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Discussion on the Approximate Controllability of Hilfer Fractional Neutral Integro-Differential Inclusions via Almost Sectorial Operators

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Abstract: This paper focuses on the approximate controllability of Hilfer fractional neutral Volterra integro-differential inclusions via almost sectorial operators. Almost sectorial operators, fractional differential, Leray-Schauder fixed point theorem and multivalued maps are used to prove the result. We start by emphasizing the existence of a mild solution and demonstrate the approximate controllability of the fractional system. In addition, an example is presented to demonstrate the principle.

Keywords: Hilfer fractional system; multivalued maps, sectorial operators; approximate controllability

MSC: 26A33; 34A08; 34K30; 47D09

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1. Introduction

Controllability is a well-known (quantitative and qualitative) feature of a control system that is important in many control issues in finite and infinite dimensional domains. In recent decades, researchers have been drawn to control problems, and substantial contributions to theory and applications have been made. Controllability is a fundamental quality of a control system that aids in solving various control problems, such as the stabilization of unstable systems via feedback control. As a result, difficulties in controllability for various linear, nonlinear stochastic, and deterministic dynamic systems have garnered much attention. Furthermore, approximate controllability is becoming increasingly common, and approximate controllability is frequently sufficient in applications. For further details, consult the articles in [1–5].

In modern mathematics, the fundamentals of fractional computation and the fractional differential equation have taken center stage. The idea of fractional computation has now been tested in various social, physical, signal, image processing, biological, control theory, engineering, etc., challenges. However, it has been demonstrated that fractional differential equations may be valuable for describing various situations. For many realistic applications, fractional-order models are superior to integer-order models. The research articles in [6–13] are concerned with the theory of fractional differential systems, and readers will find several fascinating findings about fractional dynamical systems.

Neutral functional differential systems have received a lot of interest recently since they are used in many areas of applied mathematics, biological models, electronics, fluid dynamics, and chemical kinetics. Neutral structures with delays or without delays, in particular, serve as a summary association of a vast number of partial neutral structures that emerge in issues with heat flow in substances, viscoelasticity, and a range of natural processes. The most effective neutral structures have received a lot of attention in the recent generation because neutral systems are prevalent in many applications of applied mathematics. Neutral fractional differential systems with or without delays have lately been produced by a large number of researchers who have made use of a variety of fixed-point techniques, mild solutions, noncompactness measures, nonlocal conditions and stochastic systems. We can refer to [14–20] for more information.

Other fractional derivatives were introduced by Hilfer [21], including the Riemann-Liouville (*R*–*L*) derivative and Caputo fractional derivative. Many scholars have recently shown tremendous interest in this area, which has sparked effort such as those in [22–27], where the researchers established their results with the help of the fixed point method.

Zhou et al. [28] focused on the existence of a mild solution of Hilfer fractional differential equations with the order $\lambda \in (0,1)$ and the type $\nu \in [0,1]$ in the abstract sense, as follows:

$$\begin{aligned} &{}^H D_{0^+}^{\lambda,\nu} y(t) = Ay(t) + g(t, y(t)), \quad t \in (0,T], \\ &I_{0^+}^{(1-\lambda)(1-\nu)} y(0) = y_0, \end{aligned}$$

where A denotes the almost sectorial operator of the analytic semigroup and Schauder fixed point theorem is used.

Zhang and Zhou [29] demonstrated the existence of fractional Cauchy problems using almost sectorial operators of the following type:

$$\begin{aligned} &{}^L D_{0^+}^{q} x(t) = Ax(t) + f(t, x(t)), \quad t \in [0,a], \\ &I_{0^+}^{(1-q)} x(0) = x_0, \end{aligned}$$

where ${}^L D_{0^+}^{q}$ is the $R-L$ derivative of order q, $0 < q < 1$, $I_{0^+}^{(1-q)}$ is the $R-L$ integral of order $1-q$, and A is an almost sectorial operator on a complex Banach space. We refer to [30–33] for more information.

Prior conclusions from the literature are expanded based on these discoveries to a class of Hilfer fractional differential systems where the closed operator is almost sectorial. However, few articles have reported on the study of Hilfer fractional differential (HFD_{tial}) inclusions with almost sectorial operators, so we are interested in its study.

This article will examine the Hilfer fractional neutral Volterra integro-differential inclusion, which contains almost sectorial operators

$$D_{0^+}^{\kappa,\varepsilon}[\mathfrak{q}(\mathfrak{y}) - \mathcal{K}(\mathfrak{y},\mathfrak{q}(\mathfrak{y}))] \in \mathtt{A}\mathfrak{q}(\mathfrak{y}) + \mathfrak{B}v(\mathfrak{y}) + \mathfrak{F}\left(\mathfrak{y},\mathfrak{q}(\mathfrak{y}),\int_0^{\mathfrak{y}} h(\mathfrak{y},s,\mathfrak{q}(s))ds\right), \qquad \mathfrak{y} \in \mathcal{I}' = (0,d], \tag{1}$$

$$I_{0^+}^{(1-\kappa)(1-\varepsilon)}\mathfrak{q}(0) = \mathfrak{q}_0, \tag{2}$$

where $\mathtt{A}$ is an almost sectorial operator of the analytic semigroup $\{\mathtt{T}(\mathfrak{y}), \mathfrak{y} \geq 0\}$ on Y. $D_{0^+}^{\kappa,\varepsilon}$ denotes the Hilfer fractional derivative (HFD_{ve}) of order κ, $0 < \kappa < 1$ and type ε, $0 \leq \varepsilon \leq 1$. Let $\mathfrak{q}(\cdot)$ be the state in a Banach space Y with norm $\|\cdot\|$ and $v(\cdot)$ be the control function in $L^2(\mathcal{I}, U)$, where U is the Banach space. Here, $B: U \to Y$ is an operator in the control term. Let $\mathcal{I} = [0,d]$, $\mathfrak{F}: \mathcal{I} \times Y \times Y \to 2^Y \backslash \{0\}$ be a nonempty multivalued mapping, which is closed, bounded and convex, $\mathcal{K}: \mathcal{I} \times Y \to Y$ and $h: \mathcal{I} \times \mathcal{I} \times Y \to Y$ are the required mappings.

This study aims to prove that HFD_{tial} inclusions (1) and (2) in Banach space become approximately controllable under fundamental system operator assumptions, in particular that the corresponding linear system is approximately controllable. The major goal of this work is to find enough circumstances for HFD_{tial} inclusions to be approximately controllable. The article is divided into the following sections. In Section 2, we cover the principles of fractional calculus, semi-group, sectorial operators and multivalued maps. In Section 3, initially, we establish the existence of the mild solution and then continue to examine the approximate controllability of the system. In Section 4, we cover an illustration of our key concepts. Finaly, Section 5 provides some conclusions.

2. Preliminaries

In this section, we introduce basic definitions, theorems and results applied throughout the article.

Set $d > 0$, and consider $\mathcal{I} = [0, d]$ and $\mathcal{I}' = (0, d]$. Let $C(\mathcal{I}, Y) = \mathbb{C}$ be the set of all continuous functions from $\mathcal{I}$ to Y. Let $\mathcal{X} = \{\mathfrak{q} \in C(\mathcal{I}', Y) : \lim_{\mathfrak{y}\to 0} \mathfrak{y}^{(1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathfrak{q}(\mathfrak{y})$ exists and finite$\}$, be a Banach space with the norm on $\|\cdot\|_{\mathcal{X}}$ and $\|\mathfrak{q}\|_{\mathcal{X}} = \sup_{\mathfrak{y}\in\mathcal{I}'}\{\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\|\mathfrak{q}(\mathfrak{y})\|\}$. Set $B_P(\mathcal{I}) = \{u \in \mathbb{C}$ such that $\|u\| \le P\}$. Let $\mathfrak{q}(\mathfrak{y}) = \mathfrak{y}^{-1+\varepsilon-\kappa\varepsilon+\kappa\vartheta}y(\mathfrak{y})$, $\mathfrak{y} \in (0, d]$; then, $\mathfrak{q} \in \mathcal{X}$ *iff* $y \in \mathbb{C}$ *and* $\|\mathfrak{q}\|_{\mathcal{X}} = \|y\|$.

Definition 1 ([34])**.** *The integral of fractional order κ for the function $\mathfrak{F} : [d, \infty) \to \mathbb{R}$ with the lower limit d is given by*

$$I^{\kappa}_{d+}\mathfrak{F}(\mathfrak{y}) = \frac{1}{\Gamma(\kappa)}\int_d^{\mathfrak{y}} \frac{\mathfrak{F}(\delta)}{(\mathfrak{y}-\delta)^{1-\kappa}}d\delta \quad \mathfrak{y} > 0;\ \kappa \in \mathbb{R}^+.$$

Definition 2 ([34])**.** *The R-L fractional derivative of a function $(\mathfrak{F} : [d, +\infty) \to \mathbb{R})$ with order $\kappa > 0$, $m - 1 \le \kappa < m$, $m \in \mathbb{N}$ is given by*

$${}^{L}D^{\kappa}_{d+}\mathfrak{F}(\mathfrak{y}) = \frac{1}{\Gamma(m-\kappa)}\frac{d^m}{d\mathfrak{y}^m}\int_d^{\mathfrak{y}} \frac{\mathfrak{F}(\delta)}{(\mathfrak{y}-\delta)^{\kappa+1-m}}d\delta,\ \mathfrak{y} > d,\ \delta \in \mathbb{R}^+.$$

Definition 3 ([34])**.** *The Caputo fractional derivative of a function $(\mathfrak{F} : [d, +\infty) \to \mathbb{R})$ with order $\kappa > 0$, $m - 1 \le \kappa < m$, $m \in \mathbb{N}$ is presented as*

$${}^{C}D^{\kappa}_{d+}\mathfrak{F}(\mathfrak{y}) = \frac{1}{\Gamma(m-\kappa)}\int_d^{\mathfrak{y}} \frac{\mathfrak{F}^m(\delta)}{(\mathfrak{y}-\delta)^{\kappa+1-m}}d\delta = I^{m-\kappa}_{\mathfrak{q}+}\mathfrak{F}^m(\mathfrak{y}),\ \mathfrak{y} > d,\ \delta \in \mathbb{R}^+.$$

Definition 4 ([21])**.** *The HFD$_{ve}$ of a function $(\mathfrak{F} : [d, +\infty) \to \mathbb{R})$ with order $0 < \kappa < 1$ and type $0 \le \varepsilon \le 1$ is given by*

$$D^{\kappa,\varepsilon}_{d+}\mathfrak{F}(\mathfrak{y}) = [I^{(1-\kappa)\varepsilon}_{d+}D(I^{(1-\kappa)(1-\varepsilon)}_{d+}\mathfrak{F})](\mathfrak{y}).$$

Remark 1 ([21])**.**

1. *If $\varepsilon = 0$, $0 < \kappa < 1$, and $d = 0$, then the HFD$_{ve}$ corresponds to the classical R-L fractional derivative:*

$$D^{\kappa,0}_{0+}\mathcal{G}(\mathfrak{z}) = \frac{d}{d\mathfrak{z}}I^{1-\kappa}_{0+}\mathcal{G}(\mathfrak{z}) = ^{L} D^{\kappa}_{0+}\mathcal{G}(\mathfrak{z}).$$

2. *If $\varepsilon = 1$, $0 < \kappa < 1$, and $d = 0$, then the HFD$_{ve}$ corresponds to the classical Caputo fractional derivative:*

$$D^{\kappa,1}_{0+}\mathcal{G}(\mathfrak{z}) = I^{1-\kappa}_{0+}\frac{d}{d\mathfrak{z}}\mathcal{G}(\mathfrak{z}) = ^{C} D^{\kappa}_{0+}\mathcal{G}(\mathfrak{z}).$$

Definition 5 ([35])**.** *For $0 < \vartheta < 1$, $\quad 0 < \varphi < \frac{\pi}{2}$, we denote $\Theta^{-\vartheta}_{\varphi}$ as the family of closed linear operators, the sector $S_\varphi = \{\theta \in \mathbb{C}\backslash\{0\}$ with $|\arg\theta| \le \varphi\}$ and $\mathtt{A} : D(\mathtt{A}) \subset Y \to Y$ such that*

(i) $\sigma(\mathtt{A}) \subseteq S_{\varphi}$;
(ii) There exists Δ_{δ} that is a constant such that

$$\left\|(\theta I - \mathtt{A})^{-1}\right\| \leq \Delta_{\delta}|\mathfrak{y}|^{-\vartheta}, \text{ for all } \quad \varphi < \delta < \pi;$$

then, $\mathtt{A} \in \Theta_{\varphi}^{-\vartheta}$ is called an almost sectorial operator on Y.

Lemma 1 ([35]). *Consider $0 < \vartheta < 1$ and $0 < \varphi < \frac{\pi}{2}$, $\mathtt{A} \in \Theta_{\varphi}^{-\vartheta}(Y)$. Then, we obtain*

1. $\mathtt{T}(\mathfrak{y}_1 + \mathfrak{y}_2) = \mathtt{T}(\mathfrak{y}_1) + \mathtt{T}(\mathfrak{y}_2)$, for every $\mathfrak{y}_1, \mathfrak{y}_2 \in S^0_{\frac{\pi}{2}-\delta}$;
2. there exists $\Delta_0 > 0$ that is a constant such that $\|\mathtt{T}(\mathfrak{y})\|_{\mathtt{C}} \leq \Delta_0 \mathfrak{y}^{\vartheta-1}$, for any $\mathfrak{y} > 0$;
3. $R(\mathtt{T}(\mathfrak{y}))$ is the range of $\mathtt{T}(\mathfrak{y})$, $\mathfrak{y} \in S^0_{\frac{\pi}{2}-\delta}$ contained in $D(\mathtt{A}^{\infty})$. Specifically, $R(\mathtt{T}(\mathfrak{y})) \subset D(\mathtt{A}^{\theta})$ for all $\theta \in \mathbb{C}$ with $Re(\theta) > 0$,
$$\mathtt{A}^{\theta}\mathtt{T}(\mathfrak{y})y = \frac{1}{2\pi i}\int_{\Gamma_{\gamma}} z^{\theta}e^{-\mathfrak{y}z}R(z;\mathtt{A})y dz, \text{ for all } y \in Y,$$
and hence, there exists a constant $\Delta' = \Delta'(\beta,\theta) > 0$ such that
$$\left\|\mathtt{A}^{\theta}\mathtt{T}(\mathfrak{y})\right\|_{B(Y)} \leq \Delta' \mathfrak{y}^{-\beta-Re(\theta)-1}, \text{ for all } \mathfrak{y} > 0;$$
4. If $\theta > 1-\vartheta$, then $D(\mathtt{A}^{\theta}) \subset \Sigma_{\mathtt{T}} = \{y \in Y : \lim_{\mathfrak{y}\to 0+} \mathtt{T}(\mathfrak{y})y = y\}$;
5. $R(\kappa', \mathtt{A}) = \int_0^{\infty} e^{-\kappa'\mathfrak{y}}\mathtt{T}(\mathfrak{y})d\mathfrak{y}$, for all $\kappa' \in C$ with $Re(\kappa') > 0$.
Consider the operator families $\{\mathcal{S}_{\kappa}(\mathfrak{y})\}_{\mathfrak{y}\in S_{\frac{\pi}{2}-\varphi}}, \{\mathcal{Q}_{\kappa}(\mathfrak{y})\}_{\mathfrak{y}\in S_{\frac{\pi}{2}-\varphi}}$ defined as follows:
$$\mathcal{S}_{\kappa}(\mathfrak{y}) = \int_0^{\infty} W_{\kappa}(\xi)T(\mathfrak{y}^{\kappa}\xi)d\xi,$$
$$\mathcal{Q}_{\kappa}(\mathfrak{y}) = \int_0^{\infty} \kappa\xi W_{\kappa}(\xi)T(\mathfrak{y}^{\kappa}\xi)d\xi.$$
We have the Wright-type function $W_{\kappa}(\beta)$:
$$W_{\kappa}(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\kappa k)(k-1)!}, \ \beta \in \mathbb{C}. \tag{3}$$
Consider $-1 < \iota < \infty$, $p > 0$, the succeeding properties are satisfied.

(a) $W_{\kappa}(\theta) \geq 0, \quad \mathfrak{y} > 0$;
(b) $\int_0^{\infty} \theta^{\iota}W_{\kappa}(\theta)d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\kappa\iota)}$;
(c) $\int_0^{\infty} \frac{\kappa}{\theta^{(\kappa+1)}}e^{-p\theta}W_{\kappa}(\frac{1}{\theta^{\kappa}})d\theta = e^{-p^{\kappa}}$.

Definition 6 ([27]). *Let $\mathfrak{F}$ be a multivalued map known as an u.s.c. on Y if, for each $\mathfrak{q}_0 \in Y$, the set $\mathfrak{F}(\mathfrak{q}_0)$ is a nonempty, closed subset of Y, and if for each open set $\mathcal{D}$ of Y containing $\mathfrak{F}(\mathfrak{q}_0)$, there exists an open neighborhood $\mathcal{W}$ of $\mathfrak{q}_0$ such that $\mathfrak{F}(\mathcal{W}) \subseteq \mathcal{D}$.*

Lemma 2. *System (1) and (2) is identical to an integral inclusion presented by*

$$\mathfrak{q}(\mathfrak{y}) \in \frac{\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))}{\Gamma(\varepsilon(1-\kappa)}\mathfrak{y}^{((1-\kappa)(\varepsilon-1)+\kappa)} + \mathcal{K}(\mathfrak{y},\mathfrak{q}(\mathfrak{y})) + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathtt{A}\mathcal{K}K(0,\mathfrak{q}(0))d\delta$$
$$+ \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\left[\mathtt{A}\mathfrak{q}(\delta) + \mathtt{B}v(\delta) + \mathfrak{F}\left(\delta, \mathfrak{q}(\delta), \int_0^{\delta} h(\delta,s,\mathfrak{q}(s))ds\right)\right]d\delta.$$

Definition 7. *The mild solution of the Cauchy problem in (1) and (2) is a function* $\mathfrak{q}(\mathfrak{y}) \in C(\mathcal{I}', Y)$ *that satisfies*

$$\mathfrak{q}(\mathfrak{y}) =\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0, \mathfrak{q}(0))\big] + \mathcal{K}(\mathfrak{y}, \mathfrak{q}(\mathfrak{y})) + \int_0^{\mathfrak{y}} \mathcal{K}_\kappa(\mathfrak{y} - \delta)\mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta))d\delta$$
$$+ \int_0^{\mathfrak{y}} \mathcal{K}_\kappa(\mathfrak{y} - \delta)\mathtt{B}v(\delta)d\delta + \int_0^{\mathfrak{y}} \mathcal{K}_\kappa(\mathfrak{y} - \delta)\mathfrak{F}\bigg(\delta, \mathfrak{q}(\delta), \int_0^{\delta} h(\delta, s, \mathfrak{q}(s))ds\bigg)d\delta, \ \mathfrak{y} \in \mathcal{I},$$

where $\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}) = I_0^{\varepsilon(1-\kappa)}\mathcal{K}_\kappa(\mathfrak{y}), \ \mathcal{K}_\kappa(\mathfrak{y}) = \mathfrak{y}^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y})$.

Proposition 1 ([27]). *Let* $0 < \kappa < 1, \ 0 < \mu \leq 1$, *and for all* $\mathfrak{q} \in D(\mathtt{A})$, *there then exists a constant* $\Delta_\mu > 0$ *such that*

$$\left\|\mathtt{A}^\mu \mathcal{Q}_\kappa(\mathfrak{y})\mathfrak{q}\right\| \leq \frac{\kappa\Delta_\mu\Gamma(2-\mu)}{\mathfrak{y}^{\kappa\mu}\Gamma(1+\kappa(1-\mu))}\|\mathfrak{q}\|, \ \mathfrak{y} \in (0, d).$$

Lemma 3 ([28]). *For any fixed* $\nu > 0, \ \mathcal{Q}_\kappa(\mathfrak{y}), \ \mathcal{K}_\kappa(\mathfrak{y})$ *and* $\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})$ *are linear operators, and for any* $\mathfrak{q} \in Y$,

$$\|\mathcal{Q}_\kappa(\mathfrak{y})\mathfrak{q}\| \leq L'\mathfrak{y}^{-\kappa+\kappa\vartheta}, \ \|\mathcal{K}_\kappa(\mathfrak{y})\mathfrak{q}\| \leq L'\mathfrak{y}^{-1+\kappa\vartheta}\|\mathfrak{q}\|, \ \|\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\mathfrak{q}\| \leq L''\mathfrak{y}^{-1+\varepsilon-\kappa\varepsilon+\kappa\vartheta}\|\mathfrak{q}\|,$$

where

$$L' = \Delta_0\frac{\Gamma(\vartheta)}{\Gamma(\kappa\vartheta)}, \ L'' = \Delta_0\frac{\Gamma(\vartheta)}{\Gamma(\varepsilon(1-\kappa)+\kappa\vartheta)}.$$

Lemma 4 ([28]). *Let* $\{T(\mathfrak{y})\}_{\mathfrak{y}>0}$ *be equicontinuous; then,* $\{\mathcal{Q}_\kappa(\mathfrak{y})\}_{\mathfrak{y}>0}, \ \{\mathcal{K}_\kappa(\mathfrak{y})\}_{\mathfrak{y}>0}$, *and* $\{\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\}_{\mathfrak{y}>0}$ *are strongly continuous, i.e., for any* $\mathfrak{q} \in Y$ *and* $\nu_2 > \mathfrak{y}_1 > 0$,

$$\left\|\mathcal{Q}_\kappa(\mathfrak{y}_2)\mathfrak{q} - \mathcal{Q}_\kappa(\mathfrak{y}_1)\mathfrak{q}\right\| \to 0, \ \left\|\mathcal{K}_\kappa(\mathfrak{y}_2)\mathfrak{q} - \mathcal{K}_\kappa(\mathfrak{y}_1)\mathfrak{q}\right\| \to 0,$$
$$\left\|\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_2)\mathfrak{q} - \mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_1)\mathfrak{q}\right\| \to 0, \ \text{as } \mathfrak{y}_2 \to \mathfrak{y}_1.$$

Lemma 5 ([4]). *Suppose that* $\mathcal{P}_{bd,cv,cl}(Y)$ *is denoted by the collections of all nonempty, bounded, closed and convex subset of* Y. *Let* $\mathfrak{F}$ *be the* L^1*-Caratheodory multivalued map, measurable to* $\mathfrak{y}$ *for each* $\mathfrak{q} \in Y$, *u.s.c. to* $\mathfrak{q}$ *for each* $\mathfrak{y} \in C(\mathcal{I}, Y)$, *where the set*

$$S_{\mathfrak{F},\mathfrak{q}} = \bigg\{f \in L^1(\mathcal{I}, Y) : f(\mathfrak{y}) \in \mathfrak{F}\bigg(\mathfrak{y}, \mathfrak{q}(\mathfrak{y}), \int_0^{\mathfrak{y}} h(\mathfrak{y}, s, \mathfrak{q}(\mathfrak{y}))ds\bigg), \quad \mathfrak{y} \in \mathcal{I}\bigg\}, \tag{4}$$

is nonempty. Consider the linear continuous function Ξ *from* $L^1(\mathcal{I}, Y)$ *to* $\mathtt{C}$; *then,*

$$\Xi \circ S_{\mathfrak{F}} : \mathtt{C} \to \mathcal{P}_{bd,cv,cl}(\mathtt{C}), \quad \mathfrak{q} \to (\Xi \circ S_{\mathfrak{F}})(\mathfrak{q}) = \Xi(S_{\mathfrak{F},\mathfrak{q}}), \tag{5}$$

is a closed graph operator in $\mathtt{C} \times \mathtt{C}$.

Lemma 6 ([36]). *[Non-Linear Alternative Leray–Schauder Fixed Point Theorem]*

Let Y *be a Banach space,* $\mathcal{C}$ *be a closed convex subset of* Y, *and* $\mathcal{D}$ *be an open subset of* $\mathcal{C}$ *and* $0 \in \mathcal{D}$; *suppose that* $\mathfrak{F} : \bar{\mathcal{D}} \to \mathcal{P}_{cp,cv}(\mathcal{C})$ *is an u.s.c. compact map. Then, either*

(a) $\mathfrak{F}$ *has a fixed point in* $\bar{\mathcal{D}}$, *or*
(b) *there is a* $\mathfrak{q} \in \partial\mathcal{D}$ *and* $0 < \lambda < 1$ *with* $\mathfrak{q} \in \lambda\mathfrak{F}(\mathfrak{q})$.

3. Approximate Controllability

We need the following succeeding hypotheses:

Hypotheses (H1). *The almost sectorial operator* $\mathtt{A}$ *generates an analytic semi group* $\mathtt{T}(\mathfrak{y})$, $\mathfrak{y} \geq 0$ *in* Y *such that* $\|\mathtt{T}(\mathfrak{y})\| \leq M$, *for some positive value* M *and* $\|\alpha R(\alpha, \mathfrak{T}_0^d)\| \leq 1$ *for* $\alpha > 0$.

Hypotheses (H2). *Let* $\mathfrak{F} : \mathcal{I} \times Y \times Y \to \mathcal{P}_{bd,cv,cl}(Y)$ *be measurable to* $\mathfrak{y}$ *for each fixed* $\mathfrak{q} \in Y$ *and upper semi continuous to* $\mathfrak{q}$ *for each* $\mathfrak{y} \in \mathcal{I}$, *and for each* $\mathfrak{q} \in \mathtt{C}$,

$$S_{\mathfrak{F},\mathfrak{q}} = \left\{ f \in L^1(\mathcal{I}, Y) : f(\mathfrak{y}) \in \mathfrak{F}\left(\mathfrak{y}, \mathfrak{q}(\mathfrak{y}), \int_0^{\mathfrak{y}} h(\mathfrak{y}, s, \mathfrak{q}(s)) ds\right), \ \mathfrak{y} \in \mathcal{I} \right\}$$

is nonempty.

Hypotheses (H3). *For* $\mathfrak{y} \in \mathcal{I}$, $\mathfrak{F}(\mathfrak{y}, \cdot, \cdot) : Y \times Y \to Y$, $h(\mathfrak{y}, s, \cdot) : Y \to Y$ *are continuous functions, and for each* $\mathfrak{q} \in \mathcal{X}$, $\mathfrak{F}\left(\cdot, \mathfrak{q}, \int h\right) : \mathcal{I} \to \mathcal{I}$ *and* $h(\cdot, \cdot, \mathfrak{q}) : \mathcal{I} \times \mathcal{I} \to Y$ *are strongly measurable.*

Hypotheses (H4). *Let* $P > 0$, $\mathfrak{q} \in \mathcal{X}$ *such that* $\|\mathfrak{q}\|_{\mathcal{X}} \leq P$ *and* $L_{\mathfrak{F},P}(\cdot) \in L^1(\mathcal{I}', \mathbb{R})$ *satisfying*

$$\lim_{\mathfrak{y} \to 0} \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} I^{\kappa\vartheta} L_{\mathfrak{F},P}(\mathfrak{y}) = 0,$$
$$\sup \left\{ \|\mathfrak{F}\| : \mathfrak{F}(\mathfrak{y}) \in \mathfrak{F}\left(\mathfrak{y}, u(\mathfrak{y}), \int_0^{\mathfrak{y}} h(\mathfrak{y}, s, \mathfrak{q}(s)) ds\right) \right\} \leq L_{\mathfrak{F},P}(\mathfrak{y}),$$

for all $u(\mathfrak{y}) \in B_P$.

Hypotheses (H5). *For any* $\mathfrak{y} \in \mathcal{I}$, *a multivalued map* $\mathcal{K} : \mathcal{I} \times Y \to Y$ *is a continuous function and there exists* $0 < \mu < 1$ *such that* $\mathcal{K} \in D(\mathtt{A}^{\mu})$, *and for all* $\mathfrak{q} \in Y$, $\mathfrak{y} \in \mathcal{I}$, $\mathtt{A}^{\mu}\mathcal{K}(\mathfrak{y}, \cdot)$ *satisfies the following:*

$$\left\| \mathtt{A}^{\mu}\mathcal{K}(\mathfrak{y}, \mathfrak{q}(\mathfrak{y})) \right\| \leq M_{\mathcal{K}}\left(1 + \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\|\mathfrak{q}(\mathfrak{y})\|\right), \ (\mathfrak{y}, \mathfrak{q}) \in \mathcal{I} \times Y.$$

Hypotheses (H6). *Let* $\mathcal{K}$ *be the completely continuous, and for any bounded set* D *subset of* $\mathtt{C}$, *the set* $\{\mathfrak{y} \to \mathcal{K}(\mathfrak{y}, \mathfrak{q}(\mathfrak{y})), \ \mathfrak{q} \in D\}$ *is equicontinuous in* Y.

Before looking into the approximation of controllability of the non-linear control system, we first examine its linear component in (1) and (2),

$$\begin{cases} D_{0^+}^{\kappa,\varepsilon} \in \mathtt{A}\mathfrak{q}(\mathfrak{y}) + (\mathtt{B}v)(\mathfrak{y}), \ \mathfrak{y} \in (0, d], \\ I_{0^+}^{(1-\kappa)(1-\varepsilon)}\mathfrak{q}(0) = \mathfrak{q}_0. \end{cases} \tag{6}$$

Here, $\mathtt{B} : U \to Y$ is a linear bounded operator $v \in L^2(\mathcal{I}, U)$.

Now, we go through some key terms:

$$\mathfrak{T}_0^d = \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta)\mathtt{B}\mathtt{B}^* \mathcal{Q}_\kappa^*(d-\delta) d\delta,$$
$$R(\alpha, \mathfrak{T}_0^d) = (\alpha I + \mathfrak{T}_0^d)^{-1}, \ \alpha > 0,$$

where $\mathtt{B}^*$ and $\mathcal{Q}_\kappa^*$ are the adjoint of $\mathtt{B}$ and $\mathcal{Q}_\kappa$, respectively, and $\mathfrak{T}_0^d$ is the linear bounded operator.

Now, for every $\alpha > 0$, and $\mathfrak{q}_1 \in Y$, consider

$$v(\mathfrak{y}) = \mathtt{B}^* \mathcal{Q}_\kappa^*(d-\mathfrak{y}) R(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}(\cdot)),$$

where

$$
\begin{aligned}
P(\mathfrak{q}(\cdot)) =& \mathfrak{q}_1 - \mathcal{S}_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0,\mathfrak{q}(0))] - \mathcal{K}(d,\mathfrak{q}(d)) - \int_0^d (d-\delta)^{\kappa-1}\mathcal{Q}_\kappa(d-\delta)\mathfrak{F}(\delta)d\delta \\
&- \int_0^d (d-\delta)^{\kappa-1}\mathcal{Q}_\kappa(d-\delta)\mathtt{A}\mathcal{K}(\delta,\mathfrak{q}(\delta))d\delta.
\end{aligned}
$$

Theorem 1. *Assume that $(H_1)-(H_6)$ holds; then, the HFD_{tial} system in (1) and (2) has a solution on $\mathcal{I}$, provided*

$$
\frac{M}{d^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\left[M^{**} + \frac{d^{-\kappa(1+2\vartheta)}(\Delta_p M_B)^2}{\alpha(-\kappa(1+2\vartheta))}[\mathfrak{q}_1 - M^{**}]\right]} > 1,
$$

where

$$
\begin{aligned}
M^{**} =& L'' d^{-1+\varepsilon-\kappa\varepsilon+\kappa\vartheta}\big(\mathfrak{q}(0) - M_0 M_{\mathcal{K}}\big) + M_0 M_{\mathfrak{K}}\big(1+P\big) \\
&+ \Delta_{1-\mu}\frac{d^{\kappa\mu}\Gamma(1+\mu)}{\mu\Gamma(1+\kappa\mu)}\big(M_{\mathcal{K}}(1+P)\big) + L' L_{\mathfrak{F},P}(d)\frac{d^{\kappa\vartheta}}{\kappa\vartheta}.
\end{aligned}
$$

and $\mathfrak{q}(0) \in D(\mathtt{A}^\theta)$ with $\theta > 1-\vartheta$.

Proof. Consider the multivalued map $\Psi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$, defined as

$$
\begin{aligned}
\Psi(\mathfrak{q}(\mathfrak{y})) = \Bigg\{ z \in \mathcal{X} : z(\mathfrak{y}) =& \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Bigg[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big) \\
&+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
&+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\left(\delta,\mathfrak{q}(\delta),\int_0^\delta h(\delta,s,\mathfrak{q}(s))ds\right)d\delta \\
&+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}v(\delta)d\delta\Bigg],\ \mathfrak{y} \in (0,d]\Bigg\}.
\end{aligned}
$$

Show that Ψ has a fixed point. Let

$$
\begin{aligned}
z(\mathfrak{y}) =& \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Bigg[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big) \\
&+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta + \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)R(\alpha,\mathfrak{T}_0^d) \\
&\times \Bigg(\mathfrak{q}_1 - \mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] - \mathcal{K}\big(d,\mathfrak{q}(d)\big) - \int_0^d (d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}(r)\big)dr \\
&- \int_0^d (d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathfrak{F}\left(r,\mathfrak{q}(r),\int_0^r h(d,r,\mathfrak{q}(r))\right)dr\Bigg)d\delta \\
&+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\left(\delta,\mathfrak{q}(\delta),\int_0^\delta h(\delta,s,\mathfrak{q}(s))ds\right)d\delta\Bigg].
\end{aligned}
$$

Step 1: For every $\mathfrak{q} \in B_P(\mathcal{I})$, $\Psi(\mathfrak{q})$ is convex.

Consider $z_1, z_2 \in \{\Psi\mathfrak{q}(\mathfrak{y})\}$ and $\mathfrak{F}_1, \mathfrak{F}_2 \in S_{\mathtt{F},\mathfrak{q}}$ such that $\mathfrak{y} \in \mathcal{I}$. We know

$$
\begin{aligned}
z_i =& \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg(\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)R(\alpha,\mathfrak{T}_0^d)\\
&\times\bigg[\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]-\mathcal{K}\big(d,\mathfrak{q}(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}(r)\big)dr\\
&-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathfrak{F}_i(r)dr\bigg]d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}_i(\delta)d\delta\bigg),\ i=1,2.
\end{aligned}
$$

Take $0\le\chi\le1$; then, for each $\mathfrak{y}\in\mathcal{I}$, we have

$$
\begin{aligned}
&\chi z_1+(1-\chi)z_2(\mathfrak{y})\\
&=\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg(\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(d,\mathfrak{q}(d)\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)R(\alpha,\mathfrak{T}_0^d)\\
&\times\bigg[\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]-\mathcal{K}\big(d,\mathfrak{q}(d)\big)\\
&-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}(r)\big)dr-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\big[\chi\mathfrak{F}_1+(1-\chi)\mathfrak{F}_2\big]dr\bigg]d\delta\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\big[\chi\mathfrak{F}_1+(1-\chi)\mathfrak{F}_2\big]d\delta\bigg).
\end{aligned}
$$

We know that $S_{\mathfrak{F},\mathfrak{q}}$ is convex. Therefore, $\chi\mathfrak{F}_1+(1-\chi)\mathfrak{F}_2\in S_{\mathfrak{F},\mathfrak{q}}$. Furthermore,

$$\chi z_1+(1-\chi)z_2\in\Psi\mathfrak{q}(\mathfrak{y}).$$

Hence, Ψ is convex.

Step 2: Ψ is bounded on bounded sets of $B_P(\mathcal{I})$. It is adequate to show that there exists a $\mathcal{I}_*>0$ that is a constant such that, for each $\mathfrak{q}\in B_P(\mathcal{I})$, one has $\|z(\mathfrak{y})\|\le\mathbb{M}_*$. Consider that, for every $\mathfrak{q}\in B_P(\mathcal{I})$, we have

$$
\begin{aligned}
\|z(\mathfrak{y})\| \leq \sup\ & \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg\|\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\left[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\right]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big) \\
& +\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
& +\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\left(\delta,\mathfrak{q}(\delta),\int_0^{\delta}h\big(\delta,s,u(s)\big)ds\right)d\delta \\
& +\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{BB}^*\mathcal{Q}_\kappa^*(d-\delta)R(\alpha,\mathfrak{T}_0^d)\Big[\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\left[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\right] \\
& -\mathcal{K}\big(d,\mathfrak{q}(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}(r)\big)dr \\
& -\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathfrak{F}\left(r,\mathfrak{q}(r),\int_0^{\delta}h\big(\delta,s,\mathfrak{q}(s)\big)ds\right)dr\Big]d\delta\bigg\| \\
\leq\ & d^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg(\sup\left\|\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\left[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\right]\right\|+\left\|\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big)\right\| \\
& +\sup\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\left\|\mathtt{A}^{1-\mu}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\right\|\left\|\mathtt{A}^{\mu}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)\right\|d\delta \\
& +\sup\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\left\|\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\right\|\left\|\mathfrak{F}\left(\delta,\mathfrak{q}(\delta),\int_0^{\delta}h\big(\delta,s,\mathfrak{q}(s)\big)\right)\right\|d\delta \\
& +\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{2\kappa\vartheta-\kappa-1}L'^2M_B^2\frac{1}{\alpha}\Big[\mathfrak{q}_1-\sup\left\|\mathcal{S}_{\kappa,\varepsilon}(d)\left[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\right]\right\| \\
& -\left\|\mathcal{K}\big(d,\mathfrak{q}(d)\big)\right\|-\int_0^d(d-r)^{\kappa-1}\left\|\mathtt{A}^{1-\mu}\mathcal{Q}_\kappa(d-r)\right\|\left\|\mathtt{A}^{\mu}\mathcal{K}\big(r,\mathfrak{q}(r)\big)\right\|dr \\
& -\int_0^d(d-r)^{\kappa-1}\left\|\mathcal{Q}_\kappa(d-r)\right\|\left\|\mathfrak{F}\left(r,\mathfrak{q}(r),\int_0^{r}h\big(d,r,\mathfrak{q}(r)\big)\right)\right\|dr\Big]d\delta \\
\leq\ & d^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\left[M^{**}+\frac{d^{\kappa(2\vartheta-1)}(L'M_B)^2}{\kappa(2\vartheta-1)}\left[\mathfrak{q}_1-M^{**}\right]\right] \\
<\ & \mathbb{M}_*,
\end{aligned}
$$

where

$$
\begin{aligned}
M^{**}=\ & L''d^{-1+\varepsilon-\kappa\varepsilon+\kappa\vartheta}\big(\mathfrak{q}(0)-M_0M_{\mathcal{K}}\big)+M_0M_{\mathcal{K}}(1+P) \\
& +\Delta_{1-\mu}\frac{d^{\kappa\mu}\Gamma(1+\mu)}{\mu\Gamma(1+\kappa\mu)}\big(M_{\mathcal{K}}(1+P)\big)+L'L_{\mathfrak{F},P}(d)\frac{d^{\kappa\vartheta}}{\kappa\vartheta}.
\end{aligned}
$$

Hence, it is bounded.

Step 3: Equicontinuity of Ψ on the set of $B_P(\mathcal{I})$. Consider $0<\mathfrak{y}_1<\mathfrak{y}_2\leq d$ and that there exists $\mathfrak{F}\in S_{\mathfrak{F},\mathfrak{q}}$; we have

$$
\begin{aligned}
&\left\|z(\mathfrak{y}_2) - z(\mathfrak{y}_1)\right\| \\
&\quad \leq \bigg\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg(\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_2)\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}\big(\mathfrak{y}_2,\mathfrak{q}(\mathfrak{y}_2)\big) \\
&\qquad + \int_0^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
&\qquad + \int_0^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\delta \\
&\qquad + \int_0^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\bigg) \\
&\qquad - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg(\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_1)\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}\big(\mathfrak{y}_1,\mathfrak{q}(\mathfrak{y}_1)\big) \\
&\qquad + \int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
&\qquad + \int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\delta \\
&\qquad + \int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{B}v(\delta)d\delta\bigg)\bigg\| \\
&\quad \leq \Big\|\big[\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_2) - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_1)\big]\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big]\Big\| \\
&\qquad + \Big\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{K}(\mathfrak{y}_2,\mathfrak{q}(\mathfrak{y}_2)) - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{K}(\mathfrak{y}_1,\mathfrak{q}(\mathfrak{y}_1))\Big\| \\
&\qquad + \bigg\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
&\qquad + \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta \\
&\qquad - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\bigg\| \\
&\qquad + \bigg\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\delta \\
&\qquad + \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\delta \\
&\qquad - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)\bigg)d\delta\bigg\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\right.\\
&+\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{B}v(\delta)d\delta\right\|\\
\leq&\left\|\left[\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_2)-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}_1)\right]\left[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\right]\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{K}(\mathfrak{y}_2,\mathfrak{q}(\mathfrak{y}_2))-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\mathcal{K}(\mathfrak{y}_1,\mathfrak{q}(\mathfrak{y}_1))\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\right\|\\
&+\left\|\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\Big)d\delta\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\Big)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\Big)d\delta\right\|\\
&+\left\|\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\Big)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\Big)d\delta\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\right\|\\
&+\left\|\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_2-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\right\|\\
&+\left\|\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta)\mathtt{B}v(\delta)d\delta\right.\\
&\left.-\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\int_0^{\mathfrak{y}_1}(\mathfrak{y}_1-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)\mathtt{B}v(\delta)d\delta\right\|\\
=&\sum_{i=1}^{11}I_i.
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})(\mathfrak{q}(0)-M_0M_{\mathcal{K}})$, we have I_1 *tends to* 0 *as* $\mathfrak{y}_2\to\mathfrak{y}_1$. The equicontinuity of $\mathcal{K}$ ensures that I_2 *tends to* 0*, as* $\mathfrak{y}_2\to\mathfrak{y}_1$.

$$\begin{aligned}
I_3 &= \left\| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} (\mathfrak{y}_2-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta)) d\delta \right\| \\
&\leq \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \Delta_{1-\mu} M_{\mathcal{K}}(1+P) \frac{\Gamma(1+\mu)}{\mu\Gamma(1+\kappa\mu)} (\mathfrak{y}_2-\mathfrak{y}_1)^{\kappa\mu}.
\end{aligned}$$

Then, I_3 *tends* 0 *as* $\mathfrak{y}_2 \to \mathfrak{y}_1$.

$$\begin{aligned}
I_4 &\leq \left\| \left(\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right) \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta)) d\delta \right\| \\
&\leq \kappa \Delta_{1-\mu} M_{\mathcal{K}}(1+P) \frac{\Gamma(1+\mu)}{\mu\Gamma(1+\kappa\mu)} \\
&\quad \left\| \int_0^{\mathfrak{y}_1} \left(\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right) (\mathfrak{y}_2-\delta)^{\kappa(\mu-1)} d\delta \right\|.
\end{aligned}$$

We havethat I_4 *tendstoward* 0 *as* $\mathfrak{y}_2 \to \mathfrak{y}_1$*. Additionally,*

$$\begin{aligned}
I_5 &\leq \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa-1} \| [\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)] \| \| \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta)) \| \\
&\leq M_0' M_{\mathcal{K}}(1+P) \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa-1} \| [\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta)] \|.
\end{aligned}$$

By Theorem 4 and strong continuity of $\mathcal{Q}_\kappa(\mathfrak{y})$*,* I_5 *tends to* 0*, as* $\mathfrak{y}_2 \to \mathfrak{y}_1$.

$$\begin{aligned}
I_6 &\leq L' \left| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta) d\delta \right| \\
&\leq L' \left| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_2} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta) d\delta - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta) d\delta \right| \\
&\leq L' \int_0^{\mathfrak{y}_1} \left[\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} - \mathfrak{y}_2^{(1+\kappa\vartheta)(1-\kappa)} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} \right] L_{\mathfrak{F},P}(\delta)] d\delta.
\end{aligned}$$

Then, I_6 *tends to* 0 as $\mathfrak{y}_2 \to \mathfrak{y}_1$ by using (H_4) and the Lebesgue's dominated convergent theorem.

$$I_7 \leq L' \int_0^{\mathfrak{y}_1} (\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} \left| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right| L_{\mathfrak{F},P}(\delta) d\delta.$$

Consider

$$\begin{aligned}
&(\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} \left| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right| L_{\mathfrak{F},P}(\delta) \\
&\quad \leq \left[\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon+\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} + \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} (\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} \right] L_{\mathfrak{F},P}(\delta) \\
&\quad \leq \left[\mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} + \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} \right] L_{\mathfrak{F},P}(\delta) \\
&\quad \leq 2\mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta)
\end{aligned}$$

and that $\int_0^{\mathfrak{y}_1} 2\mathfrak{y}_1^{(1+\kappa\vartheta)(1-\kappa)} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta) d\delta$ exists ($\delta \in (0, \mathfrak{y}_1]$); then, using the dominated convergence theorem, we have

$$\int_0^{\mathfrak{y}_1} (\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} \left| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right| L_{\mathfrak{F},P}(\delta) d\delta \to 0 \text{ as } \mathfrak{y}_2 \to \mathfrak{y}_1,$$

so we conclude $\lim\limits_{\mathfrak{y}_2 \to \mathfrak{y}_1} I_7 = 0$.

For any $\varsigma > 0$, we have

$$
\begin{aligned}
I_8 \leq & \int_0^{\mathfrak{y}_1-\varsigma} \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \left\| \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta) \right\| (\mathfrak{y}_1-\delta)^{\kappa-1} L_{\mathfrak{F},P}(\delta) d\delta \\
& + \int_{\mathfrak{y}_1-\varsigma}^{\mathfrak{y}_1} \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \left\| \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta) \right\| (\mathfrak{y}_1-\delta)^{\kappa-1} L_{\mathfrak{F},P}(\delta) d\delta \\
\leq & \, \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1-\varsigma} (\mathfrak{y}_1-\delta)^{\kappa-1} L_{\mathfrak{F},P}(\delta) d\delta \sup_{w \in [0,\mathfrak{y}_1-\varsigma]} \left\| \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta) \right\| \\
& + L' \int_{\mathfrak{y}_1-\varsigma}^{\mathfrak{y}_1} \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} [(\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} + (\mathfrak{y}_1-\delta)^{-\kappa+\kappa\vartheta}] (\mathfrak{y}_1-\delta)^{\kappa-1} L_{\mathfrak{F},P}(\delta) d\delta \\
\leq & \, \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa-1} L_{\mathfrak{F},P}(\delta) d\delta \sup_{\delta \in [0,\mathfrak{y}_1-\varsigma]} \left\| \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta) \right\| \\
& + 2L' \int_{\mathfrak{y}_1-\varsigma}^{\mathfrak{y}_1} \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa\vartheta-1} L_{\mathfrak{F},P}(\delta) d\delta.
\end{aligned}
$$

From Theorem 4 and $\lim_{\mathfrak{y}_2 \to \mathfrak{y}_1} I_6 = 0$, we have $I_8 \to 0$ independently of $\mathfrak{q} \in B_P(\mathcal{I})$ as $\mathfrak{y}_2 \to \mathfrak{y}_1$, $\varsigma \to 0$.

$$
\begin{aligned}
I_9 = & \left\| \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} (\mathfrak{y}_2-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) \mathrm{B} v(\delta) d\delta \right\| \\
\leq & \, \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} L' M_B \left\| \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} (\mathfrak{y}_2-\delta)^{\kappa\vartheta-1} v(\delta) d\delta \right\|.
\end{aligned}
$$

I_9 tends to zero as $\mathfrak{y}_2 \to \mathfrak{y}_1$.

$$
\begin{aligned}
I_{10} = & \left\| \int_0^{\mathfrak{y}_1} \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \left[(\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right] \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) \mathrm{B} v(\delta) \right\| \\
\leq & \, L' M_B \int_0^{\mathfrak{y}_1} \mathfrak{y}_2^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \left[(\mathfrak{y}_2-\delta)^{\kappa-1} - \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} (\mathfrak{y}_1-\delta)^{\kappa-1} \right] (\mathfrak{y}_2-\delta)^{-\kappa+\kappa\vartheta} v(\delta) d\delta, \\
I_{11} = & \left\| \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa-1} \left[\mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-1) \right] \mathrm{B} v(\delta) d\delta \right\| \\
\leq & \, M_B \, \mathfrak{y}_1^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}_1} (\mathfrak{y}_1-\delta)^{\kappa-1} v(\delta) d\delta \, \sup \left\| \mathcal{Q}_\kappa(\mathfrak{y}_2-\delta) - \mathcal{Q}_\kappa(\mathfrak{y}_1-\delta) \right\|.
\end{aligned}
$$

Similar to the proof of I_7 and I_8, we have that I_{10} and I_{11} tend to zero.

Hence, $\|z(\mathfrak{y}_2) - z(\mathfrak{y}_1)\| \to 0$ independently of $\mathfrak{q} \in B_P(\mathcal{I})$ as $\mathfrak{y}_2 \to \mathfrak{y}_1$. Therefore, $\{\Psi\mathfrak{q}(\mathfrak{y}) : \mathfrak{q} \in B_P(\mathcal{I})\}$ is equicontinuous on $\mathcal{I}$.

Step 4: The relatively compact of $E(\mathfrak{y}) = \{z(\mathfrak{y}) : z \in \Psi(\mathfrak{q}(\mathfrak{y})),\ \mathfrak{y} \in B_P(\mathcal{I})\}$ in Y

Set $0 < \alpha < \mathfrak{y}$ and a positive value q, and let $z(\mathfrak{y})$ be the operator from $B_P(\mathcal{I})$, defined as

$$
\begin{aligned}
z_{\alpha,q}(\mathfrak{y}) &= \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big) \\
&\quad+\int_0^{\mathfrak{y}-\alpha}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\delta+\int_0^{\mathfrak{y}-\alpha}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}v(\delta)d\delta \\
&\quad+\int_0^{\mathfrak{y}-\alpha}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\delta\bigg] \\
&= \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big) \\
&\quad+\int_0^{\mathfrak{y}-\alpha}\int_q^\infty \kappa\theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1}T((\mathfrak{y}-\delta)^\kappa\theta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)d\theta d\delta \\
&\quad+\int_0^{\mathfrak{y}-\alpha}\int_q^\infty \kappa\theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1}T((\mathfrak{y}-\delta)^\kappa\theta)\mathtt{B}v(\delta)d\theta d\delta \\
&\quad+\int_0^{\mathfrak{y}-\alpha}\int_q^\infty \kappa\theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1}T((\mathfrak{y}-\delta)^\kappa\theta)\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)d\theta d\delta\bigg] \\
&= \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\bigg[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}(\mathfrak{y})\big)\bigg] \\
&\quad+\kappa\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}T(\alpha^\kappa q)\int_0^{\mathfrak{y}-q}\int_q^\infty \theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1}T((\mathfrak{y}-\delta)^\kappa\theta-\alpha^\kappa q) \\
&\quad\times\bigg[\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}(\delta)\big)+\mathfrak{F}\bigg(\delta,\mathfrak{q}(\delta),\int_0^\delta h\big(\delta,s,\mathfrak{q}(s)\big)ds\bigg)+\mathtt{B}v(\delta)\bigg]d\theta d\delta.
\end{aligned}
$$

Therefore, $E_{\alpha,\vartheta}(\mathfrak{y})=\big\{(z(\mathfrak{y}))_{\alpha,q}\ :\mathfrak{y}\in B_P(\mathcal{I})\big\}$ is precompact in Y for all $\alpha\in(0,\mathfrak{y})$, since $T(\alpha^\kappa q)$ is compact. For every $\mathfrak{q}\in B_P(\mathcal{I})$, we have

$$
\begin{aligned}
&\Big\| z(\mathfrak{y}) - z_{\alpha,q}(\mathfrak{y}) \Big\| \\
&\leq \Big\| \kappa \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_0^{\mathfrak{y}} \int_0^q \theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1} T((\mathfrak{y}-\delta)^\kappa \theta) \\
&\quad \Big[\mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta)) + \mathfrak{F}\Big(\delta, \mathfrak{q}(\delta), \int_0^\delta h(\delta, s, \mathfrak{q}(s)) ds\Big) + \mathtt{B}v(\delta) \Big] d\theta d\delta \Big\| \\
&\quad + \Big\| \kappa \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \int_{\mathfrak{y}-\alpha}^{\mathfrak{y}} \int_q^\infty (\mathfrak{y}-\delta)^{\kappa-1} \theta M_\kappa(\theta) T((\mathfrak{y}-\delta)^\kappa \theta) \\
&\qquad\qquad \big[\mathtt{A}\mathcal{K}(\delta, \mathfrak{q}(\delta)) + \mathfrak{F}\Big(\delta, \mathfrak{q}(\delta), \int_0^\delta h(\delta, s, \mathfrak{q}(s)) ds\Big) + \mathtt{B}v(\delta) \big] d\theta d\delta \Big\| \\
&\leq \kappa \Delta_0 \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \Big(\int_0^{\mathfrak{y}} \int_0^q \theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa-1} (\mathfrak{y}-\delta)^{\kappa\vartheta-\kappa} \theta^{\vartheta-1} \\
&\quad \times \big[M_0' M_{\mathcal{K}}(1+P) + L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\theta d\delta \\
&\quad + \int_{\mathfrak{y}-\alpha}^{\mathfrak{y}} \int_q^\infty (\mathfrak{y}-\delta)^{\kappa-1} \theta M_\kappa(\theta)(\mathfrak{y}-\delta)^{\kappa\vartheta-\kappa} \theta^{\vartheta-1} \big[M_0' M_{\mathcal{K}}(1+P) + L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\delta \Big) \\
&\leq \kappa \Delta_0 \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \Big(\int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa\vartheta-1} \big[M_0' M_{\mathcal{K}}(1+P) + L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\delta \int_0^q \theta^\vartheta M_\kappa(\theta) d\theta \\
&\quad + \int_{\mathfrak{y}-\alpha}^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa\vartheta-1} \big[M_0' M_{\mathcal{K}}(1+P) + L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\delta \int_0^\infty \theta^\vartheta M_\kappa(\theta) d\theta \Big) \\
&\leq \kappa \Delta_0 \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \Big(\int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa\vartheta-1} \big[L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\delta \int_0^q \theta^\vartheta M_\kappa(\theta) d\theta \\
&\quad + \frac{\Gamma(1-\vartheta)}{\Gamma(1-\kappa\vartheta)} \int_{\mathfrak{y}-\alpha}^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa\vartheta-1} \big[M_0' M_{\mathcal{K}}(1+P) + L_{\mathfrak{F},P}(\delta) + M_B\|v\| \big] d\delta \Big) \\
&\to 0 \text{ as } \alpha \to 0, q \to 0.
\end{aligned}
$$

So, $E_{\alpha,q}(\mathfrak{y}) = \{z_{\alpha,q}(\mathfrak{y}) : \mathfrak{y} \in B_P(\mathcal{I})\}$ is arbitrary closed to $E(\mathfrak{y}) = \{z(\mathfrak{y}) : \mathfrak{y} \in B_P(\mathcal{I})\}$. Therefore, using the Arzela–Ascoli Theorem, $\{z(\mathfrak{y}) : \mathfrak{y} \in B_P(\mathcal{I})\}$ is relatively compact. The continuity and relatively compactness of $\{z(\mathfrak{y}) : \mathfrak{y} \in B_P(\mathcal{I})\}$ imply that $z(\mathfrak{y})$ is a completely continuous operator.

Step 5: Ψ has a closed graph.

Considering that $\mathfrak{q}_k \to \mathfrak{q}_*$ as $k \to \infty$, $z_k(\mathfrak{y}) \in \Psi(\mathfrak{q}_k)$ and $z_k \to z_*$ as $k \to \infty$, we have to show $z_* \in \Psi(\mathfrak{q}_*)$. Since $z_k \in \Psi(\mathfrak{q}_k)$, then there exists a function $\mathfrak{F}_k \in S_{\mathfrak{F},\mathfrak{q}_k}$ such that

$$
\begin{aligned}
z_k(\mathfrak{y}) &= \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta} \Big[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y}) \big[\mathfrak{q}(0) - \mathcal{K}(0, \mathfrak{q}(0))\big] + \mathcal{K}(\mathfrak{y}, \mathfrak{q}_k(\mathfrak{y})) \\
&\quad + \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}_k(\delta)) d\delta + \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathfrak{F}_k(\delta) d\delta \\
&\quad + \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathtt{B}\mathtt{B}^* \mathcal{Q}_\kappa^*(d-\delta) R(\alpha, \mathfrak{T}_0^d) \Big[\mathfrak{q}_1 - \mathcal{S}_{\kappa,\varepsilon}(d) \big[\mathfrak{q}(0) - \mathcal{K}(0, \mathfrak{q}(0))\big] \\
&\quad - \mathcal{K}(d, \mathfrak{q}_k(d)) - \int_0^d (d-r)^{\kappa-1} \mathcal{Q}_\kappa(d-r) \big[\mathtt{A}\mathcal{K}(r, \mathfrak{q}_k(r)) + \mathfrak{F}_k(r) \big] dr \Big] d\delta \Big].
\end{aligned}
$$

We need to show that there exists $\mathfrak{F}_* \in S_{\mathfrak{F},\mathfrak{q}(0)}$,

$$
\begin{aligned}
z_*(\mathfrak{y}) = \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[&\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}_*(\mathfrak{y})\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}_*(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}_*(\delta)d\delta\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)R(\alpha,\mathfrak{T}_0^d)\Big[\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]\\
&-K\big(d,\mathfrak{q}_*(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\big[\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}_*(r)\big)+\mathfrak{F}_*(r)\big]dr\Big]d\delta\Big].
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
&\Bigg\|z_k(\mathfrak{y})-\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}_k(\mathfrak{y})\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\mathfrak{y}-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}_k(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)\\
&\times R(\alpha,\mathfrak{T}_0^d)\Big(\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]-\mathcal{K}\big(d,\mathfrak{q}(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}_k(r)\big)dr\Big)\Big]\\
&-\Big[z_*(\mathfrak{y})-\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)-K(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}_*(\mathfrak{y})\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\mathfrak{y}-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}_*(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)\\
&\times R(\alpha,\mathfrak{T}_0^d)\Big(\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]\\
&-\mathcal{K}\big(d,\mathfrak{q}(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}_*(r)\big)dr\Big)\Big]\Big]\Bigg\|\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Next, we define an operator $\Xi : L^1(\mathcal{I}, Y) \to \mathcal{X}$,

$$
\begin{aligned}
\Xi(\mathfrak{F})(\mathfrak{y}) = &\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}(\delta))d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*R(0,\mathfrak{T}_0^d)\\
&\times\Big(\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathfrak{F}_k(r)dr\Big).
\end{aligned}
$$

We have, by Lemma 5, that $(\Xi \circ S_{\mathfrak{F},\mathfrak{q}})$ is a closed graph operator. Therefore, by referring to Ξ, we have

$$
\begin{aligned}
\Big[z_k(\mathfrak{y})-\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big(&\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0)+\mathcal{K}(0,\mathfrak{q}(0))\big]+\mathcal{K}\big(\mathfrak{y},\mathfrak{q}_k(\mathfrak{y})\big)\\
&+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\mathfrak{y}-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}\big(\delta,\mathfrak{q}_k(\delta)\big)d\delta+\int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)\\
&\times R(\alpha,\mathfrak{T}_0^d)\Big[\mathfrak{q}_1-\mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0)-\mathcal{K}(0,\mathfrak{q}(0))\big]\\
&-\mathcal{K}\big(d,\mathfrak{q}_k(d)\big)-\int_0^d(d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}\big(r,\mathfrak{q}_k(r)\big)dr\Big)\Big]\in\Xi(S_{\mathfrak{F},\mathfrak{q}_k}),
\end{aligned}
$$

since $\mathfrak{F}_k \to \mathfrak{F}_*$, it follows from Lemma 5 that

$$\Big[z_*(\mathfrak{y}) - \mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big(\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}(\mathfrak{y},\mathfrak{q}_*(\mathfrak{y}))$$
$$+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\mathfrak{y}-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}(\delta,\mathfrak{q}_*(\delta))d\delta + \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\kappa^*(d-\delta)$$
$$\times R(\alpha,\mathfrak{T}_0^d)\Big[\mathfrak{q}_1 - \mathcal{S}_{\kappa,\varepsilon}(d)\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big]$$
$$- \mathcal{K}(d,\mathfrak{q}_*(d)) - \int_0^d (d-r)^{\kappa-1}\mathcal{Q}_\kappa(d-r)\mathtt{A}\mathcal{K}(r,\mathfrak{q}_*(r))dr\Big)\Big] \in \Xi(S_{\mathfrak{F},u(0)}).$$

Hence, Ψ is a closed graph.

Step 6: The operator Ψ has a solution. It is enough to prove that the given set is bounded.

$$\Lambda = \Big\{\mathfrak{q} \in \partial B_P(\mathcal{I}) : \mathfrak{q} = \lambda\Psi(\mathfrak{q}) \text{ for some } \lambda \in (0,1)\Big\}.$$

Let $\mathfrak{q} \in \Lambda$. Then, $\mathfrak{q} \in \lambda\Psi(\mathfrak{q})$ for some $0 < \lambda < 1$. Thus, there exists $\mathfrak{F} \in S_{\mathfrak{F},\mathfrak{q}}$ in ways that, for each $\mathfrak{y} \in [0,d]$, we have

$$\begin{aligned}\mathfrak{q}(\mathfrak{y}) =& \lambda\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}(\mathfrak{y},\mathfrak{q}(\mathfrak{y}))\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}(\delta,\mathfrak{q}(\delta))d\delta\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h(\delta,s,\mathfrak{q}(s))ds\Big)d\delta\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}v(\delta)d\delta\Big].\end{aligned}$$

By assumptions $(H_3)-(H_6)$, we have

$$\begin{aligned}|\mathfrak{q}(\mathfrak{y})| =& \Big|\lambda\mathfrak{y}^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[\mathcal{S}_{\kappa,\varepsilon}(\mathfrak{y})\big[\mathfrak{q}(0) - \mathcal{K}(0,\mathfrak{q}(0))\big] + \mathcal{K}(\mathfrak{y},\mathfrak{q}(\mathfrak{y}))\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{A}\mathcal{K}(\delta,\mathfrak{q}(\delta))d\delta\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathfrak{F}\Big(\delta,\mathfrak{q}(\delta),\int_0^\delta h(\delta,s,\mathfrak{q}(s))ds\Big)\Big]d\delta\\ &+ \int_0^{\mathfrak{y}}(\mathfrak{y}-\delta)^{\kappa-1}\mathcal{Q}_\kappa(\mathfrak{y}-\delta)\mathtt{B}v(\delta)d\delta\Big|.\end{aligned}$$

From step 2, we have

$$|\mathfrak{q}(\mathfrak{y})| \le d^{1-\varepsilon+\kappa\varepsilon-\kappa\vartheta}\Big[M^{**} + \frac{d^{\kappa(2\vartheta-1)}(L'M_B)^2}{a(-\kappa(1+2\vartheta))}[\mathfrak{q}_1 - M^{**}]\Big].$$

Then, by our assumption, there exists $M > 0$ as a constant such that $\mathfrak{q}(\mathfrak{y}) \neq M$. Set $\overline{\mathcal{D}} = \{\mathfrak{q} \in B_P(\mathcal{I}) : \|\mathfrak{q}\| \le M+1\}$. Comprehensibly, $\overline{\mathcal{D}}$ is a closed subset of $B_P(\mathcal{I})$. Based on $\mathcal{D}$'s selection, there is no $\mathfrak{q} \in \partial\mathcal{D}$ such that $\mathfrak{q} = \lambda\Psi(\mathfrak{q})$ for some $0 < \lambda < 1$. Then, the statement (ii) in Lemma (6) does not hold. As a result of the Leray–Schauder type's nonlinear alternative, we are able to determine that the statement (i) of Lemma (6) is true. Hence, the operator Ψ has a fixed point, which is the mild solution of the systems (1) and (2). $\square$

Definition 8 ([27]). *The Cauchy problem (1) and (2) is known to be approximately controllable on $\mathcal{I}$ for all $\mathfrak{q}_0 \in Y$; there is some control $v \in L^2(\mathcal{I}, U)$, $\overline{\mathcal{R}(d, \mathfrak{q}_0)} = Y$, where $\mathcal{R}(d, \mathfrak{q}_0) = \{\mathfrak{q}(d, v); v \in L^2(\mathcal{I}, U), \mathfrak{q}(0, v) = \mathfrak{q}_0\}$, which is the reachable set of system (1) and (2) with the initial value $\mathfrak{q}_0$ at the terminal time d.*

Theorem 2. *Suppose that $(H_1) - (H_6)$ hold and multivalued mapping $\mathfrak{F}$ is uniformly bounded. Furthermore, assume that the corresponding linear system (6) is approximately controllable on $\mathcal{I}$; then, the system in (1) and (2) is approximately controllable on $\mathcal{I}$.*

Proof. Assume that $\mathfrak{q}^\alpha(\cdot)$ is a fixed point of Ψ in $B_P(\mathcal{I})$. From (1), any fixed point of Ψ is the mild solution of (1)–(2). Furthermore, from the results on the Dunford–Pettis Theorem, we conclude that there is a subsequence $\{\mathfrak{F}^\alpha(\delta)\}$ that converges weakly to $\mathfrak{F}(\delta)$ in $L^1(\mathcal{I}, Y)$. For every $\alpha > 0$, there exists $\mathfrak{F}^\alpha \in S_{\mathfrak{F},\mathfrak{q}}$,

$$\begin{aligned}\mathfrak{q}^\alpha(\mathfrak{y}) =& S_{\kappa,\varepsilon}(\mathfrak{y})[\mathfrak{q}_0 - \mathcal{K}(0, u^\alpha(0))] + \mathcal{K}(\mathfrak{y}, u^\alpha(\mathfrak{y})) \\ &+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta)) d\delta + \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathfrak{F}^\alpha(\delta) d\delta \\ &+ \int_0^{\mathfrak{y}} (\mathfrak{y}-\delta)^{\kappa-1} \mathcal{Q}_\kappa(\mathfrak{y}-\delta) \mathtt{B}\mathtt{B}^* \mathcal{Q}_\kappa^*(d-\delta) R(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha(\mathfrak{y})) d\delta,\end{aligned}$$

where

$$V^\alpha(\mathfrak{y}) = B^* \mathcal{Q}_\kappa^*(d-\mathfrak{y}) R(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha)(\mathfrak{y}),$$

and

$$\begin{aligned}P(\mathfrak{q}^\alpha) =& \mathfrak{q}_1 - \mathcal{S}_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0, \mathfrak{q}^\alpha(0))] - \mathcal{K}(d, \mathfrak{q}^\alpha(d)) - \int_0^d (d-r)^{\kappa-1} \mathcal{Q}_\kappa(d-r) \mathfrak{F}^\alpha(r) dr \\ &- \int_0^d (d-r)^{\kappa-1} \mathcal{Q}_\kappa(d-r) \mathtt{A}\mathcal{K}(r, \mathfrak{q}^\alpha(r)) dr.\end{aligned}$$

Taking note of $\left(I - \mathfrak{T}_0^d R(\alpha, \mathfrak{T}_0^d)\right) = \alpha \mathcal{R}(\alpha, \mathfrak{T}_0^d)$, we obtain

$$\begin{aligned}\mathfrak{q}^\alpha(d) &= S_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0, u^\alpha(0))] + \mathcal{K}(d, u^\alpha(d)) \\ &\quad + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta)) d\delta + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathfrak{F}^\alpha(\delta) d\delta \\ &\quad + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathtt{B}\mathtt{B}^* \mathcal{Q}_\kappa^*(d-\delta) \mathcal{R}(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha(d)) d\delta \\ &= S_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0, u^\alpha(0))] + \mathcal{K}(d, u^\alpha(d)) + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta)) d\delta \\ &\quad + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathfrak{F}^\alpha(\delta) d\delta + \mathfrak{T}_0^d \mathcal{R}(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha) \\ &= S_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0, u^\alpha(0))] + \mathcal{K}(d, u^\alpha(d)) + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathtt{A}\mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta)) d\delta \\ &\quad + \int_0^d (d-\delta)^{\kappa-1} \mathcal{Q}_\kappa(d-\delta) \mathfrak{F}^\alpha(\delta) d\delta + P(\mathfrak{q}^\alpha) - \alpha \mathcal{R}(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha) \\ &= \mathfrak{q}_1 - \alpha \mathcal{R}(\alpha, \mathfrak{T}_0^d) P(\mathfrak{q}^\alpha), \text{ for all } \mathfrak{F} \in S_{\mathfrak{F},\mathfrak{q}}.\end{aligned}$$

Furthermore, by our assumptions, there exists a constant $L_{\mathfrak{F},P} < \infty$ such that $\|\mathfrak{F}^\alpha(\delta)\| \leq L_{\mathfrak{F},P}$. Consequently, the sequence $\{\mathfrak{F}^\alpha(\delta)\}$ has a subsequence still denoted by $\{\mathfrak{F}^\alpha(\delta)\}$, that weakly converges to $\mathfrak{F}(\delta)$.

Take

$$W = \mathfrak{q}_1 - S_{\kappa,\varepsilon}(d)[\mathfrak{q}_0 - \mathcal{K}(0, u^\alpha(0))] + \mathcal{K}(d, u_1)$$
$$- \int_0^d (d-\delta)^{\kappa-1}\mathcal{Q}_\kappa(d-\delta)\mathtt{A}\mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta))d\delta - \int_0^d (d-\delta)^{\kappa-1}\mathcal{Q}_\kappa(d-\delta)\mathfrak{F}(\delta)d\delta,$$

and

$$\|P(\mathfrak{q}^\alpha) - \delta\| = \|\mathcal{K}(d, \mathfrak{q}_1^\alpha) - \mathcal{K}(d, \mathfrak{q}_1)\| + \left\| \int_0^d (d-\delta)^{\kappa-1}\mathcal{Q}_\kappa(t-\delta)[\mathfrak{F}^\alpha(\delta) - \mathfrak{F}(\delta)]d\delta \right\|$$
$$+ \left\| \int_0^d (d-\delta)^{\kappa-1}\mathtt{A}\mathcal{Q}_\kappa(d-\delta)[\mathcal{K}(d, \mathfrak{q}_1^\alpha) - \mathcal{K}(\delta, \mathfrak{q}^\alpha(\delta))]d\delta \right\|.$$

The compactness of the operator $\{\mathcal{Q}_\kappa(\mathfrak{y}),\ \mathfrak{y} > 0\}$ is deduced, and the uniform boundedness of $\mathfrak{F}^\alpha(\delta)$ suggests that there exists some $\mathfrak{F}(\delta) \in L^(\mathcal{I}, Y)$ such as $\alpha \to 0^+$,

$$\mathcal{Q}_\kappa(d-\delta)\mathfrak{F}^\alpha(\delta) \to \mathcal{Q}_\kappa(d-\delta)\mathfrak{F}(\delta).$$

Hence, for every $\mathfrak{y} \in [0, d]$, we have $\|P(\mathfrak{q}^\alpha) - \delta\| \to 0$. Additionally, by approximate controllability of system (6), we have $\alpha\mathcal{R}(\alpha, \mathfrak{T}_0^d) \to 0$ as $\alpha \to 0^+$ in the strong topology. As a result, we have that $\alpha \to 0^+$,

$$\begin{aligned}
\|\mathfrak{q}^\alpha(d) - \mathfrak{q}_1\| &= \left\|\alpha\mathcal{R}(\alpha, \mathfrak{T}_0^d)P(\mathfrak{q}^\alpha)\right\| \\
&\le \left\|\alpha\mathcal{R}(\alpha, \mathfrak{T}_0^d)\delta\right\| + \left\|\alpha\mathcal{R}(\alpha, \mathfrak{T}_0^d)\right\|\left\|(P(\mathfrak{q}^\alpha) - \delta)\right\| \\
&\le \left\|\alpha\mathcal{R}(\alpha, \mathfrak{T}_0^d)\delta\right\| + \left\|\left(P(\mathfrak{q}^\alpha) - \delta\right)\right\| \to 0.
\end{aligned}$$

As a result, the system in (1) and (2) is approximately controllable on $\mathcal{I}$. $\square$

4. Example

Consider the following HFD_{tial} system:

$$\begin{cases}
D_{0^+}^{\frac{2}{3},\varepsilon}[\rho(\mathfrak{y},\tau) - \overline{\mathcal{K}}(\mathfrak{y}, \rho(\mathfrak{y},\tau))] \in \rho_{\mathfrak{y}\mathfrak{y}}(\mathfrak{y},\tau) + \beta(\mathfrak{y},\tau) + \frac{e^{-\mathfrak{y}}}{1+e^{-\mathfrak{y}}}\Big[\sin(\rho(\mathfrak{y},\tau)) \\
\qquad + \int_0^{\mathfrak{y}} \sin(\mathfrak{y}s)\rho(\mathfrak{y},\tau)ds\Big] & \mathfrak{y} \in (0, d], \quad \tau \in [0, \pi], \\
\rho(\mathfrak{y}, 0) = \rho(\mathfrak{y}, \pi) = 0, & \mathfrak{y} \in [0, d], \\
I^{(1-\frac{2}{3})(1-\varepsilon)}\mathfrak{q}(\rho, 0) = \mathfrak{q}_0(\tau),
\end{cases} \tag{7}$$

where $D_{0^+}^{\frac{2}{3},\varepsilon}$ is the HFD_{ve} of order $\frac{2}{3}$ and type ε, $I^{(1-\frac{2}{3})(1-\varepsilon)}$ is the R-L integral of order $\frac{3}{7}(1-\varepsilon)$, and $\beta(\mathfrak{y},\tau)$ *and* $\bar{\mathcal{K}}(\mathfrak{y}, \rho(\mathfrak{y},\tau))$ are the required functions. Let $U = Y = L^2[0, \pi]$ and $\mathtt{A}$ be an almost sectorial operator defined by $\mathtt{A}\rho = \rho_{\mathfrak{y}\mathfrak{y}}$ with the domain

$$D(\mathtt{A}) = \Big\{\rho \in Y : \rho_{\mathfrak{y}}, \rho_{\mathfrak{y}\mathfrak{y}} \text{ being absolutely continuous in Y, such that } \rho(\mathfrak{y}, 0) = \rho(\mathfrak{y}, \pi) = 0\Big\}.$$

The operator $\mathtt{A}$ generates an analytic semigroup $T(\mathfrak{y})$ and is defined by

$$\mathtt{T}(\mathfrak{y})\mathfrak{q} = \sum_{k=0}^{\infty} e^{-k^2\mathfrak{y}}\langle \mathfrak{q}, e_k\rangle,\ \mathfrak{q} \in Y.$$

Moreover, $\mathtt{A}$ has a discrete spectrum, the eigen values $k^2, k \in \mathbb{N}$ agree with orthogonal eigen vectors $\mathfrak{e}_k(\mathfrak{q}) = \sqrt{\frac{2}{\pi}}\sin(k\mathfrak{q})$. Then,

$$\mathtt{A}\mathfrak{q} = \sum_{k=0}^{\infty} k^2 \langle \mathfrak{q}, \mathfrak{e}_k \rangle \mathfrak{e}_k, \ \mathfrak{q} \in Y.$$

Specifically, $\mathtt{T}(\cdot)$ is a uniformly stable semigroup and $\|\mathtt{T}(\mathfrak{y})\| \leq e^{-\mathfrak{y}}$.

Consider $\mathfrak{q}(\mathfrak{y})(\tau) = \rho(\mathfrak{y}, \tau)$, $\mathfrak{y} \in \mathcal{I} = [0, d]$, $a \in [0, \pi]$ and bounded linear operator $\mathtt{B}v(\mathfrak{y})(\tau) = \beta(v, \tau)$, $0 < \tau < \pi$, where $\beta : \mathcal{I} \times [0, \pi] \to [0, \pi]$ is continuous in $\mathfrak{y}$. Now, for any $\mathfrak{q} \in Y = L^2[0, \pi]$, $\tau \in [0, \pi]$, we define the function $\mathfrak{F} : \mathcal{I} \times Y \times Y \to Y$,

$$\mathfrak{F}\big(\mathfrak{y}, \mathfrak{q}(\mathfrak{y}), (\mathcal{H}\mathfrak{q})(\mathfrak{y})\big) = \frac{e^{-\mathfrak{y}}}{1 + e^{-\mathfrak{y}}} \left[\sin\big(\rho(\mathfrak{y}, \tau)\big) + \int_0^{\mathfrak{y}} \sin(\mathfrak{y}s) \rho(s, \tau) ds \right],$$

where

$$(\mathcal{H}\mathfrak{q})(\mathfrak{y})(\tau) = \frac{e^{-\mathfrak{y}}}{1 + e^{-\mathfrak{y}}} \left[\int_0^{\mathfrak{y}} \sin(\mathfrak{y}s) \rho(s, \tau) ds \right].$$

Additionally, $\mathcal{K} : \mathcal{I} \times Y \to Y$ is completely continuous mapping, defined as $\mathcal{K}(\mathfrak{y}, \mathfrak{q}(\mathfrak{y})) = \overline{\mathcal{K}}(\mathfrak{y}, \rho(\mathfrak{y}, \tau))$. Therefore, fractional system (7) is written as the nonlinear Cauchy problems (1) and (2).

Clearly, $\mathfrak{F}\big(\mathfrak{y}, \rho(\mathfrak{y}, \tau), (H\rho)(\mathfrak{y}, \tau)\big)$ is uniformly bounded. Then, the hypotheses $(H_1) - (H_6)$ are satisfied. However, the linear system that corresponds to (7) is approximately controllable; thus, the Theorem 1 is true. As a result, the requirements of Theorem 2 are met in full. Therefore, the HFD_{tial} inclusion in (1) and (2) is therefore approximately controllable on $[0, d]$.

5. Conclusions

This paper concentrated on the approximate controllability of Hilfer fractional neutral Volterra integro-differential inclusions via an approximately sectorial operator. The major conclusions are established by applying the results and ideas belonging to almost sectorial operators, fractional differential, multivalued map and fixed point method. We first proved the existence of the mild solution of the fractional system and then looked into the approximate controllability. Finally, to explain the principle, we offered an example. In the future, the authors will use a fixed point technique to study the exact controllability of the Hilfer fractional derivative using almost sectorial operators and will try to develop some real-life applications related to Hilfer fractional differential systems because there are only a few studies with real-life applications.

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Abbreviations

The following abbreviations are used in this manuscript:

HFD_{ve}	Hilfer fractional derivative
HFD_{tial}	Hilfer fractional differential
HF	Hilfer fractional

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