_2 \in \bar{E}_{T_{l+1}}$, we have $d_1(t) - d_2(t) = 0$ and $G_{d_1}(t) - G_{d_2}(t) = 0$ for $t \in [0,T_l]$. Then

$$\mathcal{T}d_1(t) - \mathcal{T}d_2(t) = \frac{1}{\Gamma(\alpha)}\int_{T_l}^t g_\alpha(t-s)[G_{d_1}(s) - G_{d_2}(s)]ds.$$

This follows from (11) that

$$\|\mathcal{T}d_1 - \mathcal{T}d_2\|_{C[T_l,T_{l+1}]} \le \frac{\|\mathcal{L}^n\|}{\Gamma(1+\alpha)}\|d_1 - d_2\|_{C[T_l,T_{l+1}]}(T_{l+1} - T_l)^\alpha.$$

Similarly, we obtain

$$\begin{aligned}
&\|(\mathcal{T}d_1)' - (\mathcal{T}d_2)'\|_{C[T_l,T_{l+1}]} \\
\le& \frac{1}{\Gamma(1+\alpha)}\big(\|(\mathcal{L}^n)'\|\|d_1 - d_2\|_{C[T_l,T_{l+1}]} + \|\mathcal{L}^n\|\|d'_1 - d'_2\|_{C[T_l,T_{l+1}]}\big)(T_{l+1} - T_l)^\alpha,
\end{aligned}$$

and

$$\|(\mathcal{T}d_1)'' - (\mathcal{T}d_2)''\|_{C[T_l,T_{l+1}]} \leq \frac{(T_{l+1}-T_l)^\alpha}{\Gamma(1+\alpha)} \Big[\|(\mathcal{L}^n)''\| \|d_1 - d_2\|_{C[T_l,T_{l+1}]} + 2\|(\mathcal{L}^n)'\| \|d_1' - d_2'\|_{C[T_l,T_{l+1}]} + \|\mathcal{L}^n\| \|d_1'' - d_2''\|_{C[T_l,T_{l+1}]} \Big].$$

Therefore,

$$\|\mathcal{T}d_1 - \mathcal{T}d_2\|_{\bar{E}_{T_{l+1}}} \leq \frac{4\|\mathcal{L}^n\|_{C^2[0,T]}}{\Gamma(1+\alpha)} \|d_1 - d_2\|_{\bar{E}_{T_{l+1}}} (T_{l+1} - T_l)^\alpha.$$

Moreover, we can choose one $T_{l+1} \in \left(T_l, T_l + \left(\frac{\Gamma(1+\alpha)}{4\|\mathcal{L}^n\|_{C^2[0,T]}} \right)^{\frac{1}{\alpha}} \right)$ such that $T_{l+1} - T_l$ is small enough. It also ensures that

$$0 < \frac{4\|\mathcal{L}^n\|_{C^2[0,T]}}{\Gamma(1+\alpha)} (T_{l+1} - T_l)^\alpha < 1.$$

Hence, the operator $\mathcal{T}$ is a strict contraction on $\bar{E}_{T_{l+1}}$, this also shows that Equation (8) has a unique solution on the interval $[T_l, T_{l+1}]$. We proceed to repeat the process on the intervals $[T_{l+1}, T_{l+2}], \cdots$, until Equation (8) has a unique solution on the interval $[0, T]$. The claim then follows. □

4. Energy Estimates

The purpose of this section is to establish some a priori estimates of approximation solutions through a mathematical analysis, which plays an important role in obtaining the main results. We can accomplish this with the following lemma.

Lemma 5. *Assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and recall the condition imposed to the parameters $a_{i,j}$, b_j, c, and f. Then, for every $n \in \mathbb{N}$ and $t \in (0,T]$ the approximate solution u_n given by (4) and (8) satisfies the inequality*

$$\begin{aligned} &{}_0I_t^{2-\alpha} \|\partial_t u_n(t,\cdot)\|^2 + \int_0^t \|\partial_s u_n(s,\cdot)\|^2 ds + \|\nabla u_n(t,\cdot)\|^2 \\ \leq &\widetilde{M}_1 \left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha} \right) + \widetilde{M}_2 \int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds, \end{aligned}$$

where $\widetilde{M}_1$ and $\widetilde{M}_2$ are positive constants.

Proof. Multiply Equation (6) by $d'_{n,l}(t)$, sum it up from 1 to n and recall (4) to discover

$$\begin{cases} \left(\partial_t^\alpha u_n(t,\cdot), \partial_t u_n(t,\cdot)\right) + \mathcal{B}^n[u_n, \partial_t u_n; t] = \left(f_{\frac{1}{n}}(t,\cdot), \partial_t u_n(t,\cdot)\right), \quad (t,x) \in (0,T] \times \Omega, \\ u_n(0,\cdot) = u_{n0},\ \partial_t u_n(0,\cdot) = u_{n1}. \end{cases} \tag{15}$$

Taking into account Lemma 2, we have

$$\partial_t^\alpha u_n(t,\cdot) = \frac{\partial}{\partial t}[\partial_t^{\alpha-1} u_n(t,\cdot)] - \frac{u_{n1}}{\Gamma(2-\alpha)} t^{1-\alpha}.$$

Using Example 1, it follows that

$$\begin{aligned}
&\int_0^t \big(\partial_s^\alpha u_n(s,\cdot), \partial_s u_n(s,\cdot)\big)ds \\
=&\int_0^t \big(\frac{\partial}{\partial s}[\partial_s^{\alpha-1} u_n(s,\cdot)], \partial_s u_n(s,\cdot)\big)ds - \int_0^t \big(\frac{u_{n1}}{\Gamma(2-\alpha)} s^{1-\alpha}, \partial_s u_n(s,\cdot)\big)ds \\
\geq& \frac{1}{2} {}_0I_t^{2-\alpha}\|\partial_t u_n(t,\cdot)\|^2 + \frac{1}{2}\int_0^t g_{2-\alpha}(s)\|\partial_s u_n(s,\cdot)\|^2 ds - \frac{1}{\Gamma(2-\alpha)}\int_0^t s^{1-\alpha}\big(u_{n1}, \partial_s u_n(s,\cdot)\big)ds.
\end{aligned}$$

Therefore, we integrate the first equality of Equation (15) with respect to the time variable from 0 to t to obtain that

$$\begin{aligned}
&\frac{1}{2} {}_0I_t^{2-\alpha}\|\partial_t u_n(t,\cdot)\|^2 + \frac{1}{2}\int_0^t g_{2-\alpha}(s)\|\partial_s u_n(s,\cdot)\|^2 ds \\
&+ \sum_{i,j=1}^N \int_0^t \int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i\partial_s u_n(s,x)dxds \\
\leq& \sum_{j=1}^N \int_0^t \int_\Omega b_j^n(s,x)\partial_j u_n(s,x)\partial_s u_n(s,x)dxds + \int_0^t\int_\Omega c^n(s,x)u_n(s,x)\partial_s u_n(s,x)dxds \\
&+ \int_0^t \big(f_{\frac{1}{n}}(s,\cdot), \partial_s u_n(s,\cdot)\big)ds + \frac{1}{\Gamma(2-\alpha)}\int_0^t s^{1-\alpha}\big(u_{n1}, \partial_s u_n(s,\cdot)\big)ds \\
=:& J_1(t) + J_2(t).
\end{aligned} \tag{16}$$

First, we estimate the third term of the left-hand side of the above inequality. Using the integration by parts with respect to s, we derive that

$$\begin{aligned}
&\sum_{i,j=1}^N \int_0^t \int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i\partial_s u_n(s,x)dxds \\
=&\sum_{i,j=1}^N \int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i u_n(s,x)dx\Big|_0^t \\
&- \sum_{i,j=1}^N \int_0^t \int_\Omega [\partial_s a_{i,j}^n(s,x)\partial_j u_n(s,x) + a_{i,j}^n(s,x)\partial_j\partial_s u_n(s,x)]\partial_i u_n(s,x)dxds.
\end{aligned}$$

It follows from $a_{i,j}^n = a_{j,i}^n$ that

$$\begin{aligned}
&\sum_{i,j=1}^N \int_0^t \int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i\partial_s u_n(s,x)dxds \\
=&\sum_{i,j=1}^N \int_0^t \int_\Omega a_{i,j}^n(s,x)\partial_j\partial_s u_n(s,x)\partial_i u_n(s,x)dxds.
\end{aligned}$$

In addition, in view of the definition of $a_{i,j}^n$, we know that $\partial_s a_{i,j}^n(s,x) = \big(\varrho_{\frac{1}{n}} * \partial_t a_{i,j}(\cdot,x)\big)(s)$; this yields

$$|\partial_s a_{i,j}^n(s,x)| \leq \|A\|_{W^{1,\infty}}.$$

Therefore, using (2) again, one can obtain that

$$
\begin{aligned}
&\sum_{i,j=1}^{N}\int_0^t\int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i\partial_s u_n(s,x)dxds\\
=&\frac{1}{2}\sum_{i,j=1}^{N}\int_\Omega a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i u_n(s,x)dx\Big|_0^t\\
&-\frac{1}{2}\sum_{i,j=1}^{N}\int_0^t\int_\Omega \partial_s a_{i,j}^n(s,x)\partial_j u_n(s,x)\partial_i u_n(s,x)dxds\\
\geq&\frac{\mu}{2}\int_\Omega|\nabla u_n(t,x)|^2dx-\frac{\nu}{2}\int_\Omega|\nabla u_n(0,x)|^2dx\\
&-\frac{1}{2}\|A\|_{W^{1,\infty}}\int_0^t\int_\Omega\sum_{i,j=1}^{N}|\partial_j u_n(s,x)\partial_i u_n(s,x)|dsdx\\
\geq&\frac{\mu}{2}\|\nabla u_n(t,\cdot)\|^2-\frac{\nu}{2}\|\nabla u_n(0,\cdot)\|^2-\frac{1}{2}\|A\|_{W^{1,\infty}}\int_0^t\|\nabla u_n(s,\cdot)\|^2ds.
\end{aligned}
$$

Next we estimate the upper bound of the right-handed side of (16). For $J_1(t)$, we use Hölder's inequality and Young's inequality to obtain

$$
\begin{aligned}
J_1(t)=&\sum_{j=1}^{N}\int_0^t\int_\Omega b_j^n(s,x)\partial_j u_n(s,x)\partial_s u_n(s,x)dxds+\int_0^t\int_\Omega c^n(s,x)u_n(s,x)\partial_s u_n(s,x)dxds\\
\leq&\int_0^t\|\nabla u_n(s,\cdot)\|\|\partial_s u_n(s,\cdot)\|\|\mathbf{b}^n(s,\cdot)\|_{L^\infty}ds\\
&+\int_0^t\|u_n(s,\cdot)\|_{L^q}\|\partial_s u_n(s,\cdot)\|\|c^n(s,\cdot)\|_{L^{\frac{2q}{q-2}}}ds\\
\leq&\frac{C_\varepsilon}{2}\int_0^t\|\nabla u_n(s,\cdot)\|^2ds+\frac{\varepsilon}{2}\int_0^t\|\partial_s u_n(s,\cdot)\|^2\|\mathbf{b}^n(s,\cdot)\|_{L^\infty}^2ds\\
+&\frac{\varepsilon}{2}\int_0^t\|\partial_s u_n(s,\cdot)\|^2ds+\frac{C_\varepsilon}{2}\int_0^t\|u_n(s,\cdot)\|_{L^q}^2\|c^n(s,\cdot)\|^2_{L^{\frac{2q}{q-2}}}ds\\
\leq&\frac{C_\varepsilon}{2}(1+C^2(q,N,\partial\Omega)\|c^n\|^2_{L^\infty(0,T,L^{\frac{2q}{q-2}})})\int_0^t\|\nabla u_n(s,\cdot)\|^2ds\\
&+\frac{\varepsilon}{2}\left(\|\mathbf{b}^n\|^2_{L^\infty((0,T)\times\Omega)}+1\right)\int_0^t\|\partial_s u_n(s,\cdot)\|^2ds,
\end{aligned}
$$

where we have used $\|u_n(s,\cdot)\|_{L^q}\leq C(q,N,\Omega)\|\nabla u_n(s,\cdot)\|$ for $q\in[2,\frac{2N}{N-2})$ obtained by Evans [29]. Moreover, $J_2(t)$ can be estimated by Young's inequality

$$
\begin{aligned}
J_2(t)=&\int_0^t(f_{\frac{1}{n}}(s),\partial_s u_n(s,\cdot))ds+\frac{1}{\Gamma(2-\alpha)}\int_0^t s^{1-\alpha}(u_{n1},\partial_s u_n(s,\cdot))ds\\
\leq&\int_0^t\|f_{\frac{1}{n}}(s,\cdot)\|\|\partial_s u_n(s,\cdot)\|ds+\frac{1}{\Gamma(2-\alpha)}\int_0^t s^{1-\alpha}\|u_{n1}\|\|\partial_s u_n(s,\cdot)\|ds\\
\leq&\int_0^t(\frac{C_\varepsilon}{2}\|f_{\frac{1}{n}}(s,\cdot)\|^2+\frac{\varepsilon}{2}\|\partial_s u_n(s,\cdot)\|^2)ds\\
&+\frac{1}{\Gamma(2-\alpha)}\int_0^t s^{1-\alpha}(\frac{C_\varepsilon}{2}\|u_{n1}\|^2+\frac{\varepsilon}{2}\|\partial_s u_n(s,\cdot)\|^2)ds\\
\leq&\frac{C_\varepsilon}{2}\frac{\|u_{n1}\|^2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{C_\varepsilon}{2}\int_0^t\|f_{\frac{1}{n}}(s,\cdot)\|^2ds+\frac{\varepsilon}{2}\int_0^t(g_{2-\alpha}(s)+1)\|\partial_s u_n(s,\cdot)\|^2ds.
\end{aligned}
$$

Let

$$
Q_n=\|\mathbf{b}^n\|^2_{L^\infty((0,T)\times\Omega)}+2\ \text{ and }\ \tilde{Q}_n=1+C^2(q,N,\Omega)\|c^n\|^2_{L^\infty(0,T,L^{\frac{2q}{q-2}})}.
$$

Then $Q_n \le C\left(\|\mathbf{b}\|^2_{L^\infty((0,T)\times\Omega)} + 2\right) := Q$ and $\tilde{Q}_n \le C\left(1 + C^2(q,N,\Omega)\|c\|^2_{L^\infty(0,T,L^{\frac{2q}{q-2}})}\right) := \tilde{Q}$ for each n.

We use the above inequalities in (16) and the decreasing property of $g_{2-\alpha}$ to obtain that

$$
\begin{aligned}
&\frac{1}{2}{}_0I_t^{2-\alpha}\|\partial_t u_n(t,\cdot)\|^2 + \frac{(1-\varepsilon)g_{2-\alpha}(T)}{2}\int_0^t \|\partial_s u_n(s,\cdot)\|^2 ds + \frac{\mu}{2}\|\nabla u_n(t,\cdot)\|^2 \\
\le&\frac{\nu}{2}\|u_{n0}\|^2_{H_0^1} + \frac{C_\varepsilon}{2}\frac{\|u_{n1}\|^2}{\Gamma(3-\alpha)}t^{2-\alpha} + \frac{1}{2}\left(C_\varepsilon\tilde{Q} + \|A\|_{W^{1,\infty}}\right)\int_0^t \|\nabla u_n(s,\cdot)\|^2 ds \\
&+ \frac{\varepsilon Q}{2}\int_0^t \|\partial_s u_n(s,\cdot)\|^2 ds + \frac{C_\varepsilon}{2}\int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds.
\end{aligned}
$$

For fixed $0 < \varepsilon < \frac{g_{2-\alpha}(T)}{g_{2-\alpha}(T)+Q}$, it follows that

$$
\|\nabla u_n(t,\cdot)\|^2 \le M_1\left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha}\right) + M_1^*\int_0^t \|\nabla u_n(s,\cdot)\|^2 ds + \frac{C_\varepsilon}{\mu}\int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds,
$$

where $M_1 = \max\left\{\frac{\nu}{\mu}, \frac{C_\varepsilon}{\mu\Gamma(3-\alpha)}\right\}$ and $M_1^* = \frac{1}{\mu}(C_\varepsilon\tilde{Q} + \|A\|_{W^{1,\infty}})$, it results from using Gronwall's inequality that

$$
\|\nabla u_n(t,\cdot)\|^2 \le M_2\left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha} + \int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds\right) \text{ for } t \in [0,T],
$$

where M_2 is a positive constant depending on M_1, M_1^*, and T. Therefore, we have for $t \in [0,T]$

$$
\begin{aligned}
&{}_0I_t^{2-\alpha}\|\partial_t u_n(t,\cdot)\|^2 + [(1-\varepsilon)g_{2-\alpha}(T) - \varepsilon Q]\int_0^t \|\partial_s u_n(s,\cdot)\|^2 ds + \mu\|\nabla u_n(t,\cdot)\|^2 \\
\le& M_1\left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha}\right) + C_\varepsilon\int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds \\
&+ M_2 t\left(C_\varepsilon\tilde{Q} + \|A\|_{W^{1,\infty}}\right)\left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha} + \int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds\right).
\end{aligned}
$$

The claim then follows. □

Lemma 6. *Assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and recall the condition imposed to the parameters $a_{i,j}$, b_j, c, and f. Then, for every $n \in \mathbb{N}$ and for every $t \in (0,T]$ the approximate solution u_n given by (4) and (8) satisfies the inequality*

$$
\int_0^t \|\partial_s^\alpha u_n(s,\cdot)\|^2_{H^{-1}} ds \le 2\widetilde{M_1}M_3^2 t\left(\|u_{n0}\|^2_{H_0^1} + \|u_{n1}\|^2 t^{2-\alpha}\right) + 2(\widetilde{M_2}M_3^2 t + 1)\int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds
$$

for $t \in (0,T]$.

Proof. For fixed $v \in H_0^1(\Omega)$, $\|v\|_{H_0^1} \le 1$, rewrite $v = v_1 + v_2$, where $v_1 \in \mathrm{span}\{e_k\}_{k=1}^n$ and $(v_2, e_k) = 0$ $(k = 1, ..., n)$. Observe $\|v_1\|_{H_0^1} \le 1$. Then (4) and (6) imply

$$
\langle \partial_t^\alpha u_n(t,\cdot), v\rangle = (\partial_t^\alpha u_n(t,\cdot), v) = (\partial_t^\alpha u_n(t,\cdot), v_1) = (f_{\frac{1}{n}}(t,\cdot), v_1) - \mathcal{B}^n[u_n, v_1; t].
$$

On the other hand, from the definition of $\mathcal{B}^n$ and Sobolev imbedding, we know that

$$
\begin{aligned}
&\left|\mathcal{B}^n[u_n, v_1; t]\right| \\
\leq&\|A\|_{W^{1,\infty}}\|\nabla u_n(t,\cdot)\|\|\nabla v_1(\cdot)\| + \|\nabla u_n(t,\cdot)\|\|v_1(\cdot)\|\|\mathbf{b}^n(t,\cdot)\|_{L^\infty} \\
&+\|u_n(t,\cdot)\|\|v_1(\cdot)\|_q\|c^n(t,\cdot)\|_{L^{\frac{2q}{q-2}}} \\
\leq&\|A\|_{W^{1,\infty}}\|\nabla u_n(t,\cdot)\|\|v_1(\cdot)\|_{H_0^1} + \|\nabla u_n(t,\cdot)\|\|v_1(\cdot)\|_{H_0^1}\|\mathbf{b}^n\|_{L^\infty((0,T)\times\Omega)} \\
&+\|u_n(t,\cdot)\|\|v_1(\cdot)\|_{H_0^1}\|c^n\|_{L^\infty(0,T,L^{\frac{q}{q-2}})} \\
\leq&M_3\|\nabla u_n(t,\cdot)\|\|v_1(\cdot)\|_{H_0^1},
\end{aligned} \tag{17}
$$

where $M_3 = C\big(\|A\|_{W^{1,\infty}} + \|\mathbf{b}\|_{L^\infty((0,T)\times\Omega)} + \|c\|_{L^\infty(0,T,L^{\frac{q}{q-2}})}\big)$. Moreover, we have $|(f_{\frac{1}{n}}(t,x), v_1)| \leq \|f_{\frac{1}{n}}(t,\cdot)\|\|v_1\| \leq \|f_{\frac{1}{n}}(t,\cdot)\|\|v_1\|_{H_0^1}$. Thus,

$$
\left|\langle \partial_t^\alpha u_n(t,\cdot), v\rangle\right| \leq \|f_{\frac{1}{n}}(t,\cdot)\| + M_3\|\nabla u_n(t,\cdot)\|
$$

for $\|v_1\|_{H_0^1} \leq 1$. Consequently, $\|\partial_t^\alpha u_n(t,\cdot)\|_{H^{-1}} \leq \|f_{\frac{1}{n}}(t)\| + M_3\|\nabla u_n(t,\cdot)\|$, from Lemma 5 we can show

$$
\begin{aligned}
\int_0^t \|\partial_s^\alpha u_n(s,\cdot)\|_{H^{-1}}^2 ds \leq& 2\int_0^t \|f_{\frac{1}{n}}(s,\cdot)\|^2 ds + 2M_3^2 t\big[\widetilde{M_1}\big(\|u_{n0}\|_{H_0^1}^2 + \|u_{n1}\|^2 t^{2-\alpha}\big) \\
&+ \widetilde{M_2}\int_0^t \|f_{\frac{1}{n}}(s)\|^2 ds\big].
\end{aligned}
$$

The claim then follows. □

5. Well-Posedness and Regularity

In this section, we take the limit in approximate sequences and present the existence and uniqueness of weak solutions, and then we show the regularity results.

Theorem 2. *Suppose that $T > 0, u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$ and let $a_{i,j}$, b_j, c, and f satisfy (3). Then there exists a weak solution $u \in C([0,T], L^2(\Omega)) \cap L^\infty(0,T,H_0^1(\Omega))$ of Equation (1) satisfying $u' \in L^2(0,T,L^2(\Omega))$, $\partial_t^\alpha u \in L^2(0,T,H^{-1}(\Omega))$. Moreover, u also satisfies the following estimate*

$$
\begin{aligned}
\max_{t\in[0,T]}\|u(t)\|_{H_0^1} + \|\partial_t u\|_{L^2(0,T,L^2)} + \|\partial_t^\alpha u\|_{L^2(0,T,H^{-1})}& \\
\leq \widetilde{M}\big(\|u_0\|_{H_0^1}^2 + \|u_1\|^2 + \|f\|_{L^2(0,T,L^2)}\big)&,
\end{aligned} \tag{18}
$$

where $\widetilde{M}$ is a positive constant.

Proof. Step 1. According to the energy estimate in Lemma 5, we see that the sequence $\{u_n(t)\}$ is bounded in $H_0^1(\Omega)$ for $t \in [0,T]$, $\{\partial_t u_n\}$ is bounded in $L^2(0,T,L^2(\Omega))$, and Lemma 6 implies that the sequence $\partial_t^\alpha u_n$ is bounded in $L^2(0,T,H^{-1}(\Omega))$. This also implies that ${}_0I_t^{2-\alpha}(u - u_0 - u_1 t)$ is uniformly bounded in ${}_0H^2(0,T,H^{-1}(\Omega))$. As a consequence there exist $u \in C([0,T], L^2(\Omega)) \cap L^\infty(0,T,H_0^1(\Omega))$ with $u' \in L^2(0,T,L^2(\Omega))$, $v \in L^2(0,T,H^{-1}(\Omega))$, and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$
\begin{aligned}
&u_n \to u \text{ in } C([0,T], L^2(\Omega)), \quad u_n(t) \rightharpoonup u(t) \text{ in } L^2(0,T,H_0^1(\Omega)), \\
&\partial_t u_n \rightharpoonup \partial_t u \text{ in } L^2(0,T,L^2(\Omega)), \quad \partial_t^\alpha u_n \rightharpoonup v \text{ in } L^2(0,T,H^{-1}(\Omega)).
\end{aligned} \tag{19}
$$

Since the continuity of ${}_0I_t^{2-\alpha}$ in $L^2(0,T)$ implies weak continuity, it follows that

$$
{}_0I_t^{2-\alpha}\partial_t u_n \rightharpoonup {}_0I_t^{2-\alpha}\partial_t u \text{ in } L^2(0,T,L^2(\Omega)). \tag{20}
$$

Next we would like to prove that $\partial_t^\alpha u = v$ in a weak sense. We take $\varphi \in C_0^\infty(0,T)$ and $\psi \in H_0^1(\Omega)$. Then

$$\begin{aligned}
\int_0^T \varphi(t)\langle v,\psi\rangle_{H^{-1}\times H_0^1} dt &= \lim_{n\to\infty}\int_0^T \varphi(t)\langle \partial_t^\alpha u_n,\psi\rangle_{H^{-1}\times H_0^1} dt \\
&= \lim_{n\to\infty}\int_0^T \varphi(t)\Big\langle \partial_{tt}[{}_0I_t^{2-\alpha}(u_n-u_{n0}-u_{n1}t)],\psi\Big\rangle_{H^{-1}\times H_0^1} dt \\
&= \lim_{n\to\infty}\int_\Omega \psi(x)dx\int_0^T \varphi(t)\partial_{tt}[{}_0I_t^{2-\alpha}(u_n-u_{n0}-u_{n1}t)]dt \\
&= -\lim_{n\to\infty}\int_\Omega \psi(x)dx\int_0^T \varphi'(t)\partial_t[{}_0I_t^{2-\alpha}(u_n-u_{n0}-u_{n1}t)]dt \\
&= -\lim_{n\to\infty}\int_\Omega \psi(x)dx\int_0^T \varphi'(t){}_0I_t^{2-\alpha}(\partial_t u_n-u_{n1})dt \\
&= -\int_0^T \varphi'(t)\langle {}_0I_t^{2-\alpha}(\partial_t u-u_1),\psi\rangle\, dt \\
&= \int_0^T \varphi(t)\langle \partial_t^\alpha u,\psi\rangle_{H^{-1}\times H_0^1} dt,
\end{aligned}$$

where we have used Lemma 2. Therefore, $\partial_t^\alpha u = v$ in a weak sense.

Step 2. Fix an integer Λ and choose a function $w \in H_0^1(\Omega)$ of the form

$$\omega(x) = \sum_{k=1}^{\Lambda} \gamma_k e_k(x), \tag{21}$$

where $\{\gamma_k\}$ are arbitrary numbers. We select $n \geq \Lambda$, multiply (6) by γ_k and sum it up from 1 to Λ. Then we proceed to multiply the equation by $\varrho_\varepsilon(t+\tau)$ for fixed $\tau \in (0,T)$ and integrate with respect to t to discover

$$\begin{aligned}
&\int_0^T \varrho_\varepsilon(t+\tau)\int_\Omega \partial_t^\alpha u_n(t,x)\omega(x)dxdt + \int_0^T \varrho_\varepsilon(t+\tau)\mathcal{B}^n[u_n,\omega;t]dt \\
&= \int_0^T \varrho_\varepsilon(t+\tau)\int_\Omega f_{\frac{1}{n}}(t,x)\omega(x)dxdt.
\end{aligned} \tag{22}$$

For $\varepsilon < T-\tau$, we recall (20) to find that for a.e. $\tau \in (0,T)$,

$$\begin{aligned}
&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T \varrho_\varepsilon(t+\tau)\int_\Omega \partial_t^\alpha u_n(t,x)\omega(x)dxdt \\
&= -\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T {\varrho_\varepsilon}'(t+\tau)\int_\Omega \partial_t[{}_0I_t^{2-\alpha}(u_n-u_{n0}-u_{n1}t)]\omega(x)dx\,dt \\
&= -\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T {\varrho_\varepsilon}'(t+\tau)\int_\Omega {}_0I_t^{2-\alpha}(\partial_t u_n-u_{n1})\omega(x)dx\,dt \\
&= -\lim_{\varepsilon\to0}\int_0^T {\varrho_\varepsilon}'(t+\tau)\int_\Omega {}_0I_t^{2-\alpha}(\partial_t u-u_1)\omega(x)dx\,dt \\
&= \lim_{\varepsilon\to0}\int_0^T \varrho_\varepsilon(t+\tau)\int_\Omega \partial_t^\alpha u(t,x)\omega(x)dx\,dt \\
&= \int_\Omega \partial_t^\alpha u(\tau,x)\omega(x)dx.
\end{aligned}$$

We proceed similarly with the remaining terms. We see that $\varrho_\varepsilon(t+\tau)\partial_i\omega(x)$ is smooth in $(0,T)\times\Omega$. From assumptions $a^n_{i,j}\in L^\infty((0,T)\times\Omega)$ thus $a^n_{i,j}\to a_{i,j}$ in $L^2((0,T)\times\Omega)$, and $\partial_j u_n(x,t)\rightharpoonup\partial_j u(x,t)$ in $L^2((0,T)\times\Omega)$ when $n\to\infty$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega a^n_{i,j}(t,x)\partial_j u_n(t,x)\cdot\partial_i\omega(x)dx\,dt\\=&\lim_{\varepsilon\to0}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega a_{i,j}(t,x)\partial_j u(t,x)\cdot\partial_i\omega(x)dx\,dt\\=&\int_\Omega a_{i,j}(\tau,x)\partial_j u(\tau,x)\cdot\partial_i\omega(x)dx.\end{aligned}$$

Similarly, since $b^n_j(t)\to b_j$ in $L^\infty((0,T)\times\Omega)$ and $c^n(t)\to c$ in $L^\infty(0,T,L^{\frac{2q}{q-2}}(\Omega))$, thus $b^n_j(t)\to b_j$ and $c^n(t)\to c$ in $L^2((0,T)\times\Omega)$, it, together with (19), follows that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega b^n_j(t,x)\partial_j u_n(t,x)\cdot\omega(x)dxdt=\int_\Omega b_j(\tau,x)\partial_j u(\tau,x)\omega(x)dx,$$
$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega c^n(t,x)u_n(t,x)\cdot\omega(x)dxdt=\int_\Omega c(\tau,x)u(\tau,x)\omega(x)dx.$$

Therefore, one can find that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T\varrho_\varepsilon(t+\tau)\mathcal{B}^n[u_n,\omega;t]dt=\mathcal{B}[u,\omega;\tau].$$

Moreover, we can derive that for a.e. $\tau\in(0,T)$,

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega f_{\frac1n}(t,x)\omega(x)dxdt=&\lim_{\varepsilon\to0}\int_0^T\varrho_\varepsilon(t+\tau)\int_\Omega f(t,x)\omega(x)dxdt\\=&(f(\tau,\cdot),\omega).\end{aligned}$$

Therefore the following equality holds

$$\langle\partial_t^\alpha u(t,\cdot),\omega\rangle+\mathcal{B}[u,\omega;t]=(f(t,\cdot),\omega)\tag{23}$$

for $\omega=\sum_{k=1}^\Lambda\gamma_k e_k(x)$ and a.e. $t\in(0,T)$, since functions of the form (21) are dense in $H^1_0(\Omega)$, then the above equality also holds for all $\omega\in H^1_0(\Omega)$ and a.e. $t\in(0,T)$.

Finally, we note that

$$\int_0^t\|f_{\frac1n}(s,\cdot)\|^2ds\le\int_0^t\|f(s,\cdot)\|^2ds+\int_t^{t+\frac1n}\|f(s,\cdot)\|^2ds,$$

by the assumption of f, we know that $\sup_{t\in(0,T-\frac1n)}\int_t^{t+\frac1n}\|f(s,\cdot)\|^2ds\to0$ uniformly with respect to t as $n\to\infty$. Therefore, Lemmas 5 and 6 produce estimate (18). □

Remark 2. *If we assume that the coefficients $a_{i,j},b_j,c\in C^2((0,T],L^\infty(\Omega))$ and $f\in C^2((0,T],L^2(\Omega))$, then the mollification arguments imposed to the coefficients a,b,c, and f can be avoided. Similar to the proof that we derived in Theorem 2, the existing result is obtained. For similar results, we also refer to [27].*

Theorem 3. *Under the assumptions of Theorem 2, we suppose that $b_j\in W^{1,\infty}(0,T,W^{1,\infty}(\Omega))$, $c\in W^{1,\infty}(0,T,L^{\frac{2q}{q-2}}(\Omega))$. Then a weak solution u of Equation (1) is unique.*

Proof. It suffices to show that the only weak solution of (1) with $f \equiv u_0 \equiv u_1 \equiv 0$ is $u \equiv 0$. To verify this, fix $0 \le t \le T$ and set $\omega(\tau) = \int_\tau^t u(s,\cdot)ds$ if $0 \le \tau \le t$ and $\omega(t) = 0$ if $0 \le t \le \tau \le T$. Then $\omega(\tau) \in H_0^1(\Omega)$ for each $\tau \in [0,T]$ and we have

$$\int_0^t \langle \partial_\tau^\alpha u(\tau,\cdot), \omega(\tau)\rangle + \mathcal{B}[u,\omega;\tau]d\tau = 0.$$

Since $\partial_t u(0,\cdot) = \omega(t) = 0$, then $\partial_t^\alpha u = \partial_t \partial_t^{\alpha-1} u$, and so we obtain after integrating by parts in the first term above

$$\int_0^t -(\partial_\tau^{\alpha-1} u(\tau,\cdot), \omega'(\tau)) + \mathcal{B}[u,\omega;\tau]d\tau = 0.$$

Now $\omega' = -u$ for $0 \le \tau \le t \le T$, and then

$$\int_0^t (\partial_\tau^{\alpha-1} u(\tau,\cdot), u(\tau,\cdot)) - \mathcal{B}[\omega',\omega;\tau]d\tau = 0.$$

From Lemma 3 and the decreasing property of $g_{2-\alpha}$ we know

$$\begin{aligned}\int_0^t (\partial_\tau^{\alpha-1} u(\tau,\cdot), u(\tau,\cdot))d\tau \ge& \frac{1}{2}[g_{2-\alpha} * \|u(\tau,\cdot)\|^2](t) + \frac{g_{2-\alpha}(t)}{2}\int_0^t \|u(\tau,\cdot)\|^2 d\tau \\ \ge& \frac{g_{2-\alpha}(t)}{2}\int_0^t \|u(\tau,\cdot)\|^2 d\tau.\end{aligned}$$

Thus,

$$\frac{g_{2-\alpha}(t)}{2}\int_0^t \|u(\tau,\cdot)\|^2 d\tau - \frac{1}{2}\int_0^t \partial_\tau \mathcal{B}[\omega,\omega;\tau]d\tau \le \frac{1}{2}\int_0^t -\mathcal{C}[\omega,\omega;\tau] + \mathcal{D}[u,\omega;\tau]d\tau,$$

due to $2\mathcal{B}[\omega',\omega;\tau] = \partial_\tau \mathcal{B}[\omega,\omega;\tau] - \mathcal{C}[\omega,\omega;\tau] - \mathcal{D}[u,\omega;\tau]$, where

$$\begin{aligned}\mathcal{C}[u,v;\tau] &= \int_\Omega \sum_{i,j=1}^N \partial_\tau a_{i,j}(\tau,x)\partial_j u \cdot \partial_i v - \sum_{j=1}^N \partial_\tau b_j(\tau,x)\partial_j uv - \partial_\tau c(\tau,x)uvdx, \\ \mathcal{D}[u,v;\tau] &= \int_\Omega \sum_{j=1}^N \partial_j b_j(\tau,x)uv + 2\sum_{j=1}^N b_j(\tau,x)\partial_j vudx\end{aligned}$$

for $u,v \in H_0^1(\Omega)$. Since

$$\begin{aligned}|\mathcal{C}[\omega,\omega;\tau]| \le& \|\partial_\tau a_{i,j}(\tau)\|_{L^\infty}\|\nabla\omega\|^2 + \|\nabla\omega\|\|\omega\|\|\partial_\tau b_j(\tau)\|_{L^\infty} + \|\omega\|\|\omega\|_{L^q}\|\partial_\tau c(\tau)\|_{L^{\frac{2q}{q-2}}} \\ \le& \|\partial_\tau a_{i,j}(\tau)\|_{L^\infty(\Omega)}\|\nabla\omega\|^2 + \|\nabla\omega\|^2 + \|\omega\|^2\|\partial_\tau b_j(\tau)\|_{L^\infty}^2 \\ &+ \|\omega\|^2 + C^2(q,N,\Omega)\|\nabla\omega\|^2\|\partial_\tau c(\tau)\|^2_{L^{\frac{2q}{q-2}}} \\ \le& C\|\omega\|^2_{H_0^1},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{D}[u,\omega;\tau]| \le& \|\partial_j b_j(\tau)\|_{L^\infty}\|u(\tau,\cdot)\|\|\omega\| + 2\|b_j(\tau)\|_{L^\infty}\|D\omega\|\|u(\tau,\cdot)\| \\ \le& \|u(\tau,\cdot)\|^2 + \|\partial_j b_j(\tau)\|^2_{L^\infty}\|\omega\|^2 + 2\|D\omega\|^2 + 2\|b_j(\tau)\|^2_{L^\infty}\|u(\tau,\cdot)\|^2 \\ \le& C(\|\omega\|^2_{H_0^1} + \|u(\tau,\cdot)\|^2).\end{aligned}$$

Hence,

$$\frac{g_{2-\alpha}(t)}{2}\int_0^t \|u(\tau,\cdot)\|^2 d\tau + \frac{1}{2}\mathcal{B}[\omega(0),\omega(0);t] \le C\int_0^t \|\omega(\tau)\|_{H_0^1}^2 + \|u(\tau,\cdot)\|^2 d\tau,$$

which together with

$$\begin{aligned}\mathcal{B}[\omega(0),\omega(0);t] \ge&\mu\|\nabla\omega(0)\|^2 - \|\nabla\omega(0)\|\|\omega(0)\|\|b_j(t)\|_{L^\infty} - \|\omega(0)\|\|\omega(0)\|_{L^q}\|c(t)\|_{L^{\frac{2q}{q-2}}}\\ \ge&\mu\|\nabla\omega(0)\|^2 - (\frac{\mu}{4}\|\nabla\omega(0)\|^2 + \frac{1}{\mu}\|\omega(0)\|^2\|b_j(t)\|_{L^\infty}^2)\\ &-(\frac{1}{4\varepsilon}\|\omega(0)\|^2 + \varepsilon C^2(q,N,\Omega)\|c\|^2_{L^\infty(0,T,L^{\frac{2q}{q-2}})}\|\nabla\omega(0)\|^2)\\ \ge&\frac{\mu}{2}\|\omega(0)\|_{H_0^1}^2 - C\|\omega(0)\|^2\end{aligned}$$

for $\varepsilon = \frac{\mu}{4}\frac{1}{C^2(q,N,\partial\Omega)\|c\|^2_{L^\infty(0,T,L^{\frac{2q}{q-2}})}}$, shows that

$$\begin{aligned}&g_{2-\alpha}(t)\int_0^t \|u(\tau,\cdot)\|^2 d\tau + \|\omega(0)\|_{H_0^1}^2\\ \le& C\left(\int_0^t (\|\omega(\tau)\|_{H_0^1}^2 + \|u(\tau,\cdot)\|^2)d\tau + \|\omega(0)\|^2\right).\end{aligned} \tag{24}$$

Let us write

$$W(t) := \int_0^t u(\tau,\cdot)d\tau, \;\; t\in[0,T],$$

whereupon (24) becomes

$$\begin{aligned}&g_{2-\alpha}(t)\int_0^t \|u(\tau,\cdot)\|^2 d\tau + \|W(t)\|_{H_0^1}^2\\ \le& C\left(\int_0^t (\|W(t)-W(\tau)\|_{H_0^1}^2 + \|u(\tau,\cdot)\|^2)d\tau + \|W(t)\|^2\right).\end{aligned}$$

Since $\|W(t)-W(\tau)\|_{H_0^1}^2 \le 2\|W(\tau)\|_{H_0^1}^2 + 2\|W(t)\|_{H_0^1}^2$, and $\|W(t)\| \le \int_0^t \|u(\tau,\cdot)\|d\tau$, we can derive

$$g_{2-\alpha}(t)\int_0^t \|u(\tau,\cdot)\|^2 d\tau + (1-2tC_1)\|W(t)\|_{H_0^1}^2 \le C_1\int_0^t (\|W(\tau)\|_{H_0^1}^2 + \|u(\tau,\cdot)\|^2)d\tau.$$

Choose T_1 so mall that

$$\frac{g_{2-\alpha}(T_1)}{2} \ge C_1 \;\text{ and }\; 1-2T_1C_1 \ge \frac{1}{2}.$$

Then if $0<t\le T_1$, we have

$$[g_{2-\alpha}(t) - C_1]\int_0^t \|u(\tau,\cdot)\|^2 d\tau + \frac{1}{2}\|W(t)\|_{H_0^1}^2 \le C_1\int_0^t \|W(\tau)\|_{H_0^1}^2 d\tau.$$

Consequently, Gronwall's inequality implies $W(t)\equiv 0$ on $[0,T_1]$. Then $\int_0^t \|u(\tau)\|^2 d\tau \equiv 0$ on $[0,T_1]$. This, together with the continuity of u, shows $u(t)\equiv 0$ on $[0,T_1]$.

We use the same argument on $[T_1,T_2]$, $[T_2,T_3]$,..., and then we can deduce $u\equiv 0$. □

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