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Some General Fractional Integral Inequalities Involving LR–Bi-Convex Fuzzy Interval-Valued Functions

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Abstract: The main objective of this paper is to introduce a new class of convexity called left-right-bi-convex fuzzy interval-valued functions. We study this class from the perspective of fractional Hermite–Hadamard inequalities, involving a new fractional integral called the left-right–AB fractional integral. We discuss several special cases that demonstrate that our results are quite unifying. We provide non-trivial numerical examples regarding special means for positive real numbers in order to check the validity of our outcomes.

Keywords: Hermite–Hadamard inequality; fuzzy integral; LR–convex function; LR–bi-convex fuzzy interval-valued function; LR–AB fractional integrals

MSC: 26A33 ; 26A51; 26D07; 26D10; 26D15; 26D20



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1. Introduction and Preliminaries

Convex Analysis is the branch of mathematics in which we study the properties of convex sets and convex functions. The theory of convexity has played a vital role in different branches of pure and applied sciences through its numerous applications, and has played a significant role in the development of the theory of inequalities. Here, we recall the notions of convex sets and convex functions as follows:

A set $\mathcal{C} \subseteq \mathbb{R}$ is said to be convex if

$$(1 - \tau)\varphi_1 + \tau\varphi_2 \in \mathcal{C}, \quad \forall \varphi_1, \varphi_2 \in \mathcal{C}, \tau \in [0, 1].$$

A function $\aleph_1 : \mathcal{C} \rightarrow \mathbb{R}$ is said to be convex if

$$\aleph_1((1 - \tau)\varphi_1 + \tau\varphi_2) \leq (1 - \tau)\aleph_1(\varphi_1) + \tau\aleph_1(\varphi_2), \quad \forall \varphi_1, \varphi_2 \in \mathcal{C}, \tau \in [0, 1].$$

Many famous inequalities known to us today are direct consequences of the applications of the convexity property of functions. A very useful result which has many applications in different fields of applied and engineering sciences is the Hermite–Hadamard inequality, which provides a necessary and sufficient condition for a function to be convex. This result was obtained independently by Hermite and Hadamard, and reads as follows:

Let $\aleph_1 : I := [\varphi_1, \varphi_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function; then,

$$\aleph_1\left(\frac{\varphi_1 + \varphi_2}{2}\right) \leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \aleph_1(x) dx \leq \frac{\aleph_1(\varphi_1) + \aleph_1(\varphi_2)}{2}.$$

In recent years, many different novel approaches have been used to obtain new variants of these inequalities. For example, Sarikaya et al. [1] obtained the first fractional analogue of the Hermite–Hadamard inequality. In 1985, Ramik [2] deduced inequalities through fuzzy numbers and used the inequality in fuzzy optimization. In [3], the authors established Jensen-type inequalities for the notion of fuzzy interval valued mappings. Costa et al. [4] used the concept of interval-valued fuzzy convex mappings to compute new integral inequalities. Zhao et al. [5] presented the idea of generalized interval-valued convexity to investigate inequalities of the Jensen and Hermite–Hadamard types. In [6], Liu et al. studied the modular inequalities of interval-valued soft sets with the aid of J -inclusions. In 2021, Yang et al. [7] formulated new inequalities of the Hermite–Hadamard type in association with exponential fuzzy interval-valued convex mappings. In [8], the authors used the idea of interval-valued mappings to compute Ostrowski-type inequalities and apply them to numerical integration. In [9], Santos-Gomez investigated coordinated inequalities via interval-valued fuzzy pre-invex functions. In [10], Khan et al. derived new Hermite–Hadamard-like inclusions involving harmonically interval-valued fuzzy mappings. In [11], the authors concluded that certain Hermite–Hadamard inequalities and their weighted forms, known as Fejer-type inclusions, involve generalized fractional operators with an exponential kernel. For recent developments and applications pertaining to the Hermite–Hadamard inequality, see [12–19].

Before we proceed further, let us first recall several well known concepts and results from fuzzy interval analysis.

First of all, suppose K_c is the space of all closed and bounded intervals of $\mathbb{R}$ and let $\chi \in K_c$ be defined as

$$\chi := [\chi_*, \chi^*] = \{\mu \in \mathbb{R} : \chi_* \leq \mu \leq \chi^*\}, \quad (\chi_*, \chi^* \in \mathbb{R}).$$

If $\chi_* = \chi^*$, then χ is said to be degenerate. Throughout the sequelae of this paper, all intervals are non-degenerate intervals. If $\chi_* \geq 0$, then $[\chi_*, \chi^*]$ is called a positive interval. The set of all positive intervals is denoted by K_c^+ and is defined as $K_c^+ := \{[\chi_*, \chi^*] : [\chi_*, \chi^*] \in K_c \wedge \chi_* \geq 0\}$.

Let $\alpha \in \mathbb{R}$ and $\alpha.\chi$ be defined as

$$\alpha.\chi := \begin{cases} [\alpha\chi_*, \alpha\chi^*] & \text{if } \alpha \geq 0, \\ [\alpha\chi^*, \alpha\chi_*] & \text{if } \alpha < 0. \end{cases}$$

Then, the Minkowski difference $\gamma - \chi$, addition $\chi + \gamma$, and $\chi \times \gamma$ for $\chi, \gamma \in K_c$ are defined by

$$\begin{aligned} [\gamma_*, \gamma^*] - [\chi_*, \chi^*] &:= [\gamma_* - \chi_*, \gamma^* - \chi^*] \\ [\gamma_*, \gamma^*] + [\chi_*, \chi^*] &:= [\gamma_* + \chi_*, \gamma^* + \chi^*] \end{aligned}$$

and

$$[\gamma_*, \gamma^*] \times [\chi_*, \chi^*] := [\min\{\gamma_*\chi_*, \gamma^*\chi_*, \gamma_*\chi^*, \gamma^*\chi^*\}, \max\{\gamma_*\chi_*, \gamma^*\chi_*, \gamma_*\chi^*, \gamma^*\chi^*\}].$$

Definition 1 ([20]). The relation “ $\leq_\rho$ ” defined on K_c is provided by

$$[\gamma_*, \gamma^*] \leq_\rho [\chi_*, \chi^*],$$

if and only if

$$\gamma_* \leq \chi_*, \gamma^* \leq \chi^*$$

for all $[\gamma_*, \gamma^*], [\chi_*, \chi^*] \in K_c$ is a pseudo-order relation. The relation $[\gamma_*, \gamma^*] \leq_\rho [\chi_*, \chi^*]$ is coincident to $[\gamma_*, \gamma^*] \leq [\chi_*, \chi^*]$ on K_c , where “ $\leq$ ” is partial order relation on K_c .

It can be seen that “ $\leq_\rho$ ” looks like “left and right” on the real line $\mathbb{R}$, thus, we say that “ $\leq_\rho$ ” is the “left and right” order (or “LR” order, in short).

Remark 1 ([20,21]). A fuzzy set of $\mathbb{R}$ is a function of $\tilde{\lambda} : \mathbb{R} \rightarrow [0, 1]$. For each fuzzy set $\vartheta \in (0, 1]$, ϑ -level sets of $\tilde{\lambda}$ are denoted and defined as follows: $\lambda_\vartheta = \{\omega \in \mathbb{R} : \tilde{\lambda}(\omega) \geq \vartheta\}$, and by $\text{supp}(\tilde{\lambda})$ or λ_0 , its support set, i.e., $\text{supp}(\tilde{\lambda}) = \text{cl}\{\omega \in \mathbb{R} : \tilde{\lambda}(\omega) > 0\}$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\tilde{\lambda} \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then, we define the following:

- $\tilde{\lambda}$ is said to be normal if there exists $\omega \in \mathbb{R}$ and $\tilde{\lambda}(\omega) = 1$;
- $\tilde{\lambda}$ is said to be upper semi-continuous on $\mathbb{R}$ if, for given $y \in \mathbb{R}$, there exist $\epsilon > 0$ and there exist $\delta > 0$ such that $\tilde{\lambda}(\omega) - \tilde{\lambda}(y) < \epsilon$ for all $\omega \in \mathbb{R}$ with $|\omega - y| < \delta$;
- $\tilde{\lambda}$ is said to be fuzzy convex if λ_ϑ is convex for every $\vartheta \in [0, 1]$;
- $\tilde{\lambda}$ is compactly supported if $\text{supp}(\tilde{\lambda})$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it satisfies the above-mentioned four properties, and $\mathbb{F}_0$ denotes the family of all fuzzy intervals.

Let $\lambda \in \mathbb{F}_0$ be a fuzzy interval if, and only if, ϑ -levels $[\lambda]^\vartheta$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\lambda]^\vartheta = [\lambda_*(\vartheta), \lambda^*(\vartheta)],$$

where

$$\lambda_*(\vartheta) := \inf\{\omega \in \mathbb{R} : \lambda(\omega) \geq \vartheta\}, \quad \lambda^*(\vartheta) := \sup\{\omega \in \mathbb{R} : \lambda(\omega) \geq \vartheta\}.$$

Thus, a fuzzy interval can be identified by a parameterized triplet. For more details, see [22].

$$\{\lambda_*(\vartheta), \lambda^*(\vartheta); \vartheta \in [0, 1]\}.$$

where the two end point functions $\lambda_*(\vartheta)$ and $\lambda^*(\vartheta)$ are used to characterize a real fuzzy interval.

Proposition 1. If $\gamma, \chi \in \mathbb{F}_0$, then the relation “ $\preceq$ ” is defined on $\mathbb{F}_0$ by

$$\gamma \preceq \chi \text{ if and only if } [\gamma]^\vartheta \leq_\rho [\chi]^\vartheta \text{ for all } \vartheta \in [0, 1].$$

this relation is known as the partial order relation.

For $\gamma, \chi \in \mathbb{F}_0$ and $\gamma \in \mathbb{R}$, the sum $\gamma \tilde{+} \chi$, product $\gamma \tilde{\times} \chi$, scalar product $c \cdot \gamma$, and sum with scalar are defined by

$$\begin{aligned} [\gamma \tilde{+} \chi]^\vartheta &= [\gamma]^\vartheta + [\chi]^\vartheta, \\ [\gamma \tilde{\times} \chi]^\vartheta &= [\gamma]^\vartheta \times [\chi]^\vartheta, \\ [c \cdot \gamma]^\vartheta &= c \cdot [\gamma]^\vartheta, \\ [c \tilde{+} \gamma]^\vartheta &= c + [\gamma]^\vartheta. \end{aligned}$$

For $\aleph_1 \in \mathbb{F}_0$ such that $\gamma = \chi \tilde{+} \aleph_1$, by this result we have the existence of a Hukuhara difference of γ and χ ; we can say that $\aleph_1$ is the H-difference of γ and χ , denoted by $\gamma \tilde{-} \chi$. If an H-difference exists, then

$$\begin{aligned} (\aleph_1)^*(\vartheta) &= (\gamma \tilde{-} \chi)^*(\vartheta) = \gamma^*(\vartheta) - \chi^*(\vartheta), \\ (\aleph_1)_*(\vartheta) &= (\gamma \tilde{-} \chi)_*(\vartheta) = \gamma_*(\vartheta) - \chi_*(\vartheta). \end{aligned}$$

Definition 2 ([23]). A fuzzy map $\aleph_1 : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{F}_0$ is a fuzzy interval valued function for each $\vartheta \in [0, 1]$ for which the ϑ -levels define the family of I.V.F. $\aleph_{1\vartheta} : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow K_c$ are provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_{1*}^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_2]$. Here, for each $\vartheta \in [0, 1]$, the

left and right real valued functions $\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta) : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ are called the lower and upper functions of $\aleph_1$.

Definition 3 ([24]). Let $\aleph_1 : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{F}_0$ be a fuzzy interval-valued function. Then, the fuzzy Riemann integral of $\aleph_1$ over $[\omega_1, \omega_2]$, denoted by $(FR) \int_{\omega_1}^{\omega_2} \aleph_1(\mu) d\mu$, is provided level-wise by

$$\begin{aligned} \left[(FR) \int_{\omega_1}^{\omega_2} \aleph_1(\mu) d\mu \right]^\vartheta &= (IR) \int_{\omega_1}^{\omega_2} \aleph_{1\vartheta}(\mu) d\mu \\ &= \left\{ \int_{\omega_1}^{\omega_2} \aleph_1(\mu, \vartheta) d\mu : \aleph_1(\mu, \vartheta) \in \mathcal{R}_{([\omega_1, \omega_2], \vartheta)} \right\}, \end{aligned}$$

for all $\vartheta \in [0, 1]$, where $\mathcal{R}_{([\omega_1, \omega_2], \vartheta)}$ denotes the collection of Riemannian integrable functions of interval-valued functions. $\aleph_1$ is FR-integrable over $[\omega_1, \omega_2]$ if $(FR) \int_{\omega_1}^{\omega_2} \aleph_{1\vartheta}(\mu) d\mu \in \mathbb{F}_0$.

Theorem 1 ([22]). Let $\aleph_1 : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{F}_0$ be a fuzzy interval-valued function, and for all $\vartheta \in [0, 1]$, ϑ -levels define the family of interval valued functions $\aleph_{1\vartheta} : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow K_c$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_2]$. Then, $\aleph_1$ is fuzzy Riemann integrable (FR-integrable) over $[\omega_1, \omega_2]$ if and only if $\aleph_{1*}(\mu, \vartheta)$ and $\aleph_1^*(\mu, \vartheta)$ both are Riemann integrable (R-integrable) over $[\omega_1, \omega_2]$. Moreover, if $\aleph_1$ is FR-integrable over $[\omega_1, \omega_2]$, then

$$\begin{aligned} \left[(FR) \int_{\omega_1}^{\omega_2} \aleph_1(\mu) d\mu \right]^\vartheta &= \left[(R) \int_{\omega_1}^{\omega_2} \aleph_{1*}(\mu, \vartheta), (R) \int_{\omega_1}^{\omega_2} \aleph_1^*(\mu, \vartheta) \right] \\ &= (IR) \int_{\omega_1}^{\omega_2} \aleph_{1\vartheta}(\mu) d\mu \end{aligned}$$

for all $\vartheta \in [0, 1]$, where IR represent interval Riemann integration of $\aleph_{1\vartheta}(\mu)$. For all $\vartheta \in [0, 1]$, $FR_{([\omega_1, \omega_2], \vartheta)}$ denotes the collection of all FR-integrable fuzzy interval-valued functions over $[\omega_1, \omega_2]$.

Remark 2 ([22]). If $\aleph_1 : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{F}_0$ is a fuzzy interval valued function, then $\aleph_1(\mu)$ is called a continuous function at $\mu \in [\omega_1, \omega_2]$ if, for each $\vartheta \in [0, 1]$, both the left and right real valued functions $\aleph_{1*}(\mu, \vartheta)$ and $\aleph_1^*(\mu, \vartheta)$ are continuous at $\mu \in [\omega_1, \omega_2]$.

Definition 4 ([25]). Let $\alpha \in [0, 1]$ and $L[\wp_1, \wp_2]$ be the collection of all Lebesgue measurable functions on $[\wp_1, \wp_2]$. Then, the fractional integral related to the new fractional derivative with a nonlocal kernel of a mapping is defined as follows:

$$\begin{aligned} {}^{AB}I_{\wp_1}^\alpha \aleph_1(\mu) &= \frac{1-\alpha}{B(\alpha)} \aleph_1(\mu) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\wp_1}^\mu \aleph_1(x)(\mu-x)^{\alpha-1} dx, \\ \text{where } \wp_1 &< \mu < \wp_2, \alpha \in [0, 1]. \end{aligned}$$

The right-hand side of the integral operator is as follows:

$${}^{AB}I_{\wp_2}^\alpha \aleph_1(\mu) = \frac{1-\alpha}{B(\alpha)} \aleph_1(\mu) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_\mu^{\wp_2} \aleph_1(x)(x-\mu)^{\alpha-1} dx.$$

Here, $\Gamma(\alpha)$ is the gamma function and $B(\alpha) > 0$ is called the normalization function, which satisfies the condition $B(0) = B(1) = 1$. For more details, see [26,27].

Now, we define the fuzzy left and right AB fractional integral based on the left and right end-point functions.

Definition 5. Let $\alpha \in [0, 1]$ and $L([\wp_1, \wp_2], \mathbb{R}_I^+)$ be the collection of all Lebesgue-measurable interval valued functions on $[\wp_1, \wp_2]$. Then, the fuzzy fractional integral related to the new fractional derivative with a nonlocal kernel of a mapping is defined as follows:

$$\begin{aligned} \left[{}^{AB}_{\wp_1} I_{\mu}^{\alpha} \aleph_1(\mu) \right]^{\vartheta} &= \frac{1-\alpha}{B(\alpha)} \aleph_{1\vartheta}(\mu) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\wp_1}^{\mu} \aleph_{1\vartheta}(x)(\mu-x)^{\alpha-1} dx \\ &= \frac{1-\alpha}{B(\alpha)} [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)] + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\wp_1}^{\mu} [\aleph_{1*}(x, \vartheta), \aleph_1^*(x, \vartheta)](\mu-x)^{\alpha-1} dx, \quad (\wp_2 > \wp_1), \end{aligned}$$

where

$${}^{AB}_{\wp_1} I_{\mu}^{\alpha} \aleph_{1*}(\mu, \vartheta) = \frac{1-\alpha}{B(\alpha)} \aleph_{1*}(\mu, \vartheta) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\wp_1}^{\mu} \aleph_{1*}(x, \vartheta)(\mu-x)^{\alpha-1} dx$$

and

$${}^{AB}_{\wp_1} I_{\mu}^{\alpha} \aleph_1^*(\mu, \vartheta) = \frac{1-\alpha}{B(\alpha)} \aleph_1^*(\mu, \vartheta) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\wp_1}^{\mu} \aleph_1^*(x, \vartheta)(\mu-x)^{\alpha-1} dx.$$

Here, $\Gamma(\alpha)$ is the gamma function and $B(\alpha) > 0$ is called the normalization function.

Similarly, the left and right end point functions can be used to define the right AB fractional integral.

Definition 6 ([20]). The fuzzy interval valued function $\aleph_1 : [\omega_1, \omega_2] \rightarrow \mathbb{F}_0$ is called an LR-convex fuzzy interval valued function on $[\omega_1, \omega_2]$ if

$$\aleph_1(\tau\mu + (1-\tau)y) \preceq_{\rho} \tau\aleph_1(\mu) \tilde{+} (1-\tau)\aleph_1(y), \quad (1)$$

for all $\mu, y \in [\omega_1, \omega_2], \tau \in [0, 1]$, where $\aleph_1(\mu) \succ \tilde{0}$ for all $\mu \in [\omega_1, \omega_2]$. If it is reversed, then $\aleph_1$ is called an LR-concave fuzzy interval valued function on $[\omega_1, \omega_2]$.

Definition 7 ([28]). Let K be a non-empty set in real Hilbert Space H . Let $\aleph_1 : K_{\varphi} \rightarrow \mathbb{R}$ be a continuous function and let $\varphi(\cdot - \cdot) : K_{\varphi} \times K_{\varphi} \rightarrow \mathbb{R}$ be an arbitrary continuous function. Then, the set K_{φ} in real Hilbert space H is said to be a bi-convex set with respect to an arbitrary bifunction $\varphi(\cdot, \cdot)$, if

$$\mu + \tau\varphi(y - \mu) \in K_{\varphi}, \quad \forall \mu, y \in K_{\varphi}, \tau \in [0, 1].$$

The biconvex set K_{φ} can be called a φ -connected set.

The main motivation of this paper is to derive generalization fractional integral inclusions involving a new class of convexity called LR-bi-convex fuzzy interval-valued functions. Our paper is organized as follows: in Section 2, we derive new fractional analogues of Hermite–Hadamard inequalities involving a new fractional integral called an LR–AB fractional integral. We discuss several special cases that demonstrate that our results are quite unifying. In order to check the validity of our outcomes, we discuss several non-trivial numerical examples. In Section 3, we present an application of our proposal to special means for positive real numbers. It is our hope that the technique presented in this paper will inspire interested readers and stimulate further research in following the same direction.

2. Main Results

In this section, we discuss our main results. First, we introduce the class of LR-bi-convex fuzzy interval-valued functions.

Definition 8. The fuzzy interval-valued function $\aleph_1 : [\omega_1, \omega_2] \rightarrow \mathbb{F}_0$ is called an LR-bi-convex fuzzy interval-valued function on $[\omega_1, \omega_2]$ if

$$\aleph_1(\mu + \tau\varphi(y - \mu)) \preceq_{\rho} \tau\aleph_1(\mu) \dot{+} (1 - \tau)\aleph_1(y), \quad (2)$$

for all $\mu, y \in [\omega_1, \omega_2], \tau \in [0, 1]$, where $\aleph_1(\mu) \succ \tilde{0}$ for all $\mu \in [\omega_1, \omega_2]$. If it is reversed, then $\aleph_1$ is called an LR-bi-concave fuzzy interval-valued function on $[\omega_1, \omega_2]$.

Remark 3. If $\varphi(y - \mu) = y - \mu$, then we have inequality (1).

If $\aleph_{1*}(\mu, \vartheta) = \aleph_1^*(\mu, \vartheta)$ with $\vartheta = 1$, then we have the definition of a classical bi-convex function. If $\aleph_{1*}(\mu, \vartheta) = \aleph_1^*(\mu, \vartheta)$ with $\varphi(y - \mu) = y - \mu$ and $\vartheta = 1$, then we have the definition of a classical convex function.

Therefore, we need the following condition to obtain new results.

Condition M.

Assume that the bifunction $\varphi(\cdot, \cdot)$ satisfies the following assumption:

$$\begin{aligned} \varphi(\gamma\varphi(y - \mu)) &= \gamma\varphi(y - \mu), \quad \forall \mu, y \in K_{\varphi}, \gamma \in \mathbb{R}. \\ \varphi(y - \mu - \gamma\varphi(y - \mu)) &= (1 - \gamma)\varphi(y - \mu), \quad \forall \mu, y \in K_{\varphi}. \end{aligned}$$

For more details regarding condition M, see [28].

Theorem 2. Let K_{φ} be a bi-convex set and $\aleph_1 : [\omega_1, \omega_2] \rightarrow \mathbb{F}_0$ be a fuzzy interval-valued function such that

$$\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)], \quad \forall \mu \in K_{\varphi}$$

for all $\mu \in K_{\varphi}$ and $\vartheta \in [0, 1]$. Then, $\aleph_1$ is an LR-bi-convex-fuzzy interval-valued function on K_{φ} if and only if $\aleph_{1*}(\mu, \vartheta)$ and $\aleph_1^*(\mu, \vartheta)$ both are fuzzy bi-convex functions.

Proof. Assume that $\aleph_{1*}, \aleph_1^*$ are fuzzy bi-convex functions. Then, for all $\mu, y \in K_{\varphi}, \tau \in [0, 1]$, we have

$$\aleph_{1*}(\mu + \tau\varphi(y - \mu)) \preceq \tau\aleph_{1*}(\mu) + (1 - \tau)\aleph_{1*}(y)$$

and

$$\aleph_1^*(\mu + \tau\varphi(y - \mu)) \preceq \tau\aleph_1^*(\mu) + (1 - \tau)\aleph_1^*(y).$$

From Definition 8 and order relation $\preceq_{\rho}$, we have

$$[\aleph_{1*}(\mu + \tau\varphi(y - \mu)), \aleph_1^*(\mu + \tau\varphi(y - \mu))] \preceq_{\rho} [\tau\aleph_{1*}(\mu), \tau\aleph_1^*(\mu)] + [(1 - \tau)\aleph_{1*}(y), (1 - \tau)\aleph_1^*(y)],$$

that is,

$$\aleph_1(\mu + \tau\varphi(y - \mu)) \preceq_{\rho} \tau\aleph_1(\mu) + (1 - \tau)\aleph_1(y), \quad \forall \mu, y \in K_{\varphi}, \tau \in [0, 1].$$

Hence, $\aleph_1$ is an LR-bi-convex fuzzy interval-valued function.

Conversely, let $\aleph_1$ be an LR-bi-convex fuzzy interval-valued function. Then, for all $\mu, y \in K_{\varphi}$ and $\tau \in [0, 1]$, we have

$$\aleph_1(\mu + \tau\varphi(y - \mu)) \preceq_{\rho} \tau\aleph_1(\mu) + (1 - \tau)\aleph_1(y),$$

that is,

$$[\aleph_{1*}(\mu + \tau\varphi(y - \mu)), \aleph_1^*(\mu + \tau\varphi(y - \mu))] \preceq_\rho [\tau\aleph_{1*}(\mu), \tau\aleph_1^*(\mu)] + [(1 - \tau)\aleph_{1*}(y), (1 - \tau)\aleph_1^*(y)],$$

thus, it follows that

$$\aleph_{1*}(\mu + \tau\varphi(y - \mu)) \preceq \tau\aleph_{1*}(\mu) + (1 - \tau)\aleph_{1*}(y)$$

and

$$\aleph_1^*(\mu + \tau\varphi(y - \mu)) \preceq \tau\aleph_1^*(\mu) + (1 - \tau)\aleph_1^*(y).$$

This completes the proof. $\square$

Now, we present the Hermite–Hadamard inequality for an LR-bi-convex fuzzy-interval valued function.

Theorem 3. Let $\aleph_1 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be an LR-bi-convex fuzzy interval-valued function on $[\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$, with its ϑ -levels defining the family of interval-valued functions $\aleph_{1\vartheta} : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \subset \mathbb{R} \rightarrow K_c^+$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)]$ and for all $\vartheta \in [0, 1]$. If φ satisfies Condition M and $\aleph_1 \in L([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$, then

$$\begin{aligned} & \aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \\ & \preceq_\rho \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \dot{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1(\omega_1) \right. \\ & \quad \left. - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1) \dot{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \} \right] \\ & \preceq_\rho \left(\frac{\aleph_1(\omega_1) \dot{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2} \right) \preceq_\rho \left(\frac{\aleph_1(\omega_1) \dot{+} \aleph_1(\omega_2)}{2} \right). \end{aligned}$$

If $\aleph_1(\mu)$ is an LR-bi-concave fuzzy interval-valued function, then

$$\begin{aligned} & \aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \\ & \succeq_\rho \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \dot{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1(\omega_1) \right. \\ & \quad \left. - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1) \dot{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \} \right] \\ & \succeq_\rho \left(\frac{\aleph_1(\omega_1) \dot{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2} \right) \succeq_\rho \left(\frac{\aleph_1(\omega_1) \dot{+} \aleph_1(\omega_2)}{2} \right). \end{aligned}$$

Proof. Let $\aleph_1 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be an LR-bi-convex fuzzy-interval valued function. If Condition M holds true, then by hypothesis, we have

$$2\aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \preceq_\rho \aleph_1(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1)) + \aleph_1(\omega_1 + \tau\varphi(\omega_2 - \omega_1)).$$

Therefore, for every $\vartheta \in [0, 1]$, we have

$$\begin{aligned} 2\aleph_{1*}\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta\right) & \leq \aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta), \\ 2\aleph_1^*\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta\right) & \leq \aleph_1^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta). \end{aligned}$$

Multiplying both sides of the above inequalities by $\tau^{\alpha-1}$ and then integrating the obtained result with respect to τ over $(0, 1)$, we have

$$\begin{aligned} 2 \int_0^1 \tau^{\alpha-1} \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) d\tau &\leq \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\ &+ \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) d\tau, \\ 2 \int_0^1 \tau^{\alpha-1} \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) d\tau &\leq \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\ &+ \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) d\tau. \end{aligned}$$

Let $\mu = \omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)$ and $y = \omega_1 + \tau\varphi(\omega_2 - \omega_1)$. Then, we have

$$\begin{aligned} \frac{2}{\alpha} \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) &\leq \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\omega_1 + \varphi(\omega_2 - \omega_1))^{\alpha-1} \aleph_{1*}(y, \vartheta) dy \\ &+ \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) d\mu, \\ \frac{2}{\alpha} \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) &\leq \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\omega_1 + \varphi(\omega_2 - \omega_1))^{\alpha-1} \aleph_1^*(y, \vartheta) dy \\ &+ \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_1^*(\mu, \vartheta) d\mu. \end{aligned}$$

By multiplying both sides of the last inequality by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$, we obtain

$$\begin{aligned} &\frac{2(\varphi(\omega_2 - \omega_1))^\alpha}{B(\alpha)\Gamma(\alpha)} \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\ &\leq {}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \\ &\quad - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \} \end{aligned}$$

and

$$\begin{aligned} &\frac{2(\varphi(\omega_2 - \omega_1))^\alpha}{B(\alpha)\Gamma(\alpha)} \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\ &\leq {}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \\ &\quad - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \}. \end{aligned}$$

That is,

$$\begin{aligned} &\left[\aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right), \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \right] \\ &\leq {}_\rho \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \right. \\ &\quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \}, \right. \\ &\quad \left. {}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \right. \\ &\quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \\ & \preceq_{\rho} \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^{\alpha}} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1(\omega_1) \right. \\ & \quad \left. \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \} \right]. \end{aligned} \quad (3)$$

In a similar way as above, we have

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^{\alpha}} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1(\omega_1) \right. \\ & \quad \left. \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \} \right] \\ & \preceq_{\rho} \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2} \preceq_{\rho} \left(\frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2)}{2} \right). \end{aligned} \quad (4)$$

Combining inequalities (3) and (4), we obtain the result. $\square$

Remark 4. From Theorem 3, we can clearly see that

If $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$, then from Theorem 3, we have the following result in fuzzy fractional calculus:

$$\aleph_1\left(\frac{\omega_1 + \omega_2}{2}\right) \preceq_{\rho} \frac{B(\alpha)\Gamma(\alpha)}{2(\omega_2 - \omega_1)^{\alpha}} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1(\omega_2) \tilde{+} {}^{AB}I_{\omega_2}^{\alpha} \aleph_1(\omega_1) \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2) \} \right] \preceq_{\rho} \left(\frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2)}{2} \right).$$

Taking $\alpha = 1$ in Theorem 3, we obtain the result for an LR-bi-convex fuzzy interval-valued function:

$$\begin{aligned} & \aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \\ & \preceq_{\rho} \frac{1}{2(\varphi(\omega_2 - \omega_1))} \left(\int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} \aleph_1(y) dy \tilde{+} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} \aleph_1(w) dw \right) \\ & \preceq_{\rho} \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2}. \end{aligned}$$

Choosing $\alpha = 1$ and $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$ in Theorem 3, we then obtain the result for an LR-convex fuzzy interval-valued function:

$$\aleph_1\left(\frac{\omega_1 + \omega_2}{2}\right) \preceq_{\rho} \frac{1}{2(\omega_2 - \omega_1)} \left(\int_{\omega_1}^{\omega_2} \aleph_1(y) dy \tilde{+} \int_{\omega_1}^{\omega_2} \aleph_1(w) dw \right) \preceq_{\rho} \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2)}{2}.$$

Example 1. Let $\alpha = 1$ and consider a fuzzy interval valued function with $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] = [1, 1 + \varphi(4 - 1)] \rightarrow \mathbb{F}_0$, where $\mathbb{F}_0$ is the family of all fuzzy intervals. Now, we can define the fuzzy interval valued function by

$$\aleph_1(\mu)(\vartheta) = \begin{cases} \frac{\vartheta - e^{\mu}}{e^{\mu}}, & \vartheta \in [e^{\mu}, 2e^{\mu}], \\ \frac{4e^{\mu} - \vartheta}{2e^{\mu}}, & \vartheta \in (2e^{\mu}, 4e^{\mu}], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\vartheta \in [0, 1]$, we have $\aleph_{1\vartheta}(\mu) = [(1 + \vartheta)e^\mu, 2(2 - \vartheta)e^\mu]$. Because the end point functions $\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)$ are bi-convex functions with respect to $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$, for each $\vartheta \in [0, 1]$, $\aleph_1(\mu)$ is bi-convex fuzzy interval-valued function. Then,

$$\begin{aligned}\aleph_{1*}\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta\right) &= \aleph_{1*}\left(\frac{5}{2}, \vartheta\right) = 12.18(1 + \vartheta), \\ \aleph_1^*\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta\right) &= \aleph_1^*\left(\frac{5}{2}, \vartheta\right) = 24.36(2 - \vartheta)\end{aligned}$$

and

$$\begin{aligned}& \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \right] \\ &= \frac{1}{(2)(3)} \left[\int_1^{1+\varphi(4-1)} (1 + \vartheta)e^\mu d\mu + \int_1^{1+\varphi(4-1)} (1 + \vartheta)e^\mu d\mu \right] \\ &= 17.29(1 + \vartheta), \\ & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \right] \\ &= \frac{1}{(2)(3)} \left[\int_1^{1+\varphi(4-1)} (2 - \vartheta)e^\mu d\mu + \int_1^{1+\varphi(4-1)} (2 - \vartheta)e^\mu d\mu \right] \\ &= 34.58(2 - \vartheta),\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2}\right) &= 28.66(1 + \vartheta), \\ \left(\frac{\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2}\right) &= 57.32(2 - \vartheta).\end{aligned}$$

Therefore,

$$[12.18(1 + \vartheta), 24.36(2 - \vartheta)] \leq_\rho [17.29(1 + \vartheta), 34.58(2 - \vartheta)] \leq_\rho [28.66(1 + \vartheta), 57.32(2 - \vartheta)].$$

Hence, Theorem 3 is verified.

Theorem 4. Let $\aleph_1 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be an LR-bi-convex fuzzy interval-valued function with $\omega_1 < \omega_2$, the ϑ -levels of which define the family of interval valued functions $\aleph_{1\vartheta} : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \subset \mathbb{R} \rightarrow K_c^+$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)]$ and for all $\vartheta \in [0, 1]$. Let $\aleph_1 \in L([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$; $\aleph_2 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{R}$, $\aleph_2(\mu) \geq 0$, and is symmetric with respect to $\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}$. If φ satisfies Condition M, then

$$\begin{aligned}& \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1 \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1 \aleph_2(\omega_1) \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1 \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1 \aleph_2(\omega_1) \} \right] \\ & \leq_\rho \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_2(\omega_1) \right. \\ & \quad \left. \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\ & \leq_\rho \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2)}{2} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_2(\omega_1) \right. \\ & \quad \left. \sim \frac{1 - \alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right].\end{aligned}$$

If $\aleph_1$ is an LR-bi-concave fuzzy interval-valued function, then the above inequalities are reversed.

Proof. Let $\aleph_1$ be an LR-bi-convex fuzzy interval-valued function. Then, for each $\vartheta \in [0, 1]$, we have

$$\begin{aligned} & \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) \\ & \leq \tau^{\alpha-1} (\tau \aleph_{1*}(\omega_1, \vartheta) + (1-\tau) \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)), \\ & \tau^{\alpha-1} \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) \\ & \leq \tau^{\alpha-1} (\tau \aleph_1^*(\omega_1, \vartheta) + (1-\tau) \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) \end{aligned} \quad (5)$$

and

$$\begin{aligned} & \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \\ & \leq \tau^{\alpha-1} (\tau \aleph_{1*}(\omega_1, \vartheta) + \tau \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)), \\ & \tau^{\alpha-1} \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \\ & \leq \tau^{\alpha-1} (\tau \aleph_1^*(\omega_1, \vartheta) + \tau \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)). \end{aligned} \quad (6)$$

After adding the above inequalities (5) and (6) and integrating with respect to τ over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) d\tau \\ & + \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\ & \leq \int_0^1 \left[\tau^{\alpha-1} \aleph_{1*}(\omega_1, \vartheta) \{ \tau \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) + (1-\tau) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \} \right. \\ & \quad \left. + \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \{ (1-\tau) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) + \tau \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \} \right] d\tau \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) d\tau \\ & + \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\ & \leq \int_0^1 \left[\tau^{\alpha-1} \aleph_1^*(\omega_1, \vartheta) \{ \tau \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) + (1-\tau) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \} \right. \\ & \quad \left. + \tau^{\alpha-1} \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \{ (1-\tau) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) + \tau \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) \} \right] d\tau. \end{aligned}$$

Because $\aleph_2$ is symmetric, we have the following successive equalities:

$$\begin{aligned}
&= [\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)] \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau, \\
&= [\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)] \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau. \\
&= \frac{\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \right. \\
&\quad \left. + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right], \\
&= \frac{\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \right. \\
&\quad \left. + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right].
\end{aligned} \tag{7}$$

Because

$$\begin{aligned}
&\int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1)) d\tau \\
&\quad + \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\
&= \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \left[\int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(2\omega_1 + \varphi(\omega_2 - \omega_1) - \mu, \vartheta) \aleph_2(\mu) d\mu \right. \\
&\quad \left. + \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(\mu) d\mu \right] \\
&= \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \left[\int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(2\omega_1 + \varphi(\omega_2 - \omega_1) - \mu) d\mu \right. \\
&\quad \left. + \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(\mu) d\mu \right].
\end{aligned}$$

By multiplying both sides of the last inequality by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$ and then adding the term $\frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1) + \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \}$, we obtain

$$\begin{aligned}
&= \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_{1*} \aleph_2(\omega_1) \right. \\
&\quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_{1*} \aleph_2(\omega_1) \} \right] \\
&= \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_1^* \aleph_2(\omega_1) \right. \\
&\quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \right].
\end{aligned} \tag{8}$$

From equalities (7) and (8), we have

$$\begin{aligned}
&{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_{1*} \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_{1*} \aleph_2(\omega_1) \} \\
&\leq \frac{\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \left[{}^{AB}_{\omega_1} I_\mu^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}_{\omega_1 + \varphi(\omega_2 - \omega_1)} I_\mu^\alpha \aleph_2(\omega_1) \right. \\
&\quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right]
\end{aligned}$$

and

$$\begin{aligned}
& {}^{AB}I_{\omega_1}^{\alpha} \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1^* \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \\
& \leq \frac{\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right].
\end{aligned}$$

That is,

$$\begin{aligned}
& \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1^* \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \}, \right. \\
& \quad \left. {}^{AB}I_{\omega_1}^{\alpha} \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1^* \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \right] \\
& \leq_{\rho} \left[\frac{\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2}, \frac{\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \right] \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \right. \\
& \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1 \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1 \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1 \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1 \aleph_2(\omega_1) \} \right] \\
& \preceq_{\rho} \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1))}{2} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \preceq_{\rho} \frac{\aleph_1(\omega_1) \tilde{+} \aleph_1(\omega_2)}{2} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right].
\end{aligned}$$

□

Example 2. Let $\alpha = 1$ and consider a fuzzy interval valued function with $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] = [1, 1 + \varphi(4 - 1)] \rightarrow \mathbb{F}_0$, defined by

$$\aleph_1(\mu)(\vartheta) = \begin{cases} \frac{\vartheta - e^{\mu}}{e^{\mu}}, & \vartheta \in [e^{\mu}, 2e^{\mu}], \\ \frac{4e^{\mu} - \vartheta}{2e^{\mu}}, & \vartheta \in (2e^{\mu}, 4e^{\mu}], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\vartheta \in [0, 1]$, we have $\aleph_{1\vartheta}(\mu) = [(1 + \vartheta)e^{\mu}, 2(2 - \vartheta)e^{\mu}]$. Because end point functions $\aleph_{1*}(\mu, \vartheta)$, $\aleph_1^*(\mu, \vartheta)$ are bi-convex functions with respect to $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$ for each $\vartheta \in [0, 1]$, $\aleph_1(\mu)$ is a bi-convex fuzzy interval-valued function. If

$$\aleph_2(\mu) = \begin{cases} \mu - 1, & \vartheta \in [1, \frac{5}{2}], \\ 4 - \mu, & \vartheta \in (\frac{5}{2}, 4]. \end{cases}$$

then $\aleph_2(2 - \mu) = \aleph_2(\mu) \geq 0$ for all $\mu \in [1, 4]$. If $\alpha = 1$, then

$$\begin{aligned}
& {}_{\omega_1}^{AB}I_{\mu}^{\alpha}\aleph_{1*}\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{AB}I_{\mu}^{\alpha}\aleph_{1*}\aleph_2(\omega_1) \\
& = 65.90(1 + \vartheta), \\
& {}_{\omega_1}^{AB}I_{\mu}^{\alpha}\aleph_1^*\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{AB}I_{\mu}^{\alpha}\aleph_1^*\aleph_2(\omega_1) \\
& = 131.81(2 - \vartheta)
\end{aligned}$$

and

$$\begin{aligned}
& \left(\frac{\aleph_{1*}(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \right) = 28.66(1 + \vartheta), \\
& \left(\frac{\aleph_1^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)}{2} \right) = 57.32(2 - \vartheta), \\
& {}_{\omega_1}^{AB}I_{\mu}^{\alpha}\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{AB}I_{\mu}^{\alpha}\aleph_2(\omega_1) = 4.5.
\end{aligned}$$

Therefore,

$$[65.90(1 + \vartheta), 131.81(2 - \vartheta)] \leq_{\rho} [128.97(1 + \vartheta), 257.94(2 - \vartheta)].$$

Hence, Theorem 4 is verified.

Theorem 5. Let $\aleph_1 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be an LR-bi-convex fuzzy interval-valued function with $\omega_1 < \omega_2$, the ϑ -levels of which define the family of interval valued functions $\aleph_{1\vartheta} : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \subset \mathbb{R} \rightarrow K_c^+$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)]$ and for all $\vartheta \in [0, 1]$. Let $\aleph_1 \in L([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$ and $\aleph_2 : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{R}$, $\aleph_2(\mu) \geq 0$, and is symmetric with respect to $\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}$. If φ satisfies Condition M, then

$$\begin{aligned}
& \aleph_1 \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2} \right) \left[{}_{\omega_1}^{AB}I_{\mu}^{\alpha}\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{AB}I_{\mu}^{\alpha}\aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \preceq_{\rho} \left[{}_{\omega_1}^{AB}I_{\mu}^{\alpha}\aleph_1\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \dot{+} {}_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{AB}I_{\mu}^{\alpha}\aleph_1\aleph_2(\omega_1) - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1\aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \dot{+} \aleph_1\aleph_2(\omega_1) \} \right].
\end{aligned}$$

If $\aleph_1$ is LR-bi-concave-fuzzy interval valued function, then above inequality is reversed.

Proof. Because $\aleph_1$ is an LR-bi-convex fuzzy interval-valued function, then for $\vartheta \in [0, 1]$, we have

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \leq \frac{1}{2} (\aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)), \\
& \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \leq \frac{1}{2} (\aleph_1^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)).
\end{aligned}$$

Because $\aleph_2(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1)) = \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1))$, by multiplying above inequalities by $\tau^{\alpha-1}\aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1))$ and integrating it with respect to τ over $[0, 1]$, we obtain

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\
& \leq \frac{1}{2} \left[\int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \right. \\
& \quad \left. + \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \right],
\end{aligned}$$

$$\begin{aligned}
& \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\
& \leq \frac{1}{2} \left[\int_0^1 \tau^{\alpha-1} \aleph_1^* (\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \right. \\
& \quad \left. + \int_0^1 \tau^{\alpha-1} \aleph_1^* (\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \right].
\end{aligned}$$

Let $\mu = \omega_1 + \tau\varphi(\omega_2 - \omega_1)$. Then, we have

$$\begin{aligned}
& \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau + \int_0^1 \tau^{\alpha-1} \aleph_2(\omega_1 + \tau\varphi(\omega_2 - \omega_1)) d\tau \\
& = \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2, \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(2\omega_1 + \varphi(\omega_2 - \omega_1) - \mu, \vartheta) \aleph_2(\mu) d\mu \\
& \quad + \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2, \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(\mu) d\mu \\
& = \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2, \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(2\omega_1 + \varphi(\omega_2 - \omega_1) - \mu) d\mu \\
& \quad + \frac{1}{(\varphi(\omega_2 - \omega_1))^\alpha} \int_{\omega_1}^{\omega_1 + \varphi(\omega_2, \omega_1)} (\mu - \omega_1)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \aleph_2(\mu) d\mu.
\end{aligned}$$

By multiplying both sides of the last inequality by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$, and then adding the term $\frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1)) \}$, we obtain

$$\begin{aligned}
& = \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*} \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_{1*} \aleph_2(\omega_1) \} \right] \\
& = \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^* \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \right].
\end{aligned}$$

Now,

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \left[{}^{AB}I_{\omega_1}^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \leq \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*} \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_{1*} \aleph_2(\omega_1) \} \right], \\
& \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \left[{}^{AB}I_{\omega_1}^\alpha \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \leq \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^* \aleph_2(\omega_1) - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \right],
\end{aligned}$$

from which we have

$$\begin{aligned}
& \left[\aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right), \aleph_{1*}^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \right] \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \right. \\
& \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \leq_{\rho} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_{1*} \aleph_2(\omega_1) - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_{1*} \aleph_2(\omega_1) \}, \right. \\
& \quad \left. {}^{AB}I_{\omega_1}^{\alpha} \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1^* \aleph_2(\omega_1) - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_1^* \aleph_2(\omega_1) \} \right],
\end{aligned}$$

that is,

$$\begin{aligned}
& \aleph_1 \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2} \right) \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) \right. \\
& \quad \left. - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + \aleph_2(\omega_1) \} \right] \\
& \preceq_{\rho} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_{1*} \aleph_2(\omega_1) \tilde{-} \frac{1 - \alpha}{B(\alpha)} \{ \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_{1*} \aleph_2(\omega_1) \} \right].
\end{aligned}$$

This completes the proof. $\square$

Remark 5. If $\aleph_2(\mu) = 1$, then from Theorem 4 and Theorem 5, we obtain Theorem 3.

Example 3. Let $\alpha = 1$ and consider a fuzzy interval-valued function with $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] = [1, 1 + \varphi(4 - 1)] \rightarrow \mathbb{F}_0$, defined by

$$\aleph_1(\mu)(\vartheta) = \begin{cases} \frac{\vartheta - e^{\mu}}{e^{\mu}}, & \vartheta \in [e^{\mu}, 2e^{\mu}], \\ \frac{4e^{\mu} - \vartheta}{2e^{\mu}}, & \vartheta \in (2e^{\mu}, 4e^{\mu}], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\vartheta \in [0, 1]$, we have $\aleph_{1\vartheta}(\mu) = [(1 + \vartheta)e^{\mu}, 2(2 - \vartheta)e^{\mu}]$. Because end point functions $\aleph_{1*}(\mu, \vartheta)$, $\aleph_{1*}^*(\mu, \vartheta)$ are bi-convex functions with respect to $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$ for each $\vartheta \in [0, 1]$, $\aleph_1(\mu)$ is bi-convex fuzzy interval-valued function. If

$$\aleph_2(\mu) = \begin{cases} \mu - 1, & \vartheta \in [1, \frac{5}{2}], \\ 4 - \mu, & \vartheta \in (\frac{5}{2}, 4], \end{cases}$$

then $\aleph_2(2 - \mu) = \aleph_2(\mu) \geq 0$ for all $\mu \in [1, 4]$. If $\alpha = 1$, then

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) = \aleph_{1*} \left(\frac{5}{2}, \vartheta \right) = 12.18(1 + \vartheta), \\
& \aleph_{1*}^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) = \aleph_{1*}^* \left(\frac{5}{2}, \vartheta \right) = 24.36(2 - \vartheta), \\
& {}^{AB}I_{\omega_1}^{\alpha} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_2(\omega_1) = 4.5
\end{aligned}$$

and

$$\begin{aligned} & {}^{AB}I_{\omega_1}^{\alpha} \aleph_{1*} \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_{1*} \aleph_2(\omega_1) \\ &= 65.90(1 + \vartheta), \\ & {}^{AB}I_{\omega_1}^{\alpha} \aleph_1^* \aleph_2(\omega_1 + \varphi(\omega_2 - \omega_1)) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1^* \aleph_2(\omega_1) \\ &= 131.81(2 - \vartheta). \end{aligned}$$

Therefore,

$$[54.81(1 + \vartheta), 108.81(2 - \vartheta)] \leq_{\rho} [65.90(1 + \vartheta), 131.81(2 - \vartheta)].$$

Hence, Theorem 5 is verified.

Theorem 6. Let $\aleph_1, Y : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be two LR-bi-convex fuzzy interval-valued functions with $\omega_1 < \omega_2$, the ϑ -levels of which define the family of interval valued functions $\aleph_{1\vartheta}, Y_{\vartheta} : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \subset \mathbb{R} \rightarrow K_c^+$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_{1*}^*(\mu, \vartheta)]$ and $Y_{\vartheta}(\mu) = [Y_*(\mu, \vartheta), Y^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)]$ and for all $\vartheta \in [0, 1]$. Let $\aleph_1 \tilde{\times} Y \in L([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$ and let φ satisfy Condition M; then,

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^{\alpha}} \left[{}^{AB}I_{\omega_1}^{\alpha} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^{\alpha} \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \right. \\ & \quad \left. - \frac{1 - \alpha}{B(\alpha)} \{ \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1(\omega_1) \tilde{\times} Y_*(\omega_1) \} \right] \\ & \preceq_{\rho} \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \left(\frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \end{aligned}$$

where

$$\begin{aligned} \Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) &:= \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)), \\ \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) &:= \aleph_1(\omega_1) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1), \\ \Delta_{\vartheta}(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) &:= [\Delta_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta), \Delta^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)], \\ \nabla_{\vartheta}(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) &:= [\nabla_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta), \nabla^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)]. \end{aligned}$$

Proof. Because $\aleph_1, Y$ are both LR-bi-convex fuzzy interval-valued functions and Condition M holds for φ , for each $\vartheta \in [0, 1]$ we have

$$\begin{aligned} \aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) &\leq \tau \aleph_{1*}(\omega_1, \vartheta) + (1 - \tau) \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta), \\ \aleph_{1*}^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) &\leq \tau \aleph_{1*}^*(\omega_1, \vartheta) + (1 - \tau) \aleph_{1*}^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \end{aligned}$$

and

$$\begin{aligned} Y_*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) &\leq \tau Y_*(\omega_1, \vartheta) + (1 - \tau) Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta), \\ Y^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) &\leq \tau Y^*(\omega_1, \vartheta) + (1 - \tau) Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta). \end{aligned}$$

From the definition of LR-bi-convex fuzzy interval-valued functions, it follows that $\tilde{0} \preceq \aleph_1(\mu)$ and $\tilde{0} \preceq Y(\mu)$, thus,

$$\begin{aligned}
& \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq (\tau \aleph_{1*}(\omega_1, \vartheta) + (1-\tau)\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta))(\tau Y_*(\omega_1, \vartheta) + (1-\tau)Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \\
& = \tau^2 \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) + (1-\tau)^2 \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \tau(1-\tau)\aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \tau(1-\tau)\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1, \vartheta), \quad (9) \\
& \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq (\tau \aleph_1^*(\omega_1, \vartheta) + (1-\tau)\aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta))(\tau Y^*(\omega_1, \vartheta) + (1-\tau)Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \\
& = \tau^2 \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) + (1-\tau)^2 \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \tau(1-\tau)\aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \tau(1-\tau)\aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1, \vartheta).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
& \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq (1-\tau)^2 \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) + \tau^2 \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \tau(1-\tau)\aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \tau(1-\tau)\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1, \vartheta), \quad (10) \\
& \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq (1-\tau)^2 \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) + \tau^2 \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \tau(1-\tau)\aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \tau(1-\tau)\aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1, \vartheta).
\end{aligned}$$

Adding the above inequalities (9) and (10), we have

$$\begin{aligned}
& \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq [\tau^2 + (1-\tau)^2][\aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + 2\tau(1-\tau)[\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1, \vartheta) + \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)], \\
& \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \\
& \leq [\tau^2 + (1-\tau)^2][\aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + 2\tau(1-\tau)[\aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)].
\end{aligned}$$

By multiplying the above inequality by $\tau^{\alpha-1}$ and integrating the obtained result with respect to τ over $(0, 1)$, we have

$$\begin{aligned}
& \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\
& \quad + \int_0^1 \tau^{\alpha-1} \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\
& \leq \Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \int_0^1 \tau^{\alpha-1} [\tau^2 + (1-\tau)^2] d\tau \\
& \quad + 2\nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \int_0^1 \tau^{\alpha-1} \tau(1-\tau) d\tau
\end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1-\tau)\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\ & + \int_0^1 \tau^{\alpha-1} \aleph_1^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) d\tau \\ & \leq \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \int_0^1 \tau^{\alpha-1} [\tau^2 + (1-\tau)^2] d\tau \\ & + 2\nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \int_0^1 \tau^{\alpha-1} \tau(1-\tau) d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\ & + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \\ & \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \} \right] \\ & \leq \frac{2}{\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\ & + \frac{2}{\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \end{aligned}$$

and

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\ & + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \\ & \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \} \right] \\ & \leq \frac{2}{\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\ & + \frac{2}{\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta). \end{aligned}$$

That is,

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\ & + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \\ & - \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \}, \\ & {}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \\ & \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \} \right] \\ & \leq_\rho \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [\Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta), \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta)] \\ & + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [\nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta), \nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta)]. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_\mu^\alpha \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \right. \\ & \quad \left. - \frac{1-\alpha}{B(\alpha)} \{ \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1(\omega_1) \tilde{\times} Y_*(\omega_1) \} \right. \\ & \quad \left. \preceq_\rho \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)). \right. \end{aligned}$$

This completes the proof. $\square$

Example 4. Let $\alpha = 1$ and consider a fuzzy interval-valued function with $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] = [0, 1 + \varphi(2 - 0)] \rightarrow \mathbb{F}_0$, and define $\aleph_1(\mu) = [\mu, 2\mu]$, and $Y(\mu) = [\mu, 3\mu]$ by

$$\aleph_1(\mu)(\vartheta) = \begin{cases} \frac{\vartheta}{\mu}, & \vartheta \in [0, \mu], \\ \frac{2\mu - \vartheta}{\mu}, & \vartheta \in (\mu, 2\mu], \\ 0, & \text{otherwise,} \end{cases}$$

$$Y(\mu)(\vartheta) = \begin{cases} \frac{\vartheta}{2\mu}, & \vartheta \in [0, 2\mu], \\ \frac{4\mu - \vartheta}{2\mu}, & \vartheta \in (2\mu, 4\mu], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\vartheta \in [0, 1]$, we have $\aleph_{1\vartheta}(\mu) = [\vartheta\mu, (2 - \vartheta)\mu]$ and $Y_\vartheta(\mu) = [2\vartheta\mu, 2(2 - \vartheta)\mu]$. Because left and right end point functions $\aleph_{1*}(\mu, \vartheta) = \vartheta\mu$, $\aleph_1^*(\mu, \vartheta) = (2 - \vartheta)\mu$ and $Y_*(\mu, \vartheta) = 2\vartheta\mu$, $Y^*(\mu, \vartheta) = 2(2 - \vartheta)\mu$ are bi-convex functions with respect to $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$ for each $\vartheta \in [0, 1]$, $\aleph_1(\mu)$ and $Y(\mu)$ are bi-convex fuzzy interval-valued functions. Then,

$$\begin{aligned} & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_\mu^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\ & \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \right] = 2.67\vartheta^2, \\ & \frac{B(\alpha)\Gamma(\alpha)}{2(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_\mu^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\ & \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \right] = 2.67(2 - \vartheta)^2 \end{aligned}$$

and

$$\begin{aligned} \Delta_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) &= \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\ &= \aleph_{1*}(0, \vartheta) \times Y_*(0, \vartheta) + \aleph_{1*}(0 + \varphi(2 - 0), \vartheta) \times Y_*(0 + \varphi(2 - 0), \vartheta) \\ &= 8\vartheta^2, \\ \Delta^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) &= \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\ &= \aleph_1^*(0, \vartheta) \times Y^*(0, \vartheta) + \aleph_1^*(0 + \varphi(2 - 0), \vartheta) \times Y^*(0 + \varphi(2 - 0), \vartheta) \\ &= 8(2 - \vartheta)^2, \end{aligned}$$

$$\begin{aligned}
\nabla_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) &= \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1, \vartheta) \\
&= \aleph_{1*}(0, \vartheta) \times Y_*(0 + \varphi(2 - 0), \vartheta) + \aleph_{1*}(0 + \varphi(2 - 0), \vartheta) \times Y_*(0, \vartheta) \\
&= 0, \\
\nabla^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) &= \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1, \vartheta) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) \Delta_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right) \nabla_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
&= \frac{1}{3}(8\vartheta^2) + \frac{1}{6}(0) = 2.67\vartheta^2, \\
&\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) \Delta^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right) \nabla^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
&= \frac{1}{3}(8(2-\vartheta)^2) + \frac{1}{6}(0) = 2.67(2-\vartheta)^2.
\end{aligned}$$

Thus,

$$[2.67\vartheta^2, 2.67(2-\vartheta)^2] \leq_\rho [2.67\vartheta^2, 2.67(2-\vartheta)^2].$$

Hence, Theorem 6 is verified.

Theorem 7. Let $\aleph_1, Y : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ be two LR-bi-convex fuzzy interval-valued function with $\omega_1 < \omega_2$, the ϑ -levels of which define the family of interval valued functions $\aleph_{1\vartheta}, Y_\vartheta : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \subset \mathbb{R} \rightarrow K_c^+$ provided by $\aleph_{1\vartheta}(\mu) = [\aleph_{1*}(\mu, \vartheta), \aleph_1^*(\mu, \vartheta)]$ and $Y_\vartheta(\mu) = [Y_*(\mu, \vartheta), Y^*(\mu, \vartheta)]$ for all $\mu \in [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)]$ and for all $\vartheta \in [0, 1]$. Let $\aleph_1 \tilde{\times} Y \in L([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$ and let φ satisfy Condition M; then,

$$\begin{aligned}
&\aleph_1\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \tilde{\times} Y\left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}\right) \\
&\preceq_\rho \frac{B(\alpha)\Gamma(\alpha)}{4(\varphi(\omega_2 - \omega_1)^\alpha)} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \right] \\
&\quad \simeq \frac{1-\alpha}{B(\alpha)} \{ \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \} \\
&\quad \tilde{+} \frac{1}{2} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \frac{1}{2} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)),
\end{aligned}$$

where $\Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \Delta_\vartheta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \nabla_\vartheta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1))$ are defined as in Theorem 6.

Proof. Consider $\aleph_1, Y : [\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)] \rightarrow \mathbb{F}_0$ are LR-bi-convex fuzzy interval-valued functions. Then, by hypothesis, for each $\vartheta \in [0, 1]$ we have

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y_* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\
& \leq \frac{1}{4} [\aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) Y_*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [\aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) Y_*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)], \\
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\
& \leq \frac{1}{4} [\aleph_{1*}^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) Y^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [\aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) Y^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)]. \\
& \leq \frac{1}{4} [\aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [(\tau\aleph_{1*}(\omega_1, \vartheta) + (1 - \tau)\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \times ((1 - \tau)Y_*(\omega_1, \vartheta) + \tau Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \\
& \quad + ((1 - \tau)\aleph_{1*}(\omega_1, \vartheta) + \tau\aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \times (\tau Y_*(\omega_1, \vartheta) + (1 - \tau)Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta))]. \\
& \leq \frac{1}{4} [\aleph_{1*}^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times \aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [(\tau\aleph_{1*}^*(\omega_1, \vartheta) + (1 - \tau)\aleph_{1*}^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \times ((1 - \tau)Y^*(\omega_1, \vartheta) + \tau Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \\
& \quad + ((1 - \tau)\aleph_{1*}^*(\omega_1, \vartheta) + \tau\aleph_{1*}^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta)) \times (\tau Y^*(\omega_1, \vartheta) + (1 - \tau)Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta))]. \\
& = \frac{1}{4} [\aleph_{1*}(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times \aleph_{1*}(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [\tau^2 + (1 - \tau)^2] \nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) + [\tau(1 - \tau) + (1 - \tau)\tau] \Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta). \\
& = \frac{1}{4} [\aleph_{1*}^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + (1 - \tau)\varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad + \aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta) \times \aleph_{1*}^*(\omega_1 + \tau\varphi(\omega_2 - \omega_1), \vartheta)] \\
& \quad + \frac{1}{4} [\tau^2 + (1 - \tau)^2] \nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) + [\tau(1 - \tau) + (1 - \tau)\tau] \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta).
\end{aligned}$$

By multiplying the last inequalities by $\tau^{\alpha-1}$ and integrating over $(0, 1)$, we obtain

$$\begin{aligned}
& \frac{1}{\alpha} \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y_* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\
& \leq \frac{1}{4(\varphi(\omega_2 - \omega_1))^\alpha} \left[\int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\omega_1 + \varphi(\omega_2 - \omega_1) - \mu)^{\alpha-1} \aleph_{1*}(\mu, \vartheta) \times Y_*(\mu, \vartheta) d\mu \right. \\
& \quad + \left. \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (y - \omega_1)^{\alpha-1} \aleph_{1*}(y, \vartheta) \times Y(y, \vartheta) dy \right] \\
& \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& \quad + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& = \frac{B(\alpha)\Gamma(\alpha)}{4\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\
& \quad + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \\
& \quad - \left. \frac{1-\alpha}{B(\alpha)} \{ \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \} \right] \\
& \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& \quad + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta_*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\alpha} \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \\
& \leq \frac{1}{4(\varphi(\omega_2 - \omega_1))^\alpha} \left[\int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (\omega_1 + \varphi(\omega_2 - \omega_1) - \mu)^{\alpha-1} \aleph_1^*(\mu, \vartheta) \times Y^*(\mu, \vartheta) d\mu \right. \\
& \quad + \left. \int_{\omega_1}^{\omega_1 + \varphi(\omega_2 - \omega_1)} (y - \omega_1)^{\alpha-1} \aleph_1^*(y, \vartheta) \times Y(y, \vartheta) dy \right] \\
& \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& \quad + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& = \frac{B(\alpha)\Gamma(\alpha)}{4\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\
& \quad + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \\
& \quad - \left. \frac{1-\alpha}{B(\alpha)} \{ \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \} \right] \\
& \quad + \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta) \\
& \quad + \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta^*((\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)), \vartheta).
\end{aligned}$$

That is,

$$\begin{aligned}
& \frac{1}{\alpha} \aleph_1 \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2} \right) \tilde{\times} Y \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2} \right) \\
& \preccurlyeq \frac{B(\alpha)\Gamma(\alpha)}{4\alpha(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \right] \\
& \simeq \frac{1-\alpha}{B(\alpha)} \{ \aleph_1(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{\times} Y(\omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \aleph_1(\omega_1) \tilde{\times} Y(\omega_1) \} \\
& \tilde{+} \frac{1}{2\alpha} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)) \tilde{+} \frac{1}{2\alpha} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)).
\end{aligned}$$

This completes the proof. $\square$

Example 5. Under the assumptions of Example 4, we have

$$\begin{aligned}
& \aleph_{1*} \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y_* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) = 2\vartheta^2, \\
& \aleph_1^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) \times Y^* \left(\frac{2\omega_1 + \varphi(\omega_2 - \omega_1)}{2}, \vartheta \right) = 2(2 - \vartheta)^2
\end{aligned}$$

and

$$\begin{aligned}
& \frac{B(\alpha)\Gamma(\alpha)}{4(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\
& \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) \right] = 1.335\vartheta^2, \\
& \frac{B(\alpha)\Gamma(\alpha)}{4(\varphi(\omega_2 - \omega_1))^\alpha} \left[{}^{AB}I_{\omega_1}^\alpha \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \right. \\
& \quad \left. + {}^{AB}I_{\omega_1 + \varphi(\omega_2 - \omega_1)}^\alpha \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) \right] = 1.335(2 - \vartheta)^2,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) = \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1, \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad = 8\vartheta^2, \\
& \Delta^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) = \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1, \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad = 8(2 - \vartheta)^2, \\
& \nabla_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) = \aleph_{1*}(\omega_1, \vartheta) \times Y_*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_{1*}(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y_*(\omega_1, \vartheta) \\
& \quad = 0, \\
& \nabla^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) = \aleph_1^*(\omega_1, \vartheta) \times Y^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \aleph_1^*(\omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \times Y^*(\omega_1, \vartheta) \\
& \quad = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad = \frac{1}{3}(8\vartheta^2) + \frac{1}{6}(0) = 2.67\vartheta^2, \\
& \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \Delta^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) + \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*(\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1), \vartheta) \\
& \quad = \frac{1}{3}(8(2 - \vartheta)^2) + \frac{1}{6}(0) = 2.67(2 - \vartheta)^2.
\end{aligned}$$

Therefore,

$$[2.67\theta^2, 2.67(2-\theta)^2] \leq_p [2.67\theta^2, 2.67(2-\theta)^2].$$

Hence, Theorem 7 is verified.

Remark 6. If we take $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$, $\Theta = 1 = \alpha$ and $\Lambda_*(\mu, \theta) = \Lambda^*(\mu, \theta)$ in our results, then we obtain classical results.

3. Applications to Special Means

The following notations are used for special means of two non-negative numbers ω_1, ω_2 with $\omega_1 < \omega_2$

1. The arithmetic mean $\mathcal{A} := \mathcal{A}(\omega_1, \omega_2) = \frac{\omega_1 + \omega_2}{2}$
2. The p -logarithmic mean $\mathcal{L}_p := \mathcal{L}_p(\omega_1, \omega_2) = \left(\frac{\omega_2^{p+1} - \omega_1^{p+1}}{(p+1)(\omega_2 - \omega_1)} \right)^{\frac{1}{p}}$, $p \in \mathbb{R} \setminus \{-1, 0\}$

Proposition 2. Because $\Lambda \in \mathcal{L}([\omega_1, \omega_1 + \varphi(\omega_2 - \omega_1)], \mathbb{F}_0)$, then

$$e^{\mathcal{A}(\omega_1, \omega_2)} \preceq_p \mathcal{L}(e^{\omega_1}, e^{\omega_2}) \preceq_p \mathcal{A}(e^{\omega_1}, e^{\omega_2}).$$

Proof. The assertion follows from Theorem 3 for $\alpha = 1$ and the fuzzy interval valued function $\Lambda_\theta(\mu) = [(1+\theta)e^\mu, 2(2-\theta)e^\mu]$ for each $\theta \in [0, 1]$. End point functions $\Lambda_*(\mu, \theta), \Lambda^*(\mu, \theta)$ are bi-convex functions with respect to $\varphi(\omega_2 - \omega_1) = \omega_2 - \omega_1$ for each $\theta \in [0, 1]$. We have

$$\Lambda\left(\frac{\omega_1 + \omega_2}{2}\right) \preceq_p \frac{1}{2(\omega_2 - \omega_1)} \left(\int_{\omega_1}^{\omega_2} \Lambda(y) dy \tilde{+} \int_{\omega_1}^{\omega_2} \Lambda(w) dw \right) \preceq_p \frac{\Lambda(\omega_1) \tilde{+} \Lambda(\omega_2)}{2},$$

then

$$\begin{aligned} & \left[(1+\theta)e^{\frac{\omega_1 + \omega_2}{2}}, 2(2-\theta)e^{\frac{\omega_1 + \omega_2}{2}} \right] \\ & \leq \frac{1}{\omega_2 - \omega_1} [(1+\theta)(e^{\omega_2} - e^{\omega_1}), 2(2-\theta)(e^{\omega_2} - e^{\omega_1})] \\ & \leq \left[\frac{(1+\theta)(e^{\omega_1} + e^{\omega_2})}{2}, \frac{2(2-\theta)(e^{\omega_1} + e^{\omega_2})}{2} \right] \end{aligned}$$

implying that

$$\begin{aligned} & [(1+\theta)e^{\mathcal{A}(\omega_1, \omega_2)}, 2(2-\theta)e^{\mathcal{A}(\omega_1, \omega_2)}] \\ & \leq [(1+\theta)\mathcal{L}(e^{\omega_1}, e^{\omega_2}), 2(2-\theta)\mathcal{L}(e^{\omega_1}, e^{\omega_2})] \\ & \leq [(1+\theta)\mathcal{A}(e^{\omega_1}, e^{\omega_2}), 2(2-\theta)\mathcal{A}(e^{\omega_1}, e^{\omega_2})]. \end{aligned}$$

Thus,

$$e^{\mathcal{A}(\omega_1, \omega_2)} \preceq_p \mathcal{L}(e^{\omega_1}, e^{\omega_2}) \preceq_p \mathcal{A}(e^{\omega_1}, e^{\omega_2}).$$

□

4. Conclusions

In this paper, we have introduced a new class of convexity called LR-bi-convex fuzzy interval-valued functions. We studied this class from the perspective of fractional Hermite–Hadamard inequalities, involving a new fractional integral called the LR–AB fractional integral. We discussed several special cases, demonstrating that our results are quite unifying. The novelty of our work is that we have derived new fractional Hermite–Hadamard and Fejér-type fractional Hermite–Hadamard inequalities by introducing the

concept of a new generalized class of convexity called LR-bi-convex fuzzy interval-valued functions, fuzzy interval valued-functions, and fuzzy AB-fractional integrals based on left and right end points. Moreover, we provide several non-trivial numerical examples to check the validity of our outcomes. We would like to mention here that the AB-fractional derivatives were introduced by Atangana and Baleanu to overcome the deficiency of Caputo–Fabrizio fractional derivatives in terms of not meeting initial conditions at $\alpha = 1$, making AB-fractional derivatives and integrals quite different from classical Riemann–Liouville fractional operators. Because the class of fuzzy interval-valued functions has large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, related optimization theory, and mathematical inequalities, and may stimulate further research in different areas of both pure and applied sciences. In future, we shall try to obtain new variants of Hermite–Hadamard-type inequalities using a modified form of the AB-fractional operator involving different kernels. Studies relating convexity and preinvex functions (as contractive operators) may have useful applications in complex interdisciplinary studies, including maximizing likelihood from multiple linear regressions involving Gauss–Laplace distributions. For more details, see [29–33].

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