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Extended Comparison between Two Derivative-Free Methods of Order Six for Equations under the Same Conditions

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Abstract: Under the same conditions, we propose the extended comparison between two derivative free schemes of order six for addressing equations. The existing convergence technique used the standard Taylor series approach, which requires derivatives up to order seven. In contrast to previous researchers, our convergence theorems only demand the first derivative. In addition, formulas for determining the region of uniqueness for the solution, convergence radii, and error estimations are suggested. As a consequence, we broaden the utility of these productive schemes. Moreover, we present a comparison of attraction basins for these schemes to obtain roots of complex polynomial equations. The confirmation of our convergence findings on application problems brings this research to a close.

Keywords: nonlinear models; divided difference; derivative free method; sixth order convergence; convergence ball

MSC: 37N30; 47J25; 49M15; 65H10; 65J15

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1. Introduction

Let T_1 and T_2 denote normed linear spaces which are complete. Suppose $A \subseteq T_1$ is non-null, open and convex. Nonlinear equations of the type [1–4]

$$L(t) = 0, \tag{1}$$

where $L : A \subseteq T_1 \to T_2$ is derivable as per Fréchet, may be used to simulate a wide range of complex scientific and engineering issues. The closed version of the solution t_* can be determined only in some special cases. The employment of iterative algorithms to conclude is common among scientists and researchers because of this. Newton's method is a popular iterative process for dealing with nonlinear equations. Many novels and higher-order iterative strategies for dealing with nonlinear equations have been discovered and are currently being used in recent years [5–8]. However, the theorems on the convergence of these schemes in most of these publications are derived by applying high-order derivatives. Furthermore, no results are discussed regarding the error distances, radii of convergence, or the region in which the solution is the only one.

In research work of iterative procedures, it is crucial to determine the region where convergence is possible. Most of the time, the convergence zone is rather small. It is required to broaden the convergence domain without making any extra assumptions. Likewise, while investigating the convergence of iterative algorithms, exact error distances must be estimated. Taking these points into consideration, we develop convergence theorems for two methods GM_6 (2) and SM_6 (3) proposed in [9,10], respectively. Let

$$u_n = t_n + L(t_n),$$
$$s_n = t_n - L(t_n)$$

and $[u_n, s_n; L]$ is a first order divided difference [2,11], i.e., $[\cdot,\cdot;L] : A \times A \to L(T_1, T_2)$ denoted the space of continuous linear operators mapping T_1 to T_2, $[\cdot,\cdot;L] : A \to L(T_1, T_2)$ and for I denoting the identity operator on T_1,

$$\begin{aligned} y_n &= t_n - [u_n, s_n; L]^{-1} L(t_n), \\ z_n &= y_n - A_n^{-1} L(y_n), \\ t_{n+1} &= z_n - A_n^{-1} L(z_n), \\ A_n &= 2[y_n, t_n; L] - [u_n, s_n; L], \end{aligned} \tag{2}$$

The convergence order is four for the two-step methods (2) and (3) and the order is six for the complete three step methods (2) and (3). The development, comparison, and performance of the four and six-order methods were also given in [9,10]

$$\begin{aligned} y_n &= t_n - [u_n, s_n; L]^{-1} L(t_n), \\ z_n &= y_n - (3I - 2[u_n, s_n; L]^{-1}[y_n, t_n; L])[u_n, s_n; L]^{-1} L(y_n), \\ t_{n+1} &= z_n - (3I - 2[u_n, s_n; L]^{-1}[y_n, t_n; L])[u_n, s_n; L]^{-1} L(z_n). \end{aligned} \tag{3}$$

Convergence works of these algorithms [9,10] are based on derivatives of L up to order seven and offer only a convergence rate. As a consequence, the productivity of these schemes is limited. To observe this, we define L on $A = [-\frac{1}{2}, \frac{3}{2}]$ by

$$L(t) = \begin{cases} 0, & \text{if } \ t = 0 \\ t^3 \ln(t^2) + t^5 - t^4, & \text{if } \ t \neq 0 \end{cases}. \tag{4}$$

Due to the unboundedness of L''' the results on the convergence of GM_6 [9] and SM_6 [10] do not stand true for this example. Furthermore, these articles do not produce any formula for approximating the error $\|t_n - t_*\|$, the convergence region, or the uniqueness and accurate location of t_*. The same approach applies to other methods with inverses such as [8,12–19]. This encourages us to develop the ball convergence theorems and hence compare the convergence domains of GM_6 and SM_6 by considering assumptions only on L'. Our research provides important formulas for the estimation of $\|t_n - t_*\|$ and convergence radii. This study also discusses an exact location and the uniqueness of t_*. Furthermore, a visual process, called the attraction basin, is utilized to compare the convergence regions of these algorithms.

The other contents include follow: In Section 2, theorems on GM_6 and SM_6 are given. Section 3 describes the comparison of the attraction basins. Numerical testing of convergence outcomes is placed in Section 4. Concluding remarks are also stated.

2. Local Analysis

The local analysis is presented in this section for the methods GM_6 and SM_6, respectively. This analysis used real parameters and real functions. Let $D = [0, \infty)$ and $a \geq 0$, $b \geq 0$, $c \geq 0$.

Assume function:

(i)

$$B_0(ax, bx) - 1$$

has a minimal root $R_0 \in D_0 \setminus \{0\}$ for some continuous and non-decreasing function $B_0 : D \times D \to D$. Let $D_0 = [0, R_0)$.

(ii)
$$N_1(x) - 1$$
has a minimal root $r_1 \in D_0 \setminus \{0\}$, with the function $B : D_0 \times D_0 \to D$ being non-decreasing and continuous and non-decreasing, and $N_1 : D_0 \to D$ is given as
$$N_1(x) = \frac{B(cx, bx)}{1 - B_0(ax, bx)}.$$

(iii)
$$p(x) - 1$$
has a minimal root $R_p \in D_0 \setminus \{0\}$, where $p : D_0 \to D$ is given as
$$p(x) = 2B_0(N_1(x)x, x) + B_0(ax, bx).$$
Set $R_1 = \min\{R_0, R_p\}$ and $D_1 = [0, R_1)$.

(iv)
$$N_2(x) - 1$$
has a minimal root $r_2 \in D_1 \setminus \{0\}$, where $N_2 : D_1 \to D$ is given as
$$N_2(x) = \left[\frac{B((a + N_1(x))x, bx)}{1 - B_0(ax, bx)} + \frac{2cB_1((a + N_1(x))x, cx)}{(1 - B_0(ax, bx))(1 - p(x))}\right] N_1(x)$$
for some function $B_1 : D_1 \times D_1 \to D$ which is continuous and non-decreasing.

(v)
$$N_3(x) - 1$$
has a minimal root $r_3 \in D_1 \setminus \{0\}$, where $N_3 : D_1 \to D$ is given as
$$N_3(x) = \left[\frac{B((a + N_2(x))x, bx)}{1 - B_0(ax, bx)} + \frac{2cB_1((a + N_1(x))x, cx)}{(1 - B_0(ax, bx))(1 - p(x))}\right] N_2(x)$$
Let
$$r = \min\{r_m\}, m = 1, 2, 3. \tag{5}$$
Let $D_2 = [0, r)$. Notice that for each $x \in D_2$
$$0 \le B_0(ax, bx) < 1 \tag{6}$$
$$0 \le p(x) < 1 \tag{7}$$
and
$$0 \le N_m(x) < 1. \tag{8}$$

We utilize the condition (C) provided x_* is a simple root of L and the functions "B" is as given above.

(C_1)
$$\|I + [t, x_*; L]\| \le a$$
$$\|I - [t, x_*; L]\| \le b$$
and
$$\|L'(x_*)^{-1}([t, y; L] - L'(x_*))\| \le B_0(\|x_* - t\|, \|x_* - y\|)$$
hold for each $t, y \in A$. Let $A_1 = D(x_*, R_0) \cap A$.

(C_2)
$$\|L'(x_*)^{-1}([t, x_*; L] - [u, s; L])\| \le B(\|u - t\|, \|s - x_*\|),$$
$$\|L'(x_*)^{-1}([t, y; L] - [u, s; L])\| \le B_1(\|u - t\|, \|s - y\|)$$

and

$$\|L'(x_*)^{-1}L(t)\| \le c\|x_* - t\|$$

hold for $t, u, s, y \in A_0$.

(C_3) $\overline{D}(x_*, R) \subset A$, where parameter $R = \max\{\tilde{r}, a\tilde{r}, b\tilde{r}, c\tilde{r}\}$ and $\tilde{r}$ is given later.

(C_4) There exist $r_* \ge \tilde{r}$ satisfying $B_0(0, \tilde{r}) < 1$ or $B_0(\tilde{r}, 0) < 1$.

Let $A_1 = \overline{D}(x_*, \tilde{r}) \cap A$.

Next, conditions (C) are needed to prove the local convergence analysis of method GM_6.

Theorem 1. *Assume conditions (C) hold for $\tilde{r} = r$. Then, we have $\lim_{n\to\infty} t_n = x_*$, provided $t_0 \in D(x_*, \tilde{r}) \setminus \{x_*\}$ and the only root of L in the set A_1 is x_*.*

Proof. Items

$$\|y_n - x_*\| \le N_1(\|t_n - x_*\|)\|x_* - t_n\| \le \|x_* - t_n\| < r \tag{9}$$

$$\|z_n - x_*\| \le N_2(\|x_* - t_n\|)\|x_* - t_n\| \le \|x_* - t_n\| \tag{10}$$

and

$$\|t_{n+1} - x_*\| \le N_3(\|x_* - t_n\|)\|x_* - t_n\| \le \|x_* - t_n\|, \tag{11}$$

shall be proven, where the radius r is given in (2) and function N_m are as previously defined. By hypothesis $t_0 \in D(x_*, r) \setminus \{x_*\}$.

It follows by (C_1), and (C_3) that

$$\begin{aligned}\|u_0 - x_*\| &= \|x_* - t_0 + L(t_0)\| \\ &= \|(I + [t_0, x_*; L])(x_* - t_0)\| \\ &\le \|I + [t_0, x_*; L]\|\ \|x_* - t_0\| \le a\|x_* - t_0\| < ar < R \end{aligned}\tag{12}$$

$$\begin{aligned}\|s_0 - x_*\| &= \|x_* - t_0 - L(t_0)\| \\ &= \|(I - [t_0, x_*; L])(x_* - t_0)\| \\ &\le \|I - [t_0, x_*; L]\|\ \|x_* - t_0\| \le b\|x_* - t_0\| < br < R \end{aligned}\tag{13}$$

and

$$\begin{aligned}\|L'(x_*)^{-1}([u_0, s_0; L] - L'(x_*))\| &\le B_0(\|u_0 - x_*\|, \|s_0 - x_*\|) \\ &\le B_0(a\|x_* - t_0\|, b\|x_* - t_0\|) \le B_0(ar, br) < 1. \end{aligned}\tag{14}$$

Estimate (14) with a lemma due to Banach on linear operators with inverses [2,11] give $[u_0, s_0; L]^{-1} \in L(T_2, T_1)$, and

$$\|[u_0, s_0; L]^{-1}L'(x_*)\| \le \frac{1}{1 - B_0(\|u_0 - x_*\|, \|s_0 - x_*\|)}. \tag{15}$$

It also follows by (15) and the first substep of method GM_6 that iterate y_0 is well defined, and

$$\begin{aligned} y_0 - x_* &= x_* - t_0 - [u_0, s_0; L]^{-1}L(t_0) \\ &= [u_0, s_0; L]^{-1}([u_0, s_0; L] - [t_0, x_*; L])(x_* - t_0). \end{aligned}\tag{16}$$

Using (2), (8) (for $m = 1$), (C_2), (15) and (16),

$$\begin{aligned} \|y_0 - x_*\| &\le \frac{B(\|u_0 - t_0\|, \|s_0 - x_*\|)\|x_* - t_0\|}{1 - B_0(a\|x_* - t_0\|, b\|x_* - t_0\|)} \\ &\le N_1(\|x_* - t_0\|)\|x_* - t_0\| \le \|x_* - t_0\| < r, \end{aligned}\tag{17}$$

proving (9) if $n = 0$ and that the iterate $y_0 \in D(x_*, r)$.

Next, we prove $A_0^{-1} \in L(T_2, T_1)$. By (2), (7) and (17),

$$\begin{aligned}
\|L'(x_*)^{-1}(A_0 - L'(x_*))\| &\leq 2\|L'(x_*)^{-1}([y_0, t_0; L] - L'(x_*))\| \\
&+ \|L'(x_*)^{-1}([u_0, s_0; L] - L'(x_*))\| \\
&\leq 2B_0(\|y_0 - x_*\|, \|x_* - t_0\|) + B_0(\|u_0 - x_*\|, \|s_0 - x_*\|) \\
&\leq 2B_0(N_1(\|x_* - t_0\|)\|x_* - t_0\|, \|x_* - t_0\|) \\
&+ B_0(a\|x_* - t_0\|, b\|x_* - t_0\|) = p(\|x_* - t_0\|) \\
&\leq p(r) < 1,
\end{aligned} \tag{18}$$

so

$$\|A_0^{-1}L'(x_*)\| \leq \frac{1}{1 - p(\|x_* - t_0\|)}. \tag{19}$$

Hence, the iterate z_0 exists given GM_6. Moreover, we get

$$\begin{aligned}
z_0 - x_* &= y_0 - x_* - [u_0, s_0; L]^{-1}L(y_0) \\
&+ [u_0, s_0; L]^{-1}(A_0 - [u_0, s_0; L])A_0^{-1}L(y_0).
\end{aligned} \tag{20}$$

Then, it follows by (2), (8) (for $m = 2$), (15), (C_2), (C_3), (17), (19) and (20),

$$\begin{aligned}
\|z_0 - x_*\| &\leq \Bigg[\frac{B(\|u_0 - y_0\|, \|s_0 - x_*\|)}{1 - B_0(a\|x_* - t_0\|, b\|x_* - t_0\|)} \\
&+ \frac{2cB_1(\|y_0 - u_0\|, \|t_0 - s_0\|)}{(1 - B_0(a\|x_* - t_0\|, b\|x_* - t_0\|))(1 - p(\|x_* - t_0\|)}\Bigg]\|y_0 - x_*\| \\
&\leq N_2(\|x_* - t_0\|)\|x_* - t_0\| \leq \|x_* - t_0\|,
\end{aligned} \tag{21}$$

proving (10) if $n = 0$ and the iterates $z_0 \in D(x_*, r)$. The iterate t_1 is well defined by the third substep of method GM_6. Furthermore, as in (20) and (21)), we write

$$\begin{aligned}
t_1 - x_* &= z_0 - x_* - [u_0, s_0; L]^{-1}L(z_0) \\
&+ ([u_0, s_0; L]^{-1} - A_0^{-1})L(z_0) \\
&= z_0 - x_* - [u_0, s_0; L]^{-1}L(z_0) \\
&+ [u_0, s_0; L]^{-1}(A_0 - [u_0, s_0; L])A_0^{-1})L(z_0).
\end{aligned} \tag{22}$$

By using (2), (8) (for $m = 2$), (15), (17), (19), (21) and (22),

$$\begin{aligned}
\|t_1 - x_*\| &\leq \Bigg[\frac{B(\|u_0 - z_0\|, \|s_0 - x_*\|)}{1 - B_0(a\|x_* - t_0\|, b\|x_* - t_0\|)} \\
&+ \frac{2cB_1(\|y_0 - u_0\|, \|t_0 - s_0\|)}{(1 - B_0(a\|x_* - t_0\|, b\|x_* - t_0\|))(1 - p(\|x_* - t_0\|)}\Bigg]\|z_0 - x_*\| \\
&\leq N_3(\|x_* - t_0\|)\|x_* - t_0\| \leq \|x_* - t_0\|,
\end{aligned} \tag{23}$$

proving (11) if $n = 0$ and that the iterate $t_1 \in D(x_*, r)$. Simply exchange u_0, s_0, t_0, y_0, z_0, t_1 by u_i, s_i, t_i, y_i, z_i, t_{i+1} in the above calculations, the induction for (9)–(11) is done. Then, from the inequality

$$\|t_{i+1} - x_*\| \leq \rho\|t_i - x_*\| < r, \tag{24}$$

we get $\lim_{i \to \infty} t_i = x_*$ and $t_{i+1} \in D(x_*, r)$.

Let $T = [x_*, q; L]$ for some $q \in A_1$ and $L(q) = 0$. Then, by (C_1) and (C_4), it follows that

$$\|L'(x_*)^{-1}(x_* - T))\| \leq B_0(0, \|x_* - q\|) \leq B_0(0, r_*) < 1,$$

which implies $q = x_*$, since $T^{-1} \in L(B_2, B_1)$ and $0 = L(x_*) - L(q) = T(x_* - q)$. $\square$

Next, the local analysis of method SM_6 follows analogously. However, this time the "N_i" functions are given as

$$\overline{N_2}(x) = \left[\frac{B((a+N_1(x))x, bx)}{1 - B_0(ax, bx)} + \frac{2cB_1(cx, cx)}{(1 - B_0(ax, bx))^2}\right] N_1(x)$$

and

$$\overline{N_3}(x) = \left[\frac{B((a+\overline{N_2}(x))x, cx)}{1 - B_0(ax, bx)} + \frac{2cB_1((a+N_1(x))x, cx)}{(1 - B_0(ax, bx))^2}\right] \overline{N_2}(x),$$

and

$$\bar{r} = \min\{r_1, \overline{r_2}, \overline{r_3}\}, \tag{25}$$

where $\overline{r_2}, \overline{r_3}$ are the minimal positive roots of $\overline{N_2}(x) - 1$, $\overline{N_3}(x) - 1$ (assumed to exist). These functions are motivated by the estimations (under conditions (C) with $\tilde{r} = \bar{r}$):

$$\begin{aligned} z_n - x_* &= y_n - x_* - [u_n, s_n; L]^{-1} L(y_n) \\ &\quad + 2(I - [u_n, s_n; L]^{-1}[y_n, t_n; L])[u_n, s_n; L]^{-1} L(y_n) \\ &= y_n - x_* - [u_n, s_n; L]^{-1} L(y_n) \\ &\quad + 2[u_n, s_n; L]^{-1}([u_n, s_n; L] - [y_n, t_n; L])[u_n, s_n; L]^{-1} L(y_n), \end{aligned}$$

so

$$\begin{aligned} \|z_n - x_*\| &\le \left[\frac{B(\|u_n - y_n\|, \|s_n - t_n\|)}{1 - B_0(a\|t_n - x_*\|, b\|t_n - x_*\|)} + \frac{2cB_1(\|u_n - t_n\|, \|s_n - t_n\|)}{(1 - B_0(a\|t_n - x_*\|, b\|t_n - x_*\|))^2}\right] \|y_n - x_*\| \\ &\le \overline{N_2}(\|t_n - x_*\|)\|t_n - x_*\| \le \|t_n - x_*\| < \bar{r}, \end{aligned}$$

and

$$\begin{aligned} t_{n+1} - x_* &= z_n - x_* - [u_n, s_n; L]^{-1} L(z_n) \\ &\quad + 2[u_n, s_n; L]^{-1}([u_n, s_n; L] - [y_n, t_n; L])[u_n, s_n; L]^{-1} L(z_n), \end{aligned}$$

so

$$\begin{aligned} \|t_{n+1} - x_*\| &\le \left[\frac{B(\|u_n - z_n\|, \|s_n - t_n\|)}{1 - B_0(a\|t_n - x_*\|, b\|t_n - x_*\|)} + \frac{2cB_1(\|u_n - y_n\|, \|s_n - t_n\|)}{(1 - B_0(a\|t_n - x_*\|, b\|t_n - x_*\|))^2}\right] \|z_n - x_*\| \\ &\le \overline{N_3}(\|t_n - x_*\|)\|t_n - x_*\| \le \|t_n - x_*\|. \end{aligned}$$

That is we have proven the corresponding local convergence analysis for method SM_6.

Theorem 2. *Assume conditions (C) hold for $\tilde{r} = \bar{r}$ provided that $t_0 \in D(x_*, \tilde{r}) \setminus \{x_*\}$. Then, the items of Theorem 2 hold for method SM_6 with $\bar{r}$, $\overline{N_2}$, $\overline{N_3}$ replacing r, N_2, N_3, respectively.*

3. Attraction Basins Comparison

For evaluating the convergence zones of iterative algorithms the basin of attraction is a valuable geometrical tool. These basins illustrate all the initial estimations that imply convergence to a root of an equation when an iterative approach is used, allowing us to see visually which places are suitable starters and which are not. Using this excellent tool, we compare the convergence areas of GM_6 and SM_6 for a variety of complex polynomials. With the starting point $z_0 \in W = [-2, 2] \times [-2, 2] \subset \mathbb{C}$, GM_6 and SM_6 used on polynomials with complex coefficients. The starter z_0 is in the basin of a root z_* of a test polynomial if $\lim_{m \to \infty} z_m = z_*$ and then a typical color associated with z_* is applied on z_0. Black color is applied on $z_0 \in W$ if $\{z_m\}$ diverges. To end the iteration process, the conditions $\|z_m - z_*\| < 10^{-6}$ or the maximum of 100 iterations is used. The fractal figures are created in MATLAB 2019a.

The experiment begins with the polynomials $F_1(z) = z^2 + 1$ and $F_2(z) = z^2 + z$ to design the basins of their roots. In Figures 1 and 2, yellow and magenta colors are associated with the roots i and $-i$, of $F_1(z)$, respectively. Figures 3 and 4 offer basins of roots -1 and 0 of $F_2(z)$ in magenta and green colors, respectively. Next, the polynomials $F_3(z) = z^3 + 1$ and $F_4(z) = z^3 + z$ are picked. Figures 5 and 6 give the attraction basins of roots $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, -1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ of $F_3(z)$ in cyan, yellow and magenta, respectively. In Figures 7 and 8, the basins of the roots 0, $-i$, and i of $F_4(z)$ are painted in cyan, yellow and magenta colors, respectively. Further, $F_5(z) = z^4 + 1$ and $F_6(z) = z^4 + z$ are chosen to decorate the attraction basins of their roots. In Figures 9 and 10, the basins of the roots $0.707106 + 0.707106i$, $0.707106 - 0.707106i$, $-0.707106 - 0.707106i$ and $-0.707106 + 0.707106i$ of $F_5(z) = 0$ are, respectively, indicated in green, blue, red and yellow zones. In Figures 11 and 12, convergence to the roots -1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and 0 of the polynomial $F_6(z)$ is presented in yellow, blue, green and red, respectively. Furthermore, $F_7(z) = z^5 + 1$ and $F_8(z) = z^5 + z$ are taken. In Figures 13 and 14, magenta, green, yellow, blue and red colors are applied to the basins of roots $0.809016 + 0.587785i$, $0.809016 - 0.587785i$, $-0.309016 - 0.951056i$, -1 and $-0.309016 + 0.951056i$, respectively, of $F_7(z)$. Figures 15 and 16 display the basins of the roots $0.707106 + 0.707106i$, 0, $-0.707106 + 0.707106i$, $-0.707106 - 0.707106i$ and $0.707106 - 0.707106i$ of $F_8(z) = 0$ in blue, green, magenta, yellow, and red colors, respectively. Lastly, we select $F_9(z) = z^6 + 1$ and $F_{10}(z) = z^6 + z$. In Figures 17 and 18, the basins of the roots $0.500000 - 0.866025i$, $0.500000 + 0.866025i$, $1i$, $-0.500000 - 0.866025i$, $-0.500000 + 0.866025i$ and $-1i$ of $F_9(z) = 0$ are illustrated in yellow, blue, green, magenta, cyan and red, respectively. Figures 19 and 20 give the basins of the roots -1, $-0.3090169 + 0.951056i$, 0, $-0.3090169 - 0.951056i$, $0.809016 + 0.587785i$ and $0.809016 - 0.587785i$ of $F_{10}(z) = 0$ in green, yellow, red, cyan, magenta and blue colors, respectively.

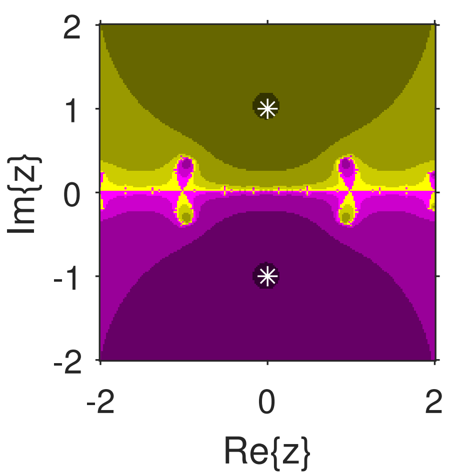


Figure 1. GM_6.

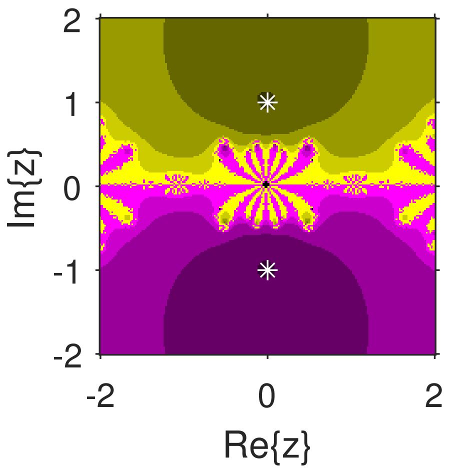


Figure 2. SM_6.

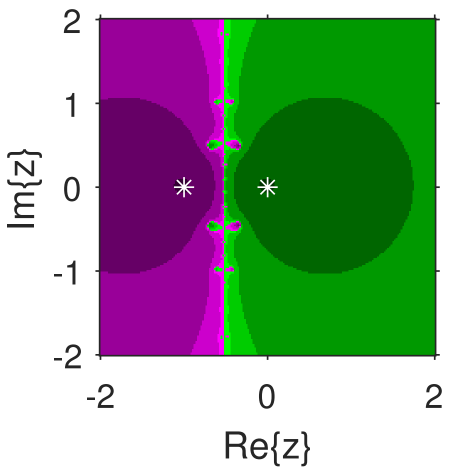


Figure 3. GM_6.

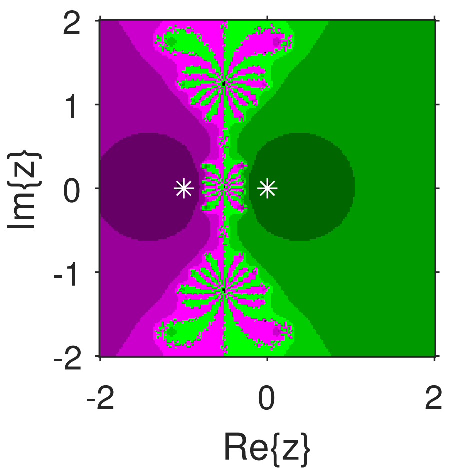


Figure 4. SM_6.

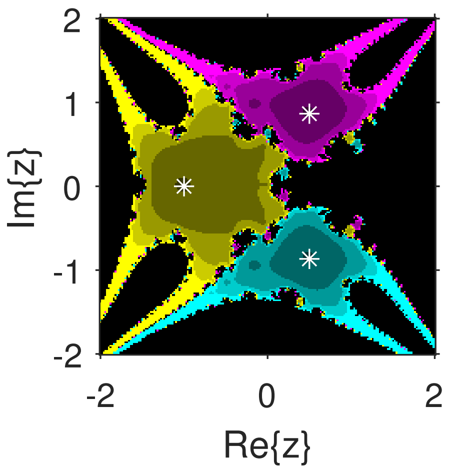


Figure 5. GM_6.

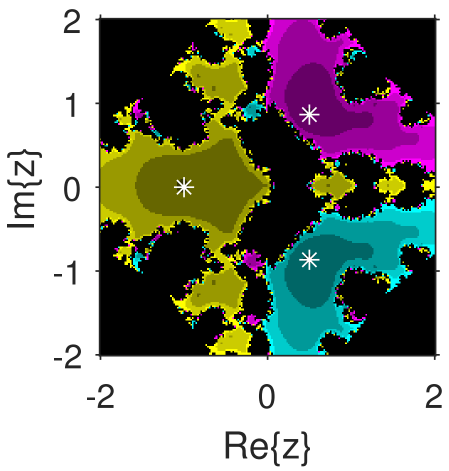


Figure 6. SM_6.

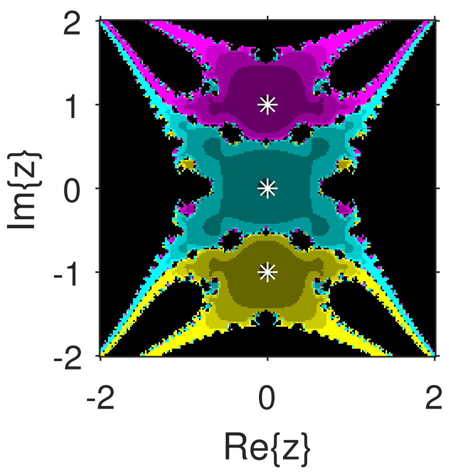


Figure 7. GM_6.

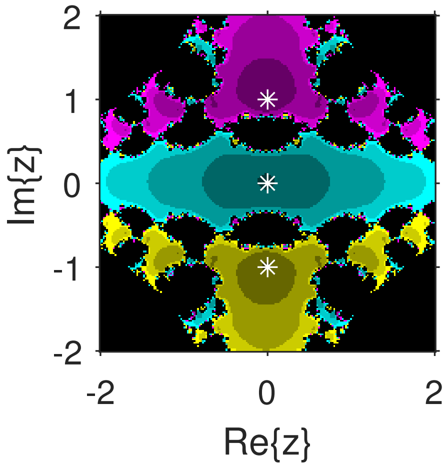


Figure 8. SM_6.

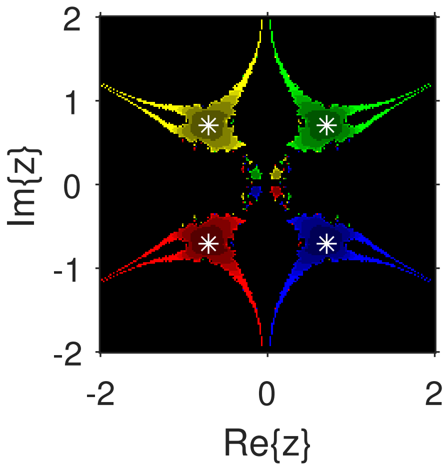


Figure 9. GM_6.

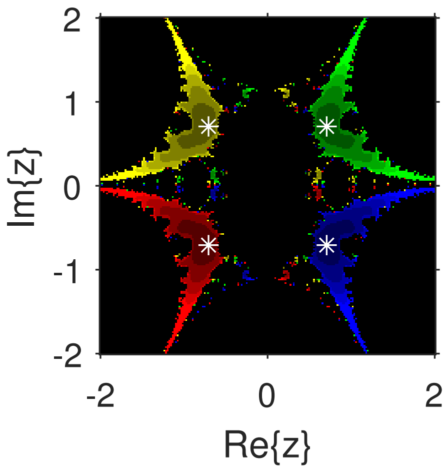


Figure 10. SM_6.

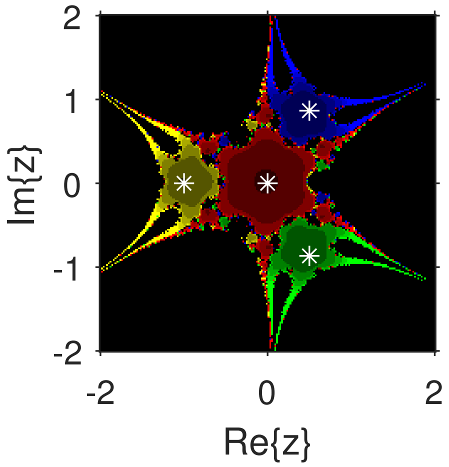


Figure 11. GM_6.

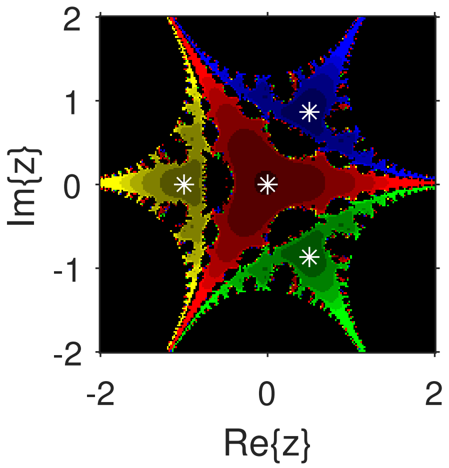


Figure 12. SM_6.

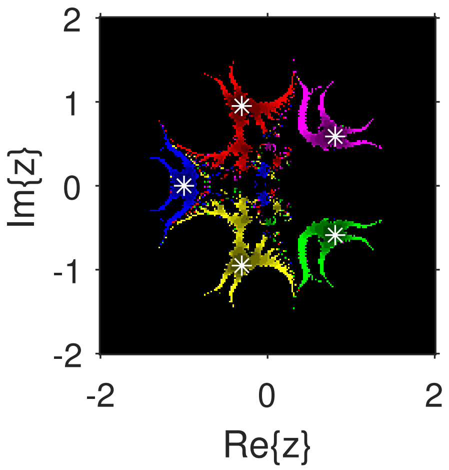


Figure 13. GM_6.

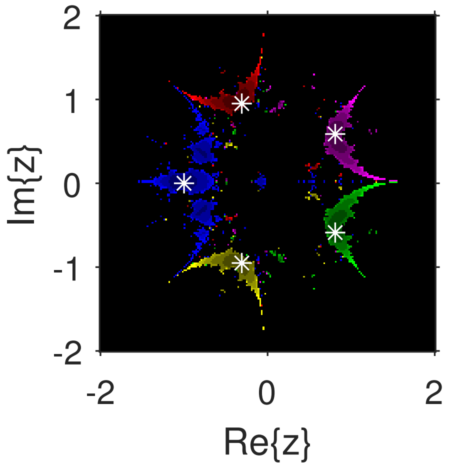


Figure 14. SM_6.

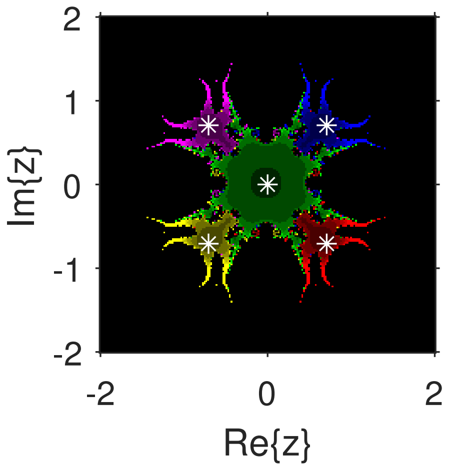


Figure 15. GM_6.

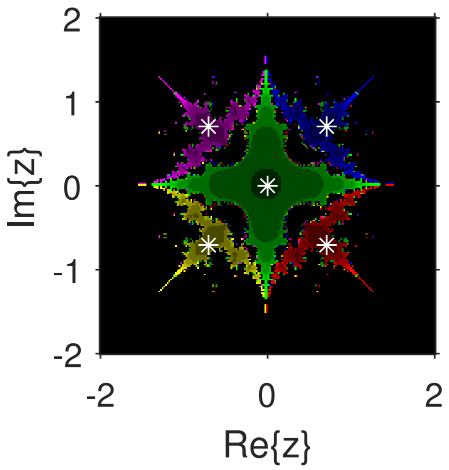


Figure 16. SM_6.

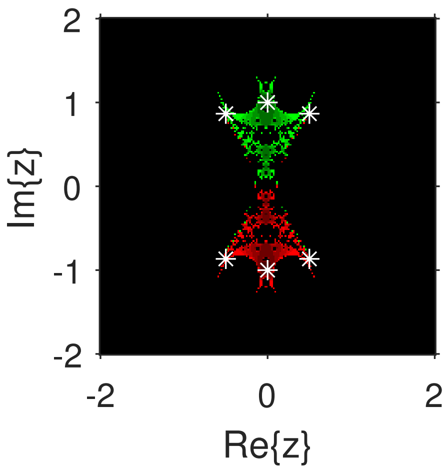


Figure 17. GM_6.

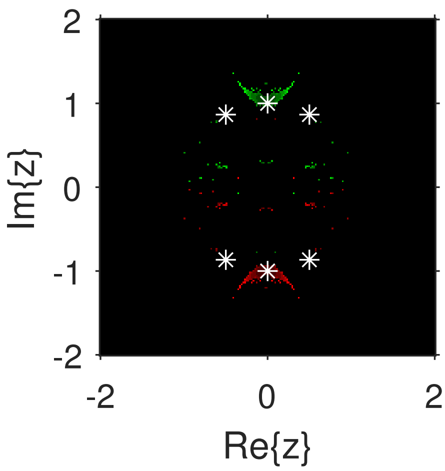


Figure 18. SM_6.

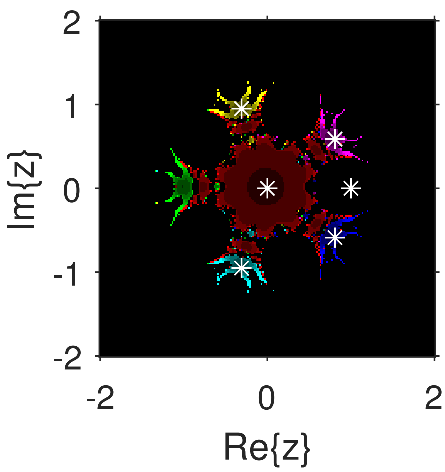


Figure 19. GM_6.

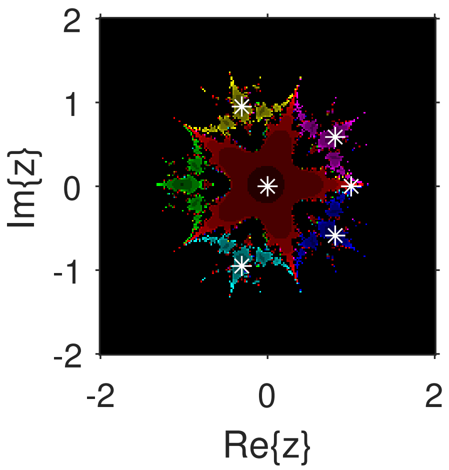
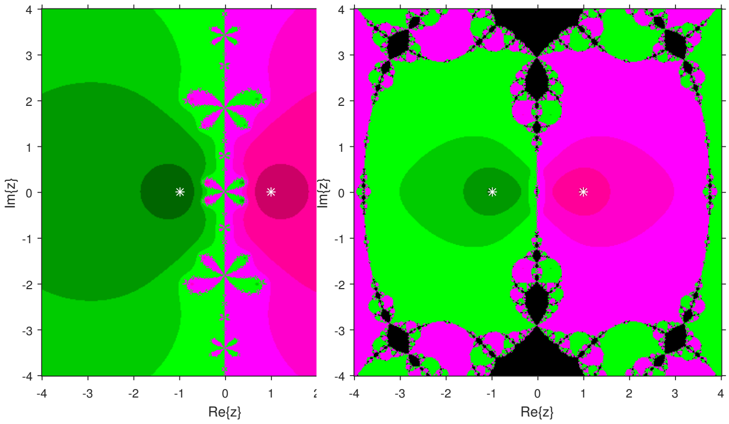


Figure 20. SM_6.

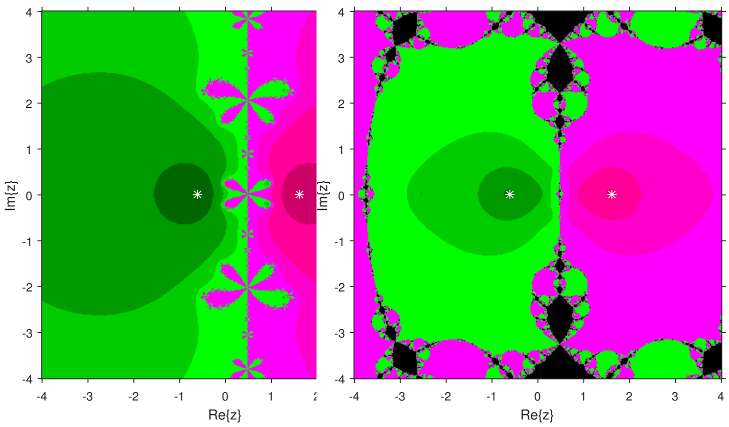
We consider polynomials $W_1(z) = z^2 - 1$ and $W_2(z) = z^2 - z - 1$ of degree two. The results of the comparison between attraction basins for (2) and (3) are displayed in Figures 21 and 22. In Figure 21, green and pink areas show the attraction basins corresponding to the roots -1 and 1, respectively, of $W_1(z)$. The basins of the roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ of $W_2(z) = 0$ are shown in Figure 22 by using pink and green colors, respectively. Figures 23 and 24 determine the attraction basins for (2) and (3) associated with the roots of $W_3(z) = z^3 + (-0.7250 + 1.6500i)z - 0.2750 - 1.6500i$ and $W_4(z) = z^3 - z$. The basins for (2) and (3) associated with the roots $1, -1.401440 + 0.915201i$ and $0.4014403 - 0.915201i$ of $W_3(z)$ are given in Figure 23 by means of green, pink and blue domains, respectively. In Figure 24, the basins of the roots 0, 1, and -1 of $W_4(z) = 0$ are painted in yellow, magenta and cyan, respectively. Next, we use polynomials $W_5(z) = z^4 - 10z^2 + 9$ and $W_6(z) = z^4 - z$ of degree four to compare the attraction basins for (2) and (3). The basins for (2) and (3) corresponding to the roots $-1, 3, -3$ and 1 of $W_5(z)$ are illustrated in Figure 25 using yellow, pink, green and blue colors, respectively. Figure 26 gives the comparison of basins for these schemes associated with the roots $0, 1, -\frac{1}{2} - \frac{3}{2}i$ and $-\frac{1}{2} + \frac{3}{2}i$ of $W_6(z) = 0$, which are denoted in green, blue, yellow and red regions, respectively. Moreover, we select polynomials $W_7(z) = z^5 + z$ and $W_8(z) = z^5 - 5z^3 + 4z$ of degree five to give and compare the attraction basins for (2) and (3). In Figure 27, green, cyan, red, pink and yellow regions illustrate the attraction basins of the roots $-0.707106 - 0.707106i, -0.707106 + 0.707106i, 0.707106 + 0.707106i, 0.707106 - 0.707106i$ and 0, respectively, of $W_7(z) = 0$. Figure 28 gives the basins of roots $0, 2, -1, -2$ and 1 of $W_8(z)$ in yellow, magenta, red, green and cyan colors, respectively. Lastly, sixth degree complex polynomials $W_9(z) = z^6 + z - 1$ and $W_{10}(z) = z^6 - 0.5z^5 + \frac{11}{4}(1+i)z^4 - \frac{1}{4}(19+3i)z^3 + \frac{1}{4}(11+i)z^2 - \frac{1}{4}(19+3i)z + \frac{3}{2} - 3i$ are considered. In Figure 29, green, pink, red, yellow, cyan and blue colors are used to give the basins related to the roots $-1.134724, 0.629372 - 0.735755i, 0.7780895, -0.451055 - 1.002364i, 0.629372 + 0.735755i$ and $-0.451055 + 1.002364i$ of $W_9(z) = 0$, respectively. In Figure 30, the attraction basins for (2) and (3) corresponding to the roots $1 - i, -\frac{1}{2} - \frac{i}{2}, -\frac{3}{2}i, 1, i$ and $-1 + 2i$ of $W_{10}(z)$ are provided in blue, yellow, green, magenta, cyan and red colors, respectively.

From Figures 21–30, we deduce that (2) has the wider basins in comparison to (3). as it can be seen that the black zones that appear in Figures 21, 25 and 28 only appear in (3) method and not in (2). Furthermore, (2) is better than (3) in terms of less chaotic behavior as it can be seen that basins are bigger in (2) and there are fewer changes of basin than in (3) in each Figure, which means that the fractal dimension is lower in (2) and consequently less chaotic. Hence, the overall conclusion of this comparison is that the numerical stability of (2) is higher than (3). This means that (2) is the preferable alternative for solving real problems. Moreover, related to the patterns that appear in the basin of attraction, it is clear that the (2) is similar to third-order methods such us Halley or Chebyshev and the immediate basin of attraction is big and black zones are avoided. On the other hand, in the (3) everything seems more independent with different structures, for example in Figure 29 where the roots are bounded by a small basin and then a really big one in red appears or Figures 21, 25 and 28 where zones with no convergence appear, especially in Figure 25 where almost the half of the plane is black. Finally, in Figures 24, 26, 27, and 29 it seems that a compactification appears in the roots but one of the basins is much bigger than the rest, and this behavior is really interesting and can be considered in the future.



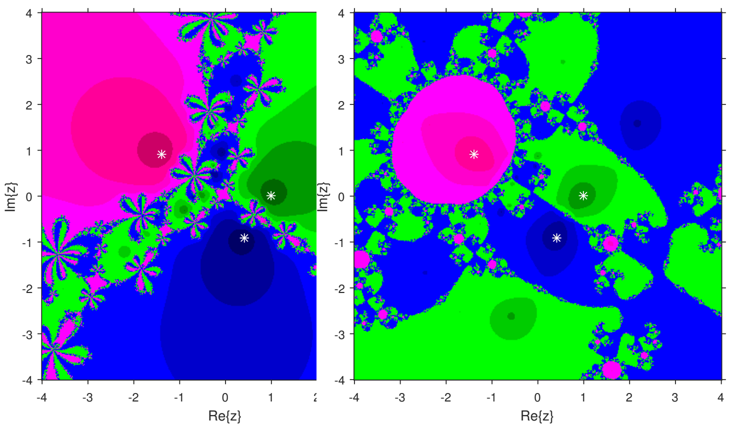
(2) (3)

Figure 21. Attraction basins comparison between (2) and (3) (related to $W_1(z)$).



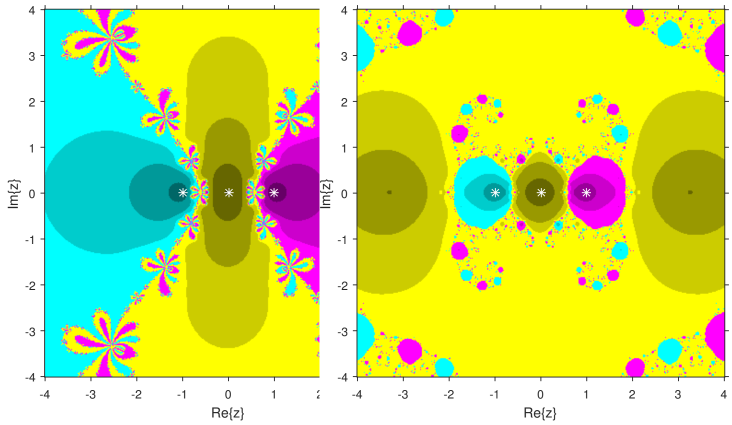
(2) (3)

Figure 22. Attraction basins comparison between (2) and (3) (related to $W_2(z)$).



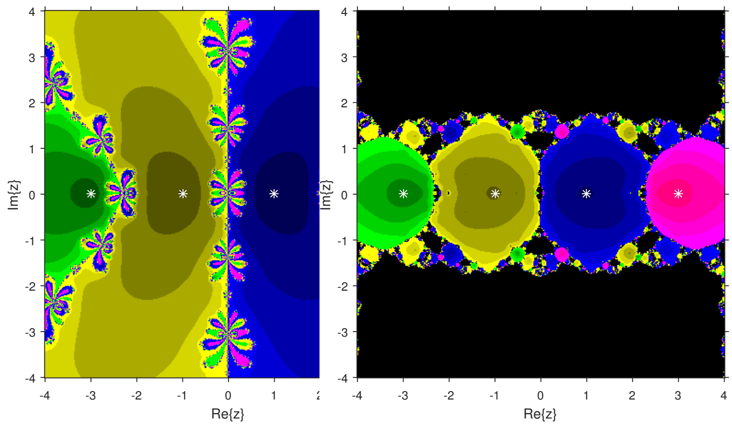
(2) (3)

Figure 23. Attraction basins comparison between (2) and (3) (related to $W_3(z)$).



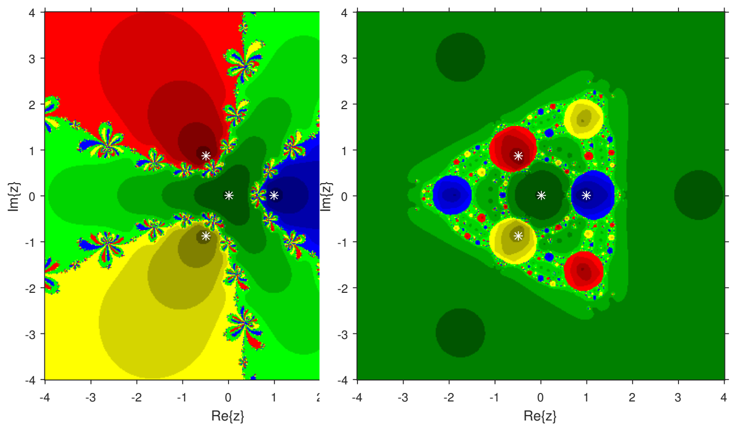
(2) (3)

Figure 24. Attraction basins comparison between (2) and (3) (related to $W_4(z)$).



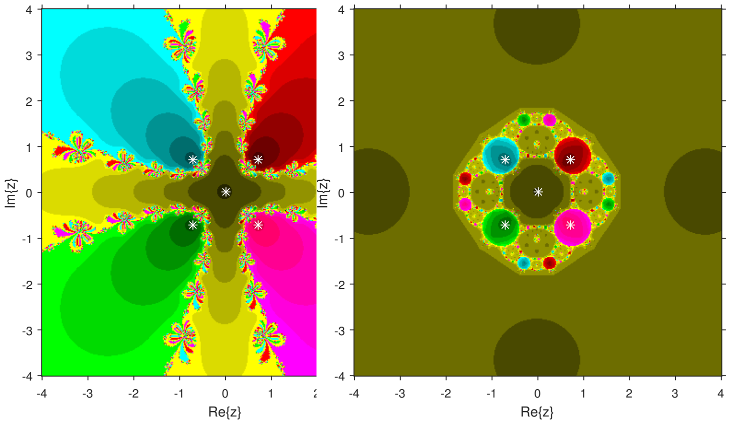
(2) (3)

Figure 25. Attraction basins comparison between (2) and (3) (related to $W_5(z)$).



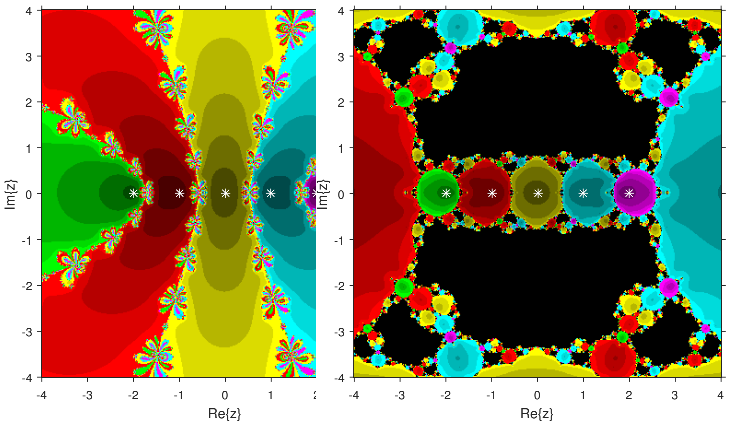
(2) (3)

Figure 26. Attraction basins comparison between (2) and (3) (related to $W_6(z)$).



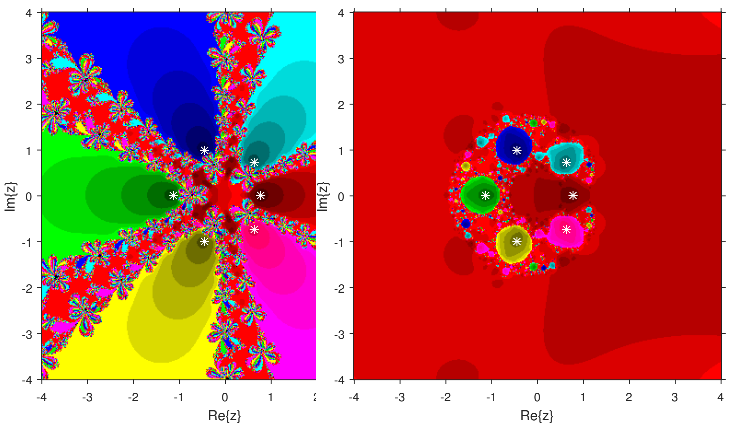
(2) (3)

Figure 27. Attraction basins comparison between (2) and (3) (related to $W_7(z)$).



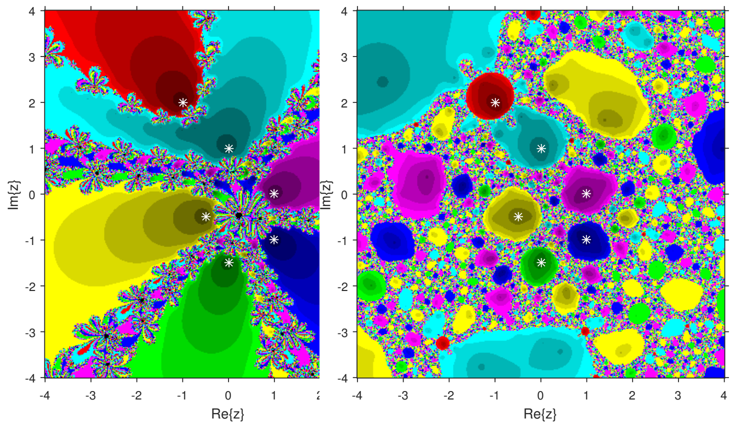
(2) (3)

Figure 28. Attraction basins comparison between (2) and (3) (related to $W_8(z)$).



(2) (3)

Figure 29. Attraction basins comparison between (2) and (3) (related to $W_9(z)$).



(2) (3)

Figure 30. Attraction basins comparison between (2) and (3) (related to $W_{10}(z)$).

4. Examples

The convergence radii are determined for the iterative procedures GM_6 and SM_6.

Example 1. *Let $T_1 = T_2 = \mathbb{R}^3$ and $A = \overline{D}(0,1)$. Consider L on A for $t = (t_1, t_2, t_3)^T$ as*

$$L(t) = (e^{t_1} - 1, \frac{e-1}{2}t_2^2 + t_2, t_3)^T$$

Notice that

$$t_* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Also,

$$\begin{aligned} a = b &= \frac{1}{2}(3 + e^{\frac{1}{e-1}}),\ c = 2, \\ B_0(v, v_1) &= \frac{(e-1)}{2}(v + v_1), \\ B(v, v_1) &= \frac{e^{\frac{1}{e-1}}}{2}(v + v_1), \\ B_1(v, v_1) &= \frac{1}{2}(e^{\frac{1}{e-1}}v + (e-1)v_1). \end{aligned}$$

Using proposed theorems, we obtained r and $\bar{r}$. These radii are given in Table 1.

Table 1. Comparison of the radii in Example 1.

SM_6	GM_6
$r_1 = 0.124265$	$r_1 = 0.124265$
$\overline{r_2} = 0.071733$	$r_2 = 0.064445$
$\overline{r_3} = 0.057472$	$r_3 = 0.052507$
$\bar{r} = 0.057472$	$r = 0.052507$

Example 2. *Let us choose $T_1 = T_2 = C[0,1]$ and $A = \overline{D}(0,1)$. Consider the operator L on A defined as*

$$L(t)(v) = t(v) - 5\int_0^1 vs.y\ t(y)^3\, dy,$$

where $t(v) \in C[0,1]$. *We have* $t_* = 0$. *Furthermore,*

$$\begin{aligned} a &= b = 9, \ c = 2, \\ B_0(x,y) &= \frac{7.5}{2}(x+y), \\ B(x,y) &= \frac{15}{2}(x+y), \\ B_1(x,y) &= \frac{1}{2}(15x+7.5y). \end{aligned}$$

The convergence radii r and $\bar{r}$ *are obtained using the suggested theorems and presented in Table 2.*

Table 2. Comparison of the radii in Example 2.

SM_6	GM_6
$r_1 = 0.066667$	$r_1 = 0.066667$
$\overline{r_2} = 0.004614$	$r_2 = 0.003524$
$\overline{r_3} = 0.003497$	$r_3 = 0.002828$
$\bar{r} = 0.003497$	$r = 0.002828$

Example 3. *We solve the nonlinear systems*

$$\begin{aligned} y_m y_{m+1} - 1 = 0, &\quad 1 \le m \le 18 \\ y_m y_1 - 1 = 0, &\quad m = 19. \end{aligned}$$

The initial point is chosen to be $t_0 = [1.5, 1.5, \ldots, 1.5]^T$. *Then, the two-step method (2) and (3) after three iterations as well as the three-step methods (2) and (3) after two iterations given the solution vector* $t_* = [1, 1, \ldots, 1]^T$.

Example 4. *Let us consider the two-point differential equation*

$$w'' + 3ww' = 0, \quad w(0) = 0, \quad w(2) = 1.$$

Moreover, let

$$p_0 = 0 < p_1 < p_2 \cdots < p_{j-1} < p_j = 2, p_{i+1} = p_i + h, h = \frac{2}{i}.$$

Set $p_0 = w(p_0) = 0$, $p_1 = w(p_1) = \cdots = p_{i-1} = w(p_{i-1})$ *and* $p_i = w(p_i) = 1$.
The central difference for first, as well as the second order derivative discretion, give

$$\begin{aligned} w_i'' &= \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}, \quad i = 1, 2, \ldots, j-1 \\ w_i' &= \frac{w_{i+1} - w_{i-1}}{h^2}, \quad i = 1, 2, \ldots, j-1 \\ w_i &= \frac{w_{i+1} + w_{i-1}}{2}, \quad i = 1, 2, \ldots, j-1. \end{aligned}$$

By these substitutions, we obtain that $(j-1)(j-1)$ *nonlinear system.*

$$4(w_{i-1} - 2w_i + w_{i+1}) + 3h(w_{i+1}^2 - w_{i-1}^2) = 0, \quad i = 1, 2, \ldots, j-1.$$

This system is solved for $j = 6$. *Let* $t_0 = [0.5, 0.5, 0.5, 0.5, 0.5]^T$. *Then, if we apply the two-step methods (2) and (3) after three iterations or the three-step methods (2) and (3) after two iterations we obtain the solution vector* t_* *with five entries given by*

$$t_* = [0.4524453796, 0.760355997, 0.9116610553, 0.9715327158, 0.9927818807]^T.$$

5. Conclusions

A comparison is made between the convergence balls and dynamical behaviors of two derivative free equation solvers that are similar in their efficiency. The ball convergence of GM_6 and SM_6 solely require generalized Lipschitz continuity of L'. Then, the convergence zones of GM_6 and SM_6 are presented using the geometric tool attraction basins. Finally, our analytical conclusions are validated against real-world application challenges. The scheme SM_6 is discovered to have bigger convergence balls than the solver GM_6. Future research will involve the extension of this technique to multipoint method [1,4].

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