fractal and fractional

MDPI

Article

Nontrivial Solutions of a Class of Fourth-Order Elliptic Problems with Potentials

Jiabin Zuo [1,*], Zakaria El Allali [2] and Said Taarabti [3]

[1] College of Mathematics, Changchun Normal University, Changchun 130032, China
[2] Team of Modeling and Scientific Computing, LAMAO, Department of Mathematics, Multidisciplinary Faculty of Nador, Mohammed First University, Oujda 60000, Morocco
[3] LISTI, National School of Applied Sciences of Agadir, Ibn Zohr University, Agadir 80000, Morocco
* Correspondence: zuojiabin88@163.com

Abstract: This paper deals with a fourth-order elliptic equation with Dirichlet boundary conditions. Using a variant form of the mountain pass theorem, we prove the existence of nontrivial solutions to this problem. Furthermore, we discuss the fundamental properties of the representation of the solution by considering two cases. Our results not only make previous results more general but also show new insights into fourth-order elliptic problems.

Keywords: nontrivial solutions; fourth-order; mountain pass lemma; existence

MSC: 35J65; 58E05; 35J60

check for updates

Citation: Zuo, J.; El Allali, Z.; Taarabti, S. Nontrivial Solutions of a Class of Fourth-Order Elliptic Problems with Potentials. *Fractal Fract.* **2022**, *6*, 568. https://doi.org/10.3390/fractalfract6100568

Academic Editor: Jocelyn Sabatier

Received: 20 August 2022
Accepted: 30 September 2022
Published: 6 October 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Since the pioneering studies of Ambrosetti, Brezis, and Cerami (see [1,2]), there has been a significant amount of interest in second-order elliptic problems with mixed nonlinearities. However, as far as the authors are aware, the case of the fourth-order elliptic issue appears to receive very few results. With regard to u, the classical criteria of sublinearity at 0 and superlinearity at ∞ are introduced here as local counterparts.

We now provide a brief summary of the relevant findings. In [3], Ying and Jiang investigated the following nonlinear fourth-order elliptic equations

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = g(u), & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N) \end{cases}$$

where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator, V is an indefinite potential, and g grows subcritically and satisfies the Ambrosetti–Rabinowitz type condition $g(t)t \geq \mu G(t) \geq 0$ with $\mu > 3$. Using Morse theory, they obtained nontrivial solutions of the above equations.

Moreover, Mao and Wang [4] proved the multiplicity solutions of the following nonlocal fourth-order elliptic problem

$$\Delta^2 u - \left(1 + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^3,$$

where $\Delta^2 = \Delta(\Delta)$ is the bi-harmonic operator, and $\lambda \geq 0$ is a constant. They focused on the case in which $f(x,u)$ involves a combination of convex and concave terms, and the potential $V(x)$ is allowed to be sign-changing.

In this work, we study the following fourth-order elliptic equations

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda V(x)|u|^{p-2}u + g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, Δ^2 denotes a biharmonic operator, $1 < p < 2$, λ is a non-negative parameter, c is a constant, and $V \in L^\infty(\Omega)$. The potential $V(x) \geq 0$ and the function $g(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are both continuous.

An and Liu used the well-known mountain pass theorem in [5] to demonstrate the existence of nontrivial solutions to (1). They provide the following assumptions for $g(x,t)$:

(A_1) $g(x,t) \in C(\bar{\Omega} \times \mathbb{R}); f(x,t) \geq 0, \forall x \in \bar{\Omega}, t > 0; f(x,t) \equiv 0, \forall x \in \bar{\Omega}, t \leq 0$.

(A_2) $|g(x,t)| \leq a(x) + b|t|^p$, for a.e. $x \in \Omega$ and $\forall t \in \mathbb{R}$, where $a(x) \in L^q(\Omega), b \in \mathbb{R}$ and $1 < p < \frac{N+4}{N-4}, N > 4$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(A_3) For a.e. $x \in \Omega$, $\frac{g(x,t)}{t}$ is nondecreasing with respect to $t > 0$.

(A_4) $\lim_{t\to 0^+} \frac{g(x,t)}{t} = \mu, \lim_{t\to+\infty} \frac{g(x,t)}{t} = l (0 < l < +\infty)$ uniformly a.e. $x \in \Omega$, where $\mu < \Lambda_1 < l, l \neq \Lambda_k = \lambda_k(\lambda_k - c)$ are constants, and $\lambda_k (k = 1, 2, \ldots)$ are the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$.

They demonstrated that Problem (1) has at least one nontrivial solution if $c < \lambda_1$ and Conditions $(A_1) - (A_4)$ hold. On the one hand, we use a variation of the mountain pass theorem in place of the standard one and discover that Conditions (A_2) and (A_3) in [5] can be omitted and that $l \neq \Lambda_k = \lambda_k(\lambda_k - c)$ in Condition (A_4) in [5] can also be omitted. We demonstrate that the result mentioned above is likewise accurate (see Theorem 1). Additionally, our findings in the manuscript outperform a comparable finding in [5] (see Theorems 1 and 2).

Fourth-order elliptic equations are commonly used to describe a variety of occurrences in physics, engineering, and other disciplines. A great number of authors have focused their attention on fourth-order problems in recent years (see [6–15] and references therein).

Problem (1) presents a model for studying traveling waves in suspension bridges, according to Lazer and McKenna in [16]. According to the authors of [17,18], Problem (1) also occurs in other types of spacecraft, such as communication satellites, space shuttles, and space stations, all of which are outfitted with huge antennas installed on long flexible masts (beams). A considerable number of studies [5,19–21] on Problem (1) have been undertaken in recent years. For example, Yin and Jiang [3] prove the existence of nontrivial solutions of a modified fourth-order elliptic equation by applying the Morse theory. Wu and Chen [22] investigated the multiplicity result for a fourth-order elliptic problem in whole space $\mathbb{R}^N$. Moreover, they showed that this problem has a ground state by using variational methods.

We see that the trivial solution $u = 0$ is always allowed for Problem (1). Therefore, it makes sense to inquire as to whether there are any other solutions. We have further theorems in this direction.

The present paper is motivated by [23]. The goal of this note is to prove the existence of nontrivial solutions to (1) under suitable conditions on $g(x,t)$ [24–30].

The following assumptions about $g(x,t) \in C(\overline{\Omega} \times \mathbb{R})$ will be considered in this paper:

$$(F_1) : \begin{cases} g(x,t) \geq 0, & \text{if} \quad \forall x \in \overline{\Omega}, t > 0, \\ g(x,t) \equiv 0, & \text{if} \quad \forall x \in \overline{\Omega}, t \leq 0. \end{cases}$$

$$(F_2) : \lim_{t\to 0^+} \frac{g(x,t)}{t} = q(x), \quad \lim_{t\to+\infty} \frac{g(x,t)}{t} = l \ (0 < l \leq +\infty) \text{ uniformly a.e. } x \in \Omega,$$

where $0 \leq q(x) \in L^\infty(\Omega)$, $\|q(x)\|_\infty < \lambda_1$, and λ_1 is the first eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$.

(F_3): For a.e. $x \in \Omega$, $\frac{g(x,t)}{t}$ is nondecreasing with respect to $t > 0$.

Now, we disclose our primary findings, which pertain to the existence results for Problem (1).

Theorem 1. *If (F_1) and (F_2) hold, then Equation (1) admits a nontrivial solution when $\lambda_1 < l < +\infty$.*

Theorem 2. *If (F_1)–(F_3) hold, then Equation (1) admits a nontrivial solution when $l = +\infty$ and $g(x,t)$ is subcritical, i.e.,*

$$\lim_{t\to+\infty}\frac{g(x,t)}{t^{r-1}}=0$$

uniformly a.e. $x\in\Omega$ *for some* $r\in(2,\frac{2N}{N-4})$.

Remark 1. *If* $\lambda=0$ *in Problem (1), then we obtain the same results as in [23] (see Theorems 1(ii) and 2).*

Remark 2. *If* $c=0$ *and* $\lambda=0$ *in Problem (1), then we obtain the same results as in [11] (see Theorems 1.2 and 1.2′).*

2. Preliminaries

The Hilbert space $H=H^2(\Omega)\cap H_0^1(\Omega)$ with the inner product

$$\langle u,v\rangle_H=\int_\Omega(\Delta u\Delta v+\nabla u.\nabla v)dx$$

as well as the norm $\|u\|_H^2=\int_\Omega|\Delta u|^2dx+\int_\Omega|\nabla u|^2dx$. Denote $\mu_k(k\in\mathbb{N})$ as the eigenvalue and $u_k(k\in\mathbb{N})$ as the associated eigenfunctions of

$$\begin{cases}-\Delta u=\mu u, & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{2}$$

where each eigenvalue μ_k appears as many times as the multiplicity. Note that $0<\mu_1<\mu_2\le\mu_3\le\ldots\le\mu_k\to+\infty$ and $u_1>0$ for $x\in\Omega$. It is clear that $\lambda_k=\mu_k(\mu_k-c)$ are the following problem's eigenvalues

$$\begin{cases}\Delta^2u+c\Delta u=\lambda u, & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega,\end{cases}\tag{3}$$

and the associated eigenfunctions are u_k.

Suppose that $c<\mu_1$. We introduce a new norm:

$$\|u\|^2=\int_\Omega|\Delta u|^2dx-c\int_\Omega|\nabla u|^2dx,\ \forall u\in H.$$

Obviously, the norm $\|.\|$ with a norm on H are equivalent due to the [11]. Furthermore, for all $\forall u\in H$, Poincaré inequality holds (for more details see [9,10]):

$$\|u\|^2\ge\lambda_1\|u\|_{L^2}^2.\tag{4}$$

Next, we give a definition of a weak solution of Equation (1), if u satisfies

$$\int_\Omega\Big(\Delta u\Delta v-c\nabla u.\nabla v-\lambda V(x)|u|^{p-2}uv-g(x,u)v\Big)dx=0,\quad\forall v\in H^*,$$

where H^* is the dual space of H.

The weak solution of Equation (1) is equivalent to the critical point of the Euler–Lagrange functional

$$I(u)=\frac{1}{2}\int_\Omega\Big(|\Delta u|^2-c|\nabla u|^2\Big)dx-\frac{\lambda}{p}\int_\Omega V(x)|u(x)|^pdx-\int_\Omega G(x,u)dx,\tag{5}$$

where $G(x,u)=\int_0^u g(x,t)dt$.

Obviously, $I\in C^1(H,\mathbb{R})$, and

$$\langle\nabla I(u),v\rangle=\int_\Omega\Big(\Delta u\Delta v-c\nabla u.\nabla v-\lambda V(x)|u|^{p-2}uv-g(x,u)v\Big)dx,\quad\forall u,v\in H.$$

Lemma 1 (see [23]). *If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then the two norms $\|u\|$ and $\|\Delta u\|_2$ are equivalent where $\|u\|_2 = \|u\|_{L^2} = \left(\int_\Omega |u|^2\right)^{\frac{1}{2}}$.*

Lemma 2 (see [23]). *If $u_n \to u$ in $L^p(\Omega)$ for $1 \leq p < +\infty$, then $u_n^+ \to u^+$ in $L^p(\Omega)$.*

Lemma 3. *Let (F_1) and (F_2) be true.*

(i) There exist $\rho, R > 0$ such that $I(u) \geq R$ for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\|u\| = \rho$.

(ii) If $\varphi_1 > 0$ is the λ_1-eigenfunction of $\left(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega)\right)$, and $\lambda_1 < l < +\infty$, then $I(t\varphi_1) \to -\infty$ as $t \to +\infty$.

Proof. (i) In light of the conditions (F_1) and (F_2), it follows that for any $\varepsilon > 0$, there exists $A = A(\varepsilon) > 0$, such that, for all $(x,t) \in \Omega \times \mathbb{R}$ and $\alpha \in (1, \frac{N+4}{N-4})$,

$$G(x,t) \leq \frac{1}{2}(\|q(x)\|_\infty + \varepsilon)t^2 + A|t|^{\alpha+1}. \tag{6}$$

Let $\varepsilon > 0$, such that $\|q(x)\|_\infty + \varepsilon < \lambda_1$. We have $\|u\|_{\alpha+1}^{\alpha+1} \leq K\|u\|^{\alpha+1}$. According to Lemma 1 and the Sobolev inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_\Omega V(x)|u(x)|^p dx - \int_\Omega G(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda|h|_\infty}{p}\|u\|_{L^p}^p - \frac{1}{2}(\|q(x)\|_\infty + \varepsilon)\int_\Omega |u|^2 dx - A\int_\Omega |u|^{\alpha+1}dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\lambda|h|_\infty}{p}\|u\|_{L^p}^p - \frac{1}{2}(\|q(x)\|_\infty + \varepsilon)\|u\|_{L^2}^2 - A\|u\|_{L^{\alpha+1}}^{\alpha+1} \\ &\geq \frac{1}{2}\left(1 - \frac{\|q(x)\|_\infty + \varepsilon}{\lambda_1}\right)\|u\|^2 - \frac{\lambda|h|_\infty}{p}K\|u\|^p - AK\|u\|^{\alpha+1} \\ &= \left[\frac{1}{2}\left(1 - \frac{\|q(x)\|_\infty + \varepsilon}{\lambda_1}\right) - \lambda C\|u\|^{p-2} - C_s\|u\|^{\alpha-1}\right]\|u\|^2. \end{aligned} \tag{7}$$

Let

$$G(t) = \lambda C t^{p-2} + C_s t^{\alpha-1}.$$

We prove that t_0 exists in the sense that

$$G(t_0) < \frac{1}{2}\left(1 - \frac{\|q(x)\|_\infty + \varepsilon}{\lambda_1}\right).$$

Indeed,

$$G'(t) = \lambda C(p-2)t^{p-3} + C_s(\alpha-1)t^{\alpha-2}.$$

Setting

$$G'(t) = 0,$$

we obtain

$$t_0 = \left(\frac{\lambda C(2-p)}{C_s(\alpha-1)}\right)^{\frac{1}{\alpha-p+1}}.$$

At $t = t_0$, $G(t)$ reaches its minimum.
Let $\beta = \frac{C(2-p)}{C_s(\alpha-1)}$, $\widetilde{p} = \frac{p-2}{\alpha-p+1}$, $\widetilde{q} = \frac{\alpha-1}{\alpha-p+1}$, $k = \frac{1}{2}\left(1 - \frac{\|q(x)\|_\infty + \varepsilon}{\lambda_1}\right)$.
Substituting t_0 in $Q(t)$, we have

$$G(t_0) < \frac{1}{2}\left(1 - \frac{\|q(x)\|_\infty + \varepsilon}{\lambda_1}\right), \quad 0 < \lambda < \lambda^*,$$

where

$$\lambda^* = \left(\frac{k}{C^{\widetilde{p}} + C_s\beta^{\widetilde{q}}}\right)^{\frac{1}{\widetilde{q}}}.$$

Take $\rho = t_0$, then there exists $R > 0$ such that the lemma holds. This completes the proof.

(ii) According to Fatou's lemma, if we have $\lambda_1 < l < +\infty$, we obtain

$$\begin{aligned}\lim_{t\to+\infty}\frac{I(t\varphi_1)}{t^2} &\leq \frac{1}{2}\|\varphi_1\|^2 - \lim_{t\to+\infty}\frac{\lambda|t|^{p-2}}{p}\int_\Omega V(x)|\varphi_1|^p dx - \int_\Omega \lim_{t\to+\infty}\frac{G(x,t\varphi_1)}{(t\varphi_1)^2}|\varphi_1|^2 dx\\ &\leq \frac{\lambda_1}{2}\|\varphi_1\|_2^2 - \lim_{t\to+\infty}\frac{\lambda|t|^{p-2}}{p}\int_\Omega V(x)|\varphi_1|^p dx - \frac{l}{2}\|\varphi_1\|_2^2\\ &\leq \frac{1}{2}(\lambda_1 - l)\|\varphi_1\|_2^2 - \lim_{t\to+\infty}\frac{\lambda|t|^{p-2}}{p}\int_\Omega V(x)|\varphi_1|^p dx < 0 \quad (1<p<2).\end{aligned}$$

Then,

$$\lim_{t\to+\infty} I(t\varphi_1) = -\infty.$$

The proof is complete. □

Lemma 4. *Assuming (F_1) and (F_3) hold, if $\langle I^{'}(u_n), u_n\rangle \to 0$ as $n \to \infty$, then there exists a subsequence (u_n) such that $I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n)$ for all $t > 0$ and $n \in \mathbb{N}$.*

Proof. By $\langle I^{'}(u_n), u_n\rangle \to 0$ as $n \to \infty$, we obtain

$$\frac{-1}{n} < \langle I^{'}(u_n), u_n\rangle = \|u_n\|^2 - \int_\Omega g(x,u_n(x))u_n(x)dx - \lambda\int_\Omega V(x)|u_n(x)|^p dx < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

We argue that for all $t > 0$ and $n \in \mathbb{N}$,

$$I(tu_n) \leq \frac{t^2}{2n} + \int_\Omega\left(\frac{1}{2}g(x,u_n(x))u_n(x)dx - G(x,u_n(x))\right)dx + \lambda\left(\int_\Omega V(x)|u_n|^p - \frac{1}{p}V(x)|u_n|^p\right)dx.$$

In fact, a fixed $n \in \mathbb{N}$ and $x \in \Omega$, if we let

$$h(t) = \frac{t^2}{2}g(x,u_n)u_n(x) - G(x,tu_n(x)) + \frac{\lambda t^2}{2}V(x)|u_n|^p - \frac{\lambda t^p}{p}V(x)|u_n|^p \ \forall t > 0,$$

then

$$\begin{aligned}h^{'}(t) &= tg(x,u_n)u_n(x) - g(x,tu_n)u_n(x) + \lambda tV(x)|u_n|^p - \lambda t^{p-1}V(x)|u_n|^p\\ &= t\left(\frac{g(x,u_n)}{u_n}|u_n|^2 - \frac{g(x,tu_n)}{tu_n}|u_n|^2\right) + t\lambda V(x)|u_n|^p(1-t^{p-1}),\end{aligned} \tag{8}$$

hence,

$$\begin{cases} h^{'}(t) \geq 0 & \text{if} \quad 0 < t < 1,\\ h'(t) \leq 0 & \text{if} \quad t \geq 1.\end{cases} \tag{9}$$

Thus, we have

$$h(t) \leq h(1) \quad \forall t > 0.$$

Therefore, we obtain

$$\begin{aligned} I(tu_n) &= \frac{t^2}{2}\|u_n\|^2 - \int_\Omega G(x,tu_n)dx - \frac{\lambda t^p}{p}\int_\Omega V(x)|u_n|^p dx \\ &< \frac{t^2}{2}\left[\frac{1}{n} + \int_\Omega g(x,u_n)u_n dx + \lambda\int_\Omega V(x)|u_n|^p dx\right] - \int_\Omega G(x,tu_n)dx - \frac{\lambda t^p}{p}\int_\Omega V(x)|u_n|^p dx \\ &\leq \frac{t^2}{2n} + \int_\Omega\left[\frac{t^2}{2}g(x,u_n)u_n - G(x,tu_n)\right]dx + \int_\Omega\left[\frac{\lambda t^2}{2}\int_\Omega V(x)|u_n|^p dx - \frac{\lambda t^p}{p}\int_\Omega V(x)|u_n|^p dx\right] \\ &\leq \frac{t^2}{2n} + \int_\Omega\left[\frac{1}{2}g(x,u_n)u_n - G(x,u_n)\right]dx + \int_\Omega\left[\frac{\lambda}{2}\int_\Omega V(x)|u_n|^p dx - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p dx\right]. \end{aligned}$$

Our assertion is proved.
However, one has

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_\Omega G(x,u_n)dx - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p dx \\ &\geq \frac{1}{2}\left[-\frac{1}{n} + \int_\Omega g(x,u_n)u_n dx + \lambda\int_\Omega V(x)|u_n|^p dx\right] - \int_\Omega G(x,u_n)dx - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p dx, \end{aligned}$$

i.e.,

$$\int_\Omega\left[\frac{1}{2}g(x,u_n)u_n - G(x,u_n)\right]dx + \int_\Omega\left[\frac{\lambda}{2}V(x)|u_n|^p - \frac{\lambda}{p}V(x)|u_n|^p\right]dx \leq \frac{1}{2n} + I(u_n).$$

From the above, we can find that if

$$I(tu_n) \leq \frac{t^2}{2n} + \frac{1}{2n} + I(u_n),$$

then

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n), \quad \forall t > 0 \quad \text{and} \quad n \in \mathbb{N}.$$

□

3. The Proofs of Main Results

Proof of Theorem 1. With $\lambda_1 < l < +\infty$, using Lemma 3 (i) and (ii), we may obtain a $t_0 > 0$ large enough to satisfy $I(t_0\varphi_1) < 0$. Define

$$\begin{aligned} \Gamma &= \left\{\gamma \in C\Big([0,1], H^2(\Omega)\cap H_0^1(\Omega)\Big) : \gamma(0) = 0, \gamma(1) = t_0\varphi_1\right\} \\ c_0 &= \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)). \end{aligned}$$

Then, $c_0 > R > 0$ and, according to Proposition 2.1 in [23], there exists a sequence $(u_n) \subset H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p dx - \int_\Omega G(x,u_n(x))dx = c_0 + o(1), \tag{10}$$

and

$$(1+\|u_n\|)\|I'(u_n)\|_{H^{-2}(\Omega)} \underset{n\to\infty}{\to} 0, \tag{11}$$

where $H^{-2}(\Omega)$ represents the dual space of $H^2(\Omega)\cap H_0^1(\Omega)$. We denote by $o(1)$ any quantity that approaches zero as $n \to +\infty$. By (11), it follows that

$$\langle I'(u_n), u_n\rangle = \|u_n\|^2 - \lambda\int_\Omega V(x)|u_n|^p dx - \int_\Omega g(x,u_n)u_n dx = o(1). \tag{12}$$

It follows from standard argument and the theory of Sobolev embedding that there exists a subsequence (u_n) that strongly converges to u, which is a critical point of functional I if (u_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$.

In order to prove Theorem 1, we first show that (u_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. In contradiction, we assume $\|u_n\| \to +\infty$ and take

$$t_n = \frac{1}{\|u_n\|}, \quad v_n = t_n u_n = \frac{u_n}{\|u_n\|}. \tag{13}$$

Clearly, (v_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. We can deduce that by taking a subsequence from Lemma 2,

$$\begin{aligned} &v_n \underset{n\to\infty}{\rightharpoonup} w \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega), \\ &v_n \underset{n\to\infty}{\to} w \quad \text{in} \quad L^2(\Omega), \quad v_n \underset{n\to\infty}{\to} w \quad \text{a.e. in} \quad \Omega, \\ &v_n^+ \underset{n\to\infty}{\to} v^+ \quad \text{in} \quad L^2(\Omega), \quad v_n^+ \underset{n\to\infty}{\to} v^+ \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

We argue that

$$w \neq 0. \tag{14}$$

In addition, we can show from Conditions (F_1) and (F_2) that there exists $M > 0$ such that $\frac{|g(x,t)|}{|t|} \leq M$ for all $x \in \Omega$ and $t \neq 0$. Then, using (12) and (13), we obtain

$$\int_\Omega \frac{g(x,u_n)}{u_n}|v_n|^2 dx = \frac{\lambda}{\|u_n\|^2}\int_\Omega V(x)|u_n|^p dx + \frac{\int_\Omega g(x,u_n)u_n dx}{\|u_n\|^2} = 1 + o(1).$$

This means that

$$\begin{aligned} 1 + o(1) &= \frac{\lambda}{\|u_n\|^2}\int_\Omega V(x)|u_n|^p dx + \int_\Omega \frac{g(x,u_n)}{u_n}|v_n|^2 dx \\ &\leq \frac{\lambda|h|_\infty}{\|u_n\|^2}\int_\Omega |u_n|^p dx + M\int_\Omega |v_n|^2 dx \to M\int_\Omega |w|^2 dx, \end{aligned}$$

so $w \neq 0$.

By $I'(u_n) \underset{n\to\infty}{\to} 0$, for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\int_\Omega (\Delta u_n \Delta\varphi - c\nabla u_n \nabla\varphi)dx = \int_\Omega g(x,u_n)\varphi dx + \lambda\int_\Omega V(x)|u_n|^{p-2}u_n\varphi dx + o(1).$$

Then, by (13) and (F_1), we have

$$\int_\Omega (\Delta v_n \Delta\varphi - c\nabla v_n \nabla\varphi)dx = \int_\Omega \frac{g(x,u_n^+)}{u_n^+}v_n^+\varphi dx + \frac{\lambda}{\|u_n\|}\int_\Omega V(x)|u_n|^{p-2}u_n\varphi dx + o(1). \tag{15}$$

Since $\frac{|g(x,t)}{|t|} \leq M$,

$$\frac{g(x,u_n^+)}{u_n^+}v_n^+ \to 0 = lv^+ \quad \text{a.e.} \quad x \in \Omega, \quad \text{if} \quad v^+(x) = 0.$$

If $v^+(x) > 0$, then we obtain that $u_n^+(x) = v_n^+(x)\|u_n\| \underset{n\to\infty}{\to} +\infty$, a.e. $x \in \Omega$, so by (F_2) we have

$$\frac{g(x,u_n^+)}{u_n^+}v_n^+ \to lv^+ \text{ a.e.} \quad x \in \Omega.$$

$\left(\frac{g(x,u_n^+)}{u_n^+}v_n^+\right)$ is bounded in $L^2(\Omega)$ since $\frac{|g(x,t)|}{|t|} \leq M$, hence there exists a subsequence such that

$$\frac{g(x,u_n^+)}{u_n^+}v_n^+ \rightharpoonup lv^+ \quad \text{weakly in} \quad L^2(\Omega).$$

Therefore, we have

$$\int_\Omega \frac{g(x,u_n^+)}{u_n^+}v_n^+\varphi dx \underset{n\to\infty}{\to} l\int_\Omega v^+\varphi dx, \ \ \forall \varphi \in H^2(\Omega)\cap H_0^1(\Omega). \tag{16}$$

Since $v_n \underset{n\to\infty}{\rightharpoonup} w$ weakly in $H^2(\Omega)\cap H_0^1(\Omega)$, by Lemma 1, we obtain

$$\int_\Omega (\Delta v_n\Delta\varphi - c\nabla v_n\nabla\varphi)dx \to \int_\Omega (\Delta w\Delta\varphi - c\nabla w\nabla\varphi)dx, \ \ \forall \varphi \in H^2(\Omega)\cap H_0^1(\Omega). \tag{17}$$

Using (15)–(17), we obtain

$$\int_\Omega (\Delta w\Delta\varphi - c\nabla w\nabla\varphi)dx = l\int_\Omega v^+\varphi dx. \tag{18}$$

Taking $\varphi = \varphi_1(x)$ in (18), we obtain

$$\int_\Omega (\Delta w\Delta\varphi_1 - c\nabla w\nabla\varphi_1)dx = l\int_\Omega v^+\varphi_1 dx. \tag{19}$$

Observe that φ_1 is positive first eigenfunction of

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda_1 u, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{20}$$

then for each $e \in H^2(\Omega)\cap H_0^1(\Omega)$, we have

$$\int_\Omega (\Delta e\Delta\varphi_1 - \nabla e\nabla\varphi_1)dx = \lambda_1\int_\Omega e\varphi_1 dx.$$

Using $e = w(x)$, we obtain

$$\int_\Omega (\Delta w\Delta\varphi_1 - \nabla w\nabla\varphi_1)dx = \lambda_1\int_\Omega w\varphi_1 dx, \tag{21}$$

thus combining (19) and (21), we have

$$l\int_\Omega v^+\varphi_1 dx = \lambda_1\int_\Omega w\varphi_1 dx. \tag{22}$$

Notice that

$$\int_\Omega v^+\varphi_1 dx - \int_\Omega w\varphi_1 dx = \int_{\Omega_-} -w\varphi_1 dx \geq 0,$$

where $\Omega_- = \{z \in \Omega : v(z) < 0\}$. □

Now, we assert that the two situations $|\Omega_-| = 0$ and $|\Omega_-| > 0$ are incompatible:

Case 1: If $|\Omega_-| = 0$, then $w(x) \geq 0$ a.e. x in Ω. Since $w(x) \neq 0$, we obtain $\int_\Omega w\varphi_1 dx > 0$. Then (22) means that

$$l\int_\Omega w\varphi_1 dx = l\int_\Omega v^+\varphi_1 dx = \lambda_1\int_\Omega w\varphi_1 dx,$$

which is impossible due to the fact that $l > \lambda_1$.

Case 2: If $|\Omega_-| > 0$, then $\int_{\Omega_-} -w\varphi_1 dx > 0$ and $\int_\Omega v^+\varphi_1 dx > \int_\Omega w\varphi_1 dx$.

According to (22), we have

$$l\int_\Omega v^+\varphi_1 dx = \lambda_1\int_\Omega w\varphi_1 dx < \lambda_1\int_\Omega v^+\varphi_1 dx,$$

which is also impossible due to the fact that $l > \lambda_1$, if $\int_\Omega v^+\varphi_1 dx > 0$, which also is in contradiction to $0 < 0$, if $\int_\Omega v^+\varphi_1 dx = 0$.

Thus, we show the boundedness of sequence (u_n) in $H^2(\Omega)\cap H_0^1(\Omega)$.

Proof of Theorem 2. Similarly to the proof of Theorem 1, we can also identify a subsequence $(u_n)\subset H^2(\Omega)\cap H_0^1(\Omega)$ such that (10) and (12) hold for the case of $l = +\infty$. As a result, to establish Theorem 2, we only need to show that (u_n) is bounded in $H^2(\Omega)\cap H_0^1(\Omega)$. If not, we can assume that $\|u_n\|\to+\infty$ and let

$$t_n = \frac{2\sqrt{c_0}}{\|u_n\|}, \qquad v_n = t_n u_n = \frac{2\sqrt{c_0}u_n}{\|u_n\|}. \tag{23}$$

Obviously, (v_n) is bounded in $H^2(\Omega)\cap H_0^1(\Omega)$. Similarly, we also have

$$\begin{aligned} v_n &\underset{n\to\infty}{\rightharpoonup} w \quad \text{in} \quad H^2(\Omega)\cap H_0^1(\Omega),\\ v_n &\underset{n\to\infty}{\to} w \quad \text{in} \quad L^2(\Omega), \quad v_n\to w \quad \text{a.e. in} \quad \Omega,\\ v_n^+ &\underset{n\to\infty}{\to} v^+ \quad \text{in} \quad L^2(\Omega), \quad v_n^+\to v^+ \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

Observing that $l\equiv+\infty$ in (F_2), we have for each given $K>0$,

$$\lim_{n\to+\infty}\frac{g(x,u_n^+)}{u_n^+}|v_n^+|^2 \ge K|v^+|^2 \quad \text{almost everywhere} \quad x\in\Omega. \tag{24}$$

In fact, for almost everywhere $x\in\Omega$, if $v^+(x)=0$, then the equality is true. If $v^+(x)>0$, it follows from (23) that $u_n^+(x)\to+\infty$. With $|v_n^+(x)|^2 \underset{n\to\infty}{\to} |v^+(x)|^2$, let $\epsilon_0 = \frac{1}{2}|v^+(x)|^2$, thus there exists a positive N_1 such that

$$-\epsilon_0 < |v_n^+(x)|^2 - |v^+(x)|^2 < \epsilon_0 \quad \text{for} \quad n\ge N_1, \tag{25}$$

which means that

$$|v_n^+(x)|^2 > |v^+(x)|^2 - \epsilon_0 = \frac{1}{2}|v^+|^2. \tag{26}$$

With $l\equiv+\infty$ in (F_2), and $u_n^+(x)\underset{n\to\infty}{\to}+\infty$, we obtain

$$\lim_{n\to+\infty}\frac{g(x,u_n^+(x))}{u_n^+(x)} = +\infty.$$

As a result, for any given $K>0$, we can find a positive $N_2(x,K)$ such that

$$\frac{g(x,u_n^+(x))}{u_n^+(x)} \ge 2K \quad \text{for} \quad n\ge N_2. \tag{27}$$

Set $N(x,K)=\max\{N_1,N_2\}$, then, by (26) and (27), we have

$$\frac{g(x,u_n^+)}{u_n^+}|v_n^+|^2 \ge K|v^+|^2 \quad \text{for} \quad n\ge N.$$

Therefore, (24) is true.

In view of (12), (23) and (24), we obtain

$$\begin{aligned} 4c_0 &= \lim_{n\to+\infty} \|v_n\|^2 \\ &= \lim_{n\to+\infty} 4c_0 \left[\frac{\int_\Omega g(x,u_n^+)u_n^+ dx}{\|u_n\|^2} + \frac{\lambda}{\|u_n\|^2}\int_\Omega V(x)|u_n^+|^p dx \right] \\ &= \lim_{n\to+\infty} \left[\int_\Omega \frac{g(x,u_n^+)}{u_n^+}|v_n^+|^2 dx \right] \geq K\int_\Omega |v^+|^2 dx. \end{aligned}$$

Choose $K > 0$ big enough such that $\|v^+\|_2 = 0$, so $v^+(x) \equiv 0$. Therefore, we have

$$\lim_{n\to+\infty}\int_\Omega G(x,v_n(x))dx = \lim_{n\to+\infty}\int_\Omega G(x,v_n^+(x)) = 0$$

and

$$I(v_n) = \frac{1}{2}\|v_n\|^2 + o(1) = 2c_0 + o(1). \tag{28}$$

Using $\|u_n\| \to +\infty$ as $n \to +\infty$ and, with (23), we know that $t_n \to 0$ as $n \to +\infty$, which implies that according to Lemma 4 and (10), we have

$$I(v_n) = I(t_n u_n) \leq \frac{1+t_n^2}{2n} + I(u_n) \to c_0 > 0. \tag{29}$$

It is obvious that (28) and (29) are incongruous, which reveals that (u_n) is bounded, and the proof of Theorem 2 is complete. □

Remark 3. *If we assume the following conditions :*
(F_0) $g(x,0) = 0$ *and* $g(x,-t) = -g(x,t)$ *for all* $x \in \Omega, t \in \mathbb{R}$.
(F_4) $\lim\limits_{t\to|\infty|}(g(x,t)t - 2G(x,t)) = -\infty$ *uniformly in* Ω, *and (F_2) with* $c < \lambda_1$; $\lambda_k(\lambda_1 - c) < l$,
we can prove that there exists $l^\star > 0$ *such that for* $l \in (0, l^\star)$. *Problem (1) has infinitely many solutions by employing the same method used by C. Liu and J. Wang (see [24]).*

4. Conclusions

In our work, we showed the existence of nontrivial solutions to Problem (1) through a variation of the mountain pass theorem. We also discussed the main characteristics of the solution representation by looking at two cases. We discovered that Conditions (A2) and (A3) in [5] can be removed, and we established that our results are also accurate by using a variation of the mountain pass theorem rather than the usual one (see Theorem 1). Furthermore, in comparison to the findings that are in [5], our findings in this paper performed significantly better (see Theorems 1 and 2).

This finding supports some previously published research. In Theorem 1, we studied the case of $\lambda_1 < l < +\infty$, and we found that our conditions are not sufficient to study the cases if $0 < l < \lambda_1$ and if $l = \lambda_1$. This issue has been left as an open question for researchers who are interested in the subject.

Moreover, the presence of the term V in our problem makes it more challenging to investigate the uniqueness and convergence of solutions. Therefore, we leave this topic as an unsolved question for experts in this field.

Author Contributions: Writing-original draft, J.Z., Z.E.A. and S.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ambrosetti, A.; Brezis, H.; Cerami, G. Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **1994**, *122*, 519–543. [CrossRef]
2. Figueiredo, D.G.D.; Gossez, J.P.; Ubilla, P. Local superlinearity and sublinearity for indefinite semilinear elliptic problems. *J. Funct. Anal.* **2003**, *199*, 452–467. [CrossRef]
3. Yin, L.F.; Jiang, S. Existence of nontrivial solutions for modified nonlinear fourth-order elliptic equations with indefinite potentia. *J. Math. Anal. Appl.* **2022**, *505*, 125459. [CrossRef]
4. Mao, A.; Wang, W. Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^3$. *J. Math. Anal. Appl.* **2018**, *459*, 556–563. [CrossRef]
5. An, Y.; Liu, R. Existence of nontrivial solutionsof an asymptotically linear fourth-order elliptic equation. *Nonlinear Anal.* **2008**, *68*, 3325–3331. [CrossRef]
6. Zhang, J.H. Existence results for some fourth-order elliptic problem. *Nonlinear Anal.* **2001**, *41*, 29–36.
7. Micheletti, A.M.; Pistoia, A. Three solutions of a fourth order elliptic problem via variational theorems of mixed type. *Appl. Anal.* **2000**, *75*, 43–59. [CrossRef]
8. Ahmed, N.U.; Harbi, H. Mathematical analysis of dynamic models of suspension bridges. *SIAM J. Appl. Math.* **1998**, *58*, 853–874.
9. MacManus, P. Poincaré inequalities and Sobolev spaces. *Publ. Mat.* **2002**, 181–197. [CrossRef]
10. Alasz, P.H.; Koskela, P. *Sobolev Met Poincaré*; Number 688; American Mathematical Society: Providence, RI, USA, 2000; Volume 145.
11. Liu, Y.; Wang, Z.P. Biharmonic equations with asymptotically linear nonlinearities. *Acta Math. Sci. Ser. B Engl. Ed.* **2007**, *27*, 549–560. [CrossRef]
12. Zuo, J.; Taarabti, S.; An, T.; Repovŝ, D.D. On Nonlinear Biharmonic Problems on the Heisenberg Group. *Symmetry* **2022**, *14*, 705. [CrossRef]
13. Haddouch, K.B.; Allali, Z.E.; Mermri, E.; Tsouli, N. Strict monotonicity and unique continuation for the third-order spectrum of biharmonic operator. *Abstr. Appl. Anal.* **2012**, *2012*, 571951. [CrossRef]
14. Wu, Y.; Taarabti, S.; Allali, Z.E.; Hadddouch, K.B.; Zuo, J. A Class of Fourth-Order Symmetrical Kirchhoff Type Systems. *Symmetry* **2022**, *14*, 1630. [CrossRef]
15. Junior, J.C.O. A class of modified nonlinear fourth-order elliptic equations with unbounded potential. *Complex Var. Elliptic Equ.* **2020**, *66*, 876–891. [CrossRef]
16. Lazer, A.C.; McKenna, P.J. Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Rev.* **1990**, *32*, 537–578. [CrossRef]
17. Biswas, S.K.; Ahmed, N.U. Optimal control of large space structures governed by a coupled system of ordinary and partial differential equations. *J. Math. Control Signals Syst.* **1989**, *2*, 1–18. [CrossRef]
18. Ahmed, N.U.; Biswas, S.K. Mathematical modeling and control of large space structures with multiple appendages. *J. Math. Comput. Model.* **1988**, *10*, 891–900. [CrossRef]
19. Tarantello, G. A note on a semilinear problem. *Differ. Integral Equ.* **1992**, *5*, 561–565.
20. Pao, C.V. On fourth-order elliptic boundary value problems. *Proc. Am. Math. Soc.* **2000**, *128*, 1023–1030. [CrossRef]
21. Zhou, J.; Wu, X. Sign-changing solutions for some fourth-order nonlinear elliptic problems. *J. Math. Anal. Appl.* **2008**, *342*, 542–558. [CrossRef]
22. Wu, Z.; Chen, H. A class of fourth-order elliptic equations with concave and convex nonlinearities in RN. *Electron. J. Qual. Theory Differ. Equ.* **2021**, *71*, 16.
23. Hu, S.; Wang, L. Existence of nontrivial solutions for fourth-order asymptotically linear elliptic equations. *Nonlinear Anal.* **2014**, *94*, 120–132. [CrossRef]
24. Liu, C.; Wang, J. Existence of multiple solutions for a class of biharmonic equations. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 809262. [CrossRef]
25. Gu, H.; An, T. Infinitely many solutions for a class of fourth-order partially sublinear elliptic problem. *Bound. Value Probl.* **2017**, *2017*, 1. [CrossRef]
26. Alsaedi, R. Combined perturbation effects for a class of nonlinear fourth-order Navier problems. *Complex Var. Elliptic Equ.* **2018**, *63*, 463–471. [CrossRef]
27. Pu, Y.; Wu, X.-P.; Tang, C.-L. Fourth-order Navier boundary value problem with combined nonlinearities. *J. Math. Anal. Appl.* **2013**, *398*, 798–813. [CrossRef]
28. Wei, Y. Multiplicity results for some fourth-order elliptic equations. *J. Math. Anal. Appl.* **2012**, *385*, 797–807. [CrossRef]
29. Pei, R.; Zhang, J. Non-uniformly asymptotically linear fourth-order elliptic problems. *Bound. Value Probl.* **2015**, *2015*, 209. [CrossRef]
30. Moaaz, O.; Kumam, P.; Bazighifan, O. On the Oscillatory Behavior of a Class of Fourth-Order Nonlinear Differential Equation. *Symmetry* **2020**, *12*, 524. [CrossRef]